Element-by-element construction of wavelets satisfying stability and moment conditions

Report No. 9725, Department of Mathematics, University of Nijmegen, December 1997, submitted to *SIAM J. Numer. Anal.*

ELEMENT-BY-ELEMENT CONSTRUCTION OF WAVELETS SATISFYING STABILITY AND MOMENT CONDITIONS

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ABSTRACT. In this paper, we construct a class of locally supported wavelet bases for C^0 Lagrange finite element spaces on possibly non-uniform meshes on n-dimensional domains or manifolds. The wavelet bases are stable in the Sobolev spaces $\mathcal{H}^s$ for $|s| < \frac{3}{2}$ ($|s| \leq 1$ on Lipschitz' manifolds), and the wavelets can, in principal, be arranged to have any desired order of vanishing moments. As a consequence, these bases can be used e.g. for constructing an optimal solver of discretized $\mathcal{H}^s$-elliptic problems for s in above ranges.

The construction of the wavelets consists of two parts: An implicit part involves some computations on a reference element which, for each type of finite element space, have to be performed only once. In addition there is an explicit part which takes care of the necessary adaptations of the wavelets to the actual mesh. The only condition we need for this construction to work is that the refinements of initial elements are uniform.

We will show that the wavelet bases can be implemented efficiently.

1. INTRODUCTION

This paper is concerned with the construction of *finite element* based wavelet bases with respect to arbitrary initial triangulations. This introductory section is devoted to a brief summary of relevant background information which, in particular, motivates specific requirements on the wavelet bases concerning stability in Sobolev spaces and moment conditions.

1.1. Motivation and background.

Let us denote by $\mathcal{H}^s$, $s \in \mathbb{R}$ (or $|s| \leq t$) a scale of Sobolev spaces on an n-dimensional domain or sufficiently smooth manifold. When $s < 0$ the space $\mathcal{H}^s$ is understood to be the *dual* of $\mathcal{H}^{-s}$ (whose precise structure depends, of course, also on the boundary conditions incorporated in $\mathcal{H}^s$). Consider the variational problem: Given $f \in \mathcal{H}^{-r}$, find $u \in \mathcal{H}^r$ such that

$$a(u,v) = f(v) \qquad (v \in \mathcal{H}^r), \tag{1.1}$$

where a is a scalar product satisfying

$$a(v,v) \eqsim \|v\|^2_{\mathcal{H}^r},$$

i.e., the problem (1.1) is symmetric and *elliptic of order* $2r$. In order to avoid the repeated use of generic but unspecified constants, by $C \lesssim D$ we mean that C can be

Date: November 25, 1997.

1991 *Mathematics Subject Classification.* 65N30, 65N55, 65F35, 65R20.

Key words and phrases. wavelets, stable biorthogonal space decompositions, operator equations, vanishing moments, matrix compression, optimal multi-level preconditioners, non-uniform meshes.

bounded by a multiple of D, independently of parameters on which C and D may depend. Obviously, $C \gtrsim D$ is defined as $D \lesssim C$, and $C \eqsim D$ as $C \lesssim D$ and $C \gtrsim D$.

As typical model examples for (1.1) we have in mind (variational formulations of)

- the *differential equation* $-\nabla \cdot B\nabla u + cu = f$ on a domain Ω, where $B(x) \eqsim I$ and $0 \leq c(x) \lesssim 1$, supplemented with suitable boundary conditions ($r = 1$),
- reformulations of Laplace's equation on Ω or $\mathbb{R}^n \backslash \Omega$ as an *integral equation* on $\partial\Omega$, like the single layer potential equation ($r = -\frac{1}{2}$), the hypersingular equation ($r = \frac{1}{2}$), or the double layer potential equation ($r = 0$).

Suppose we are given a sequence of nested closed subspaces, also called a *multiresolution analysis*,

$$\mathcal{S}_0 \subset \mathcal{S}_1 \subset \cdots \mathcal{S}_j \subset \cdots \subset \mathcal{H}^r \cap L_2.$$

Then the Galerkin discretization of (1.1) reads as follows: Find $u_j \in \mathcal{S}_j$ such that

$$a(u_j, v_j) = f(v_j) \qquad (v_j \in \mathcal{S}_j). \tag{1.2}$$

By fixing a basis $\{\varphi_{j,x} : x \in I_j\}$ of $\mathcal{S}_j \subset \mathcal{H}^r$, (1.2) leads to a linear system of equations

$$\mathbf{A}_j \mathbf{u}_j = \mathbf{f}_j. \tag{1.3}$$

The size of the matrix $\mathbf{A}_j$ in realistic applications often excludes the use of *direct solvers*. In order to solve (1.3) iteratively in an efficient or rather optimal way, the following two questions are relevant:

- Can we select $\{\varphi_{j,x} : x \in I_j\}$ such that $\mathbf{A}_j$, preconditioned by its diagonal, is well-conditioned, i.e. the spectral condition number is bounded uniformly as a function of j. It is easily verified that this property is equivalent to the uniform $\mathcal{H}^r$*-stability* of the bases, defined as

$$\|\sum_{x\in I_j} c_{j,x}\varphi_{j,x}\|^2_{\mathcal{H}^r} \eqsim \sum_{x\in I_j} |c_{j,x}|^2 \|\varphi_{j,x}\|^2_{\mathcal{H}^r}. \tag{1.4}$$

  (For convenience, in the sequel we often refer to the property (1.4) by saying that $\{\varphi_{j,x} : x \in I_j\}$ is an $\mathcal{H}^r$-stable basis of $\mathcal{S}_j$, where we thus mean the uniform $\mathcal{H}^r$-stability of the *sequence* of bases $\{\varphi_{j,x} : x \in I_j\}$ for $j = 0, 1, \cdots$).

Secondly, considering integral equations which generally lead to dense matrices:

- Can we select $\{\varphi_{j,x} : x \in I_j\}$ so that it is possible to find a well-conditioned and sparse approximation of $\mathbf{A}_j$, meaning that $\mathbf{A}_j$ has only $\mathcal{O}(\dim\mathcal{S}_j)$ nonzero entries, such that the resulting approximate solution has, as function of j, the same order of convergence as the exact Galerkin solution. This process of finding a sparse, accurate approximation is usually called *matrix compression*.

In a sequence of papers (see [Dah97] and the references cited there), it was shown that both questions concerning stability and compression can be answered affirmatively when suitable wavelet bases are employed. Let us briefly point out what is meant by *suitable wavelet bases* in this context. Suppose that $\mathcal{W}_{k+1} \subset \mathcal{S}_{k+1}$ is chosen such that

$$\mathcal{S}_{k+1} = \mathcal{W}_{k+1} \oplus \mathcal{S}_k$$

and let $\Psi_{k+1} = \{\psi_{k+1,x} : x \in J_{k+1}\}$ be a basis of $\mathcal{W}_{k+1} \subset L_2$. For convenience let us also set $\mathcal{W}_0 = \mathcal{S}_0$ equipped with some basis Ψ_0. This gives rise to the direct sum decomposition

$$\mathcal{S}_j = \oplus_{k=0}^{j} \mathcal{W}_k,$$

which is sometimes called a *multiscale decomposition*. Consequently, $\Psi^j := \cup_{k=0}^{j} \Psi_k$ is a basis of $\mathcal{S}_j \subset L_2$, called *multiscale-*, or *wavelet* basis.

Aiming at computationally optimal implementations, we will always assume the availability of L_2-stable *single-scale* bases $\Phi_k = \{\phi_{k,x} : x \in I_k\}$ of the spaces $\mathcal{S}_k$, which are *locally finite*, i.e., $\sup_{x \in I_k} \#\{y \in I_k : \mathrm{supp}\phi_{k,x} \cap \mathrm{supp}\phi_{k,y} \neq \emptyset\} \lesssim 1$. Viewing bases as (column) vectors whose components are the basis functions, the nesting $\mathcal{S}_k \subset \mathcal{S}_{k+1}$ implies the existence of a refinement relation

$$\Phi_k^T = \Phi_{k+1}^T \mathbf{P}_{k+1,0}. \tag{1.5}$$

Here the xth column of the $\#I_{k+1} \times \#I_k$ matrix $\mathbf{P}_{k+1,0}$ contains the coefficients for the linear combination of $\phi_{k,x}$ in terms of the $\phi_{k+1,y}$. We will assume that Φ_k is *local with respect to* Φ_{k+1}, with which we mean that the rows and columns of $\mathbf{P}_{k+1,0}$ have a uniformly bounded number of non-vanishing entries. This situation is encountered for any standard finite element discretization based on nested partitions. Clearly the matrix $\mathbf{P}_{k+1,0}$ represents a *prolongation operator* in multigrid terminology.

Given the single-scale bases Φ_k, we put $\Psi_0 = \Phi_0$ and search for complement bases Ψ_{k+1} , such that Ψ_{k+1} is local with respect to Φ_{k+1}. In terms of two-scale relations this means that we have to find $\#I_{k+1} \times \#J_{k+1}$ matrices $\mathbf{P}_{k+1,1}$ which are *sparse* in the above sense so that

$$\Psi_{k+1}^T = \Phi_{k+1}^T \mathbf{P}_{k+1,1}. \tag{1.6}$$

In that case $\Psi_{k+1} \cup \Phi_k$ is also local with respect to Φ_{k+1}. The fact that $\Phi_k \cup \Psi_{k+1}$ is also a basis for $\mathcal{S}_k$ is equivalent to saying that the matrix $\mathbf{P}_{k+1} := [\mathbf{P}_{k+1,0}, \mathbf{P}_{k+1,1}]$ is invertible. The matrices $\mathbf{P}_{k+1}, k = 0, \ldots, j-1$ are the core ingredients of the basis transformation $\mathbf{T}_j$ in $\mathcal{S}_j$ that takes the multiscale coefficients of an element in $\mathcal{S}_j$ into its single-scale coefficients. It can be implemented recursively from bottom-to-top as a pyramid scheme which, on account of the sparseness of the $\mathbf{P}_{k+1}$, requires $\mathcal{O}(\dim \mathcal{S}_j)$ operations.

The question of $\mathcal{H}^r$ -stability of wavelet bases can be separated into two issues:

Remark 1.1. *If*

$$\Psi_k \text{ is an } L_2\text{-stable basis of } \mathcal{W}_k. \tag{1.7}$$

and

$$\|\sum_{k=0}^{\infty} v_k\|_{\mathcal{H}^r}^2 \eqsim \sum_{k=0}^{\infty} \lambda_{k,r} \|v_k\|_{L_2}^2 \qquad (v_k \in \mathcal{W}_k) \tag{1.8}$$

for some arbitrary constants $\lambda_{k,r}$, then $\Psi^j = \cup_{k=0}^{j} \Psi_k$ is an $\mathcal{H}^r$-stable basis of $\mathcal{S}_j$.

Indeed, from (1.8) and (1.7), we infer

$$\begin{aligned}\|\sum_{k,x} c_{k,x}\psi_{k,x}\|^2_{\mathcal{H}^r} &\eqsim \sum_k \lambda_{k,r}\|\sum_x c_{k,x}\psi_{k,x}\|^2_{L_2} \eqsim \sum_{k,x}|c_{k,x}|^2\lambda_{k,r}\|\psi_{k,x}\|^2_{L_2}\\ &\eqsim \sum_{k,x}|c_{k,x}|^2\|\psi_{k,x}\|^2_{H^r}.\end{aligned}$$

The stability (1.7) on each refinement level is, in principal, relatively easy to check. For example, assuming a normalization such that $\|\phi_{k,y}\|_{L_2}$, $\|\psi_{k,x}\|_{L_2} \eqsim 1$ (uniformly in k, $x \in I_k$, $y \in J_k$), the L_2-stability of the basis $\Phi_{k-1} \cup \Psi_k$ is equivalent to the fact that

$$\|\mathbf{P}_k\|,\ \|\mathbf{P}_k^{-1}\| \eqsim 1, \tag{1.9}$$

where $\|\cdot\|$ denotes the spectral norm.

By contrast, it is usually much less apparent how to assert the stability (1.8) across all levels. In [Dah96], it was proved that for (1.8), it is sufficient that $\mathcal{W}_{k+1} = (\mathcal{S}'_k)^{\perp_{L_2}} \cap \mathcal{S}_{k+1}$, where

$$\mathcal{S}'_0 \subset \mathcal{S}'_1 \subset \cdots \mathcal{S}'_j \subset \cdots \subset L_2,$$

is a "dual" sequence of nested closed subspaces, such that both $(\mathcal{S}_k)_k$ as $(\mathcal{S}'_k)_k$ satisfy certain *direct* (or Jackson) and *inverse* (or Bernstein) estimates with respect to suitable Sobolev norms. The precise formulation and a new proof of this statement can be found in Section 2 and Appendix A respectively. In this case, we also have $\mathcal{S}'_{k+1} = \mathcal{W}'_{k+1} \oplus \mathcal{S}'_k$, where $\mathcal{W}'_{k+1} = (\mathcal{S}_k)^{\perp_{L_2}} \cap \mathcal{S}'_{k+1}$. Since therefore

$$\mathcal{S}_k \perp_{L_2} \mathcal{W}'_{k+1},\ \mathcal{S}'_k \perp_{L_2} \mathcal{W}_{k+1}, \tag{1.10}$$

and so $\mathcal{W}'_\ell \perp_{L_2} \mathcal{W}_k$ $(\ell \neq k)$, the space decompositions $\oplus_k \mathcal{W}_k$ and $\oplus_k \mathcal{W}'_k$ are called *biorthogonal space decompositions*.

Now we turn to the question of matrix compression. Using the Bramble-Hilbert lemma, aforementioned direct estimates are usually enforced by demanding that, relative to their underlying meshes, $\mathcal{S}_k$ and $\mathcal{S}'_k$ contain all piecewise polynomials, possibly satisfying some global smoothness conditions, of sufficiently high degree, say degree $d-1$, and $d'-1$, respectively. As a consequence, then the wavelets are L_2-orthogonal to all polynomials of degree less than d', or the wavelets are said to have *vanishing moments* of order d'. As first observed in [BCR91], in case of an integral operator it is this property that ensures that the stiffness matrix with respect to the wavelet basis is close to a sparse one.

For this biorthogonal setting, at the end of a sequence of papers ([DKPS94, DPS94, PS95]), it was proved in [DPS94, Sch95] that if

$$d' > d - 2r, \tag{1.11}$$

and if the Schwarz kernel of the integral operator is smooth off the diagonal and exhibits a certain asymptotic behavior under differentiation, then the stiffness matrix can be compressed to a sparse one by dropping small elements in an à priori way, i.e., without computing these elements, in such a way that the order of convergence is not reduced. More precisely, this was shown in [DPS94] where sparse meant that the order of non-vanishing entries is $\dim \mathcal{S}_j(\log \dim \mathcal{S}_j)^b$ for some $b > 0$, whereas the

analysis in [Sch95] could even remove the logarithmic factor. Note that for $r \leq 0$, condition (1.11) rules out the orthogonal space decompositions, i.e., $\mathcal{S}_k' = \mathcal{S}_k$ or $\mathcal{W}_{k+1} = \mathcal{S}_{k+1} \ominus^{\perp_{L_2}} \mathcal{S}_k$.

1.2. **Construction of suitable wavelet bases.** So far, for $d' > d$, which is needed for integral operators of non-positive order, wavelet bases satisfying all desirable conditions have been constructed primarily only for spaces $\mathcal{S}_j$ that are spanned by cardinal B-splines on uniform partitions of $\mathbb{R}$ in [CDF92], and of $[0,1]$ in [DKU96]. Clearly, by taking tensor products, these constructions can be extended to $\mathbb{R}^n$ or $[0,1]^n$.

Using these wavelet bases as building blocks, in [DS96] and later in [CTU97], wavelet bases were constructed on manifolds that can be represented as disjoint unions of smooth parametric images of the unit cube. With both approaches, $\mathcal{H}^r$-stability is restricted to $r > -\frac{1}{2}$. Recently, in [DS97], see also [Dah97], this restriction was overcome yielding $\mathcal{H}^r$-stability in principal for any r. The key in this latter approach is a characterization of function spaces such as Sobolev or Besov spaces on such manifolds in terms of partitions of that type by establishing isomorphisms to product spaces whose components satisfy certain boundary conditions.

Although the above approaches cover, in principle, fairly general situations there still appears to be a strong need for alternative concepts for the following reasons. The fact that the above mentioned constructions are based solely on smooth parametrizations of the unit cube may have several severe disadvantages from a practical point of view. There is little chance to make use of existing software packages, and so essentially all algorithmic ingredients have to be put together from scratch. More importantly, constants in the norm equivalences will strongly depend on the parametric mappings so that strong distortions imposed by the domain geometry will have a quantitative effect. Furthermore, the only scheme that also covers norm equivalences for Sobolev spaces of regularity index $r \leq -1/2$ involves certain extension operators which have to be carefully chosen depending on the problem at hand.

As an alternative approach, for C^0 Lagrange finite element spaces of order $d \geq 2$ based on subdivisions into n-simplices ("triangulations") of domains in n dimensional Euclidean space, we introduce in this paper a construction of wavelet bases that meets all aforementioned requirements in the following sense: The wavelet bases are $\mathcal{H}^r$-stable for $|r| < \frac{3}{2}$, the wavelets are local with respect to the nodal bases, and they have, in principal, any order $d' \geq 2$ of vanishing moments. The construction is applicable to arbitrary initial meshes and for arbitrary boundary conditions. The only condition we need is that of *uniform* dyadic *refinements*. Apart from this condition, the appreciated flexibility of finite elements is fully retained.

The whole construction carries directly over to finite element type spaces on certain Lipschitz manifolds. More precisely, those manifolds are covered that consist of patches, each of them the parametric image of a domain with triangulations as above, such that the images of the triangulations match at the interfaces, and on each domain the Jacobian determinant is piecewise constant with respect to the initial triangulation.

This means that we can handle manifolds consisting of patches that are for example parts of hyperplanes, spheres or cylinders. We stress that the construction yields also in this case $\mathcal{H}^r$-stable wavelet bases for the *full* range $r \in [-1, 1]$ which appeared to pose essential difficulties in the above mentioned other approaches. Although some of the resulting wavelets may have support which intersects more than a single patch so that, due to the involved different parametric mappings, one cannot resort to standard polynomial moment conditions we are still able to confirm certain cancellation properties needed to establish optimal decay estimates for matrix compression.

Our construction requires solving once and for all some system on a reference element, which depends on d, d' and n. Having solved this system, the necessary element-by-element adaptations of the wavelets to the mesh on the domain or manifold, and possibly to boundary conditions are given *explicitly*. We will perform the computations on the reference element for $d = 2$ (piecewise linears) in combination with $d' = 2$ ($n = 1$, 2 or 3) or $d' = 3$ and $d' = 5$ ($n = 1$ or 2). Note that $d' = 2$, 3 and 5 suffices for the hypersingular equation, the double layer potential equation and the single layer potential equation, respectively. The results for $d = d' = 2$ were published earlier in [Ste97b].

Compared to existing approaches, our construction seems easy and, as we will show, it can be implemented efficiently. The clue of our approach is that we drop one condition that is usually imposed, viz. the existence of a well localized dual wavelet basis. Whereas for other applications of wavelets, such as signal analysis, the availability of such a dual basis is essential and enters the computations, for our goals viz. stability and matrix compression, there is no need for explicitly knowing such dual wavelets. It appears that without this requirement the possibilities of constructing efficient "flexible" wavelet bases are increased dramatically.

The remainder of this paper is organized as follows: In Section 2, we formulate a somewhat modified version of a crucial theorem concerning stability of biorthogonal space decompositions, that was first proved in a somewhat more general Hilbert space setting in [Dah96]. We have included a new proof of this theorem in Appendix A for the following reasons. As indicated before, the formulation of the result is different and in this form essential for application in the present setting. It avoids making any assumptions on dual bases, that in our applications will not be accessible. Moreover, its proof for the corresponding slight specialization is shorter and, as we think, better accessible.

The construction of stable bases of the subspaces generating a biorthogonal space decomposition, i.e., the construction of the wavelets, is treated in Section 3.

In Section 4, the approach from Section 3 is specialized to C^0 Lagrange finite element spaces. It will appear that in this case the construction of the wavelets can be splitted into two parts: An implicit part involving only some computations on a reference element and an explicit part taking care of the adaptations of the wavelets to the actual mesh. We will perform the computations on the reference element for linear finite element spaces and the order two, three and five of vanishing moments. Special attention will be paid to showing that the wavelet construction carries over to compact manifolds, and that resulting wavelets have cancellation

properties essential for matrix compression. We give a quantitative analysis of the costs of performing the wavelet transform.

2. Biorthogonal space decompositions

In this section, we formulate sufficient conditions for existence and stability of biorthogonal space decompositions. Partly based on arguments from [Sch95], we give a shorter proof of a theorem concerning these properties (Theorem 2.1), that in a more general setting was first proved in [Dah96]. Compared to [Dah96], in Part (a) an assumption about uniform boundedness of certain projectors is replaced by sufficient (and necessary) conditions for this property which are better suited for the present context.

As pointed out in the introduction, we are primarily concerned with $\mathcal{H}^r$-elliptic problems, where throughout this paper $\mathcal{H}^s$, $s \in \mathbb{R}$ (or $|s| \le t$) denotes a scale of Sobolev spaces, possibly incorporating boundary conditions, on an n-dimensional domain or sufficiently smooth manifold. For $s < 0$, $\mathcal{H}^s$ is to be understood as $(\mathcal{H}^{-s})'$ with respect to the dual pairing $(u, v) \mapsto (u, v)_{L_2}$.

Theorem 2.1 (cf. [Dah96]). *Let $\mathcal{S}_0 \subset \mathcal{S}_1 \subset \mathcal{S}_2 \subset \cdots$ and $\mathcal{S}'_0 \subset \mathcal{S}'_1 \subset \mathcal{S}'_2 \subset \cdots$ be sequences of nested closed subspaces of L_2, and let $\rho > 1$ be some constant.*

(a) Suppose that

$$\text{(C1)} \qquad \inf_{0 \neq u_k \in \mathcal{S}_k} \sup_{0 \neq u'_k \in \mathcal{S}'_k} \frac{|(u_k, u'_k)_{L_2}|}{\|u_k\|_{L_2} \|u'_k\|_{L_2}} \gtrsim 1,$$

as well as the analogous condition (C1)$'$ with interchanged roles of $(\mathcal{S}_k)_k$ and $(\mathcal{S}'_k)_k$ hold.

Then, there exists a sequence $(Q_k)_k$ of uniformly bounded projectors $Q_k : L_2 \to L_2$, with $\Im(Q_k) = \mathcal{S}_k$, $\Im(I - Q_k) = (\mathcal{S}'_k)^{\perp_{L_2}}$, and likewise for the dual projectors, $\Im(Q'_k) = \mathcal{S}'_k$, $\Im(I - Q'_k) = (\mathcal{S}_k)^{\perp_{L_2}}$. Furthermore, one has $Q_k Q_{k+1} = Q_k$ and $Q'_k Q'_{k+1} = Q'_k$.

(b) In addition, assume that there exists $0 < \gamma < d$ such that

$$\text{(C2)} \quad \inf_{u_k \in \mathcal{S}_k} \|u - u_k\|_{L_2} \lesssim \rho^{-sk} \|u\|_{\mathcal{H}^s} \quad (u \in \mathcal{H}^s,\ 0 \le s \le d) \ (\textit{direct or Jackson estimate}),$$

$$\text{(C3)} \quad \|u_k\|_{\mathcal{H}^s} \lesssim \rho^{sk} \|u_k\|_{L_2} \quad (u_k \in \mathcal{S}_k,\ 0 \le s < \gamma) \quad (\textit{inverse or Bernstein estimate}),$$

and that analogous assumptions (C2)$'$ and (C3)$'$ with constants $0 < \gamma' < d'$ hold for $(\mathcal{S}'_k)_k$.

Then, with $Q_{-1} := 0$, one has

$$\text{(2.1)} \qquad \|\sum_{k=0}^{\infty} v_k\|^2_{\mathcal{H}^s} \lesssim \sum_{k=0}^{\infty} \rho^{2sk} \|v_k\|^2_{L_2} \qquad (v_k \in \Im(Q_k - Q_{k-1}),\ s \in (-d', \gamma))$$

and

$$\text{(2.2)} \qquad \sum_{k=0}^{\infty} \rho^{2sk} \|(Q_k - Q_{k-1})u\|^2_{L_2} \lesssim \|u\|^2_{\mathcal{H}^s} \qquad (u \in \mathcal{H}^s,\ s \in (-\gamma', d)).$$

For $s \in (-\gamma', \gamma)$, $u \mapsto ((Q_k - Q_{k-1})u)_k$ is a bounded mapping from $\mathcal{H}^s$ onto $\ell_{2,s}(Q) := \{(v_k)_k : v_k \in \Im(Q_k - Q_{k-1}),\ \|(v_k)_k\|_{\ell_{2,s}(Q)} := (\sum_{k=0}^{\infty} \rho^{2sk} \|v_k\|^2_{L_2})^{\frac{1}{2}} < \infty\}$, with bounded

inverse $(v_k)_k \mapsto \sum_{k=0}^{\infty} v_k$. *That is, for* $s \in (-\gamma', \gamma)$, *the symbols "*$\lesssim$*" in (2.1) and (2.2) can be replaced by "*$=$*" symbols.*

Analogous results $(2.1)'$, $(2.2)'$ *are valid with* $(\mathcal{Q}_k)_k$ *replaced by* $(\mathcal{Q}'_k)_k$ *and with interchanged roles of* (γ, d) *and* (γ', d').

Remark 2.2. *To relate the results of this theorem to the concepts discussed in Section 1, let us denote by* $\mathcal{W}_k := \Im(\mathcal{Q}_k - \mathcal{Q}_{k-1})$ *the range of the difference operators* $\mathcal{Q}_k - \mathcal{Q}_{k-1}$. *Then* $\mathcal{W}_0 = \mathcal{S}_0$, *and by* $\mathcal{Q}_k\mathcal{Q}_{k+1} = \mathcal{Q}_k$, *one easily confirms that* $\mathcal{S}_{k+1} = \mathcal{W}_{k+1} \oplus \mathcal{S}_k$ *and* $\mathcal{W}_{k+1} = (\mathcal{S}'_k)^{\perp_{L_2}} \cap \mathcal{S}_{k+1}$. *Under its conditions, Theorem 2.1 shows that (1.8) holds for* $r \in (-\gamma', \gamma)$, *where* $\lambda_{k,r} = \rho^{2kr}$.

Analogous results hold for $\mathcal{W}'_k := \Im(\mathcal{Q}'_k - \mathcal{Q}'_{k-1})$, *i.e., (1.10) is valid here.*

The proof of Theorem 2.1 is given in Appendix A. Here we briefly comment only on the hypotheses.

As spaces $\mathcal{S}_k$ and $\mathcal{S}'_k$, we have finite element spaces in mind. In that context the validity of direct and inverse estimates are standard facts (cf. e.g. [Osw94]).

When the functions from the spaces $\mathcal{S}_k$ and $\mathcal{S}'_k$ are piecewise smooth and globally t respectively t' times continuously differentiable, under the usual assumptions for quasi-uniform meshes with mesh sizes $\sim \rho^{-k}$ on level k, the inverse estimates (C2) and $(C2)'$ are valid with $\gamma = t + \frac{3}{2}$ and $\gamma' = t' + \frac{3}{2}$, respectively ($t\,(t') := -1$ in case of no continuity between elements).

The direct estimates (C3) and $(C3)'$ are usually enforced by demanding that, relative to their meshes, $\mathcal{S}_k$ and $\mathcal{S}'_k$ contain all t or t' times continuously differentiable piecewise polynomials of degree $d-1$ and $d'-1$ respectively. In that case, since by (1.10) $\mathcal{W}_{k+1} \subset (\mathcal{S}'_k)^{\perp_{L_2}}$, basis functions of $\mathcal{W}_{k+1}$, i.e., the wavelets, will have d' vanishing moments. As pointed out in Section 1, the property of having sufficiently many vanishing moments is essential for the use of the wavelet basis for optimal compression [DPS94, Sch95].

Finally, (C1) or $(C1)'$ are conditions concerning invertibility of certain linear operators that are usually encountered in connection with saddle point problems. Their validity will be checked using the following criterion.

Lemma 2.3. *Let* $\Xi_k = \{\xi_{k,x} : x \in I_k\}$ *and* $\Xi'_k = \{\xi'_{k,x} : x \in I_k\}$ *be* L_2*-stable bases (cf. (1.4)) of* $\mathcal{S}_k$ *and* $\mathcal{S}'_k$ *respectively. (Note that (C1) and* $(C1)'$ *can be valid simultaneously only if* $\mathcal{S}_k$ *and* $\mathcal{S}'_k$ *have the same cardinality, so it is no restriction to use the same index set* I_k *for both bases.) Define the possibly infinite matrix*

$$\mathbf{M}_k = \left(\frac{(\xi_{k,y}, \xi'_{k,x})_{L_2}}{\|\xi_{k,y}\|_{L_2}\|\xi'_{k,x}\|_{L_2}} \right)_{x \in I_k, y \in I_k},$$

Then (C1) is equivalent to $\|\mathbf{M}_k\mathbf{c}_k\|_{\ell_2(I_k)} \gtrsim \|\mathbf{c}_k\|_{\ell_2(I_k)}$ *(*$\mathbf{c}_k = (c_{k,x})_{x \in I_k}$*), and analogously* $(C1)'$ *is equivalent to* $\|\mathbf{M}_k^T\mathbf{c}_k\|_{\ell_2(I_k)} \gtrsim \|\mathbf{c}_k\|_{\ell_2(I_k)}$, *where* $\|\mathbf{c}_k\|_{\ell_2(I_k)} = (\sum_{x \in I_k} |c_{k,x}|^2)^{\frac{1}{2}}$.

Proof. For the convenience of the reader we include the argument. Write $u_k = \sum_{x \in I_k} c_{k,x} \frac{\xi_{k,x}}{\|\xi_{k,x}\|_{L_2}} \in \mathcal{S}_k$ and $u'_k = \sum_{x \in I_k} d_{k,x} \frac{\xi'_{k,x}}{\|\xi'_{k,x}\|_{L_2}} \in \mathcal{S}'_k$. The stability of Ξ'_k and Ξ_k shows that $\sup_{0 \neq u'_k \in \mathcal{S}'_k} \frac{|(u_k, u'_k)_{L_2}|}{\|u'_k\|_{L_2}} \eqsim \sup_{0 \neq \mathbf{d}_k \in \ell_2(I_k)} \frac{|(\mathbf{M}_k\mathbf{c}_k, \mathbf{d}_k)_{\ell_2(I_k)}|}{\|\mathbf{d}_k\|_{\ell_2(I_k)}} = \|\mathbf{M}_k\mathbf{c}_k\|_{\ell_2(I_k)}$ and

$\|u_k\|_{L_2} \eqsim \|\mathbf{c}_k\|_{\ell_2(I_k)}$ respectively. From these results the proof follows immediately. □

3. A general construction of wavelets

Let $(\mathcal{S}_k)_k$, $(\mathcal{S}'_k)_k$ be nested sequences of closed subspaces as in Theorem 2.1. In this section, we present a way of constructing L_2-stable bases of the spaces $\mathcal{W}_{k+1} = (\mathcal{S}'_k)^{\perp_{L_2}} \cap \mathcal{S}_{k+1}$. Having these stable bases at hand, and assuming an L_2-stable basis of $\mathcal{W}_0 = \mathcal{S}_0$ as well, Theorem 2.1 implies that for $s \in (-\gamma', \gamma)$, the union of all basis functions, called *wavelets*, of all spaces $\mathcal{W}_k$ is a stable basis of $\mathcal{H}^s$. In particular, in this case the basis of the space $\mathcal{S}_j$ defined as the union of the bases of $\mathcal{W}_0, \dots, \mathcal{W}_j$ is (uniformly) $\mathcal{H}^s$-stable (cf. Remark 1.1).

Theorem 3.1. *Let $\Phi'_k = \{\phi'_{k,y} : y \in I_k\} \subset \mathcal{S}'_k$, $\Upsilon_{k+1} = \{\upsilon_{k+1,x} : x \in J_{k+1}\} \subset \mathcal{S}_{k+1}$ and $\Theta_{k+1} = \{\theta_{k+1,y} : y \in I_k\} \subset \mathcal{S}_{k+1}$ have the following properties:*

(a). *Φ'_k is an L_2-stable basis of $\mathcal{S}'_k$,*
(b). *$\Upsilon_{k+1} \cup \Theta_{k+1}$ is an L_2-stable basis of $\mathcal{S}_{k+1}$,*
(c). *Θ_{k+1} is a* dual basis *of Φ'_k, i.e.*

$$(\theta_{k+1,y}, \phi'_{k,z})_{L_2} \eqsim \delta_{y,z} \|\theta_{k+1,y}\|_{L_2} \|\phi'_{k,z}\|_{L_2} \qquad (y,\, z \in I_k).$$

Then, defining

$$\psi_{k+1,x} := \upsilon_{k+1,x} - \sum_{y \in I_k} \frac{(\upsilon_{k+1,x}, \phi'_{k,y})_{L_2}}{(\theta_{k+1,y}, \phi'_{k,y})_{L_2}} \theta_{k+1,y}, \tag{3.1}$$

the collection $\Psi_{k+1} := \{\psi_{k+1,x} : x \in J_{k+1}\}$ is an L_2-stable basis of $\mathcal{W}_{k+1} = (\mathcal{S}'_k)^{\perp_{L_2}} \cap \mathcal{S}_{k+1}$ and $\Psi_{k+1} \cup \Theta_{k+1}$ is an L_2-stable basis of $\mathcal{S}_{k+1}$.

Proof. Let us first show that $\Psi_{k+1} \cup \Theta_{k+1}$ is an L_2-stable basis of $\mathcal{S}_{k+1}$. To this end, consider the (possibly infinite) matrix

$$\mathbf{B}_k = \left(\frac{(\upsilon_{k+1,x}, \phi'_{k,y})_{L_2} \|\theta_{k+1,y}\|_{L_2}}{(\theta_{k+1,y}, \phi'_{k,y})_{L_2} \|\upsilon_{k+1,x}\|_{L_2}} \right)_{y \in I_k, x \in J_{k+1}}.$$

For any $\mathbf{d}_{k+1} = (d_{k+1,x})_{x \in J_{k+1}}$ and $\mathbf{c}_k = (c_{k,y})_{y \in I_k}$ the L_2-stability of Υ_{k+1} and Φ'_k and condition (c) provide

$$\begin{aligned}
(\mathbf{B}_k \mathbf{d}_{k+1}, \mathbf{c}_k)_{\ell_2(I_k)} &= \Big(\sum_{x \in J_{k+1}} d_{k+1,x} \tfrac{\upsilon_{k+1,x}}{\|\upsilon_{k+1,x}\|_{L_2}}, \sum_{y \in I_k} c_{k,y} \tfrac{\|\theta_{k+1,y}\|_{L_2} \|\phi'_{k,y}\|_{L_2}}{(\theta_{k+1,y}, \phi'_{k,y})_{L_2}} \tfrac{\phi'_{k,y}}{\|\phi'_{k,y}\|_{L_2}} \Big)_{L_2} \\
&\lesssim \left(\sum_{x \in J_{k+1}} |d_{k+1,x}|^2 \right)^{\frac{1}{2}} \left(\sum_{y \in I_k} |c_{k,y}|^2 \tfrac{\|\theta_{k+1,y}\|_{L_2}^2 \|\phi'_{k,y}\|_{L_2}^2}{|(\theta_{k+1,y}, \phi'_{k,y})_{L_2}|^2} \right)^{\frac{1}{2}} \\
&\lesssim \|\mathbf{d}_{k+1}\|_{\ell_2(J_{k+1})} \|\mathbf{c}_k\|_{\ell_2(I_k)},
\end{aligned}$$

which means that

$$\|\mathbf{B}_k\|_{\ell_2(I_k) \leftarrow \ell_2(J_{k+1})} \lesssim 1. \tag{3.2}$$

The definition of $\psi_{k+1,x}$ and the L_2-stability of $\Upsilon_{k+1} \cup \Theta_{k+1}$ imply that

$$\frac{\|\psi_{k+1,x}\|_{L_2}^2}{\|\upsilon_{k+1,x}\|_{L_2}^2} \eqsim 1 + \sum_{y \in I_k} |(\mathbf{B}_k)_{y,x}|^2,$$

and so, by the boundedness of $\mathbf{B}_k$, we have $\|\psi_{k+1,x}\|_{L_2} \eqsim \|\upsilon_{k+1,x}\|_{L_2}$.

Given any $\mathbf{c}_{k+1} = (c_{k+1,x})_{x \in J_{k+1}}$, $\mathbf{d}_k = (d_{k,y})_{y \in I_k}$, the expansion

$$\sum_{x \in J_{k+1}} c_{k+1,x} \tfrac{\psi_{k+1,x}}{\|\psi_{k+1,x}\|_{L_2}} + \sum_{y \in I_k} d_{k,y} \tfrac{\theta_{k+1,y}}{\|\theta_{k+1,y}\|_{L_2}}$$

can be rewritten as

$$\sum_{x \in J_{k+1}} e_{k+1,x} \tfrac{\upsilon_{k+1,x}}{\|\upsilon_{k+1,x}\|_{L_2}} + \sum_{y \in I_k} f_{k,y} \tfrac{\theta_{k+1,y}}{\|\theta_{k+1,y}\|_{L_2}}.$$

Here the arrays $\mathbf{e}_{k+1} = (e_{k+1,x})_{x \in J_{k+1}}$ and $\mathbf{f}_k = (f_{k,y})_{y \in I_k}$ are given by $[\mathbf{e}_{k+1} \quad \mathbf{f}_k]^T = \mathbf{L}_{k+1}[\mathbf{c}_{k+1} \quad \mathbf{d}_k]^T$, where $\mathbf{L}_{k+1} = \begin{bmatrix} \Delta_{k+1} & \mathbf{0} \\ -\mathbf{B}_k\Delta_{k+1} & \mathbf{I} \end{bmatrix}$ and $\Delta_{k+1} = \operatorname{diag}\left(\frac{\|\upsilon_{k+1,x}\|_{L_2}}{\|\psi_{k+1,x}\|_{L_2}}\right)_{x \in J_{k+1}}$. From $\|\psi_{k+1,x}\|_{L_2} \eqsim \|\upsilon_{k+1,x}\|_{L_2}$ and the boundedness of the mappings $\mathbf{B}_k$ (3.2) we infer that both $\mathbf{L}_{k+1}$ as $\mathbf{L}_{k+1}^{-1}$ are uniformly bounded mappings from $\ell_2(I_k \cup J_{k+1}) \to \ell_2(I_k \cup J_{k+1})$. Since, by assumption, $\Upsilon_{k+1} \cup \Theta_{k+1}$ is an L_2-stable basis of $\mathcal{S}_{k+1}$, the boundedness of both $\mathbf{L}_{k+1}$ and $\mathbf{L}_{k+1}^{-1}$ is equivalent to the property that $\Psi_{k+1} \cup \Theta_{k+1}$ is an L_2-stable basis of $\mathcal{S}_{k+1}$.

As for the remaining part of the claim, suppose that $\upsilon_{k+1} = \sum_{x \in J_{k+1}} c_{k+1,x}\psi_{k+1,x} + \sum_{y \in I_k} d_{k,y}\theta_{k+1,y}$ belongs to $\mathcal{W}_{k+1} = (\mathcal{S}'_k)^{\perp_{L_2}} \cap \mathcal{S}_{k+1}$. One easily verifies that $\psi_{k+1,x} \in (\mathcal{S}'_k)^{\perp_{L_2}}$ $(x \in J_{k+1})$. Now by taking L_2-scalar products with basis functions $\phi'_{k,z}$, we see that $d_{k,z} = 0$ for all $z \in I_k$. Therefore Ψ_{k+1} spans $\mathcal{W}_{k+1}$ and hence is an L_2-stable basis of $\mathcal{W}_{k+1}$. □

Remark 3.2. As was already noted in Section 1, in order to get an optimal implementation, given some family of (locally finite, L_2-stable) "single-scale" bases Φ_k, so that Φ_k is local with respect to Φ_{k+1}, we need Ψ_{k+1} that is local with respect to Φ_{k+1}. Generally in the context of Theorem 3.1, this necessarily means that both Υ_{k+1} and Θ_{k+1} are local with respect to Φ_{k+1}, and that the matrix

$$\left((\phi_{k+1,x}, \phi'_{k,y})_{L_2}\right)_{x \in I_{k+1}, y \in I_k} \text{ is sparse.} \tag{3.3}$$

Remark 3.3. The construction presented in Theorem 3.1 is closely related to the concept of so-called *stable completions* proposed in [CDP96]. Differences are that in [CDP96], it is assumed that

(3.4) $\Theta_{k+1} = \Phi_k$, i.e. (Φ_k, Φ'_k) is assumed to be a so-called *biorthogonal pair* of L_2-stable bases of $\mathcal{S}_k$ and $\mathcal{S}'_k$ respectively.

(3.5) the basis functions are scaled such that $\|\phi_{k,y}\|_{L_2}$, $\|\phi'_{k,y}\|_{L_2}$, $\|\upsilon_{k,x}\| \eqsim 1$ and $(\phi_{k,y}, \phi'_{k,y})_{L_2} = 1$.

Under these two assumptions, the wavelets defined in (3.1) are exactly those yielded by [CDP96, Theorem 3.3]. In fact, since $\mathcal{W}_{k+1} = (\mathcal{S}'_k)^{\perp_{L_2}} \cap \mathcal{S}_{k+1}$ does not depend on $\mathcal{S}_k$, we may even replace in [CDP96, Theorem 3.3] $\mathcal{S}_k$ by $\operatorname{span}\Theta_{k+1}$, and in particular,

Φ_k by Θ_{k+1}, so that, upon assuming (3.5), this theorem also yields the wavelets from our Theorem 3.1.

The fact that we avoid assuming some L_2-normalization of basis functions, and correspondingly, that our definition of (1.4) of stability of a basis is independent of the scaling of the basis functions involved, is of minor importance from a mathematical point of view. However, as was noticed in [Ste97a], in practical computations, in particular cases of non-uniform meshes, the absence of scalings permits a more efficient implementation.

The generalization that we allow Θ_{k+1} to be *some* dual basis of Φ'_k in $\mathcal{S}_{k+1}$, instead of sticking to the initially given coarse generator basis Φ_k, is a crucial point of this paper. Remark 3.2 says that we need Φ_k to be local with respect to Φ_{k+1}, whereas in case $\Theta_{k+1} = \Phi_k$, and thus (Φ_k, Φ'_k) a biorthogonal pair of bases, (3.3) is equivalent to the property that Φ'_k is local with respect to Φ'_{k+1}. So far, such biorthogonal bases have been constructed for pairs $((\mathcal{S}_k),(\mathcal{S}'_k))$ satisfying the conditions of Theorem 2.1 only for spaces $\mathcal{S}_k$ which are spanned by B-splines on uniform dyadic partitions of the real line (in [CDF92]), or of the interval (in [DKU96]). In [CDF92], and as a consequence partly also in [DKU96], the construction of the dual space $\mathcal{S}'_k$ relies through the construction of biorthogonal generators still on Fourier techniques, which restricts the field of applications essentially to uniform meshes and adaptive refinements of such.

In contrast, as we will see in the next sections, our generalization concerning the choice of Θ_{k+1} permits us to take for both $\mathcal{S}_k$ and $\mathcal{S}'_k$ standard finite element spaces, also in more space dimensions and for non-uniform meshes.

On the other hand, as is known from the literature, for applications different from the generation of well-conditioned stiffness matrices or matrix compression, the wavelet construction starting with suitable biorthogonal pairs of bases offers some potential advantages as will be described in the next remark.

Remark 3.4 (cf. [CDP96]). Let (Φ_k, Φ'_k) be a biorthogonal pair of bases, i.e., suppose $\Theta_{k+1} = \Phi_k$. Suppose Φ_k is local with respect to Φ_{k+1}, and Φ'_k is local with respect to Φ'_{k+1}, i.e., both matrices $\mathbf{P}_{k+1,0}$ and $\tilde{\mathbf{P}}_{k+1,0}$ from the refinement relations $\Phi_k^T = \Phi_{k+1}^T \mathbf{P}_{k+1,0}$ and ${\Phi'_k}^T = {\Phi'_{k+1}}^T \tilde{\mathbf{P}}_{k+1,0}$ are sparse. Furthermore, assume that Υ_{k+1} can be selected such that both $\hat{\mathbf{P}}_{k+1} = [\mathbf{P}_{k+1,0} \quad \hat{\mathbf{P}}_{k+1,1}]$, defined by $[\Phi_k^T \quad \Upsilon_{k+1}^T] = \Phi_{k+1}^T \hat{\mathbf{P}}_{k+1}$, and $\hat{\mathbf{P}}_{k+1}^{-1}$ are sparse. Examples can be found in [DKU96]. Since $\Theta_{k+1} = \Phi_k$, the definition of Ψ_{k+1} from Theorem 3.1 shows that $[\Phi_k^T \quad \Psi_{k+1}^T] = \Phi_{k+1}^T \mathbf{P}_{k+1}$, where $\mathbf{P}_{k+1} = \hat{\mathbf{P}}_{k+1} \begin{bmatrix} \mathbf{I} & -\mathbf{Z}_k \\ \mathbf{0} & \mathbf{I} \end{bmatrix}$, $\mathbf{Z}_k = \left(\frac{(v_{k+1,x}, \phi'_{k,y})_{L_2}}{(\phi_{k,y}, \phi'_{k,y})_{L_2}} \right)_{y \in I_k, x \in J_{k+1}}$ $= \hat{\Delta}_k^{-1} \tilde{\mathbf{P}}_{k+1,0}^* \hat{\Delta}_{k+1} \hat{\mathbf{P}}_{k+1,1}$ and $\hat{\Delta}_k = \mathrm{diag}((\phi_{k,y}, \phi'_{k,y})_{L_2})_{y \in I_k}$. From the assumptions on $\hat{\mathbf{P}}_{k+1}$, $\hat{\mathbf{P}}_{k+1}^{-1}$ and $\tilde{\mathbf{P}}_{k+1,0}$, we conclude that in this case not only $\mathbf{P}_{k+1}$, but also $\mathbf{P}_{k+1}^{-1}$ is sparse.

As a first consequence, we note that now besides $\mathbf{T}_j$, also the basis transformation $\mathbf{T}_j^{-1}$ from single scale basis Φ_j to multi-scale basis $\Psi^j = \cup_{k=0}^j \Psi_k$ can be implemented in $\mathcal{O}(\dim \mathcal{S}_j)$ operations, by a recursive top-to-bottom application of $\mathbf{P}_{k+1}^{-1}$ $(j - 1 \geq k \geq 0)$.

For the second application, it is convenient to assume scalings as in (3.5), which means that $\hat{\Delta}_k = \mathbf{I}$, $\tilde{\mathbf{P}}^*_{k+1,0}\mathbf{P}_{k+1,0} = \mathbf{I}$ and $\|\mathbf{P}_{k+1}\|, \|\mathbf{P}^{-1}_{k+1}\| \eqsim 1$. Then, writing $(\hat{\mathbf{P}}^{-1}_{k+1})^* = [\mathbf{H}_{k+1,0} \quad \mathbf{H}_{k+1,1}]$, it is easily verified that $(\mathbf{P}^{-1}_{k+1})^* = [\tilde{\mathbf{P}}_{k+1,0} \quad \mathbf{H}_{k+1,1}]$. Defining $(\Psi'_{k+1})^T$ by $[{\Phi'_k}^T \quad {\Psi'_{k+1}}^T] = {\Phi'_{k+1}}^T(\mathbf{P}^{-1}_{k+1})^*$, we find that Ψ'_{k+1} is an L_2-stable basis of $\mathcal{W}'_{k+1} = \mathcal{S}_k^{\perp_{L_2}} \cap \mathcal{S}'_{k+1}$, which is local with respect to Φ'_{k+1} and biorthogonal to Ψ_{k+1} (with scalar products equal to one when the indices match). So in this case we obtain biorthogonal bases $\bigcup_{j\geq 0}\Psi_j$ and $\bigcup_{j\geq 0}\Psi'_j$, where the dual wavelets have similar favorable properties as the primal ones.

As a consequence, in the context of Theorem 2.1, for $s \in (-\gamma', \gamma)$ and $f \in \mathcal{H}^s$, the coefficients in the expansion $f = \sum_{k,x} c_{k,x}\psi_{k,x}$ are now given by $c_{k,x} = (f, \psi'_{k,x})_{L_2}$. Assuming that Φ'_j is locally finite, the coefficients $(f, \psi'_{k,x})_{L_2}$ for $k \leq j$ can be computed recursively from top-to-bottom in an optimal way.

4. Wavelet bases of C^0 Lagrange finite element spaces

In this section, we will choose for both $\mathcal{S}_k$ and $\mathcal{S}'_k$ C^0 Lagrange finite element spaces, on possibly non-uniform meshes. We confirm the validity of conditions (C2), (C2)′, (C3) and (C3)′. Furthermore, we reduce the problems of verifying (C1) and (C1)′, and the construction of the single-scale bases Φ_k, as well as the bases Φ'_k, Θ_{k+1} and Υ_{k+1}, to corresponding problems on the *reference element*. These questions on the reference element will be treated for a number of concrete cases.

4.1. **Finite element spaces.**

For some fixed $m \in \mathbb{N}$, let $\mathcal{T}_{-m}$ be a collection of closed n-simplices such that $\cup_{\tau\in\mathcal{T}_{-m}}\tau$ is a partition, also called triangulation, of the closure of some open domain $\Omega \subset \mathbb{R}^n$. The construction of wavelets on manifolds will be addressed in Subsection 4.3. We assume that the triangulation is conforming, i.e., the intersection of any two simplices τ, τ' in $\mathcal{T}_{-m}$ is either empty or a common lower-dimensional face.

We include the possibility of imposing essential boundary conditions on part of $\partial\Omega$. That is, we consider functions on $\overline{\Omega}$ that are zero on $\Gamma_D \subset \partial\Omega$, where Γ_D is the union of a number (possibly zero) of faces of simplices from $\mathcal{T}_{-m}$. We define $\mathcal{H}^1$ as the completion in $H^1(\Omega)$ of $\{u \in C^\infty(\Omega) \cap H^1(\Omega) : \mathrm{supp} u \cap \Gamma_D = \emptyset\}$, and for $0 \leq s \neq \frac{1}{2}$, we take $\mathcal{H}^s = \mathcal{H}^1 \cap H^s(\Omega)$. When $\Gamma_D \neq \emptyset$, a similar definition of $\mathcal{H}^{\frac{1}{2}}$ would not yield a Hilbert scale. In this case we define $\mathcal{H}^{\frac{1}{2}}$ by interpolation between L_2 and $\mathcal{H}^1$.

As we have already stated in Section 2, for $s \leq 0$, we define $\mathcal{H}^s$ as $(\mathcal{H}^{-s})'$ with respect to the dual pairing $(u, v) \mapsto (u, v)_{L_2}$. Although we do not consider this option, it will become clear that our analysis can easily be generalized to the case of a weighted L_2-scalar product, if the weight function is piecewise constant with respect to the triangulation $\mathcal{T}_{-m}$.

Let us now describe the hierarchy of meshes and associated finite element spaces. For $-m < k \in \mathbb{Z}$, let $\mathcal{T}_k$ be the collection of n-simplices generated from $\mathcal{T}_{k-1}$ by uniform, regular, dyadic refinement, i.e., each $\tau \in \mathcal{T}_{k-1}$ is subdivided into 2^n congruent n-simplices. Note that for $n > 2$, there are several possibilities to divide each simplex in this way. However, for $n = 3$ this ambiguity concerns only subsimplices whose 2-faces are not contained in any 2-face of the mother simplex. Thus the

refinement of adjacent simplices in $\mathcal{T}_{-m}$ does not affect compatibility on common faces. Therefore, to ease presentation, we will assume that $n \leq 3$, which means that refinements can be made on an element-by-element basis with automatic matching of the triangulation at interfaces of adjacent macro elements. The set of vertices of all $\tau \in \mathcal{T}_k$ is denoted by V_k.

For $\tilde{\ell} \in \mathbb{N}$, $k \in \mathbb{Z}$ with $\tilde{\ell} \leq m$, $k \geq \tilde{\ell} - m$, we define $S(\tilde{\ell}, k)$ as the C^0 Lagrange finite element space of degree $2^{\tilde{\ell}}$ corresponding to the set of "nodes" V_k and the $\tilde{\ell}$-times coarsened triangulation $\mathcal{T}_{k-\tilde{\ell}}$. Note that the dimension of $S(\tilde{\ell}, k)$ only depends on k, but not on $\tilde{\ell}$. Restricting the polynomial degrees to powers of two ensures that the sets of nodes are nested under refinement. Each $\tau \in \mathcal{T}_{k-\tilde{\ell}}$ contains $\binom{2^{\tilde{\ell}}+n}{n}$ nodes from V_k, which indeed equals the dimension of $\Pi_{2^{\tilde{\ell}}}$, the space of polynomials of total degree $2^{\tilde{\ell}}$ on $\mathbb{R}^n$. In particular, one may verify that $S(\tilde{\ell}, k)$ is well-defined (e.g. see [BS94, Sect. 3]).

Now fix $\ell, \ell' \in \mathbb{N}$ and assume throughout the following that $m = \max\{\ell, \ell'\}$. We define

$$\mathcal{S}_k = S(\ell, k) \cap \mathcal{H}^1, \ \mathcal{S}'_k = S(\ell', k) \cap \mathcal{H}^1,$$

where the intersection with $\mathcal{H}^1$ is made to impose possibly essential boundary conditions. Note that $k = 0$ is the lowest level on which both $\mathcal{S}_k$ and $\mathcal{S}'_k$ are defined. Only for matters related to obtaining an efficient implementation, sometimes it will be convenient also to consider spaces $S(\tilde{\ell}, k)$ for some negative k.

Remark 4.1. The sequences $(\mathcal{S}_k)_{k\geq 0}$, $(\mathcal{S}'_k)_{k\geq 0}$ satisfy the conditions (C2), (C2)′, (C3), (C3)′ with $\rho = 2$ (dyadic refinement), and

$$\gamma = \gamma' = \frac{3}{2} \quad d = 2^{\ell} + 1, \ d' = 2^{\ell'} + 1.$$

Assuming that (C1) and (C1)′ are satisfied as well, so that, on account of Remark 2.2, $\mathcal{W}_{k+1} = \mathcal{S}_{k+1} \cap (\mathcal{S}'_k)^{\perp_{L_2}}$ is well-defined, elements $v_{k+1} \in \mathcal{W}_{k+1}$ will have d' vanishing moments. In case of essential boundary conditions, this holds true for v_{k+1} for which $\mathrm{supp} v_{k+1} \subset \cup\{\tau \in \mathcal{T}_{k-\ell'} : \tau \cap \Gamma_D = \emptyset\}$.

As index set for bases of $S(\tilde{\ell}, k) \cap \mathcal{H}^1$, we will use

$$I_k := V_k \backslash (\Gamma_D \cap V_k). \tag{4.1}$$

4.2. **Reduction to a reference element.** In this subsection, we will reduce the problems of constructing single-scale bases Φ_k, bases Φ'_k, Θ_{k+1} and Υ_{k+1} needed in Theorem 3.1, as well as bases Ξ_k and Ξ'_k needed in Lemma 2.3 for verifying (C1) and (C1)′, to corresponding problems on some reference (macro) element $\check{\tau}$, say with vertices $v_1, \ldots, v_{n+1} \in \mathbb{R}^n$ not all lying on one hyperplane.

Starting with the collection $\check{\mathcal{T}}_{-m} = \{\check{\tau}\}$ consisting of the reference simplex only, we define in analogy to the definitions in Subsection 4.1 sequences of triangulations $(\check{\mathcal{T}}_{\check{k}})_{\check{k}\geq -m}$, nodes $(\check{V}_{\check{k}})_{\check{k}\geq -m}$, and for $\tilde{\ell} \leq m$, finite element spaces $(\check{S}(\tilde{\ell}, \check{k}))_{\check{k}\geq \tilde{\ell}-m}$. In the following we will denote $\check{V}_{\check{k}}$ by $\check{I}_{\check{k}}$, which is in agreement with (4.1), since we do not impose boundary conditions on $\check{\tau}$.

To describe certain symmetry relations which are relevant for the subsequent construction consider the mapping

$$B : \{\alpha \in I\!R^{n+1} : \sum_{j=1}^{n+1} \alpha_j = 1,\ \alpha_j \geq 0\} \to \check{\tau} : \alpha \mapsto \sum_{j=1}^{n+1} \alpha_j v_j$$

which is a bijection from the standard n-simplex to $\check{\tau}$ described in terms of barycentric coordinates. For any permutation of the coordinates $\pi : I\!R^{n+1} \to I\!R^{n+1}$, we define the bijection

$$S_\pi = B \circ \pi \circ B^{-1} : \check{\tau} \to \check{\tau}.$$

For *fixed* $\check{k} \geq \tilde{\ell} - m$, let $\{\check{\varphi}_{\check{k},\check{x}} : \check{x} \in \check{I}_{\check{k}}\} \subset \check{S}(\tilde{\ell}, \check{k})$ be a collection of functions on the reference element satisfying

(A1). $(\check{\varphi}_{\check{k},S_\pi(\check{x})} \circ S_\pi)|_{\partial\check{\tau}} = \check{\varphi}_{\check{k},\check{x}}|_{\partial\check{\tau}}$ for all permutations π and all $\check{x} \in \check{I}_{\check{k}}$,

(A2). $\check{\varphi}_{\check{k},\check{x}}(\check{y}) = 0$ if for some $i \in \{1, \ldots, n+1\}$, $(B^{-1}(\check{y}))_i = 0$, whereas $(B^{-1}(\check{x}))_i \neq 0$.

Assumption (A1) means that $\{\check{\varphi}_{\check{k},\check{x}}|_{\partial\check{\tau}} : \check{x} \in \check{I}_{\check{k}}\}$ is invariant under permutations of the barycentric coordinates, whereas (A2) says that $\check{\varphi}_{\check{k},\check{x}}$ vanishes on all faces of $\check{\tau}$ that do not contain $\check{x}$.

For $k \geq \check{k}$ and $x \in I_k$, let us now define $\varphi_{k,x} \in S(\tilde{\ell}, k) \cap \mathcal{H}^1$ by

$$\varphi_{k,x}(y) = \begin{cases} \check{\varphi}_{\check{k},T_\tau^{-1}(x)}(T_\tau^{-1}(y)) & \text{if } x, y \in \tau \in \mathcal{T}_{k-\check{k}-m} \\ 0 & \text{elsewhere} \end{cases}, \tag{4.2}$$

where $T_\tau : \check{\tau} \to \tau$ is an affine bijection. In case $n > 2$, we have to assume in addition that T_τ is a bijection between the triangulations $\check{\mathcal{T}}_{\check{k}-\tilde{\ell}}$ in the reference domain and $\{\tau_{k-\tilde{\ell}} \in \mathcal{T}_{k-\tilde{\ell}} : \tau_{k-\tilde{\ell}} \subset \tau\}$ in the physical domain. As mentioned before, for $n \leq 2$, this is ensured automatically due to the uniqueness of uniform dyadic refinements. However, for $n = 3$ there exist three possibilities to subdivide a tetrahedron into 8 congruent tetrahedra, see [DM88] for more details.

For any $\tau \neq \tilde{\tau} \in \mathcal{T}_{k-\check{k}-m}$ with $\tau \cap \tilde{\tau} \neq \emptyset$, and any T_τ, $T_{\tilde{\tau}}$ as above, there exists a permutation π such that $T_{\tilde{\tau}} \circ S_\pi \circ T_\tau^{-1}|_{\tilde{\tau}\cap\tau} = I$. Assumption (A1) now implies that $\varphi_{k,x}$ is well-defined, and that it is continuous on $\tau \cup \tilde{\tau}$ if $x \in \tau \cap \tilde{\tau}$. On the other hand, assumption (A2) shows that $\varphi_{k,x}$ vanishes on all faces of τ that do not contain x. We conclude that $\varphi_{k,x}$ is continuous on $\overline{\Omega}$, vanishes on Γ_D, and so that $\varphi_{k,x}$ belongs to $S(\tilde{\ell}, k) \cap \mathcal{H}^1$.

Remark 4.2. In our applications, we will construct $\{\check{\varphi}_{\check{k},\check{x}} : \check{x} \in \check{I}_{\check{k}}\} \subset \check{S}(\tilde{\ell}, \check{k})$ which on the *whole* of $\check{\tau}$, thus not only on $\partial\check{\tau}$ are invariant under all permutations of the barycentric coordinates that, for $n > 2$, leave the mesh $\check{\mathcal{T}}_{\check{k}-\tilde{\ell}}$ unchanged. So $\varphi_{k,x}$ defined in (4.2) will be independent of the particular choice of T_τ. Note that this construction of global bases is commonly used for affinely equivalent finite elements.

The analysis in the remainder of this subsection will be based on the following trivial relation

$$(u, v)_{L_2} = \sum_{\tau\in\mathcal{T}_k} \tfrac{\mathrm{vol}(\tau)}{\mathrm{vol}(\check{\tau})}(u \circ T_\tau, v \circ T_\tau)_{L_2(\check{\tau})} \qquad (u, v \in L_2,\ k \geq -m), \tag{4.3}$$

where $T_\tau : \check\tau \to \tau$ are affine bijections. In particular, for $\varphi_{k,x}$ defined in (4.2) this yields

$$\|\varphi_{k,x}\|_{L_2}^2 = \sum_{\{\tau\in\mathcal{T}_{k-\check k-m}:x\in\tau\}} \tfrac{\mathrm{vol}(\tau)}{\mathrm{vol}(\check\tau)} \|\check\varphi_{\check k,T_\tau^{-1}(x)}\|_{L_2(\check\tau)}^2. \tag{4.4}$$

Lemma 4.3. *For some fixed $\tilde\ell \le m$, $\check k \ge \tilde\ell - m$, let $\{\check\varphi_{\check k,\check x} : \check x \in \check I_{\check k}\}$ be a basis of $\check S(\tilde\ell,\check k)$ satisfying (A1), (A2). Then the collections $\{\varphi_{k,x} : x \in I_k\}$, defined by (4.2), form L_2-stable bases of the spaces $S(\tilde\ell,k)\cap\mathcal{H}^1$, uniformly in $k \ge \check k$.*

Proof. From the construction of $\varphi_{k,x}$ and the definition of a finite element space, it follows directly that $\{\varphi_{k,x} : x \in I_k\}$ is a basis of $M(\tilde\ell,k)\cap\mathcal{H}^1$. On account of (4.4), one has

$$\begin{aligned} \|\sum_{x\in I_k} c_{k,x}\varphi_{k,x}\|_{L_2}^2 &= \sum_{\tau\in\mathcal{T}_{k-\check k-m}} \tfrac{\mathrm{vol}(\tau)}{\mathrm{vol}(\check\tau)} \|\sum_{x\in I_k\cap\tau} c_{k,x}\check\varphi_{\check k,T_\tau^{-1}(x)}\|_{L_2(\check\tau)}^2 \\ &\eqsim \sum_{\tau\in\mathcal{T}_{k-\check k-m}} \tfrac{\mathrm{vol}(\tau)}{\mathrm{vol}(\check\tau)} \sum_{x\in I_k\cap\tau} |c_{k,x}|^2 \|\check\varphi_{\check k,T_\tau^{-1}(x)}\|_{L_2(\check\tau)}^2 \\ &= \sum_{x\in I_k} |c_{k,x}|^2 \|\varphi_{k,x}\|_{L_2}^2, \end{aligned} \tag{4.5}$$

confirms L_2-stability. □

Definition 4.4. Let $\tilde\ell \le m$ be given. For $\check k \ge \tilde\ell - m$, with $\check\Phi_{\check k}^{\tilde\ell,N} = \{\check\phi_{\check k,\check x}^{\tilde\ell,N} : \check x \in \check I_{\check k}\}$ we denote the *nodal* basis of $\check S(\tilde\ell,\check k)$ defined by $\check\phi_{\check k,\check x}^{\tilde\ell,N}(\check y) = \delta_{\check x}(\check y)$ $(\check y \in \check I_{\check k})$. Clearly, $\check\Phi_{\tilde\ell-m}^{\tilde\ell,N}$ satisfies (A1) and (A2). The corresponding *global* nodal bases of $S(\tilde\ell,k)$ $(k \ge \tilde\ell - m)$, induced by (4.2), will be denoted by $\Phi_k^{\tilde\ell,N} = \{\phi_{k,x}^{\tilde\ell,N} : x \in I_k\}$.

As "single-scale" basis Φ_k of $\mathcal{S}_k = S(\ell,k)$, we will always employ the nodal basis $\Phi_k^{\ell,N}$.

Proposition 4.5. *Let $\check\Xi_0 = \{\check\xi_{0,\check x} : \check x \in \check I_0\}$ and $\check\Xi_0' = \{\check\xi_{0,\check x}' : \check x \in \check I_0\}$ be bases of $\check S(\ell,0)$ and $\check S(\ell',0)$, respectively, satisfying (A1), (A2). Moreover, suppose that with respect to the Euclidean scalar product $(\,\cdot\,,\,\cdot\,)_{\ell_2(\check I_0)}$, the real part of the matrix $\check{\mathbf{M}}_0 := \big((\check\xi_{0,\check y},\check\xi_{0,\check x}')_{L_2(\check\tau)}\big)_{\check x\in\check I_0,\check y\in\check I_0}$ is strictly positive, i.e. $\Re(\check{\mathbf{M}}_0) := \frac12(\check{\mathbf{M}}_0 + \check{\mathbf{M}}_0^*) > 0$. Then, the pair $(\mathcal{S}_k,\mathcal{S}_k')$ satisfies (C1) and (C1)′.*

Proof. Let $\Xi_k = \{\xi_{k,x} : x \in I_k\}$ and $\Xi_k' = \{\xi_{k,x}' : x \in I_k\}$ be defined by (4.2) according to $\check\Xi_0$ and $\check\Xi_0'$, respectively. By Lemma 4.3, Ξ_k and Ξ_k' are L_2-stable bases of $\mathcal{S}_k$ and $\mathcal{S}_k'$.

Define the (possibly infinite) matrix $\tilde{\mathbf{M}}_k = \Big(\frac{(\xi_{k,y},\xi_{k,x}')_{L_2}}{\|\xi_{k,y}\|_{L_2}\|\xi_{k,x}\|_{L_2}}\Big)_{x\in I_k,y\in I_k}$. Since by (4.4), $\|\xi_{k,x}\|_{L_2} \eqsim \|\xi_{k,x}'\|_{L_2}$, for Lemma 2.3 it is sufficient to check whether $\|\tilde{\mathbf{M}}_k\mathbf{c}_k\|_{\ell_2(I_k)} \gtrsim \|\mathbf{c}_k\|_{\ell_2(I_k)}$ and $\|\tilde{\mathbf{M}}_k^T\mathbf{c}_k\|_{\ell_2(I_k)} \gtrsim \|\mathbf{c}_k\|_{\ell_2(I_k)}$ $(\mathbf{c}_k = (c_{k,x})_{x\in I_k})$. Since $\Re(\check{\mathbf{M}}_0) > 0$ one

has

$$\begin{aligned}
&\Re\Bigg(\sum_{x,y\in I_k}(\xi_{k,y},\xi'_{k,x})_{L_2}c_{k,y}\overline{c_{k,x}}\Bigg)\\
&=\sum_{\tau\in\mathcal{T}_{k-m}}\tfrac{\mathrm{vol}(\tau)}{\mathrm{vol}(\check\tau)}\,\Re\Bigg(\sum_{x,y\in I_k\cap\tau}(\check\xi_{0,T_\tau^{-1}(y)},\check\xi'_{0,T_\tau^{-1}(x)})_{L_2(\check\tau)}\,c_{k,y}\overline{c_{k,x}}\Bigg)\\
&\gtrsim\sum_{\tau\in\mathcal{T}_{k-m}}\tfrac{\mathrm{vol}(\tau)}{\mathrm{vol}(\check\tau)}\sum_{x\in I_k\cap\tau}|c_{k,x}|^2\eqsim\sum_{x\in I_k}|c_{k,x}|^2\|\xi_{k,x}\|_{L_2}^2,
\end{aligned}$$

so that $\Re(\tilde{\mathbf{M}}_k)\gtrsim\mathbf{I}$ with respect to $(\cdot\,,\cdot)_{\ell_2(I_k)}$, which implies the desired properties of $\tilde{\mathbf{M}}_k$. □

Proposition 4.6. *Let*

$$\check J_{\check k}:=\check I_{\check k}\backslash\check I_{\check k-1},\quad J_k:=I_k\backslash I_{k-1}.$$

Suppose that the collections $\check\Phi'_0=\{\check\phi'_{0,\check y}:\check y\in\check I_0\}$, $\check\Upsilon_1=\{\check\upsilon_{1,\check x}:\check x\in\check J_1\}$ *and* $\check\Theta_1=\{\check\theta_{1,\check y}:\check y\in\check I_0\}$ *have been chosen such that* $\check\Phi'_0$ *and* $\check\Upsilon_1\cup\check\Theta_1$ *are bases of* $\check S(\ell',0)$ *and* $\check S(\ell,1)$*, respectively, both satisfying the Assumptions (A1), (A2). Furthermore, assume that*

$$(\check\theta_{1,\check y},\check\phi'_{0,\check z})_{L_2(\check\tau)}=\delta_{\check y,\check z},\qquad \check y,\check z\in\check I_0,$$

that is, $\check\Theta_1$ *is a "local" dual basis to* $\check\Phi'_0$*.*

By (4.2), define for $k\geq 0$ *the corresponding "global" bases* $\Phi'_k:=\{\phi_{k,y}:y\in I_k\}$*, and for* $k\geq 1$, $\Upsilon_k:=\{\upsilon_{k,x};x\in J_k\}$ *as well as* $\Theta_k:=\{\theta_{k,y};y\in I_{k-1}\}$*. Then the triple* $(\Phi'_k,\Upsilon_{k+1},\Theta_{k+1})_{k\geq0}$ *satisfies the conditions of Theorem 3.1.*

Proof. For y, $z\in I_k$, we have

$$\begin{aligned}
(\theta_{k+1,y},\phi'_{k,z})_{L_2}&=\sum_{\{\tau\in\mathcal{T}_{k-m}:y,z\in\tau\}}\tfrac{\mathrm{vol}(\tau)}{\mathrm{vol}(\check\tau)}(\check\theta_{1,T_\tau^{-1}(y)},\check\phi'_{0,T_\tau^{-1}(z)})_{L_2(\check\tau)}\\
&\eqsim\delta_{y,z}\|\theta_{k+1,y}\|_{L_2}\|\phi'_{k,z}\|_{L_2},
\end{aligned}$$

i.e., Θ_{k+1} is a dual basis of Φ'_k. The L_2-stability of the bases Φ'_k and $\Upsilon_{k+1}\cup\Theta_{k+1}$ follows from Lemma 4.3. □

Recall from formula (3.1) that for computing the wavelets we need the quantities $\frac{(\upsilon_{k+1,x},\phi'_{k,y})_{L_2}}{(\theta_{k+1,y},\phi'_{k,y})_{L_2}}$. To this end, note that for $x\in J_{k+1}$, $y\in I_k$,

$$\frac{(\upsilon_{k+1,x},\phi'_{k,y})_{L_2}}{(\theta_{k+1,y},\phi'_{k,y})_{L_2}}=\frac{\sum_{\{\tau\in\mathcal{T}_{k-m}:x,y\in\tau\}}\mathrm{vol}(\tau)\,(\check\upsilon_{1,T_\tau^{-1}(x)},\check\phi'_{0,T_\tau^{-1}(y)})_{L_2(\check\tau)}}{\sum_{\{\tau\in\mathcal{T}_{k-m}:y\in\tau\}}\mathrm{vol}(\tau)}.\tag{4.6}$$

In summary, we conclude that we have to check $\Re(\check{\mathbf{M}}_0)>0$ (cf. Proposition 4.5) *once and for all* on the reference element. Moreover, we have to provide the collections $\check\Upsilon_1$, $\check\Theta_1$ in terms of $\check\Phi_1^{\ell,N}$ as well as the values $(\check\upsilon_{1,\check x},\check\phi'_{0,\check y})_{L_2(\check\tau)}$ ($\check x\in\check J_1$, $\check y\in\check I_0$). Apart from that, in order to compute the wavelets, i.e. the basis functions of

$\mathcal{W}_{k+1} = \mathcal{S}_{k+1} \cap (\mathcal{S}'_k)^{\perp_{L_2}}$, we only need the geometry of the initial mesh which, in particular, yields the volumes of the simplices.

Clearly, bases obtained by the above construction satisfy the locality conditions formulated in Remark 3.2. This means that the basis transformation $\mathbf{T}_j$ on $\mathcal{S}_j$ from wavelet- to nodal basis can be computed in $\mathcal{O}(\dim \mathcal{S}_j)$ operations.

4.3. **Wavelets on compact manifolds.** In this section, it will be demonstrated that the concept of an element-by-element construction of wavelets outlined above applies as well to certain types of continuous compact n-dimensional *manifolds* $\Gamma \subset \mathbb{R}^{n+1}$.

4.3.1. *Function spaces.* Suppose that $\Gamma = \cup_{i=1}^{p} \overline{\Gamma_i}$, where $\Gamma_i \cap \Gamma_j = \emptyset$ when $i \neq j$. We assume that each Γ_i is the image of a smooth regular parameterization $\kappa_i : \Omega_i \to \Gamma_i$, where $\Omega_i \subset \mathbb{R}^n$ is an open domain. For each i, we assume that $\mathcal{T}_{-m,i}$ is a collection of closed n-simplices, such that $\cup_{\tau \in \mathcal{T}_{-m,i}} \tau$ is a triangulation of $\overline{\Omega_i}$. Concerning matching of elements at the interfaces, we make the following assumption: For each maximal subset $I \subset \{1, \ldots, p\}$ for which $\cap_{i \in I} \overline{\Gamma_i} \neq \emptyset$, there exist invertible affine mappings $A_i : \mathbb{R}^n \to \mathbb{R}^n$ "gluing the Ω_i's together", such that the mapping $\cup_{i \in I} \Gamma_i \to \cup_{i \in I} A_i(\Omega_i)$ defined by

$$x \mapsto A_i(\kappa_i^{-1}(x)) \quad \text{when} \quad x \in \Gamma_i,$$

has an extension as a homeomorphic mapping between $\cup_{i \in I} \overline{\Gamma_i}$ and $\cup_{i \in I} \overline{A_i(\Omega_i)}$. We assume that $\cup_{i \in I} \cup_{\tau \in \mathcal{T}_{-m,i}} A_i(\tau)$ is a conforming triangulation of $\cup_{i \in I} \overline{A_i(\Omega_i)}$. Note that these are mild requirements that conform with typical applications, for instance, in the context of boundary element methods.

As in Section 4.1, by uniform, regular and dyadic refinement, on each Ω_i we define sequences of triangulations $(\mathcal{T}_{k,i})_{k \geq -m}$, nodes $(V_{k,i})_{k \geq -m}$, and for $\tilde{\ell} \leq m$, finite element spaces $(S_i(\tilde{\ell}, k))_{k \geq \tilde{\ell} - m}$. We define

$$S^{(\Gamma)}(\tilde{\ell}, k) = \{u \in C(\Gamma) : u \circ \kappa_i \in S_i(\tilde{\ell}, k),\ 1 \leq i \leq p\}.$$

Due to our assumptions, the dimension of $S^{(\Gamma)}(\tilde{\ell}, k)$ is equal to $\# I_k$, where

$$I_k = \cup_{i=1}^{p} \kappa_i(V_{k,i})$$

is now the set of nodes on the manifold.

Assuming that Γ is globally Lipschitz continuous, the Sobolev spaces $\mathcal{H}^s = H^s(\Gamma)$ can be defined for $|s| \leq 1$ in the usual way using a partition of unity relative to some atlas (cf. [BGZ96]).With respect to the dual pairing $(u, v) \mapsto (u, v)_{L_2(\Gamma)}$, where

$$(u, v)_{L_2(\Gamma)} = \int_\Gamma u \overline{v} d\Gamma = \sum_{i=1}^{p} \int_{\Omega_i} u(\kappa_i(z)) \overline{v(\kappa_i(z))} \sqrt{\det(D\kappa_i(z)^T D\kappa_i(z))} dz, \tag{4.7}$$

one has $\mathcal{H}^s = (\mathcal{H}^{-s})'$ $(s \in [-1, 0])$ with equivalent norms.

4.3.2. *Space decompositions.* We define the auxiliary spaces $\mathcal{H}_s$ for $s \geq 0$ as the closure of $\{u \in C(\Gamma) : u \circ \kappa_i \in H^s(\Omega_i),\ 1 \leq i \leq p\}$ with respect to the norm $\|u\|_{\mathcal{H}_s} := \sqrt{\sum_{i=1}^p \|u \circ \kappa_i\|^2_{H^s(\Omega_i)}}$, and for $s < 0$ as $(\mathcal{H}_{-s})'$ with respect to the dual pairing $(u, v) \mapsto (u, v)_{L_2(\Gamma)}$. The space $\mathcal{H}_s$ should not be confused with the product space $\prod_{i=1}^p H^s(\Gamma_i)$ that is usually associated with the norm $\|\cdot\|_{\mathcal{H}_s}$.

Fixing $\ell, \ell' \in I\!N$ with $m = \max\{\ell, \ell'\}$, we now take $\mathcal{S}_k = S^{(\Gamma)}(\ell, k)$, $\mathcal{S}'_k = S^{(\Gamma)}(\ell', k)$, $k \geq 0$. Exploiting the fact that the nodal interpolants from the spaces $\mathcal{S}_k$ and $\mathcal{S}'_k$ reproduce polynomials of degree $2^\ell, 2^{\ell'}$, respectively, as well as the locality of the nodal functionals, standard estimates confirm that, with respect to the $\mathcal{H}_s$-spaces, (C2), (C2)$'$, (C3), (C3)$'$ are valid with $d = 2^\ell + 1$, $d' = 2^{\ell'} + 1$ and $\gamma = \gamma' = \frac{3}{2}$. For this to hold true, it is sufficient that $\kappa_i \in C^{\max\{d,d'\}-1}$ with Lipschitz continuous highest order derivatives. Assuming for the moment that also (C1), (C1)$'$ are valid with $\mathcal{W}_{k+1} = \mathcal{S}_{k+1} \cap (\mathcal{S}'_k)^{\perp_{L_2(\Gamma)}}$ we conclude that for $|s| < \frac{3}{2}$ the direct sum $\oplus_{k \geq 0} \mathcal{W}_k$ is a stable decomposition of the auxiliary space $\mathcal{H}_s$ in the sense of Theorem 2.1.

We have to clarify next how the spaces $\mathcal{H}_s$ and $\mathcal{H}^s$ are interrelated. Since we admit Lipschitz manifolds our primary concern is the range $|s| \leq 1$ of Sobolev regularity. By the denseness of smooth functions in Sobolev spaces each element in $\mathcal{H}_s$ is the limit of patchwise C^∞-functions that are globally continuous. It is well known that such functions belong to $\mathcal{H}^1$. Since $\|v \circ \kappa_i\|_{H^1(\Omega_i)} \eqsim \|v\|_{H^1(\Gamma_i)}$ one concludes that on $\mathcal{H}^1$ the norms $\|\cdot\|_{\mathcal{H}_1}$ and $\|\cdot\|_{\mathcal{H}^1}$ are equivalent. Thus, by the previous remark, we infer that the spaces $\mathcal{H}^1$ and $\mathcal{H}_1$ agree as sets and have equivalent norms. By interpolation and duality we therefore conclude that

$$\mathcal{H}^s \eqsim \mathcal{H}_s, \ -1 \leq s \leq 1,$$

which establishes the desired $\mathcal{H}^s$-stability for $|s| \leq 1$ also for the case of wavelet bases on Lipschitz manifolds.

At this point some comments on the alternative approaches mentioned in the introduction are in order. In [CTU97, DS96] multiresolution spaces are constructed which are spanned by continuous bases Φ_j, Φ'_j which are biorthogonal with respect to the inner product corresponding to the norm $\|\cdot\|_{\mathcal{H}_0}$ and which both satisfy norm equivalences for the spaces $\mathcal{H}_s$ for $0 \leq s \leq 1$. However, since in contrast to the present approach biorthogonality refers to a modified inner product it is not clear that the norm equivalences extend into the negative range beyond the interesting case $s = -1/2$.

4.3.3. *Wavelets.* Now we come to the construction of the wavelets and the verification of (C1) and (C1)$'$. To reduce, as in the domain case, these questions to corresponding questions on a reference element $\check{\tau}$, we need a substitute for (4.3). We will make the assumption that the parametrizations κ_i can be chosen in such a way that for each i, $z \mapsto \sqrt{\det(D\kappa_i(z)^T D\kappa_i(z))}$ is *piecewise constant* with respect to $\mathcal{T}_{-m,i}$. Under this assumption, for $u, v \in L_2(\Gamma)$ and $k \geq -m$ we have

$$(4.3^{(\Gamma)}) \qquad (u, v)_{L_2(\Gamma)} = \sum_i \sum_{\tau \in \mathcal{T}_{k,i}} \tfrac{\mathrm{vol}(\kappa_i(\tau))}{\mathrm{vol}(\check{\tau})} (u \circ \kappa_i \circ T_\tau, v \circ \kappa_i \circ T_\tau)_{L_2(\check{\tau})},$$

where $T_\tau : \check{\tau} \to \tau$ are affine bijections. Note that this assumption on the parametrizations allows us to handle manifolds consisting of patches that are for example parts of hyperplanes, spheres or cylinders.

In fact, since the dual norms $\sup_{0\neq u\in\mathcal{H}^s} \frac{|(u,v)_{L_2(\Gamma)}|}{\|u\|_{\mathcal{H}^s}}$ $(s \geq 0)$ turn into equivalent ones when the weight function $g : \Gamma \to I\!R : x \mapsto \sqrt{\det D\kappa_i(\kappa_i^{-1}(x))^T D\kappa_i(\kappa_i^{-1}(x))}$ $(x \in \Gamma_i)$ is multiplied with a globally smooth, positive function, we can handle even somewhat more general cases. That is, when for some collection $\{\kappa_i\}$, the function g is a product of a piecewise constant and a globally smooth, positive function, we can remove this smooth factor from the scalar product before constructing the wavelets with respect to a spectrally equivalent modified inner product.

If we now replace (4.2) in the domain case, by

$$(4.2^{(\Gamma)})\quad \varphi_{k,x}(y) = \begin{cases} \check{\varphi}_{\check{k},T_\tau^{-1}(\kappa_i^{-1}(x))}(T_\tau^{-1}(\kappa_i^{-1}(y))) & \text{if } \kappa_i^{-1}(x), \kappa_i^{-1}(y) \in \tau \in \mathcal{T}_{k-\check{k}-m,i} \\ 0 & \text{elsewhere} \end{cases},$$

the construction of the bases $\Phi_k = \Phi_k^{\ell,N}$ and Ξ_k, Ξ_k', and so the verification of (C1), (C1)$'$, as well as the construction of Φ_k', Θ_k , Υ_k and hence of the wavelets Ψ_k follows exactly the same lines as in the domain case. Thus, after constructing the bases from Proposition 4.5 and 4.6 on a reference element for given n, ℓ, ℓ', we obtain an $\mathcal{H}^s$-stable wavelet basis for $|s| \leq 1$. In particular, this covers the important case of polyhedral manifolds.

4.3.4. *Cancellation property.* Recall that aside from the validity of norm equivalences the vanishing moment property is a corner stone of wavelet concepts being essential for compression and adaptivity. All wavelets which are supported inside a patch retain in essence these vanishing moments (with respect to functions whose pre-image is a polynomial) and thus unfold their usual compression power. However, this may no longer be the case for wavelets whose support intersects several patches Γ_i. Its part on each patch is not a wavelet and thus has no vanishing moments. Nevertheless, recall also that not the vanishing moments are important but the fact that integration of a wavelet against a smooth function produces something small which is perhaps more appropriately referred to as *cancellation property.* It is this fact that is used to derive the compression results in connection with boundary integral equations and we will point out next that the cancellation property remains valid in a form that gives rise to optimal compression effects in the context of singular integral operators.

Proposition 4.7. *Under the assumptions from §4.3.1 and §4.3.3, the wavelets on manifolds obtained by the element-by-element construction have the following cancellation properties: When v is a smooth function on Γ one has*

$$|(v,\psi_{k+1,x})_{L_2(\Gamma)}| \lesssim 2^{-k(d'+n/2)}|v|_{W_\infty^{d'}(\hat{\sigma}_{k+1,x})}\|\psi_{k+1,x}\|_{L_2(\Gamma)}, \tag{4.8}$$

uniformly in k, where $d' := 2^{\ell'}+1$, $\hat{\sigma}_{k+1,x}$ is some neighborhood of the support of $\psi_{k+1,x}$ of diameter $\eqsim 2^{-k}$ and $|v|_{W_\infty^{d'}(\hat{\sigma}_{k+1,x})} := \max_{|\alpha|=d'} \sup_{\eta\in\hat{\sigma}_{k+1,x}} |D^\alpha v(\eta)|$.

Proof. Consider the projector

$$P_k v := \sum_{y\in I_k} \frac{(v,\phi'_{k,y})_{L_2(\Gamma)}}{(\theta_{k+1,y},\phi'_{k,y})_{L_2(\Gamma)}}\theta_{k+1,y},$$

and note that, by construction, $P_k\psi_{k+1,x}=0$. Thus

$$(v,\psi_{k+1,x})_{L_2(\Gamma)} = (v,(I-P_k)\psi_{k+1,x})_{L_2(\Gamma)} = ((I-P'_k)v,\psi_{k+1,x})_{L_2(\Gamma)}.$$

In addition let

$$P_k^{(N)} v := \sum_{y\in I_k} v(y)\phi_{k,y}^{\ell',N}$$

denote the nodal projector associated with the nodal basis $\Phi_k^{\ell',N}$ of $\mathcal{S}'_k$. Since $P'_kP_k^{(N)} = P_k^{(N)}$, one obtains

$$(I-P'_k)v = (I-P'_k)(I-P_k^{(N)})v, \tag{4.9}$$

so that

$$|(v,\psi_{k+1,x})_{L_2(\Gamma)}| \le \|(I-P'_k)(I-P_k^{(N)})v\|_{L_2(\sigma_{k+1,x})}\|\psi_{k+1,x}\|_{L_2(\Gamma)}, \tag{4.10}$$

where $\sigma_{k+1,x} := \operatorname{supp}(\psi_{k+1,x})$. Moreover,

$$\|(I-P'_k)(I-P_k^{(N)})v\|_{L_2(\sigma_{k+1,x})} \lesssim \|(I-P_k^{(N)})v\|_{L_2(\hat\sigma_{k+1,x})},$$

where $\hat\sigma_{k+1,x}$ is the union of the supports of the $\theta_{k+1,y}$ for which the support of $\phi'_{k,y}$ intersects $\sigma_{k+1,x}$. Thus one still has that $\operatorname{diam}(\hat\sigma_{k+1,x}) \eqsim 2^{-k}$ and thus $\hat\sigma_{k+1,x}$ is comprised of a uniformly bounded number of simplices in the underlying triangulation. Hence it suffices to estimate $\|(I-P_k^{(N)})v\|_{L_2(\kappa_i(\tau))}$ for some $\tau\in\mathcal{T}_{k-\ell',i}$ such that $\kappa_i(\tau)\subset\hat\sigma_{k+1,x}$. To this end, note that any continuous function q on Γ of the form $q\,|_{\kappa_i(\tau)} = p\circ\kappa_i^{-1}$ where p is a polynomial of at most degree $d'-1$ is reproduced by $P_k^{(N)}$. Therefore for any such q one has

$$\|P_k^{(N)}v - v\|_{L_2(\kappa_i(\tau))} \le \|v-q\|_{L_2(\kappa_i(\tau))} + \|P_k^{(N)}(q-v)\|_{L_2(\kappa_i(\tau))}.$$

Writing $v = w\circ\kappa_i^{-1}$ we can choose the polynomial p as a Taylor polynomial of w and obtain a bound of the form $2^{-nk/2}2^{-kd'}|v|_{W_\infty^{d'}(\kappa_i(\tau))}$ for the first summand. Since

$$\begin{aligned}
\|P_k^{(N)}(q-v)\|_{L_2(\kappa_i(\tau))} &\lesssim 2^{-nk/2}\|P_k^{(N)}(q-v)\|_{L_\infty(\kappa_i(\tau))}\\
&\le 2^{-nk/2}\sum_{y\in\kappa_i(\tau\cap V_{k,i})}|(q-v)(y)|\|\phi_{k,y}^{\ell',N}\|_{L_\infty(\Gamma)}\\
&\le 2^{-nk/2}\#(\tau\cap V_{k,i})\|p-w\|_{L_\infty(\tau)}, \qquad (4.11)
\end{aligned}$$

the same Taylor expansion argument as before yields again a bound of the form $2^{-nk/2}2^{-kd'}|v|_{W_\infty^{d'}(\kappa_i(\tau))}$ also for the second summand whence the assertion follows. □

Let us briefly point out now the relevance of the estimate (4.8) in the context of *matrix compression*. Suppose that $a(u,v) = (Au,v)_{L_2(\Gamma)}$ where the operator $A:\mathcal{H}^r\to\mathcal{H}^{-r}$ has the form

$$Av = \int_\Gamma K(\cdot,\eta)v(\eta)d\eta,$$

and the kernel K has *global support*. We wish to estimate the size of the entries

$$(A\psi_{k+1,x},\psi_{k'+1,y})_{L_2} = \int_\Gamma\int_\Gamma K(\zeta,\eta)\psi_{k+1,x}(\eta)\psi_{k'+1,y}(\zeta)d\eta d\zeta = (K,\psi_{k+1,x}\otimes\psi_{k'+1,y})_{L_2(\Gamma\times\Gamma)}. \tag{4.12}$$

To this end, consider for any function $v:\Gamma\times\Gamma\to\mathbb{R}$ and any two projectors P,Q, acting on functions defined on Γ, the *Boolean sum*

$$(P\oplus Q)v := (P\otimes I)v + (I\otimes Q)v - (P\otimes Q)v, \tag{4.13}$$

where, for instance, $(P\otimes I)$ means that for any fixed η the projector P is applied to the first variable ζ. The remaining expressions are defined in obvious analogy. One readily infers now from (4.13) that

$$(P_k\oplus P_{k'})(\psi_{k+1,x}\otimes\psi_{k'+1,y}) = 0, \tag{4.14}$$

whereas it is straightforward to verify, on account of (4.9), that

$$(I-(P'_k\oplus P'_{k'}))K = (I-(P'_k\oplus P'_{k'}))(I-(P_k^{(N)}\oplus P_{k'}^{(N)}))K. \tag{4.15}$$

Hence, on account of (4.12), (4.14) and (4.15), the same reasoning as in the proof of Proposition 4.7 yields

$$\frac{|(A\psi_{k+1,x},\psi_{k'+1,y})_{L_2}|}{\|\psi_{k+1,x}\|_{L_2}\|\psi_{k'+1,y}\|_{L_2}} \lesssim \|(I-(P_k^{(N)}\oplus P_{k'}^{(N)}))K\|_{L_2(\hat\sigma_{k+1,x}\times\hat\sigma_{k'+1,y})}. \tag{4.16}$$

Now the Boolean sum projector is designed to ensure that the coordinate errors *multiply*, i.e.,

$$(I-(P_k^{(N)}\oplus P_{k'}^{(N)})) = (I-P_k^{(N)})\otimes(I-P_{k'}^{(N)}). \tag{4.17}$$

By definition, one has

$$((I-P_k^{(N)})\otimes(I-P_{k'}^{(N)})v)(\zeta,\eta) = (I-P_k^{(N)}\otimes I)((I-I\otimes P_{k'}^{(N)})v)(\zeta,\eta),$$

meaning that $(I-I\otimes P_{k'}^{(N)})v$ acts on η for each ζ as a parameter and then $(I-P_k^{(N)}\otimes I)$ acts on the result of the first operation but with respect to the variable ζ. Hence as above one obtains

$$\begin{aligned}&\|(I-(P_k^{(N)}\oplus P_{k'}^{(N)}))K\|_{L_2(\hat\sigma_{k+1,x}\times\hat\sigma_{k'+1,y})}\\ &\lesssim 2^{-(k+k')n/2}\|(I-P_k^{(N)}\otimes I)((I-I\otimes P_{k'}^{(N)})v)\|_{L_\infty(\hat\sigma_{k+1,x}\times\hat\sigma_{k'+1,y})}.\end{aligned} \tag{4.18}$$

Applying now the same argument used in (4.11) first with respect to the variable ζ and then with respect to η yields

$$\begin{aligned}&\max_{(\zeta,\eta)\in\hat\sigma_{k+1,x}\times\hat\sigma_{k'+1,y}}|(I-P_k^{(N)}\otimes I)((I-I\otimes P_{k'}^{(N)})v)(\zeta,\eta)|\\ &\lesssim 2^{-dk}\max_{|\alpha|=d'}\max_{(\zeta,\eta)\in\hat\sigma_{k+1,x}\times\hat\sigma_{k'+1,y}}|D_\zeta^\alpha((I-I\otimes P_{k'}^{(N)})v)(\zeta,\eta)|\\ &= 2^{-dk}\max_{|\alpha|=d'}\max_{(\zeta,\eta)\in\hat\sigma_{k+1,x}\times\hat\sigma_{k'+1,y}}|(I-I\otimes P_{k'}^{(N)})(D_\zeta^\alpha v)(\zeta,\eta)|\\ &\lesssim 2^{-d(k+k')}\max_{|\alpha|,|\beta|=d'}\max_{(\zeta,\eta)\in\hat\sigma_{k+1,x}\times\hat\sigma_{k'+1,y}}|D_\eta^\beta(D_\zeta^\alpha v)(\zeta,\eta)|,\end{aligned} \tag{4.19}$$

where we have used that taking derivatives with respect to the variable ζ commutes with the action of the nodal interpolant on the variable η.

Thus whenever K is sufficiently smooth on $\hat{\sigma}_{k+1,x}\times\hat{\sigma}_{k'+1,y}$ combining the estimate (4.16) with (4.18) and (4.19) provides

$$\frac{|(A\psi_{k+1,x},\psi_{k'+1,y})_{L_2}|}{\|\psi_{k+1,x}\|_{L_2}\|\psi_{k'+1,y}\|_{L_2}} \lesssim 2^{-(k+k')(d'+n/2)} \max_{|\alpha|,|\beta|=d'} \|D_\zeta^\alpha D_\eta^\beta K\|_{L_\infty(\hat{\sigma}_{k+1,x}\times\hat{\sigma}_{k'+1,y})}. \tag{4.20}$$

Now in many applications the kernel K is actually smooth away from its diagonal. In fact, for the previously mentioned double-, single-layer or hypersingular operators one has estimates of the form

$$|D_\zeta^\alpha D_\eta^\beta K(\zeta,\eta)| \lesssim \mathrm{dist}\,(\zeta,\eta)^{-(n+|\alpha|+|\beta|+2r)},$$

when $2r$ is the order of the operator A. Combining this with (4.20), assuming that the wavelets are normalized to $\|\psi_{k,x}\|_{L_2}\eqsim 1$, one finally obtains for wavelets with disjoint supports $\Omega_{k,x}=\mathrm{supp}\,\psi_{k,x}$ the typical decay estimate

$$|(A\psi_{k+1,x},\psi_{k'+1,y})_{L_2}| \lesssim \frac{2^{-(k+k')(d'+n/2)}}{\mathrm{dist}\,(\Omega_{k+1,x},\Omega_{k'+1,y})^{n+2d'+2r}},$$

which all the above mentioned compression approaches are based upon, see [DPS94, PS95, Sch95].

4.4. **Some comments on implementation.** Before we turn to the discussion of several concrete realizations of the above concepts we will make some preparatory comments concerning *implementation*. We will focus on the basis transformation $\mathbf{T}_j$ on $\mathcal{S}_j$ from wavelet basis $\Psi^j=\cup_{k=0}^j\Psi_k$ to nodal basis $\Phi_j^{\ell,N}$. As we already have noted, $\mathbf{T}_j$ can be applied at the cost of $\mathcal{O}(\dim\mathcal{S}_j)$ operations. Here we will make some comments which our *quantitative* complexity analysis in each case will be based upon. Before starting, we briefly explain our interest in the implementation of $\mathbf{T}_j$ and its adjoint.

Given some $f\in\mathcal{H}^{-r}$, consider the problem of finding $u_j\in\mathcal{S}_j$ such that

$$a(u_j,v_j)=f(v_j)\qquad (v_j\in\mathcal{S}_j), \tag{4.21}$$

where a is a scalar product satisfying $a(v,v)\eqsim\|v\|^2_{\mathcal{H}^r}$, i.e. the problem (4.21) is symmetric and elliptic of order $2r$. Let us denote the matrix-vector equations corresponding to (4.21) with respect to $\Phi_j^{\ell,N}$ and Ψ^j respectively by $\mathbf{A}_{\Phi_j^{\ell,N}}\mathbf{U}_{\Phi_j^{\ell,N}}=\mathbf{F}_{\Phi_j^{\ell,N}}$ and $\mathbf{A}_{\Psi^j}\mathbf{U}_{\Psi^j}=\mathbf{F}_{\Psi^j}$. When r is in the stability range $(-\gamma',\gamma)$ of the wavelet bases, the stiffness matrix $\mathbf{A}_{\Psi^j}$, preconditioned by its diagonal, is uniformly well conditioned. On the other hand, since the wavelets on lower levels have large supports, $\mathbf{F}_{\Psi^j}=(f(\psi_{k,x}))_{k=0,\dots,j,x\in J_k}$ cannot directly be computed in $\mathcal{O}(\dim\mathcal{S}_j)$ operations, and $\mathbf{U}_{\Psi^j}^T\Psi^j$ is frequently not the representation of u_j that one likes to have. However, both problems can be solved by using the relations $\mathbf{F}_{\Psi^j}=\mathbf{T}_j^*\mathbf{F}_{\Phi_j^{\ell,N}}$ and $\mathbf{U}_{\Phi_j^{\ell,N}}=\mathbf{T}_j\mathbf{U}_{\Psi^j}$. Of course, the situation requires a different appraisal when employing *adaptive* methods so that the subspace spanned by those wavelets needed to approximate the solution within a desired accuracy might have a significantly

smaller dimension than the full trial space $\mathcal{S}_j$ where j is the highest level of resolution appearing in the wavelet subspace. We will, however, not discuss this issue in this paper.

Secondly, when $a(\cdot,\cdot)$ stems from a differential equation the stiffness matrix $\mathbf{A}_{\Psi^j}$ will, in contrast to $\mathbf{A}_{\Phi_j^{\ell,N}}$, not be (fully) sparse. Nevertheless, upon using the relation $\mathbf{A}_{\Psi^j} = \mathbf{T}_j^* \mathbf{A}_{\Phi_j^{\ell,N}} \mathbf{T}_j$, the application of $\mathbf{A}_{\Psi^j}$ to a vector can be computed in $\mathcal{O}(\dim \mathcal{S}_j)$ operations. Note that in case $a(\cdot,\cdot)$ stems from an integral equation, the situation is reversed. Although both $\mathbf{A}_{\Psi^j}$ and $\mathbf{A}_{\Phi_j^{\ell,N}}$ are generally densely populated, it is $\mathbf{A}_{\Psi^j}$ that, thanks to the cancellation property, which will be close to a sparse matrix.

In the following, we order the wavelets in such a way that Ψ^j, again viewed as a column vector, can be written as $(\Psi^j)^T = (\Psi_j^T, \cdots, \Psi_0^T)^T$. From $(\Psi^j)^T = (\Phi_j^{\ell,N})^T \mathbf{T}_j$, $\Psi_0 = \Phi_0$ and $[\Psi_{k+1}^T \quad (\Phi_k^{\ell,N})^T] = (\Phi_{k+1}^{\ell,N})^T [\mathbf{P}_{k+1,1} \quad \mathbf{P}_{k+1,0}]$, we obtain the basic version of the pyramid scheme

$$\mathbf{T}_0 = \mathbf{I}, \;\; \mathbf{T}_j = [\mathbf{P}_{j,1} \quad \mathbf{P}_{j,0}\mathbf{T}_{j-1}] \; (j = 1, 2, \dots).$$

We will refer to this version as the "naive" implementation.

We discuss two improvements of this implementation. Theorem 3.1 shows that

$$\Psi_{k+1}^T = \Upsilon_{k+1}^T - \Theta_{k+1}^T \mathbf{Z}_{k+1},$$

where the $\#I_k \times \#J_{k+1}$ matrix $\mathbf{Z}_{k+1}$ is defined by $(\mathbf{Z}_{k+1})_{y,x} = \frac{(v_{k+1,x}, \phi'_{k,y})_{L_2}}{(\theta_{k+1,y}, \phi'_{k,y})_{L_2}}$. In our concrete realizations discussed below in Section 4.5, we will always choose

$$\Upsilon_{k+1} = \{\phi_{k+1,x}^{\ell,N} : x \in J_{k+1}\} \subset \Phi_{k+1}^{\ell,N}, \tag{4.22}$$

that is, $\Upsilon_{k+1}^T = (\Phi_{k+1}^{\ell,N})^T E_{k+1}$, with the $\#I_{k+1} \times \#J_{k+1}$ matrix $\mathbf{E}_{k+1}$ defined by $(\mathbf{E}_{k+1})_{x,y} = \delta_{x,y}$. Denoting by $\mathbf{G}_{k+1}$ the $\#I_{k+1} \times \#I_k$ matrix defined by $\Theta_{k+1}^T = (\Phi_{k+1}^{\ell,N})^T \mathbf{G}_{k+1}$, we obtain

$$\mathbf{P}_{k+1,1} = \mathbf{E}_{k+1} - \mathbf{G}_{k+1}\mathbf{Z}_{k+1}.$$

Thanks to the fact that the sparse matrices $\mathbf{G}_{k+1}$ and $\mathbf{Z}_{k+1}$ have smaller sizes than $\mathbf{P}_{k+1,1}$, at least for $n > 1$ making use of this "factorized" form of $\mathbf{P}_{k+1,1}$ results in a more efficient implementation.

In our realizations we will construct $\theta_{k+1,y}$ $(y \in I_k)$ as linear combinations of $\phi_{k+1-i,x}^{\ell,N}$ $(x \in I_{k+1-i})$ for $0 \le i \le m+1-\ell$, where $m = \max\{\ell, \ell'\}$. As a second improvement, instead of expressing for $i > 0$, $\phi_{k+1-i,x}^{\ell,N}$ directly in terms of $\phi_{k+1,z}^{\ell,N}$, we can exploit the fact that the prolongation operations needed for this purpose have to be executed anyway. This observation has been also made in [Swe95, LO96].

More specifically, let

$$\Theta_{k+1}^T = \sum_{i=0}^{m+1} (\Phi_{k+1-i}^{\ell,N})^T \mathbf{G}_{k+1,i}. \tag{4.23}$$

Then setting $(\mathbf{d}^j)^T = (\mathbf{d}_j^T, \cdots, \mathbf{d}_0)^T$ $(\mathbf{d}_k \in \mathbb{R}^{\#J_k})$, the arrays $\mathbf{c}_j = \mathbf{T}_j \mathbf{d}^j$ can be computed by performing the following steps:

$\mathbf{d}_{j+1} = \cdots = \mathbf{d}_{j+m-\ell} := 0$
$\mathbf{c}_{-1} = \cdots = \mathbf{c}_{-m+\ell} := 0$
$\mathbf{c}_0 := \mathbf{d}_0$
for $k = 1$ to $j + m - \ell$ do
  $\mathbf{q}_k := \mathbf{Z}_k \mathbf{d}_k$
  $\mathbf{c}_k := \mathbf{E}_k \mathbf{d}_k - \mathbf{G}_{k,0} \mathbf{q}_k$
  for $i = 1$ to $m + \ell + 1$ do
    $\mathbf{c}_{k-i} := \mathbf{c}_{k-i} - \mathbf{G}_{k,-i} \mathbf{q}_k$
  end
  $\mathbf{c}_{k-m+\ell} := \mathbf{c}_{k-m+\ell} + \mathbf{P}_{k-m+\ell,0} \mathbf{c}_{k-m+\ell-1}$
end

We will refer to this implementation as the "advanced" implementation. Ignoring, for the case $\ell \leq m-1$, a few computations on levels with negative indices, we note that besides the costs of applying the prolongation operators $\mathbf{P}_{k,0}$ $(1 \leq k \leq j)$, that are determined by the multiresolution analysis $\mathcal{S}_0 \subset \mathcal{S}_1 \subset \cdots$, and the costs of applying the $\mathbf{Z}_k$ $(1 \leq k \leq j)$, determined by the Φ'_{k-1}, the costs of the "advanced" implementation of the pyramid scheme depend on the total number of of non-zero entries in the matrices $\mathbf{G}_{k,i}$. In our realizations we will minimize this number of non-zero entries in the non-unique representation (4.23). The resulting version of "advanced" implementation will turn out to be significantly more economic than the " naive" one.

4.5. **Applications, realization of concrete cases.** We will confine the discussion to *piecewise linear* wavelets, i.e., $\ell = 0$, in combination with the following cases:

- $\ell' = 0$, $n \in \{1, 2, 3\}$,
- $\ell' = 1$, $n \in \{1, 2\}$,
- $\ell' = 2$, $n \in \{1, 2\}$.

In all cases one has $m = \max\{\ell, \ell'\} = \ell'$, so that $\check{\mathcal{T}}_{-\ell'} = \{\check{\tau}\}$. We choose

$$\check{\tau} = \{x \in I\!R^n : 0 \leq x_1 \leq \cdots \leq x_n \leq 2^{\ell'+1}\},$$

so that $\check{I}_{\check{k}} = \check{\tau} \cap 2^{1-\check{k}} I\!N^n$ $(-\ell' \leq \check{k} \leq 1)$. In particular, $\check{I}_{-\ell'}$ is the set of the $n+1$ vertices of $\check{\tau}$.

Concerning (C1) and (C1)$'$, when $\ell' = 0$ we have $\mathcal{S}_k = \mathcal{S}'_k$, and so both conditions are trivially satisfied. For the other cases we have verified the sufficient condition $\Re(\check{\mathbf{M}}_0) > 0$ from Proposition 4.5, where we took $\Xi_0 = \Phi_0^{0,N}$ and $\Xi'_0 = \Phi_0^{\ell',N}$.

As we have already announced in (4.22), in all cases we fix

$$\check{\Upsilon}_1 = \{\check{\phi}_{1,\check{x}}^{0,N} : \check{x} \in \check{J}_1\},$$

and thus

$$\Upsilon_{k+1} = \{\phi_{k+1,x}^{0,N} : x \in J_{k+1}\}.$$

For the above mentioned cases, in the following we define the remaining ingredients of the wavelet construction, viz. a basis $\check{\Phi}'_0 = \{\check{\phi}'_{0,\check{y}} : \check{y} \in \check{I}_0\}$ of $\check{S}(\ell', 0)$, and a dual set $\check{\Theta}_1 = \{\check{\theta}_{1,\check{y}} : \check{y} \in \check{I}_0\} \subset \check{S}(0,1)$ such that $\check{\Upsilon}_1 \cup \check{\Theta}_1$ is a basis of $\check{S}(0,1)$:

4.5.1. *The case* $\ell' = 0$, $n \in \{1,2,3\}$. In this case $\check{\mathcal{T}}_0 = \{\check{\tau}\}$, $\check{I}_0$ is the set of vertices of $\check{\tau}$, and $\check{J}_1$ is the set of midpoints of the edges of $\check{\tau}$.

We set

$$\check{\Phi}'_0 = \check{\Phi}_0^{0,N}$$

and search for $\check{\theta}_{1,\check{y}}$ ($\check{y} \in \check{I}_0$) as a linear combination of the fine- and coarse-grid nodal basis functions $\check{\phi}_{1,\check{y}}^{0,N}$ and $\check{\phi}_{0,\check{y}}^{0,N}$. Some computations reveal that

$$\check{\theta}_{1,\check{y}} = 2(n+1)!\,(\check{\phi}_{1,\check{y}}^{0,N} - (\tfrac{1}{2})^{n+1}\check{\phi}_{0,\check{y}}^{0,N})$$

satisfies $(\check{\theta}_{1,\check{y}}, \check{\phi}_{0,\check{z}}^{0,N})_{L_2(\check{\tau})} = \delta_{\check{y},\check{z}}$ ($\check{y},\check{z} \in \check{I}_0$), for any $n \in I\!N$.

The union $\check{\Upsilon}_1 \cup \check{\Theta}_1$ is a basis of $\check{S}(0,1)$. Both $\check{\Phi}'_0$ and $\check{\Upsilon}_1 \cup \check{\Theta}_1$ satisfy the condition (A1) of invariance on $\partial\check{\tau}$ under permutations of the barycentric coordinates, as well as the condition (A2) which says that basis functions labeled with $\check{x}$ vanish on faces that do not contain $\check{x}$. Taking $\Psi_0 = \Phi_0^{0,N}$, the application of Proposition 4.6 and Theorem 3.1 now yields a basis $\Psi^j = \cup_{k=0}^{j}\Psi_k$ of $\mathcal{S}_j$ that is uniformly stable in $\mathcal{H}^s$ for $|s| < \frac{3}{2}$ ($|s| \le 1$ for Lipschitz manifolds). Moreover, its elements, except those from the coarsest level, have vanishing moments of order $2^{\ell'} + 1 = 2$ (respectively satisfy estimates of the type (4.8)).

By construction, we have

$$\psi_{k+1,x} = \phi_{k+1,x}^{0,N} - \sum_{y\in I_k} \frac{(\phi_{k+1,x}^{0,N}, \phi_{k,y}^{0,N})_{L_2}}{(\theta_{k+1,y}, \phi_{k,y}^{0,N})_{L_2}}\theta_{k+1,y} \quad (k \ge 0,\ x \in J_{k+1}),$$

with $\theta_{k+1,y} = 2(n+1)!\,(\phi_{k+1,y}^{0,N} - (\frac{1}{2})^{n+1}\phi_{k,y}^{0,N})$. As shown in (4.6), the computation of the coefficients $\frac{(\phi_{k+1,x}^{0,N},\phi_{k,y}^{0,N})_{L_2}}{(\theta_{k+1,y},\phi_{k,y}^{0,N})_{L_2}}$ can be reduced to the computation of $(\check{\phi}_{1,\check{x}}^{0,N}, \check{\phi}_{0,\check{y}}^{0,N})_{L_2(\check{\tau})}$ for $\check{x} \in \check{J}_1$, $\check{y} \in \check{I}_0$.

For $n \le 2$, the involved basis functions on the reference element, and so the corresponding scalar products are invariant under permutations of the barycentric coordinates (cf. Remark 4.2). For $n = 1$,

$$(\check{\phi}_{1,\check{x}}^{0,N}, \check{\phi}_{0,\check{y}}^{0,N})_{L_2(\check{\tau})} = \tfrac{1}{2} \qquad (\check{x} \in \check{J}_1,\ \check{y} \in \check{I}_0).$$

An illustration of the wavelet construction in this case is given in Fig. 1.

For $n = 2$, $\check{x} \in \check{J}_1$, $\check{y} \in \check{I}_0$ one has

$$(\check{\phi}_{1,\check{x}}^{0,N}, \check{\phi}_{0,\check{y}}^{0,N})_{L_2(\check{\tau})} = \begin{cases} \frac{5}{24} & \text{if } \check{x},\check{y} \text{ share an edge} \\ \frac{1}{12} & \text{if } \check{x},\check{y} \text{ do not share an edge} \end{cases}.$$

Recall that for $n = 3$ the dyadic subdivision of tetrahedra is not unique. More precisely, the convex hull of the midpoints of the edges has yet to be decomposed into four tetrahedra. This can be done by making any two midpoints of edges that have no vertex in common the end points of a new edge. There are three such possibilities to form a dyadic triangulation $\check{\mathcal{T}}_1$. Once such a decomposition has been fixed the set $\check{J}_1$ can be splitted into the set A containing these two midpoints connected with the new edge, and the set B of the remaining midpoints. Note that the elements of A

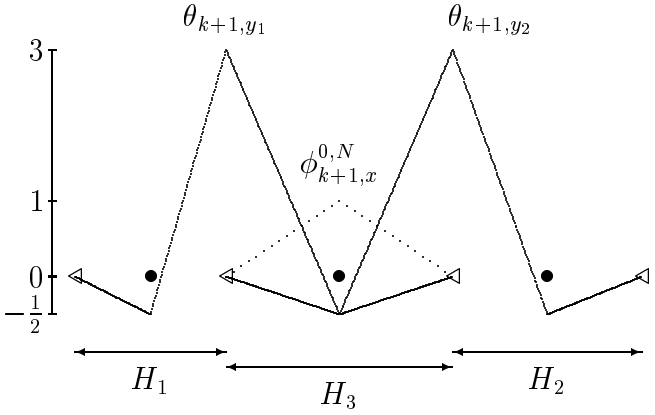

FIGURE 1. Wavelet construction for $n = 1$, $\ell = \ell' = 0$.

are vertices of six tetrahedra, while each element of B is shared by four tetrahedra. One can verify that for $\check{x} \in \check{J}_1$, $\check{y} \in \check{I}_0$ one has

$$(\check{\phi}^{0,N}_{1,\check{x}}, \check{\phi}^{0,N}_{0,\check{y}})_{L_2(\check{\tau})} = \begin{cases} \frac{4}{48} & \text{if } \check{x} \in A \text{ and } \check{x}, \check{y} \text{ share an edge} \\ \frac{2}{48} & \text{if } \check{x} \in A \text{ and } \check{x}, \check{y} \text{ do not share an edge} \\ \frac{3}{48} & \text{if } \check{x} \in B \text{ and } \check{x}, \check{y} \text{ share an edge} \\ \frac{1}{48} & \text{if } \check{x} \in B \text{ and } \check{x}, \check{y} \text{ do not share an edge.} \end{cases}$$

As an application of the complexity analysis carried out in Section 4.4, we give an asymptotic operation count (for $j \to \infty$) per unknown for performing the basis transformation on $\mathcal{S}_j$. For the case $n > 1$ we assume that the mesh is uniform, i.e., $\Omega = [0,1]^n$ and

$$\mathcal{T}_k = \{2^{-k}\alpha + \{x \in \mathbf{R}^n; 0 \le x_{\sigma(1)} \le \cdots \le x_{\sigma(n)} \le 2^{-k}\} : \alpha \in \{1, \dots, 2^k - 1\}^n, \\ \sigma \text{ a permutation of } \{1, \dots, n\}\}.$$

We do not count scalar operations when they are combined with vector operations.

For $n = 1$, $\psi_{k+1,x}$ equals $\phi^{0,N}_{k+1,x}$ minus some linear combination of two $\theta_{k+1,y}$'s, i.e., the matrices $\mathbf{Z}_{k+1}$ from Section 4.4 contain only two nonzero entries per column. By expanding each $\theta_{k+1,y}$ in terms of $\Phi^{0,N}_{k+1}$, we see that each wavelet is a linear combination of 5 nodal basis functions on its level. For both the "naive" and the "advanced" implementation, the operation count is **8** operations per unknown.

For $n = 2$, $\psi_{k+1,x} - \phi^{0,N}_{k+1,x}$ is a linear combination of four $\theta_{k+1,y}$'s, in the uniform mesh case yielding a wavelet that is a linear combination of 23 nodal basis functions on its level (see [Ste97b]). The operation count for the advanced implementation is **8** operations, compared to $25\frac{1}{3}$ for the naive implementation. Thus, the advanced implementation realizes the basis transformation at the same expense as the naive implementation of the basis transformation for a wavelet basis where the wavelets are linear combinations of an average of $5\frac{2}{3}$ nodal basis functions.

For $n = 3$, in the uniform mesh case $\psi_{k+1,x} - \phi^{0,N}_{k+1,x}$ is a linear combination of six (3 of 7 cases) or eight (4 of 7 cases) $\theta_{k+1,y}$'s, or each wavelet is a linear combination of 77 or 101 nodal basis functions on its level (see [Ste97b]). The operation count

for the advanced implementation is $\mathbf{10\frac{2}{7}}$ operations, compared to $92\frac{4}{7}$ operations for the naive implementation.

To compare this, with $\psi_{k+1,x} = \phi^{0,N}_{k+1,x}$, i.e., with the hierarchical basis (cf. [Yse86]), an analogous operation count for the basis transformation to nodal basis yields 4, $3\frac{1}{3}$ and $3\frac{1}{7}$ operations per unknown, for $n = 1$, 2 and 3, respectively. Of course, the hierarchical basis neither has any cancellation properties nor gives rise to norm equivalences within the relevant scope of Sobolev regularity.

4.5.2. *The case* $\ell' = 1$, $n = 1$. In this case $\check{\mathcal{T}}_{-1} = \{\check{\tau}\}$, $\check{\tau} = [0,4]$, $\check{I}_{-1} = \{0,4\}$, $\check{J}_0 = \{2\}$ and $\check{J}_1 = \{1,3\}$. Whenever this is relevant, we order elements from the index sets with increasing sizes.

An obvious choice for $\check{\Phi}'_0$ would be to take $\check{\Phi}^{1,N}_0$, i.e., the nodal basis of second order polynomials with respect to the nodes $\check{I}_0$. However, note that we may add a multiple of $\check{\phi}^{1,N}_{0,2}$ to $\check{\phi}^{1,N}_{0,0}$ and $\check{\phi}^{1,N}_{0,4}$, since then (A2) is retained. We use this freedom to minimize the number of non-zero coefficients $(\check{\phi}^{0,N}_{1,\check{x}}, \check{\phi}'_{0,\check{y}})_{L_2(\check{\tau})}$ ($\check{x} \in \check{J}_1$, $\check{y} \in \check{I}_0$), and thus the number of non-zero coefficients in (3.1). We take

$$\check{\Phi}'^{\,T}_0 = (\check{\Phi}^{1,N}_0)^T \begin{bmatrix} 1 & 0 & 0 \\ \frac{5}{24} & 1 & \frac{5}{24} \\ 0 & 0 & 1 \end{bmatrix}.$$

This choice yields

$$\left((\check{\phi}^{0,N}_{1,\check{x}}, \check{\phi}'_{0,\check{y}})_{L_2(\check{\tau})}\right)_{\check{x}\in\check{J}_1,\check{y}\in\check{I}_0} = \begin{bmatrix} \frac{1}{2} & \frac{17}{24} & 0 \\ 0 & \frac{17}{24} & \frac{1}{2} \end{bmatrix}.$$

We wish to represent $\check{\theta}_{1,\check{y}}$ for $\check{y} \in \{0,4\}$ as a linear combination of $\check{\phi}^{0,N}_{-1,\check{y}}$, $\check{\phi}^{0,N}_{0,\check{y}}$ and $\check{\phi}^{0,N}_{1,\check{y}}$, and $\check{\theta}_{1,2}$ as a linear combination of $\check{\phi}^{0,N}_{0,2}$ and $\check{\phi}^{0,N}_{1,2}$. Another option would be to select $\check{\theta}_{1,0}$ e.g. as a linear combination of $\check{\phi}^{0,N}_{1,0}$, $\check{\phi}^{0,N}_{1,1}$ and $\check{\phi}^{0,N}_{1,2}$, and analogously for $\check{\theta}_{1,4}$. This choice would minimize the supports of the resulting wavelets. However, note that with the latter $\check{\theta}_{1,0}$ and $\check{\theta}_{1,4}$, global dual basis functions $\theta_{k+1,y}$ with $y \in I_k$ a boundary point of a "macro element", i.e. $y \in I_{k-1}$, would be linear combinations of 5 nodal basis functions. With our choice, global dual basis functions $\theta_{k+1,y}$ are linear combinations of 3 or 2 ($y \in I_{k-1}$ or $y \in J_k$) nodal basis functions, however, belonging to different levels. Nevertheless, as explained in Section 4.4, this latter fact is harmless as far as the cost of implementation is concerned.

At this point we stress that the favorable effect on the costs of the implementation of selecting $\check{\theta}_{1,\check{y}}$ as a linear combination of nodal basis functions from different levels is more enhanced in more dimensions. On the other hand, it will appear that in more dimensions, there is less freedom in selecting a clever $\check{\Phi}'_0$ such that $\left((\check{\phi}^{0,N}_{1,\check{x}}, \check{\phi}'_{0,\check{y}})_{L_2(\check{\tau})}\right)_{\check{x}\in\check{J}_1,\check{y}\in\check{I}_0}$ has possibly many zeros.

Some computations show that

$$\begin{aligned} \check{\theta}_{1,\check{y}} &= \tfrac{1}{6}\check{\phi}^{0,N}_{-1,\check{y}} - 2\check{\phi}^{0,N}_{0,\check{y}} + \tfrac{16}{3}\check{\phi}^{0,N}_{1,\check{y}}, \qquad \check{y} \in \{0,4\}, \\ \check{\theta}_{1,2} &= -\tfrac{22}{17}\check{\phi}^{0,N}_{0,2} + \tfrac{56}{17}\check{\phi}^{0,N}_{1,2}, \end{aligned}$$

satisfy $(\check\theta_{1,\check y},\check\phi'_{0,\check z})_{L_2(\check\tau)}=\delta_{\check y,\check z}$ $(\check y,\check z\in\check I_0)$, and $\check\Upsilon_1\cup\check\Theta_1$ is a basis of $\check S(0,1)$.

Since $\check\Phi'_0$ and $\check\Upsilon_1\cup\check\Theta_1$ satisfy (A2), and are invariant under permutations of the barycentric coordinates (cf. Remark 4.2), Proposition 4.6 and Theorem 3.1 yield an $\mathcal{H}^s$-stable wavelet basis for $|s|<\frac32$ $(|s|\le 1)$, now consisting of wavelets having $2^{\ell'}+1=3$ vanishing moments. An illustration is given in Fig. 2.

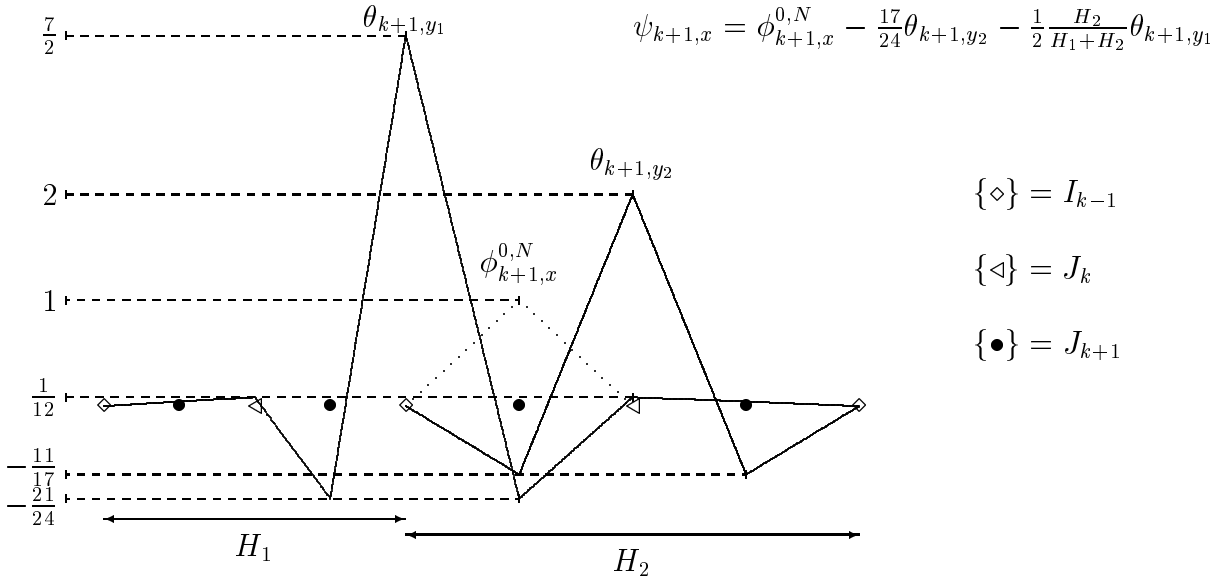

FIGURE 2. Wavelet construction for $n=1$, $\ell=0$ and $\ell'=1$.

As in the case $\ell=0$, $n=1$, $\psi_{k+1,x}-\phi^{0,N}_{k+1,x}$ is a linear combination of only two $\theta_{k+1,y}$'s, which itself are linear combinations of 3 and 2 nodal basis functions from different levels. With the "advanced" implementation as exposed in Section 4.4, for $j\to\infty$ the basis transformation $\mathbf{T}_j$ on $\mathcal{S}_j$ from wavelet- to nodal basis costs $\mathbf{8\frac12}$ operations per unknown.

4.5.3. *The case $\ell'=2$, $n=1$.* In this case $\check{\mathcal{T}}_{-2}=\{\check\tau\}$, $\check\tau=[0,8]$, $\check I_{-2}=\{0,8\}$, $\check J_{-1}=\{4\}$, $\check J_0=\{2,6\}$ and $\check J_1=\{1,3,5,7\}$.

Following the ideas described in §4.5.2, we define

$$(\check\Phi'_0)^T=(\check\Phi_0^{2,N})^T\begin{bmatrix}1&0&0&0&0\\ \frac{29781}{315868}&1&\frac{439}{1132}&\frac{-417}{5807}&\frac{2551}{315868}\\ \frac{-2585}{157934}&\frac{1420}{5807}&1&\frac{1420}{5807}&\frac{-2585}{157934}\\ \frac{2551}{315868}&\frac{-417}{5807}&\frac{439}{1132}&1&\frac{29781}{315868}\\ 0&0&0&0&1\end{bmatrix}.$$

The resulting $\check\Phi'_0$ is a basis of $\check M(2,0)$, and

$$\Big((\check\phi^{0,N}_{1,\check x},\check\phi'_{0,\check y})_{L_2(\check\tau)}\Big)_{\check x\in\check J_1,\check y\in\check I_0}=\begin{bmatrix}\frac{12}{29}&\frac{5057}{5807}&0&0&0\\ 0&\frac{11281}{17421}&\frac{2723}{3396}&0&0\\ 0&0&\frac{2723}{3396}&\frac{11281}{17421}&0\\ 0&0&0&\frac{5057}{5807}&\frac{12}{29}\end{bmatrix}.$$

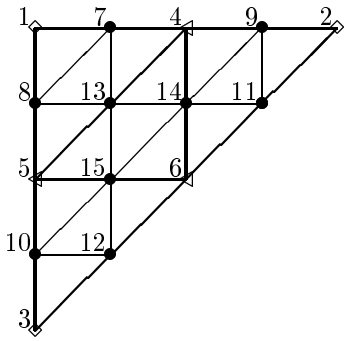

FIGURE 3. Numbering of $\check{I}_{-1} = \{\diamond\}$, $\check{J}_0 = \{\triangleleft\}$ and $\check{J}_1 = \{\bullet\}$.

With

$$\begin{aligned}
\check{\theta}_{1,0} &= \tfrac{2}{25}\check{\phi}^{0,N}_{-2,0} - \tfrac{93}{400}\check{\phi}^{0,N}_{-1,0} + \tfrac{1091}{900}\check{\phi}^{0,N}_{0,0} + \tfrac{137}{30}\check{\phi}^{0,N}_{1,0} - \tfrac{3799}{1800}\check{\phi}^{0,N}_{1,3} \\
\check{\theta}_{1,2} &= -\tfrac{5807}{5835}\check{\phi}^{0,N}_{1,1} + \tfrac{23228}{5835}\check{\phi}^{0,N}_{1,2} - \tfrac{2781553}{733265}\check{\phi}^{0,N}_{1,3} + \tfrac{4053286}{2199795}\check{\phi}^{0,N}_{1,4} - \tfrac{795559}{2199795}\check{\phi}^{0,N}_{1,5} \\
\check{\theta}_{1,4} &= \tfrac{16414}{40845}\check{\phi}^{0,N}_{-1,4} - \tfrac{535436}{106197}\check{\phi}^{0,N}_{0,4} + \tfrac{4874392}{530985}\check{\phi}^{0,N}_{1,4}\,,
\end{aligned}$$

and $\check{\theta}_{1,6}, \check{\theta}_{1,8}$ defined by permuting the barycentric coordinates, one has $(\check{\theta}_{1,\check{y}}, \check{\phi}'_{0,\check{z}})_{L_2(\check{\tau})} = \delta_{\check{y},\check{z}}$ $(\check{y}, \check{z} \in \check{I}_0)$, and $\check{\Upsilon}_1 \cup \check{\Theta}_1$ is a basis of $\check{S}(0,1)$.

Both $\check{\Phi}'_0$ and $\check{\Upsilon}_1 \cup \check{\Theta}_1$ satisfy (A2), and are invariant under permutations of the barycentric coordinates (cf. Remark 4.2), and so we obtain an $\mathcal{H}^s$-stable wavelet basis for $|s| < \frac{3}{2}$ $(|s| \le 1)$, consisting of wavelets having $2^{\ell'} + 1 = 5$ vanishing moments.

The above choice of $\check{\Theta}_1$ yields a minimizer for the cost of implementation. For $j \to \infty$, the basis transformation $\mathbf{T}_j$ on $\mathcal{S}_j$ for $j \to \infty$ from wavelet-to-nodal basis can be performed with $\mathbf{10\frac{1}{2}}$ operations per unknown.

4.5.4. *The case $\ell' = 1$, $n = 2$.* In this case $\check{\mathcal{T}}_{-1} = \{\check{\tau}\}$, $\check{\tau} = \{x \in I\!R^2 : 0 \le x_1 \le x_2 \le 4\}$. For convenience, we introduce a hierarchical numbering of elements of $\check{I}_{-1}$, $\check{J}_0$ and $\check{J}_1$ as indicated in Fig. 3, and in the following we will identify sets $\check{I}_{\check{k}}$ and $\check{J}_{\check{k}}$ with the corresponding sets of numbers.

We will define $\check{\Phi}'_0$ and $\check{\Theta}_1$ invariant under permutations of the barycentric coordinates. This means that it is sufficient to specify $\check{\phi}'_{0,\check{y}}$ and $\check{\theta}_{1,\check{y}}$ for $\check{y} \in \{1, 6\}$.

In view of (A2), we have to take

$$\check{\phi}'_{0,6} = \check{\phi}^{1,N}_{0,6}\,.$$

We may look for $\check{\phi}'_{0,1}$ as a linear combination of $\check{\phi}^{1,N}_{0,1}$ and $\check{\phi}^{1,N}_{0,4} + \check{\phi}^{1,N}_{0,5}$ (However, because of (A1), an arbitrary linear combination of $\check{\phi}^{1,N}_{0,1}$, $\check{\phi}^{1,N}_{0,4}$ and $\check{\phi}^{1,N}_{0,5}$ is not allowed). We use this freedom for making $(\check{\phi}^{0,N}_{1,11}, \check{\phi}'_{0,1})_{L_2(\tau)} = (\check{\phi}^{0,N}_{1,12}, \check{\phi}'_{0,1})_{L_2(\check{\tau})} = 0$. One may verify that it would be equally efficient to make $(\check{\phi}^{0,N}_{1,\check{x}}, \check{\phi}'_{0,1})_{L_2(\tau)} = 0$ for $\check{x} = 14, 15$. Instead, arranging this scalar product to be zero for $\check{x} = 7, 8$ or $\check{x} = 9, 10$ or $\check{x} = 13$

would save only half the number of operations. With

$$\check{\phi}'_{0,1} = \check{\phi}^{1,N}_{0,1} + \tfrac{3}{14}(\check{\phi}^{1,N}_{0,4} + \check{\phi}^{1,N}_{0,5}),$$

one has

$$\left((\check{\phi}^{0,N}_{1,\check{x}}, \check{\phi}'_{0,\check{y}})_{L_2(\check{\tau})}\right)_{\check{x}=7,9,11,13,14,\,\check{y}=1,6} = \begin{bmatrix} \frac{27}{112} & \frac{5}{336} & 0 & \frac{19}{84} & \frac{1}{21} \\ \frac{1}{32} & \frac{11}{96} & \frac{9}{32} & \frac{11}{48} & \frac{23}{48} \end{bmatrix}^T,$$

whereas the other quantities $(\check{\phi}^{0,N}_{1,\check{x}}, \check{\phi}'_{0,\check{y}})_{L_2(\check{\tau})}$ for $\check{x} \in \check{J}_1$, $\check{y} \in \check{I}_0$ follow by permuting the barycentric coordinates.

We search for $\check{\theta}_{1,1}$ as a linear combination of $\check{\phi}^{0,N}_{-1,1}$, $\check{\phi}^{0,N}_{0,1}$, $\check{\phi}^{0,N}_{1,1}$ and $\check{\phi}^{0,N}_{1,13}$, and $\check{\theta}_{1,6}$ as a linear combination of $\check{\phi}^{0,N}_{0,6}$, $\check{\phi}^{0,N}_{1,6}$, $\check{\phi}^{0,N}_{1,13}$ and $(\check{\phi}^{0,N}_{1,11} + \check{\phi}^{0,N}_{1,12})$. Although these choices do not minimize the supports of the resulting wavelets, they do minimize the costs of the implementation. Requiring that $(\check{\theta}_{1,\check{y}}, \phi'_{0,\check{z}})_{L_2(\check{\tau})} = \delta_{\check{y},\check{z}}$ $(\check{y}, \check{z} \in \check{I}_0)$, we found

$$\begin{aligned} \check{\theta}_{1,1} &= \tfrac{1}{8}\check{\phi}^{0,N}_{-1,1} - 3\check{\phi}^{0,N}_{0,1} + 16\check{\phi}^{0,N}_{1,1} + 0\check{\phi}^{0,N}_{1,13} \\ \check{\theta}_{1,6} &= \tfrac{-38}{63}\check{\phi}^{0,N}_{0,6} + \tfrac{376}{63}\check{\phi}^{0,N}_{1,6} + \tfrac{8}{63}\check{\phi}^{0,N}_{1,13} - \tfrac{4}{3}(\check{\phi}^{0,N}_{1,11} + \check{\phi}^{0,N}_{1,12}), \end{aligned}$$

whereas the other $\check{\theta}_{1,\check{y}}$ are obtained by permuting the barycentric coordinates. Note that the resulting global dual basis functions $\theta_{k+1,y}$ are linear combinations of three $(y \in I_{k-1})$ or six ($\check{\phi}^{0,N}_{1,13}$ is doubled) $(y \in J_k)$ nodal basis functions from different levels.

The constructed $\check{\Phi}'_0$ and $\check{\Upsilon}_1 \cup \check{\Theta}_1$ are bases of $\check{S}(1,0)$ and $\check{S}(0,1)$, respectively, and satisfy (A1) and (A2). By Proposition 4.6 and Theorem 3.1, an $\mathcal{H}^s$-stable wavelet basis is obtained for $|s| < \frac{3}{2}$ $(|s| \leq 1)$, consisting of wavelets having vanishing moments of order $2^{\ell'} + 1 = 3$.

For a uniform mesh and the number of levels tending to infinity, the basis transformation from wavelet to nodal basis via the "advanced" implementation as described in Section 4.4, costs $\mathbf{11\frac{7}{12}}$ operations per unknown.

4.5.5. *The case* $\ell' = 2$, $n = 2$. In this case $\check{\mathcal{T}}_{-2} = \{\check{\tau}\}$, $\check{\tau} = \{x \in \mathbb{R}^2 : 0 \leq x_1 \leq x_2 \leq 8\}$. We number the elements of $\check{I}_{-2}$, $\check{J}_{-1}$, $\check{J}_0$ and $\check{J}_1$ as indicated in Fig. 4. Based on this numbering we will identify in the following $\check{I}_{\check{k}}$ and $\check{J}_{\check{k}}$ with corresponding sets of numbers.

We will define $\check{\Phi}'_0$ and $\check{\Theta}_1$ invariant under permutations of the barycentric coordinates, and so we only have to specify $\check{\phi}'_{0,\check{y}}$ and $\check{\theta}_{1,\check{y}}$ for $\check{y} \in \{1,6,7,13\}$. Following the ideas developed in §4.5.1-4.5.4, we take

$$\begin{aligned} \check{\phi}'_{0,1} &= \check{\phi}^{2,N}_{0,1} - \tfrac{18073}{3761398}(\check{\phi}^{2,N}_{0,4} + \check{\phi}^{2,N}_{0,5}) + \tfrac{1547703}{7522796}(\check{\phi}^{2,N}_{0,7} + \check{\phi}^{2,N}_{0,8}) - \tfrac{68623}{7522796}(\check{\phi}^{2,N}_{0,9} + \check{\phi}^{2,N}_{0,10}) + \\ &\qquad \tfrac{54503}{3761398}\check{\phi}^{2,N}_{0,13} - \tfrac{6415}{7522796}(\check{\phi}^{2,N}_{0,14} + \check{\phi}^{2,N}_{0,15}) \\ \check{\phi}'_{0,6} &= \check{\phi}^{2,N}_{0,6} + \tfrac{111981}{178628}(\check{\phi}^{2,N}_{0,11} + \check{\phi}^{2,N}_{0,12}) + \tfrac{5459}{178628}\check{\phi}^{2,N}_{0,13} + \tfrac{18549}{89314}(\check{\phi}^{2,N}_{0,14} + \check{\phi}^{2,N}_{0,15}) \\ \check{\phi}'_{0,7} &= -\tfrac{32}{45}\check{\phi}^{2,N}_{0,4} + \check{\phi}^{2,N}_{0,7} - \tfrac{11}{15}\check{\phi}^{2,N}_{0,9} + \tfrac{1}{9}\check{\phi}^{2,N}_{0,13} - \tfrac{13}{45}\check{\phi}^{2,N}_{0,14} - \tfrac{2}{45}\check{\phi}^{2,N}_{0,15} \\ \check{\phi}'_{0,13} &= \check{\phi}^{2,N}_{0,13} + \tfrac{229}{527}(\check{\phi}^{2,N}_{0,14} + \check{\phi}^{2,N}_{0,15}). \end{aligned}$$

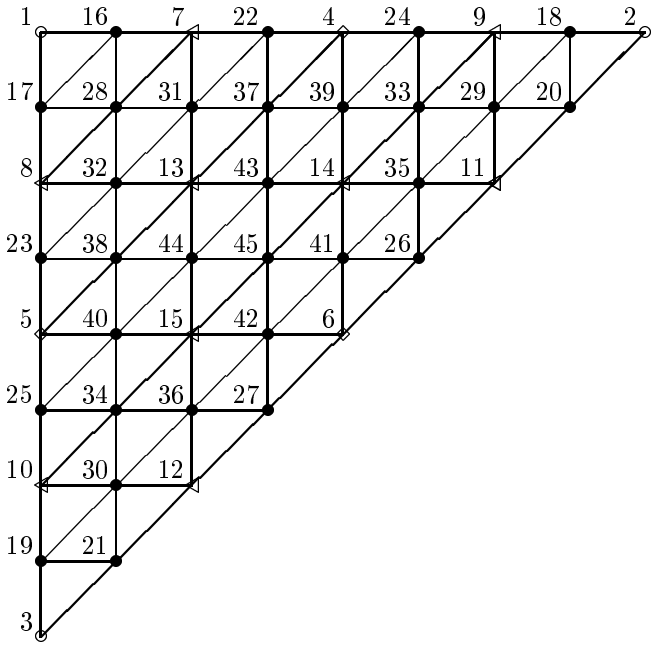

FIGURE 4. Numbering of $\check{I}_{-2} = \{\circ\}$, $\check{J}_{-1} = \{\diamond\}$, $\check{J}_0 = \{\triangleleft\}$ and $\check{J}_1 = \{\bullet\}$.

Then

$$\left((\check{\phi}^{0,N}_{1,\check{x}}\,,\,\check{\phi}'_{0,\check{y}})_{L_2(\check{\tau})}\right)_{\check{x}=16,18,20,22,24,26,28,29,31,33,35,37,39,41,43,45,\ \check{y}=1,6,13} =$$

$$\begin{bmatrix}
\frac{41058559}{180547104} & \frac{-88145}{180547104} & 0 & \frac{4733767}{180547104} & \frac{-1030601}{180547104} & 0 & \frac{427722}{1880699} & \frac{-1728}{1880699} & \\
\frac{289767}{3761398} & \frac{-8208}{1880699} & 0 & \frac{94659}{7522796} & \frac{-9936}{1880699} & \frac{864}{1880699} & 0 & 0 & \\
0 & \frac{4685}{178628} & \frac{635179}{6430608} & 0 & \frac{3263}{535884} & \frac{2337265}{6430608} & 0 & \frac{48895}{267942} & \\
\frac{4387}{1071768} & \frac{108937}{1071768} & \frac{139431}{357256} & \frac{39479}{2143536} & \frac{34971}{714512} & \frac{1125269}{2143536} & \frac{13127}{133971} & \frac{46597}{178628} & \\
\frac{3427}{37944} & \frac{1937}{303552} & 0 & \frac{49583}{303552} & \frac{13393}{151776} & \frac{11}{12648} & \frac{25655}{50592} & \frac{115}{1984} & \\
\frac{24691}{33728} & \frac{4225}{16864} & \frac{641}{5952} & \frac{11697}{16864} & \frac{50365}{101184} & \frac{263}{1984} & \frac{26059}{33728} & \frac{24263}{50592} &
\end{bmatrix}^T,$$

$$\left((\check{\phi}^{0,N}_{1,\check{x}}\,,\,\check{\phi}'_{0,7})_{L_2(\check{\tau})}\right)_{16\le\check{x}\le45} =$$

$$\begin{bmatrix}
\frac{2087}{4320} & \frac{163}{720} & \frac{-59}{864} & 0 & 0 & 0 & \frac{229}{4320} & \frac{77}{1080} & \frac{-583}{1440} & \frac{23}{2160} & 0 & 0 & \frac{1541}{2160} & \frac{-91}{540} & \frac{-1}{540} & & & \\
\frac{56}{135} & \frac{197}{540} & \frac{-1039}{2160} & \frac{1}{432} & \frac{-61}{540} & \frac{-7}{540} & \frac{-373}{2160} & \frac{329}{2160} & \frac{-115}{216} & \frac{23}{540} & \frac{-37}{540} & \frac{-19}{540} & \frac{-23}{108} & \frac{-23}{1080} & \frac{-289}{2160} &
\end{bmatrix}^T,$$

whereas the other $(\check{\phi}^{0,N}_{1,\check{x}}\,,\,\check{\phi}'_{0,\check{y}})_{L_2(\check{\tau})}$ for $\check{x} \in \check{J}_1$, $\check{y} \in \check{I}_0$ follow by permuting the barycentric coordinates.

With

$$\begin{aligned}
\check\theta_{1,1} &= \tfrac{491}{63840}\check\phi^{0,N}_{-2,1} - \tfrac{1447}{1140}\check\phi^{0,N}_{-1,1} + \tfrac{1556}{285}\check\phi^{0,N}_{0,1} + \tfrac{7936}{1995}\check\phi^{0,N}_{1,1} - \tfrac{32}{285}\check\phi^{0,N}_{1,13} - \tfrac{200}{57}\check\phi^{0,N}_{1,28}\\
&\quad + \tfrac{24}{95}\check\phi^{0,N}_{1,45} + \tfrac{40}{19}(\check\phi^{0,N}_{1,31} + \check\phi^{0,N}_{1,32}) - \tfrac{8}{19}(\check\phi^{0,N}_{1,43} + \check\phi^{0,N}_{1,44})\\
\check\theta_{1,6} &= \tfrac{39552736427842}{16415540169075}\check\phi^{0,N}_{-1,6} - \tfrac{115253331674198}{16415540169075}\check\phi^{0,N}_{0,6} - \tfrac{50086735845548}{16415540169075}\check\phi^{0,N}_{1,6} - \tfrac{15579765892424}{16415540169075}\check\phi^{0,N}_{1,13}\\
&\quad + \tfrac{1073782164671}{16415540169075}\check\phi^{0,N}_{1,28} + \tfrac{163041515506583}{16415540169075}\check\phi^{0,N}_{1,45} - \tfrac{46695093990479}{5471846723025}(\check\phi^{0,N}_{1,14} + \check\phi^{0,N}_{1,15})\\
&\quad - \tfrac{75538846833187}{32831080338150}(\check\phi^{0,N}_{1,20} + \check\phi^{0,N}_{1,21}) + \tfrac{85694636471981}{10943693446050}(\check\phi^{0,N}_{1,26} + \check\phi^{0,N}_{1,27})\\
\check\theta_{1,7} &= -\tfrac{521770882944}{316151143997}\check\phi^{0,N}_{1,4} + \tfrac{1438064282366}{316151143997}\check\phi^{0,N}_{1,7} + \tfrac{228057857077}{948453431991}\check\phi^{0,N}_{1,13} - \tfrac{2057981328101}{1264604575988}\check\phi^{0,N}_{1,16}\\
&\quad - \tfrac{652279514773}{632302287994}\check\phi^{0,N}_{1,22} + \tfrac{530136572039}{1264604575988}\check\phi^{0,N}_{1,24} - \tfrac{318838046349}{1264604575988}\check\phi^{0,N}_{1,31} + \tfrac{197161204745}{1896906863982}\check\phi^{0,N}_{1,32}\\
&\quad - \tfrac{1432064752285}{3793813727964}\check\phi^{0,N}_{1,33} + \tfrac{715847976}{316151143997}\check\phi^{0,N}_{1,34} - \tfrac{4164167977237}{3793813727964}\check\phi^{0,N}_{1,37} - \tfrac{46192346505}{632302287994}\check\phi^{0,N}_{1,38}\\
&\quad + \tfrac{5449028597573}{3793813727964}\check\phi^{0,N}_{1,39} - \tfrac{5506954693}{28312042746}\check\phi^{0,N}_{1,43} + \tfrac{18811161423}{632302287994}\check\phi^{0,N}_{1,44}\\
\check\theta_{1,13} &= -\tfrac{252514670598911312}{32123023372190565}\check\phi^{0,N}_{0,13} + \tfrac{581506357809065767}{32123023372190565}\check\phi^{0,N}_{1,13} + \tfrac{13466719718137331}{12849209348876226}\check\phi^{0,N}_{1,28}\\
&\quad + \tfrac{36849360316746824}{32123023372190565}\check\phi^{0,N}_{1,45} + \tfrac{9028736124941104 69}{5567990717846364 60}(\check\phi^{0,N}_{1,31} + \check\phi^{0,N}_{1,32})\\
&\quad - \tfrac{68672167167430439}{111359814356927292}(\check\phi^{0,N}_{1,33} + \check\phi^{0,N}_{1,34}) + \tfrac{28971942655711213}{185599690594878820}(\check\phi^{0,N}_{1,35} + \check\phi^{0,N}_{1,36})\\
&\quad + \tfrac{20805027809585291}{9279984529743941}(\check\phi^{0,N}_{1,37} + \check\phi^{0,N}_{1,38}) + \tfrac{321019232499078934}{139199767946159115}(\check\phi^{0,N}_{1,39} + \check\phi^{(N)}_{1,40}),
\end{aligned}$$

and the other $\check\theta_{1,\check y}$ for $\check y \in \check I_0$ obtained by permuting the barycentric coordinates, one has $(\check\theta_{1,\check y}, \check\phi'_{0,\check z})_{L_2(\check\tau)} = \delta_{\check y,\check z}$ $(\check y, \check z \in \check I_0)$. The constructed $\check\Phi'_0$ and $\check\Upsilon_1 \cup \check\Theta_1$ are bases of $\check S(2,0)$ and $\check S(0,1)$ and satisfy (A1) and (A2). Proposition 4.6 and Theorem 3.1 yield an $\mathcal{H}^s$-stable wavelet basis for $|s| < \frac{3}{2}$ $(|s| \le 1)$, consisting of wavelets having vanishing moments of order $2^{\ell'}+1 = 5$.

With the "advanced" implementation as described in Section 4.4, in case of a uniform mesh and the number of levels tending to infinity, performing the basis transformation from wavelet to nodal basis costs $\mathbf{24\frac{11}{48}}$ operations per unknown.

Appendix A. Proof of Theorem 2.1

We start with proving Part (a). The assertion is closely related to a well-known criterion by Fortin [F77] for the validity of the LBB-condition which is relevant in the context of saddle point problems. Let $Q'_k : L_2 \to \mathcal{S}'_k \subset L_2$ denote the orthogonal projector of L_2 onto $\mathcal{S}'_k$, defined by

$$(u, u'_k)_{L_2} = (Q'_k u, u'_k)_{L_2}, \qquad (u \in L_2,\ u'_k \in \mathcal{S}'_k), \tag{A.1}$$

and let R_k denote its restriction to $\mathcal{S}_k$. Clearly, $\|R_k\|_{L_2 \leftarrow L_2} \le 1$, whereas (C1) is equivalent to $\|R_k u_k\|_{L_2} \gtrsim \|u_k\|_{L_2}$, $(u_k \in \mathcal{S}_k)$. Since $\mathcal{S}_k$ is closed, both properties of R_k show that $\Im(R_k)$ is closed in $\mathcal{S}'_k$, and so $\Im(R_k) \ne \mathcal{S}'_k$ would mean that there exists a $0 \ne u'_k \in \mathcal{S}'_k$ such that $u'_k \perp_{L_2} \Im(R_k)$. However, this would contradict $\sup_{u_k \in \mathcal{S}_k} |(u_k, u'_k)_{L_2}| > 0$, which is a consequence of $(\mathrm{C1})'$. We conclude that the inverse $R_k^{-1} : \mathcal{S}'_k \to \mathcal{S}_k$ exists, and that it is uniformly bounded.

By substituting $u = R_k^{-1} Q'_k v$ in (A.1), we find

$$(R_k^{-1} Q'_k v, u'_k)_{L_2} = (Q'_k v, u'_k)_{L_2} = (v, u'_k)_{L_2} \qquad (v \in L_2,\ u'_k \in \mathcal{S}'_k).$$

This equation shows that $\Im(I - R_k^{-1}Q'_k) \subset (\mathcal{S}'_k)^{\perp_{L_2}}$ and, by using (C1), that $R_k^{-1}Q'_k$ is the identity operator on $\mathcal{S}_k$. We conclude that $\mathcal{Q}_k := R_k^{-1}Q'_k$ is a uniformly bounded projector satisfying $\Im(\mathcal{Q}_k) = \mathcal{S}_k$ and, because $v \in (\mathcal{S}'_k)^{\perp_{L_2}}$ implies $Q'_k v = 0$, $\Im(I - \mathcal{Q}_k) = (\mathcal{S}'_k)^{\perp_{L_2}}$. From these properties one directly infers that the adjoint projector $\mathcal{Q}'_k$ satisfies $\Im(\mathcal{Q}'_k) = \mathcal{S}'_k$ and $\Im(I - \mathcal{Q}'_k) = (\mathcal{S}_k)^{\perp_{L_2}}$. Note that $\mathcal{S}'_k \subset \mathcal{S}'_{k+1}$ implies $\mathcal{Q}'_{k+1}\mathcal{Q}'_k = \mathcal{Q}'_k$, which, in turn yields $\mathcal{Q}_k\mathcal{Q}_{k+1} = \mathcal{Q}_k$. Analogously, we have $\mathcal{Q}'_k\mathcal{Q}'_{k+1} = \mathcal{Q}'_k$, which proves the first part of the assertion.

Now we come to the proof of (2.1) and $(2.1)'$. Suppose $s \in \mathbb{R}$ is such that for some $\epsilon > 0$ one has

$$\|v_k\|_{\mathcal{H}^{s\pm\epsilon}} \lesssim \rho^{(s\pm\epsilon)k}\|v_k\|_{L_2} \qquad (v_k \in \Im(\mathcal{Q}_k - \mathcal{Q}_{k-1})). \tag{A.2}$$

Then

$$\begin{aligned} \|\sum_k v_k\|^2_{\mathcal{H}^s} &= (\sum_k v_k, \sum_\ell v_\ell)_{\mathcal{H}^s} \lesssim \sum_k \sum_{\ell\geq k} \|v_k\|_{\mathcal{H}^{s+\epsilon}}\|v_\ell\|_{\mathcal{H}^{s-\epsilon}} \\ &\lesssim \sum_k \sum_{\ell\geq k} \rho^{\epsilon k}\rho^{-\epsilon\ell}(\rho^{sk}\|v_k\|_{L_2})(\rho^{s\ell}\|v_\ell\|_{L_2}) \lesssim \sum_k \rho^{2sk}\|v_k\|^2_{L_2}. \end{aligned}$$

Condition (C1) shows inequality (A.2) for $s \pm \epsilon \in [0, \gamma)$. Now let $t = s \pm \epsilon \in [-d', 0]$. Then

$$\|v_k\|_{\mathcal{H}^t} = \sup_{0\neq w\in\mathcal{H}^{-t}} \frac{|(v_k, w)_{L_2}|}{\|w\|_{\mathcal{H}^{-t}}} = \sup_{0\neq w\in\mathcal{H}^{-t}} \frac{|(v_k, (\mathcal{Q}'_k - \mathcal{Q}'_{k-1})w)_{L_2}|}{\|w\|_{\mathcal{H}^{-t}}}, \qquad (v_k \in \Im(\mathcal{Q}_k - \mathcal{Q}_{k-1})).$$

From $\|(\mathcal{Q}'_k - \mathcal{Q}'_{k-1})w\|_{L_2} \leq \|\mathcal{Q}'_k - \mathcal{Q}'_{k-1}\|_{L_2\leftarrow L_2} \inf_{u'_{k-1}\in\mathcal{S}'_{k-1}} \|w - u'_{k-1}\|_{L_2}$ combined with $(C2)'$ we obtain inequality (A.2) for this case. We conclude that for $s \in (-d', \gamma)$, the mapping

$$G_{\mathcal{Q}} : \ell_{2,s}(\mathcal{Q}) \to \mathcal{H}^s : (v_k)_k \mapsto \sum_{k=0}^{\infty} v_k$$

is bounded, i.e., (2.1) is valid. The same proof shows $(2.1)'$.

We will now show that $(2.1)'$ implies (2.2) and, analogously that (2.1) implies $(2.2)'$. To this end we will make use of the following fact.

Lemma A.1. *With respect to the dual pairing* $<(v'_k)_k, (v_k)_k> = \sum_k (v'_k, v_k)_{L_2}$ *on* $\ell_{2,-s}(\mathcal{Q}') \times \ell_{2,s}(\mathcal{Q})$, *one has* $\ell_{2,s}(\mathcal{Q})' = \ell_{2,-s}(\mathcal{Q}')$ *with equivalent norms. The analogous result is valid with interchanged roles of* $\mathcal{Q}$ *and* $\mathcal{Q}'$.

Proof. The above assertion is a consequence of similar more general results from [T78], Section 1.11.1, see also [Dah95], Theorem 4.5.1 or [Dah96], Theorem 5.1. Since the proof for the above version is relatively short we include it for the convenience of the reader. For $(v'_k)_k \in \ell_{2,-s}(\mathcal{Q}')$, term-wise application of the Cauchy-Schwarz inequality shows that the functional $(v_k)_k \mapsto \sum_k (v'_k, v_k)_{L_2}$ is bounded on $\ell_{2,s}(\mathcal{Q})$ with norm less than or equal to $\|(v'_k)_k\|_{\ell_{2,-s}(\mathcal{Q}')}$.

If $(v'_k)_k \neq 0$, then $v'_\ell \neq 0$ for some ℓ. Defining $v_k = \begin{cases} (\mathcal{Q}_l - \mathcal{Q}_{l-1})v'_\ell & \text{if } k = \ell \\ 0 & \text{if } k \neq \ell \end{cases}$, we obtain $<(v'_k)_k, (v_k)_k> = \|v'_\ell\|^2_{L_2} \neq 0$, i.e., the mapping from $\ell_{2,-s}(\mathcal{Q}')$ to $\ell_{2,s}(\mathcal{Q})'$ is injective.

Now let $f \in \ell_{2,s}(\mathcal{Q})'$. By Riesz's representation theorem, there exists an $(w_k)_k \in \ell_{2,s}(\mathcal{Q})$, with $\|f\| = \|(w_k)_k\|_{\ell_{2,s}(\mathcal{Q})}$, satisfying

$$\begin{aligned} f((v_k)_k) &= ((w_k)_k, (v_k)_k)_{\ell_{2,s}(\mathcal{Q})} = \sum_k \rho^{2sk}(w_k, v_k)_{L_2} \\ &= <(\rho^{2sk}(\mathcal{Q}'_k - \mathcal{Q}'_{k-1})w_k)_k, (v_k)_k>, \qquad (v_k)_k \in \ell_{2,s}(\mathcal{Q}). \end{aligned}$$

The uniform boundedness of $\|\mathcal{Q}'_k\|_{L_2 \leftarrow L_2}$ shows that

$$\sum_k \rho^{-2sk} \|\rho^{2sk}(\mathcal{Q}'_k - \mathcal{Q}'_{k-1})w_k\|^2_{L_2} \lesssim \sum_k \rho^{2sk} \|w_k\|^2_{L_2},$$

i.e., $(\rho^{2sk}(\mathcal{Q}'_k - \mathcal{Q}'_{k-1})w_k)_k \in \ell_{2,-s}(\mathcal{Q}')$, its norm being bounded by some multiple of $\|f\|$. □

Now let $s \in (-\gamma', d)$. Then by $(2.1)'$, $G_{\mathcal{Q}'} : \ell_{2,-s}(\mathcal{Q}') \to \mathcal{H}^{-s} : (v'_k)_k \mapsto \sum_{k=0}^{\infty} v'_k$ is bounded. For $u \in \mathcal{H}^s$ and $(v'_k)_k \in \ell_{2,-s}(\mathcal{Q}')$, we have

$$(u, G_{\mathcal{Q}'}(v'_k)_k)_{L_2} = \sum_k (u, v'_k)_{L_2} = <((\mathcal{Q}_k - \mathcal{Q}_{k-1})u)_k, (v'_k)_k>,$$

i.e., the dual operator $G'_{\mathcal{Q}'} : \mathcal{H}^s \to \ell_{2,s}(\mathcal{Q})$ is defined by $G'_{\mathcal{Q}'}u = ((\mathcal{Q}_k - \mathcal{Q}_{k-1})u)_k$. The boundedness of $G'_{\mathcal{Q}'}$ is equivalent to (2.2).

Finally, let $s \in (-\gamma', \gamma)$. We have to show that the bounded mappings $G_{\mathcal{Q}} : \ell_{2,s}(\mathcal{Q}) \to \mathcal{H}^s$ and $G'_{\mathcal{Q}'} : \mathcal{H}^s \to \ell_{2,s}(\mathcal{Q})$ satisfy $G'_{\mathcal{Q}'}G_{\mathcal{Q}} = I$ and $G_{\mathcal{Q}}G'_{\mathcal{Q}'} = I$.

Let $(v_k)_k \in \ell_{2,s}(\mathcal{Q})$, then $\sum_k v_k$ is convergent in $\mathcal{H}^s$, so that

$$G'_{\mathcal{Q}'}G_{\mathcal{Q}}(v_k)_k = \lim_{k\to\infty}(v_0, v_1, \dots, v_k, 0, \dots) = (v_k)_k.$$

Since for $u \in \mathcal{H}^s$ one has $G_{\mathcal{Q}}G'_{\mathcal{Q}'}u = \sum_{k=0}^{\infty}(\mathcal{Q}_k - \mathcal{Q}_{k-1})u = \lim_{k\to\infty}\mathcal{Q}_k u$, we have to show that this limit equals u. For $\tilde{s} \in (-\gamma', \gamma)$, we define $F_{\mathcal{Q}',k} : \mathcal{H}^{\tilde{s}} \to \ell_{2,\tilde{s}}(\mathcal{Q})$ by $(F_{\mathcal{Q}',k}u)_\ell = \begin{cases} (G'_{\mathcal{Q}'}u)_\ell & \text{if } \ell \le k \\ 0 & \text{if } \ell > k \end{cases}$. Clearly, $F_{\mathcal{Q}',k}$ is bounded uniformly in k, and so is $\mathcal{Q}_k = G_{\mathcal{Q}}F_{\mathcal{Q}',k} : \mathcal{H}^{\tilde{s}} \to \mathcal{H}^{\tilde{s}}$. Now, let $t \in (0, \gamma)$ and $t' \in (0, \gamma')$ such that $s \in (-t', t)$. By $\|\mathcal{Q}_k\|_{L_2 \leftarrow L_2} \lesssim 1$ and (C2), $(C2)'$ we obtain that

$$\|I - \mathcal{Q}_k\|_{\mathcal{H}^{-t'} \leftarrow \mathcal{H}^t} \le \|I - \mathcal{Q}'_k\|_{L_2 \leftarrow \mathcal{H}^{t'}} \|I - \mathcal{Q}_k\|_{L_2 \leftarrow \mathcal{H}^t} \lesssim \rho^{-(t+t')k}.$$

This estimate together with $\|I - \mathcal{Q}_k\|_{\mathcal{H}^t \leftarrow \mathcal{H}^t} \lesssim 1$ shows that $\|I - \mathcal{Q}_k\|_{\mathcal{H}^s \leftarrow \mathcal{H}^t} \lesssim \rho^{-(t-s)k}$ by interpolation. Since $\mathcal{H}^t \subset \mathcal{H}^s$ is dense, and $\|\mathcal{Q}_k\|_{\mathcal{H}^s \leftarrow \mathcal{H}^s} \lesssim 1$, from

$$\|u - \mathcal{Q}_k u\|_{\mathcal{H}^s} \le \inf_{v \in \mathcal{H}^t}\{\|\mathcal{Q}_k(v-u)\|_{\mathcal{H}^s} + \|(I - \mathcal{Q}_k)v\|_{\mathcal{H}^s} + \|u - v\|_{\mathcal{H}^s}\},$$

we conclude that $\lim_{k\to\infty}\mathcal{Q}_k u = u$ on $\mathcal{H}^s$, which completes the proof.

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