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Elementary Computable Topology

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Abstract: We revise and extend the foundation of computable topology in the framework of Type-2 theory of effectivity, TTE, where continuity and computability on finite and infinite sequences of symbols are defined canonically and transferred to abstract sets by means of notations and representations. We start from a computable topological space, which is a T_0-space with a notation of a base such that intersection is computable, and define a number of multi-representations of the points and of the open, the closed and the compact sets and study their properties and relations. We study computability of boolean operations. By merely requiring "provability" of suitable relations (element, non-empty intersection, subset) we characterize in turn computability on the points, the open sets (!), computability on the open sets, computability on the closed sets, the compact sets(!), and computability on the compact sets. We study modifications of the definition of a computable topological space that do not change the derived computability concepts. We study subspaces and products and compare a number of representations of the space of partial continuous functions. Since we are operating mainly with the base elements, which can be considered as regions for points ("pointless topology"), we study to which extent these regions can be filled with points (completions). We conclude with some simple applications including Dini's Theorem as an example.
Key Words: computability, topology, computable analysis
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1 Introduction

In the various publications considering computable topology as a foundation of computable analysis the basic definitions as well as the terminology are partly inconsistent so that the comparison of results is difficult. Furthermore, some definitions are unwieldy or inappropriate [KW85, Wei87, Sch98, Wei00, Sch03, GW05, GSW07, BHW08]. Repeatedly facts from computable topology have been used in applications although they have never been proved or have not been proved in sufficient generality. In this article we try to develop a core of computable topology in a more uniform and general manner. It can be considered as a careful revision of the corresponding parts in [Wei00].

We call the basic objects in this article *computable topological spaces*. Since anyway various spaces have been called computable topological space in the literature (see the comments at the end of Section 3) and since the definition in [Wei00] is not quite reasonable, we have decided to deviate from this source. The results from this article show that the definition of a computable topological space given here is appropriate for a foundation of computable topology.

Since in the literature often "local" abbreviations are used for the important representations of the open, the closed and the compact sets and since some names from [Wei00] should be changed, in this article we use short "local" abbreviations leaving unchanged the symbols θ, ψ and κ mostly used for representations of open, closed and compact sets, respectively.

Our work is based on the representation approach, TTE (Type-2 Theory of Effectivity) [KW85, Wei00, BHW08, Wei08], which has significant advantages over other approaches for studying computability in Analysis [Wei00, Chapter 9] [BC06]. Some significant results are Theorem 13, the characterizations of the open and the compact sets and of computability on the points, the open sets and the compact sets by merely requiring "provability" of suitable relations, Theorem 22 on equivalent computable topological spaces, Lemma 23, expressing that starting the theory with a subbase is equivalent to starting with a base, Theorems 35–37 on completions of computable topological spaces under stronger and stronger restrictions, and Theorem 41, a general computable version of Dini's theorem, as an application.

In Section 2 we summarize some technical details, in particular definitions and facts from TTE [Wei00, Wei08].

In Section 3 we introduce computable topological spaces and a number of multi-representations of points and of the open, the closed and the compact sets . As an example we introduce computable predicate spaces and the derived computable topological spaces.

In Section 4 we study computability of boolean operations w.r.t. the introduced representations of subsets.

The definitions of the representations in Section 3 look reasonable but are ad hoc. In Section 5 we characterize their equivalence classes and therefore the computability concepts induced by them abstractly by a simple common principle. We show that requiring "provability" of $x \in W$, $A \cap W \neq \emptyset$ and $K \subseteq W$ (for points x, open sets W, closed sets A and compact sets K) suffices to define the open and the compact sets (!) and computability on the points and on the open, the closed and the compact sets. By Theorem 13 the objects introduced in Definition 5, including the open and the compact sets, are particulary natural.

In Section 6 we study and compare the introduced representations in more detail. We also introduce three further representations that are equivalent to the inner representation of the open sets.

In Section 7 we show that the definition of a computable topological space can be modified in various ways without changing the computability concepts.

In particular we show that introducing computability via a subbase and via a base are equivalent.

Subspaces and product spaces are considered in Section 8 and the space of continuous functions in Section 9. The use of multi-representations allows to represent the class of all (!) partial continuous functions. We introduce a number of such multi-representations and compare them.

For all the representations considered so far, names are combined from names of base sets $U \in \beta$ (or names of predicates $U \in \sigma$ for predicate spaces). Not much is known about the points in such sets U. Therefore they can be considered as "frames" or "regions" of points rather than sets ("pointless topology"). In Section 10 we fill these regions as much as possible ("completion") under the following three conditions: the domain of the subbase notation of predicate spaces is fixed, for a computable topological space the intersection of base elements is computed by a fixed program, and for a computable topological space the inclusion relation on (the names of) the open sets is fixed.

In Section 11 we show that a number of basic operations on points, sets and functions are computable w.r.t. the introduced representations. A concise proof a general computable version of Dini's theorem confirms that the concepts in this article are chosen appropriately.

2 Preliminaries

In this section we summarize some technical details, in particular definitions and facts from TTE. We will use essentially the terminology from [Wei00, BHW08, Wei08]. For more details the reader should consult these sources.

We will use the word "iff" as an abbreviation for the logical "if and only if". A *multi-function* from A to B is a triple $f = (A, B, \mathrm{R}_f)$ such that $\mathrm{R}_f \subseteq A \times B$ (the *graph* of f). We will denote it by $f : A \rightrightarrows B$. Its inverse is the multi-function $f^{-1} := (B, A, R_f^{-1})$. For $X \subseteq A$ let $f[X] := \{b \in B \mid (\exists a \in X)(a, b) \in R_f\}$, $\mathrm{dom}(f) := f^{-1}[B]$, and $\mathrm{range}(f) := f[A]$. For $a \in A$ let $f(a) := f[\{a\}]$. If for every $a \in A$, $f(a)$ contains at most one element, f can be treated as a usual partial function denoted by $f :\subseteq A \to B$. In contrast to relational composition, for multi-functions $f : A \rightrightarrows B$ and $g : B \rightrightarrows C$ define the composition $g \circ f : A \rightrightarrows C$ by $a \in \mathrm{dom}(g \circ f) :\iff f(a) \subseteq \mathrm{dom}(g)$ and $g \circ f(a) := g[f(a)]$ [Wei08, Section 3].

Let Σ be a finite alphabet such that $0, 1 \in \Sigma$. By Σ^* we denote the set of finite words over Σ and by Σ^ω the set of infinite sequences $p : \mathbb{N} \to \Sigma$ over Σ, $p = (p(0)p(1)\ldots)$. For a word $w \in \Sigma^*$ let $|w|$ be its length. Let Σ^n be the set of words of length n and let $\varepsilon \in \Sigma^0$ be the empty word. For $p \in \Sigma^\omega$ let $p^{<i} \in \Sigma^*$ be the prefix of p of length $i \in \mathbb{N}$. We write $x \sqsubseteq y$ if x is a prefix of y. We use the "wrapping function" $\iota : \Sigma^* \to \Sigma^*$, $\iota(a_1 a_2 \ldots a_k) := 110a_1 0a_2 0 \ldots a_k 011$

for coding words such that $\iota(u)$ and $\iota(v)$ cannot overlap properly. Let $\langle i,j\rangle := (i+j)(i+j+1)/2+j$ be the bijective Cantor pairing function on $\mathbb{N}$. We consider standard functions for finite or countable tupling on Σ^* and Σ^ω denoted by $\langle\cdot\rangle$ [Wei00, Definition 2.1.7], in particular, $\langle u_1,\ldots,u_n\rangle = \iota(u_1)\ldots\iota(u_n)$, $\langle u,p\rangle = \iota(u)p$, $\langle p,q\rangle = (p(0)q(0)p(1)q(1)\ldots)$ and $\langle p_0,p_1,\ldots\rangle\langle i,j\rangle = p_i(j)$ for $u,u_1,u_2,\ldots \in \Sigma^*$ and $p,q,p_0,p_1,\ldots \in \Sigma^\omega$. For $u\in\Sigma^*$ and $w\in\Sigma^*\cup\Sigma^\omega$ let $u \ll w$ iff $\iota(u)$ is a subword of w. For $w\in\Sigma^*$ let $\widehat{w}$ be the longest subword $v\in 11\Sigma^*11$ of w (and the empty word if no such subword exists). Then for $u,w_1,w_2\in\Sigma^*$, $(u\ll w_1 \vee u \ll w_2) \iff u \ll \widehat{w}_1\widehat{w}_2$.

Let $Y_0,\ldots Y_n \in \{\Sigma^*,\Sigma^\omega\}$ and $Y = Y_1\times\ldots\times Y_n$. A function $f:\subseteq Y\to Y_0$ is computable (called Turing computable in [Wei08]) if for some Type-2 machine M, f is the function f_M computed by M. For computability theory see, for example, [Rog67, Wei87, Coo04]. Informally, a Type-2 machine is a Turing machine, which reads from input files (tapes) with finite or infinite inscription, operates on work tapes and writes one-way to an output file (tape). For $Y_0=\Sigma^*$, $f_M(y)=w$, if M on input y halts with w on the output tape, and for $Y_0=\Sigma^\omega$, $f_M=q$, if M on input y computes forever and writes $q\in\Sigma^\omega$ on the output tape. The computable functions on Σ^* and Σ^ω are "essentially" closed under composition (even under programming [Wei08]): the composition of computable functions has a computable extension. For $W,Z\subseteq Y$, the set W is called "recursively enumerable in Z" if for some Type-2 machine M, M halts on input y iff $y\in W$ (for all $y\in Z$) (equivalently, if $W = Z\cap\mathrm{dom}(f)$ for some computable function $f:\subseteq Y\to\Sigma^*$). If $Z=Y$, we omit "in Z". For $p\in\Sigma^\omega$ and a Type-2 machine M with $(n+1)$ input tapes let $f_{Mp}(y) := f_M(p,y)$ (f_{Mp} is called the function computed by M with "oracle" p).

On Σ^* we consider the discrete topology and on Σ^ω the topology generated by the base $\{w\Sigma^\omega \mid w\in\Sigma^*\}$ of open sets. Every computable function is continuous and every r.e. set is open. Furthermore, a function $f:\subseteq Y\to Y_0$ is continuous iff for some Type-2 machine M and some oracle $p\in\Sigma^\omega$, f_{Mp} extends f. Finally, $W\subseteq Y$ is open iff for some Type-2 machine M with output set Σ^* and some oracle p, $W=\mathrm{dom}(f_{Mp})$.

In TTE computability on finite or infinite sequences of symbols is transferred to other sets by representations, where elements of Σ^* or Σ^ω are used as "concrete names" of abstract objects. We will need the more general concept of realization via *multi-representations*. Here we give only the definitions, for a detailed discussion see [Wei08, Section 6], see also [Sch03]. A *multi-representation* of a set M is a surjective multi-function $\gamma: Y \rightrightarrows M$ where $Y\in\{\Sigma^*,\Sigma^\omega\}$. Examples are the canonical (single-valued) notations $\nu_\mathbb{N}:\subseteq\Sigma^*\to\mathbb{N}$ and $\nu_\mathbb{Q}:\subseteq\Sigma^*\to\mathbb{Q}$ of the natural numbers and the rational numbers, respectively, and the (single-valued) representation $\rho:\subseteq\Sigma^\omega\to\mathbb{R}$ of the real numbers [Wei00]. Mathematical examples of proper multi-representations will be given below. An instructive example is the multi-representation ν of all people by their first names, for example, ν(PETER) is the set of all people with first name PETER.

For multi-representations $\gamma_i : Y_i \rightrightarrows M_i$ $(i = 0, \dots, n)$, let $Y = Y_1 \times \dots \times Y_n$, $M = M_1 \times \dots \times M_n$ and $\gamma : Y \rightrightarrows M$, $\gamma(y_1, \dots, y_n) = \gamma_1(y_1) \times \dots \times \gamma_n(y_n)$. A partial function $h :\subseteq Y \to Y_0$ realizes the multi-function $f : M \rightrightarrows M_0$ if $f(x) \cap \gamma_0 \circ h(y) \neq \emptyset$ whenever $x \in \gamma(y)$ and $f(x) \neq \emptyset$. This means that $h(y)$ is a name of some $z \in f(x)$ if y is a name of $x \in \mathrm{dom}(f)$. If $f :\subseteq M \to M_0$ is single-valued, then $h(y)$ is a name of $f(x)$ if y is a name of $x \in \mathrm{dom}(f)$. If only the representations are single-valued, $\delta_0 \circ h(y) \in f(x)$ if $\delta(y) = x$.

The multi-function f is called $(\gamma_1, \dots, \gamma_n, \gamma_0)$-continuous (-computable) if it has a continuous (computable) realization. If the multi-representations are fixed, we occasionally say that f is relatively continuous (relatively computable). The relatively continuous (computable) functions are closed under composition, even more, they are closed under GOTO-programming with indirect addressing [Wei08]. We will apply this result without further mentioning. Now we extend the definition of γ-r.e. sets [Wei00] to multi-representations:

Definition 1. With γ_i and γ from above, a point $x \in M_1$ is γ_1-computable iff $x \in \gamma_1(p)$ for some computable $p \in \mathrm{dom}(\gamma_1)$, and a set $S \subseteq M$ is $(\gamma_1, \dots, \gamma_n)$-r.e. (-open) if there is an r.e. (open) set $W \subseteq Y$ such that $(x \in S \iff y \in W)$ for all x, y such that $x \in \gamma(y)$.

Therefore, $S \subseteq M$ is $(\gamma_1, \dots, \gamma_n)$-r.e. iff there is a Type-2 machine (with oracle for the "open" case) that halts on input $y \in \mathrm{dom}(\gamma)$ iff y is a name of some $x \in S$. Notice that $\gamma\gamma^{-1}[S] = S$ if S is γ-open.

Finally, $\gamma_1 \leq \gamma_0$ (γ_1 is reducible to γ_0) if $M_1 \subseteq M_0$ and the identity id : $M_1 \to M_0$ is (γ_1, γ_0)-computable This means that some computable function h translates γ_1-names to γ_0-names, that is, $\gamma_1(p) \subseteq \gamma_0 \circ h(p)$. Continuous reducibility $\gamma_1 \leq_t \gamma_0$ is defined accordingly by means of continuous functions. Computable and continuous equivalence are defined canonically: $\gamma_1 \equiv \gamma_0 \iff \gamma_1 \leq \gamma_0 \wedge \gamma_0 \leq \gamma_1$ and $\gamma_1 \equiv_t \gamma_0 \iff \gamma_1 \leq_t \gamma_0 \wedge \gamma_0 \leq_t \gamma_1$. Two multi-representations induce the same computability (continuity) iff they are computably equivalent (continuously equivalent). For $X \subseteq M_1$, if X is γ_0-r.e. and $\gamma_1 \leq \gamma_0$, then X is γ_1-r.e., and if X is γ_0-open and $\gamma_1 \leq_t \gamma_0$, then X is γ_1-open.

From γ_1 and γ_2 a multi-representation $[\gamma_1, \gamma_2]$ of the product $M_1 \times M_2$ is defined by $[\gamma_1, \gamma_2]\langle y_1, y_2\rangle := \gamma_1(y_1) \times \gamma_2(y_2)$. Since $(x_1, x_2) \mapsto (x_1, x_2)$ is $(\gamma_1, \gamma_2, [\gamma_1, \gamma_2])$-computable and $(x_1, x_2) \mapsto x_i$ is $([\gamma_1, \gamma_2], \gamma_i)$-computable, a multi-function is $(\gamma_1, \gamma_2, \gamma_0)$-computable iff it is $([\gamma_1, \gamma_2], \gamma_0)$-computable. A set is (γ_1, γ_2)-open iff it is $[\gamma_1, \gamma_2]$-open etc. By the conjunction of two multi-representations γ and δ, defined by $(\gamma \wedge \delta)\langle p, q\rangle := \gamma(p) \cap \delta(q)$, information from two names is combined in a single name.

In [Wei00] representations $\eta^{ab} : \Sigma^\omega \to F^{ab}$ are introduced for $a, b \in \{*, \omega\}$. F^{**} is the set of all partial functions $f :\subseteq \Sigma^* \to \Sigma^*$, $F^{*\omega}$ is the set of all partial functions $f :\subseteq \Sigma^* \to \Sigma^\omega$, $F^{\omega *}$ is the set of all partial functions $f :\subseteq \Sigma^\omega \to \Sigma^*$ with open domain and $F^{\omega\omega}$ is the set of all partial functions $f :\subseteq \Sigma^\omega \to \Sigma^\omega$ with G_δ-domain (a G_δ-set is a countable intersection of open sets). While F^{**} and $F^{*\omega}$ consist of all continuous partial functions, every continuous partial function $f :\subseteq \Sigma^\omega \to \Sigma^*$ has an extension in $F^{\omega *}$, and every continuous partial function $f :\subseteq \Sigma^\omega \to \Sigma^\omega$ has an extension in $F^{\omega\omega}$. Each representation η^{ab} satisfies the "utm-theorem" and the "smn-theorem" [Wei00, Theorem 2.3.13]: the function $U :\subseteq \Sigma^\omega \times \Sigma^a \to \Sigma^b$, $U(p,x) = \eta^{ab}_p(x)$ is computable, for every computable function $f :\subseteq \Sigma^\omega \times \Sigma^a \to \Sigma^b$ there is a computable function $r : \Sigma^\omega \to \Sigma^\omega$ such that $f(p,x) = \eta^{ab}_{r(p)}(x)$.

For multi-representations $\gamma_1 : \Sigma^a \rightrightarrows M_1$ and $\gamma_2 : \Sigma^b \rightrightarrows M_1$, $a, b \in \{*, \omega\}$, a multi-representation $[\gamma_1 \rightrightarrows \gamma_2]$ of the (γ_1,γ_2)-continuous multi-functions $f : M_1 \rightrightarrows M_2$ is defined by: $f \in [\gamma_1 \rightrightarrows \gamma_2](p) \iff \eta^{ab}_p := \eta^{ab}(p)$ realizes f w.r.t. (γ_1,γ_2) [Wei08]. The restriction of $[\gamma_1 \rightrightarrows \gamma_2]$ to the single-valued functions is called $[\gamma_1 \to_p \gamma_2]$ [Wei08] or $[\gamma_1 \to \gamma_2]_{set}$ [Wei00]. The restriction of $[\gamma_1 \to_p \gamma_2]$ to the total (γ_1,γ_2)-continuous functions is called $[\gamma_1 \to \gamma_2]$ ([Sch02, Wei08], for single-valued representations [KW85, Wei00, Sch02]). The generalization of utm- and the smn-theorem from the η^{ab} to represented sets is the type conversion theorem, [Wei00, Theorem 3.3.15] for single-valued representations and total functions and [Wei08, Theorem 33]) as the most general version.

Furthermore, in this article we will use the following canonical notations and representations of finite and of countable subsets and apply Lemma 3 without further mentioning.

Definition 2. For notations $\mu :\subseteq \Sigma^* \to M$ and representations $\gamma :\subseteq \Sigma^\omega \to Y$ define notations and representations of finite and countable subsets as follows (where $w \in \Sigma^*$, $q, p_0, p_1, \ldots \in \Sigma^\omega$ and $a_0, a_1, \ldots \in \Sigma$):

$$\mu^{\mathrm{fs}}(w) = W :\Longleftrightarrow \begin{cases} (\forall v \ll w)\, v \in \mathrm{dom}(\mu), \\ W = \{\mu(v) \mid v \ll w\}\,; \end{cases} \tag{1}$$

$$\mu^{\mathrm{cs}}(q) = W :\Longleftrightarrow \begin{cases} (\forall v \ll q)\, v \in \mathrm{dom}(\mu), \\ W = \{\mu(v) \mid v \ll q\}\,; \end{cases} \tag{2}$$

$$\gamma^{\mathrm{fs}}(q) = W :\Longleftrightarrow \begin{cases} (\exists n)(\exists p_1, \ldots, p_n \in \mathrm{dom}(\gamma)) \\ \qquad q = \langle 1^n, p_1, \ldots, p_n\rangle, \\ W = \{\gamma(p_1), \ldots, \gamma(p_n)\}\,; \end{cases} \tag{3}$$

$$\gamma^{\mathrm{cs}}\langle a_0p_0, a_1p_1, \ldots\rangle = W :\Longleftrightarrow \begin{cases} (\forall i)(a_i = 0 \Longrightarrow p_i \in \mathrm{dom}(\gamma)), \\ W = \{\gamma(p_i) \mid a_i = 0\}\,. \end{cases} \tag{4}$$

If $a_i \neq 0$ for all i, then $\gamma^{\mathrm{cs}}\langle a_0p_0, a_1p_1, \ldots\rangle = \emptyset$.

Lemma 3. *For notations μ and notations or representations β, γ,*

$$\mathrm{dom}(\mu^{\mathrm{fs}}) \;\; \textit{is recursive} \quad \textit{if} \;\; \mathrm{dom}(\mu) \;\; \textit{is recursive,} \tag{5}$$

$$\mu^{\mathrm{fs}}(w) = \mu^{\mathrm{fs}}(\widehat{w}), \tag{6}$$

$$(x, y) \mapsto \{x, y\} \;\; \textit{is} \;\; (\gamma, \gamma, \gamma^{\mathrm{fs}})\textit{-computable,} \tag{7}$$

$$\gamma' \le \gamma^{\mathrm{fs}} \le \gamma^{\mathrm{cs}}, \quad \textit{where} \;\; \gamma'(w) := \{\gamma(w)\}, \tag{8}$$

$$\beta^{\mathrm{fs}} \le \gamma^{\mathrm{fs}} \;\; \textit{and} \;\; \beta^{\mathrm{cs}} \le \gamma^{\mathrm{cs}} \quad \textit{if} \quad \beta \le \gamma\,. \tag{9}$$

3 Computable Topological Spaces

In this section we introduce computable topological spaces as the basic concept for the computable topology presented in this article. We define explicitly multi-representations of points and of open, closed and compact sets, which induce computability on these classes of objects.

A topology τ on a set X is a set of subsets of X, the set of open sets, that is closed under union and finite intersection. We denote the closure of a set $A \subseteq X$ by $\overline{A}$. A base is a subset of $\beta \subseteq \tau$ such that every $U \in \tau$ is a union of base sets. (X, τ) is a T_0-space if the points can be identified by their neighborhoods, that is, for all $x, y \in X$ such that $x \neq y$, there is some $W \in \tau$ such that $(x \in W \;\wedge\; y \notin W)$ or $(x \notin W \;\wedge\; y \in W)$ [Eng89].

Definition 4 (computable topological space). An effective topological space is a 4-tuple $\mathbf{X} = (X, \tau, \beta, \nu)$ such that (X, τ) is a topological T_0-space and $\nu :\subseteq \Sigma^* \to \beta$ is a notation of a base β of τ. $\mathbf{X}$ is a *computable* topological space if $\mathrm{dom}(\nu)$ is recursive and

$$\nu(u) \cap \nu(v) = \bigcup \{\nu(w) \mid (u, v, w) \in S\} \;\; \text{for all} \;\; u, v \in \mathrm{dom}(\nu) \tag{10}$$

for some r.e. set $S \subseteq (\mathrm{dom}(\nu))^3$.

Since the base β has a notation it must be countable. T_0-spaces with countable base are called *second countable* [Eng89]. By (10) the intersection of two base elements can be computed (is (ν, ν, θ)-computable, see Definition 5). For every effective topological space there is some not necessarily r.e. set S such that (10).

Example 1 (computable topological spaces).

1. (real line) Define $\mathbf{R} := (\mathbb{R}, \tau_{\mathbb{R}}, \beta, \nu)$ such that $\tau_{\mathbb{R}}$ is the real line topology and ν is a canonical notation of the set of all open intervals with rational endpoints.
2. (lower real line) Define $\mathbf{R}_< := (\mathbb{R}, \tau_<, \beta_<, \nu_<)$ such that $\nu_<(w) := (\nu_{\mathbb{Q}}; \infty)$. The representation δ for $\mathbf{R}_<$ is called $\rho_<$ in [Wei00]. Then $\tau_< = \{(x; \infty) \mid x < \infty\} \bigcup \{\emptyset, \mathbb{R}\}$,

3. (Sierpinski space) Define $\mathbf{Si} := (\{\bot, \top\}, \tau_{\mathbf{Si}}, \beta_{\mathbf{Si}}, \nu_{\mathbf{Si}})$ such that $\nu_{\mathbf{Si}}(0) = \{\bot, \top\}$ and $\nu_{\mathbf{Si}}(1) = \{\top\}$.
4. Define $\mathbf{X} = (\mathbb{N} \cup \{-1, -2, \}, \tau, \beta, \nu)$ where β is the set of all $\{n\}$, $\{i \geq n \mid i \in \mathbb{N}\} \cup \{-1\}$ and $\{i \geq n \mid i \in \mathbb{N}\} \cup \{-2\}$ for $n \in \mathbb{N}$, and ν is some canonical notation of β.

Further examples can be found in [Wei00]. See also Lemma 9 below. For the topological space (X, τ) a set $K \subseteq X$ is *compact* iff for every set $\alpha \subseteq \tau$ such that $K \subseteq \bigcup \alpha$ there is some finite $\alpha' \subseteq \alpha$ such that $K \subseteq \bigcup \alpha'$. Some authors such as Bourbaki and Engelking [Bou66, Eng89] use the term "quasi-compact" instead, and reserve the term "compact" for topological spaces that are Hausdorff and "quasi-compact". Since every open set is a union of base elements, it suffices to consider only subsets from the base β, that is, a set K is compact iff for every set $\alpha \subseteq \beta$ such that $K \subseteq \bigcup \alpha$ there is some finite $\alpha' \subseteq \alpha$ such that $K \subseteq \bigcup \alpha'$. For a compact set K we will consider the set of all finite unions B of base elements such that $K \subseteq B$.

We define explicitly several multi-representations of points and of classes of subsets. We will use the notations $\bigcap \nu^{\mathrm{fs}}$ and $\bigcup \nu^{\mathrm{fs}}$ of the finite unions and finite intersections of base sets, respectively, see (1). As usually we assume $\bigcap \emptyset := X$ and $\bigcup \emptyset := \emptyset$.

Definition 5. Let $\mathbf{X} = (X, \tau, \beta, \nu)$ be an effective topological space.

1. Define a representation $\delta^+ :\subseteq \Sigma^\omega \to X$ of the points, a representation $\theta^+ :\subseteq \Sigma^\omega \to \tau$ of the set of open sets, a representation $\psi^+ :\subseteq \Sigma^\omega \to \mathcal{A}$ of the set of closed sets, a multi-representation $\widetilde{\psi}$ of the powerset, and a multi-representation $\kappa : \Sigma^\omega \rightrightarrows \mathcal{K}$ of the set of compact subsets of X as follows:

$$x = \delta^+(p) :\iff (\forall w \in \Sigma^*) \, (w \ll p \iff x \in \nu(w)) \, , \tag{11}$$

$$W = \theta^+(p) :\iff \begin{cases} w \ll p \Longrightarrow w \in \mathrm{dom}(\nu), \\ W = \bigcup \{\nu(w) \mid w \ll p\}, \end{cases} \tag{12}$$

$$A = \psi^+(p) :\iff (\forall w \in \Sigma^* \, (w \ll p \iff A \cap \nu(w) \neq \emptyset) \, , \tag{13}$$

$$B \in \widetilde{\psi}(p) :\iff (\forall w \in \Sigma^* \, (w \ll p \iff B \cap \nu(w) \neq \emptyset) \, , \tag{14}$$

$$K \in \kappa(p) :\iff (\forall w \in \Sigma^*) \, (w \ll p \iff K \subseteq \bigcup \nu^{\mathrm{fs}}(w)) \, . \tag{15}$$

For avoiding accumulation of indices in this article we abbreviate:
$\delta := \delta^+, \quad \theta := \theta^+, \quad \psi := \psi^+.$

2. Define a representation $\delta^- :\subseteq \Sigma^\omega \to X$ of the points, a representation $\theta^- :\subseteq \Sigma^\omega \to \tau$ of the set of open sets, and a representation $\psi^- :\subseteq \Sigma^\omega \to \mathcal{A}$ of the set of closed sets by

$$\delta^-(p) = x :\iff \theta^+(p) = X \setminus \overline{\{x\}}, \tag{16}$$
$$\theta^-(p) := X \setminus \psi(p), \tag{17}$$
$$\psi^-(p) := X \setminus \theta^+(p)\,. \tag{18}$$

Notice that names in Definitions 2 and 5 must not be "polluted" by words $w \notin \mathrm{dom}(\nu)$ since implicitly $w \in \mathrm{dom}(\nu)$ if $w \ll p \in \mathrm{dom}(\delta)$, $w \in \mathrm{dom}(\nu)$ if $w \ll p \in \mathrm{dom}(\psi)$ and $w \in \mathrm{dom}(\bigcup \nu^{\mathrm{fs}}) = \mathrm{dom}(\nu^{\mathrm{fs}})$ if $w \ll p \in \mathrm{dom}(\kappa)$. If $\mathrm{dom}(\nu)$ is recursive, these conditions can be checked easily.

A δ^+-name of a point x is a list of *all* names of *all* of its basic neighborhoods, while a δ^--name is a list of base elements exhausting the complement of $\overline{\{x\}}$. Therefore, $\delta = \delta^+$ is the "inner representation" supplying positive information and δ^- is the "outer representation" supplying negative information. A θ^+-name of an open set W is a list of base elements exhausting W, while a θ^--name is a list of *all* names of *all* basic sets intersecting its complement. Thus, $\theta = \theta^+$ is the "inner representation" supplying positive information and θ^- is the "outer representation" supplying negative information. For the closed sets, $\psi = \psi^+$ (the complement of θ^-) is the "inner representation" and ψ^- (the complement of θ^+) is the "outer representation" . Finally, $K \in \kappa(q)$ iff q is a list of *all* names of *all* finite unions of base elements that cover K. κ is the "cover representation" of the compact sets.

Lemma 6.

1. *δ is well-defined (single-valued).*
2. *ψ is well-defined (single-valued).*
3. $\mathrm{dom}(\psi) = \mathrm{dom}(\widetilde{\psi})$ *and* $B \in \widetilde{\psi}(p) \iff \overline{B} = \psi(p)$.
4. *For* $q \in \mathrm{dom}(\kappa)$ *define* $K_q := \bigcap\{\bigcup \nu^{\mathrm{fs}}(w) \mid w \ll q\}$. *Then*

$$K_q \in \kappa(q) \quad \textit{and} \quad K \subseteq K_q \quad \textit{for all} \quad K \in \kappa(q)\,.$$

5. *δ^- is well-defined (single-valued).*

Proof: 1. δ is well-defined (single-valued) since for the T_0-space $\mathbf{X}$, $\{v \mid x \in \nu(v)\} = \{v \mid y \in \nu(v)\}$ implies $x = y$.

2. Suppose $\psi(p) = A \neq B = \psi(q)$. Then w.l.o.g. $x \in A$ and $x \notin B$ for some $x \in X$. Since B is closed, its complement B^c is open, hence $x \in V \subseteq B^c$ for some $V \in \beta$. Then $A \cap V \neq \emptyset$ and $B \cap V = \emptyset$, hence $p \neq q$. Therefore, ψ is single-valued.

3. For every open set U,

$$B \cap U = \emptyset \iff B \subseteq U^c \iff \overline{B} \subseteq U^c \iff \overline{B} \cap U = \emptyset. \tag{19}$$

Therefore, $B \in \widetilde{\psi}(p) \iff (w \ll p \iff B \cap \nu(w) \neq \emptyset) \iff (w \ll p \iff \overline{B} \cap \nu(w) \neq \emptyset) \iff \overline{B} = \psi(p)$.

4. Suppose $q \in \mathrm{dom}(\kappa)$. Then $K_0 \in \kappa(q)$ for some compact set K_0. For every $K \in \kappa(q)$, $K \subseteq \bigcup \nu^{\mathrm{fs}}(w)$ for all w such that $w \ll q$, hence $K \subseteq K_q$. Therefore, $\bigcup \kappa(q) \subseteq K_q$.

Suppose $K_q \subseteq \bigcup\{\nu(v) \mid v \in I\}$. Then $K_0 \subseteq \bigcup \kappa(q) \subseteq K_q \subseteq \bigcup\{\nu(v) \mid v \in I\}$. Since K_0 is compact, $K_0 \subseteq \bigcup\{\nu(v) \mid v \in F\}$ for some finite set $F \subseteq I$. There is some u such that $\bigcup\{\nu(v) \mid v \in F\} = \bigcup \nu^{\mathrm{fs}}(u)$. Since $K_0 \subseteq \bigcup \nu^{\mathrm{fs}}(u)$ and $K_0 \in \kappa(q)$, $u \ll q$, hence $K_q \subseteq \bigcup \nu^{\mathrm{fs}}(u)$ by the definition of K_q and so $K_q \subseteq \bigcup\{\nu(v) \mid v \in F\}$, which is a finite subcover of K_q. Therefore, K_q is compact. For all $w \in \mathrm{dom}(\bigcup \nu^{\mathrm{fs}})$, $K_q \subseteq \bigcup \nu^{\mathrm{fs}}(w) \Rightarrow K_0 \subseteq \bigcup \nu^{\mathrm{fs}}(w) \Rightarrow w \ll q \Rightarrow K_q \subseteq \bigcup \nu^{\mathrm{fs}}(w)$. Therefore, $K_q \in \kappa(q)$.

5. Suppose $\overline{\{x\}} = \overline{\{y\}}$ Then by (19), $x \in \nu(u) \iff \{x\} \cap \nu(u) \neq \emptyset \iff \overline{\{x\}} \cap \nu(u) \neq \emptyset \iff \overline{\{y\}} \cap \nu(u) \neq \emptyset \iff \{y\} \cap \nu(u) \neq \emptyset \iff y \in \nu(u)$, hence $x = y$. Therefore, δ^- is single-valued. □

By Lemma 6.3, ψ can be defined by means of $\widetilde{\psi}$ and vice versa. If $A \in \widetilde{\psi}(p)$, then $\widetilde{\psi}(p) = \{B \subseteq X \mid \overline{A} = \overline{B}\}$. In particular, $\overline{A} \in \widetilde{\psi}(p)$ since $\overline{\overline{A}} = \overline{A}$. $\overline{A}$ is the greatest set in $\widetilde{\psi}(p)$. The *saturation* of a set $A \subseteq X$ is defined by $\mathrm{sat}(A) := \bigcap\{U \in \tau \mid A \subseteq U\}$ [GHK+03]. For compact sets K, $\mathrm{sat}(K) = \bigcap\{\bigcup \alpha \mid \alpha \subseteq \beta,\ \alpha \text{ finite},\ K \subseteq \bigcup \alpha\}$. By Lemma 6.4 if $K \in \kappa(q)$ then $\kappa(q) = \{L \in \mathcal{K} \mid \mathrm{sat}(K) = \mathrm{sat}(L)\}$. In particular, $\mathrm{sat}(K) = K_q \in \kappa(q)$, $\mathrm{sat}(K)$ is the greatest set in $\kappa(q)$. The restriction of κ to the saturated compact sets sets is a single-valued representation.

In general, positive information cannot be found from negative information and vice versa.

Theorem 7. *In general, ($\delta \not\leq_t \delta^-$, $\delta^- \not\leq_t \delta$), ($\theta \not\leq_t \theta^-$, $\theta^- \not\leq_t \theta$), and ($\psi \not\leq_t \psi^-$, $\psi^- \not\leq_t \psi$).*

Proof: It suffices to consider the computable topological space $\mathbf{R}_<$ from Example 1.2. Suppose, $\delta \leq_t \delta^-$. Then there are a Type-2 machine M and an "oracle" $q \in \Sigma^\omega$ such that $\delta(p) = \delta^- \circ f_M(p, q)$ for all $p \in \mathrm{dom}(\delta)$. Let $\delta(p) = 0$. Started on input (p, q) there are some time t and some word w such that the machine has written $\iota(w)$ somewhere on its output tape in t steps. In t steps the machine M has read at most some prefix $v\iota(u)$ of p. There is some $r \in \Sigma^\omega$ such that $x := \delta(v\iota(u)r) > \nu_{\mathbb{Q}}(w)$. Also on input $(v\iota(u)r, q)$ the machine M will write $\iota(w)$ in t steps somewhere on its output tape. By the definition of δ^-, $(-\infty; x] \cap (\nu_{\mathbb{Q}}(w); \infty) = \overline{\{x\}} \cap \nu_<(w) = \emptyset$, hence $x \leq \nu_{\mathbb{Q}}(w)$. Contradiction.

Statements 2, 3 and 4 can be shown in the same way, Statements 5 and 6 follow from Statements 3 and 4. □

An important class of computable topological spaces can be constructed from very simple assumptions. Let X be a set with a countable set σ of predicates

$U \subseteq X$ (the "atomic predicates"). We may say "x has property U" if $x \in U$. We assume that each point of X can be identified by its atomic predicates, see (20). For handling atomic predicates concretely we consider a notation $\lambda :\subseteq \Sigma^* \to \sigma$ assigning names to them.

Definition 8 (predicate space).

1. An *effective predicate space* is a triple $\mathbf{Z} = (X, \sigma, \lambda)$ such that $\sigma \subseteq 2^X$, λ is a notation of σ and
$$(\forall x, y \in X)\,(x = y \iff \{U \in \sigma \mid x \in U\} = \{U \in \sigma \mid y \in U\})\,. \qquad (20)$$
$\mathbf{Z}$ is called *computable predicate space* if $\mathrm{dom}(\lambda)$ is recursive.
2. Define the representation $\delta_{\mathbf{Z}}$ of X by
$$x = \delta_{\mathbf{Z}}(p) :\iff (\forall w \in \Sigma^*)\,(w \ll p \iff x \in \lambda(w))\,, \qquad (21)$$
3. Let $T(\mathbf{Z}) = (X, \tau_\lambda, \beta_\lambda, \nu_\lambda)$ where β_λ is the set of finite intersections of sets from σ, $\nu_\lambda := \bigcap \lambda^{\mathrm{fs}} :\subseteq \Sigma^* \to \beta_\lambda$ (see (1)) and τ_λ is the set of all unions of subsets from β_λ.

Since the set β_λ of the finite intersections of sets in σ is closed under intersection, it is a base of the topology τ_λ. Since $\nu_\lambda(\iota(u_1)\iota(u_2)\ldots\iota(u_k)) = \lambda(u_1) \cap \lambda(u_2) \cap \ldots \cap \lambda(u_k)$, ν_λ can be called the notation by formal finite intersection.

Lemma 9. *Let $\mathbf{Z} = (X, \sigma, \lambda)$ be an effective predicate space.*

1. *$T(\mathbf{Z})$ is an effective topological space, which is computable if $\mathbf{Z}$ is computable (that is, if $\mathrm{dom}(\lambda)$ is recursive).*
2. *Let δ_λ be the inner representation of points for $T(\mathbf{Z})$). Then $\delta_\lambda \equiv \delta_{\mathbf{Z}}$.*
3. *for every representation γ_0 of a subset of $Y \subseteq X$, $\{(x, U) \in Y \times \sigma \mid x \in U\}$ is (γ_0, λ)-r.e. iff $\{(x, V) \in Y \times \beta_\lambda \mid x \in V\}$ is (γ_0, ν_λ)-r.e.*

Proof: 1. Obviously, β_λ is a base of the topology τ_λ on X and ν_λ is a notation of β_λ that has recursive domain if λ has recursive domain. If $x \neq y$, then by (20) there is some $U \in \sigma$ such that $(x \in U \wedge y \notin U)$ or $(x \notin U \wedge y \in U)$. Since $\sigma \subseteq \tau_\lambda$, (X, τ_λ) is a T_0-space. Condition (10) holds for $S := \{(u, v, \widehat{uv}) \mid u, v \in \mathrm{dom}(\nu_\lambda)\}$.

2. There is a machine M that on input p lists all $\iota(\iota(v_1)\ldots\iota(v_k))$ such that $v_1, \ldots, v_k \ll p$. Then the function f_M translates $\delta_{\mathbf{Z}}$ to δ_λ. There is another machine N that on input q lists all $\iota(u)$ such that $u \ll v \ll q$ for some $v \in \Sigma^*$. Then f_N translates δ_λ to $\delta_{\mathbf{Z}}$.

3. From a machine M, which halts on input (p, u) iff $\gamma_0(p) \in \lambda(u)$, we can construct a machine N that halts on input (p, v) iff $\gamma_0(p) \in \nu_\lambda(v)$ and and vice versa. $\square$

Roughly speaking, a $\delta_{\mathbf{Z}}$-name of a point is a list of *all* of its *atomic predicates* while a δ_λ-name is a list of *all* finite intersections of such sets. Obviously, the two representations are equivalent.

Example 2. Define $\mathbf{Z} := (\mathbb{R}, \sigma, \lambda)$ by $\lambda(w) := (\nu_{\mathbb{Q}}(w); \nu_{\mathbb{Q}}(w) + 1)$ and $\sigma :=$ range(λ). Then $\mathbf{Z}$ is a computable predicate space and $T(\mathbf{Z}) = (\mathbb{R}, \tau_{\mathbb{R}}, \beta_\lambda, \nu_\lambda)$ is a computable topological space. The spaces $T(\mathbf{Z})$ and $\mathbf{R}$ from Definition 1 are equivalent (Definition 21).

A representation $\gamma :\subseteq \Sigma^\omega \to X$ of a topological space (X, τ) is called *admissible (with respect to τ)* if it is continuous and $\gamma' \leq_t \gamma$ for every continuous function $\gamma' :\subseteq \Sigma^\omega \to X$ [KW85, Wei00, Sch02, Sch03]. The representation δ is *admissible* [Wei00, Section 3.2] w.r.t. the topology τ. Also all the other (single-valued) representations in Definition 5 are admissible w.r.t. appropriate topologies [Sch03].

In the literature a number of variants of Definitions 4 and 8 have been introduced. In [Wei00, Section 3.2] an *effective topological space* corresponds to our effective predicate space, and for a *computable topological space* $\{(u, v) \mid \lambda(u) = \lambda(v)\}$ must be r.e. (as a consequence dom(ν) must be r.e., c.f. Theorem 24) and δ'_S is our $\delta_{\mathbf{Z}}$. The results in this article show that the condition "$\{(u, v) \mid \lambda(u) = \lambda(v)\}$ is r.e." is unnecessarily strong for a general foundation. Variants of Definition 4 are used, for example, in [KW85, GW05, GSW07]. In [GW05], ν must have recursive domain. In [GSW07], the base must have computable intersection (10). Sometimes U must be non-empty for $U \in \sigma$ or $U \in \beta$.

Representations from Definition 5 have been studied for various topological spaces under various names, for example in [KW85, KW87, WK87, Zho96, BW99, Wei00, Zie02, ZB04, Zie04, BP03, Sch03, Wei03, GW05] the correspondences of names being obvious, and have been used in many applications. The representations δ^- (which has no application so far) and $\widetilde{\psi}$ or special cases of them have not been considered before. As an application of $\widetilde{\psi}$ consider our multi-representation κ that is the T_0-version of the cover representation κ_c [Wei00, Definition 5.2.4] (equivalent to $\kappa_>$ [Wei00, Definition 5.2.1]). Then the multi-representation $\widetilde{\psi} \wedge \kappa$ is our T_0-version of $\psi_< \wedge \kappa_>$ [Wei00, Lemma 5.2.10]) that is equivalent to the minimal cover representation κ_{mc} [Wei00, Definition 5.2.4, Lemma 5.2.5]. See also Theorem 38.4 and 6. In [KW98, KW03], starting from more general concepts Kalantari and Welch arrive at special computable topological spaces, which they study in detail.

In this section, computability on a computable predicate space $\mathbf{T}$ via $T(\mathbf{Z})$ is a special case of computability on a computable topological space. In Section 7 we show that equivalently, computability on a computable topological space can be considered as a special case of computability on a computable predicate space $\mathbf{Z}$ via $T(\mathbf{Z})$.

In the following we will assume tacitly that $\delta, \theta, \psi, \widetilde{\psi}, \kappa, \delta^-, \theta^-$ and ψ^- are the representations from Definition 5 for the effective topological space $\mathbf{X} = (X, \tau, \beta, \nu)$. If not assumed differently, $\mathbf{X} = (X, \tau, \beta, \nu)$ will be a *computable* topological space since only very few results on computability remain valid for general effective topological spaces.

4 Boolean Operations

We will consider computability of union and intersection not only on pairs but on sets of sets. The following observations will be used repeatedly.

Lemma 10.

1. $\nu \leq \bigcup \nu^{\text{fs}} \leq \theta$,
2. $w \in \text{dom}(\nu^{\text{fs}})$ *if* w *is a prefix of* p *for some* $p \in \text{dom}(\delta)$.
3. $\delta[w\Sigma^\omega] = \bigcap \nu^{\text{fs}}(w)$ *for all* $w \in \text{dom}(\nu^{\text{fs}})$.

We summarize some computability results on union and intersection for computable topological spaces. By (10), intersection is (ν, ν, θ)-computable. We apply this axiom in the proof.

Theorem 11 (union and intersection).

1. *Finite intersection on open sets is* $(\nu^{\text{fs}}, \theta)$*-computable and* $(\theta^{\text{fs}}, \theta)$*-computable.*
2. *Union on open sets is* $(\theta^{\text{cs}}, \theta)$*-computable.*
3. *On closed sets, finite union is* $((\psi^-)^{\text{fs}}, \psi^-)$*-computable and intersection is* $((\psi^-)^{\text{cs}}, \psi^-)$*-computable.*
4. *On the at most countable collections* $\mathcal{B}$ *of closed sets, the function* $\mathcal{B} \mapsto \overline{\bigcup \mathcal{B}}$ *is* $((\psi)^{\text{cs}}, \psi)$*-computable.*
5. *On the compact sets finite union is* $(\kappa^{\text{fs}}, \kappa)$*-computable.*
6. *The function* $(K, A) \mapsto K \cap A$ *for compact* K *and closed* A *is* (κ, ψ^-, κ)*-computable.*

Proof: 1. For $q = \langle 1^k, p_1, \ldots, p_k\rangle$,

$$\bigcap \theta^{\text{fs}}(q) = \bigcup \{\nu(v_1) \cap \ldots \cap \nu(v_k) \mid v_1 \ll p_1, \ldots, v_k \ll p_k\}$$

and

$$\bigcap_{i=1}^{k} \nu(v_i) = \bigcup \{\nu(u) \mid u \in \text{dom}(\nu),\ (\exists_{i=0}^k u_i)(\forall_{i=1}^k (u_{i-1}, v_i, u_i) \in S \wedge u = u_k)\}\,.$$

There is a machine M that on input $\langle 1^k, p_1, \ldots, p_k\rangle \in \text{dom}(\theta^{\text{fs}})$ writes all $\iota(u)$, for which there are $v_1 \ll p_1, \ldots, v_k \ll p_k$ and $u_0, \ldots, u_k \in \text{dom}(\nu)$ such that $(u_{i-1}, v_i, u_i) \in S$ for $i = 1, \ldots, k$ and $u = u_k$). Remember that S is r.e. Then $\bigcap \theta^{\text{fs}}(q) = \theta \circ f_M(q)$ for all $q \in \text{dom}(\theta^{\text{fs}})$.

Since $\nu \leq \theta$, and hence $\nu^{\text{fs}} \leq \theta^{\text{fs}}$ by Lemma 10, intersection is also $(\nu^{\text{fs}}, \theta)$-computable.

2. There is a Machine M that on input $q = \langle a_0p_0, a_1p_1, \ldots\rangle$ lists all $\iota(v)$ such that for some i, $a_i = 0$ and $\iota(v)$ is a subword p_i (and writes 11 from time to time). Then f_M realizes union.

3. This follows from 1. and 2 by (18).

4. For every $U \in \beta$, $U \cap \overline{\bigcup \mathcal{B}} \neq \emptyset \iff U \cap \bigcup \mathcal{B} \neq \emptyset \iff (\exists A \in \mathcal{B}) U \cap A \neq \emptyset$. There is a machine that on input $q = \langle a_0 p_0, a_1 p_1, \ldots \rangle$ lists all $\iota(v)$ such that for some i, $a_i = 0$ and $\iota(v)$ is a subword p_i (and writes 11 from time to time). Then f_M realizes the function $\mathcal{B} \mapsto \overline{\bigcup \mathcal{B}}$.

5. Suppose $K_i \in \kappa(p_i)$ for $1 \leq i \leq k$. Then $K_1 \cup \ldots \cup K_k \subseteq \bigcup \nu^{\mathrm{fs}}(w)$ iff $K_i \subseteq \bigcup \nu^{\mathrm{fs}}(w)$ for $1 \leq i \leq k$ iff $w \ll p_i$ for $1 \leq i \leq k$. There is a machine M that on input $\langle 1^k, p_1, \ldots, p_k \rangle$ lists all $\iota(v)$ such that $v \ll p_i$ for all $1 \leq i \leq k$ (and writes 11 from time to time). Then f_M realizes finite union on the compact sets.

6. Observe that for sets $A, B, K \subseteq X$, $K \cap A \subseteq B \iff K \subseteq B \cup X \setminus A$. For showing that he set $K \cap A$ is compact, assume $K \cap A \subseteq \bigcup \alpha$ for some $\alpha \subseteq \tau$. Then $K \subseteq \bigcup \alpha \cup (X \setminus A)$ (an open cover) and hence $K \subseteq \bigcup \alpha' \cup \{X \setminus A\}$ for some finite $\alpha' \subseteq \alpha$, therefore $K \cap A \subseteq \bigcup \alpha'$.

For $K \in \kappa(p)$, $q \in \mathrm{dom}(\psi^-)$ and $u \in \mathrm{dom}(\bigcup \nu^{\mathrm{fs}})$, $K \cap \psi^-(q) \subseteq \bigcup \nu^{\mathrm{fs}}(u) \iff K \subseteq \bigcup \nu^{\mathrm{fs}}(u) \cup \theta(q) \iff K \subseteq \bigcup \nu^{\mathrm{fs}}(u) \cup \bigcup \{\nu(v) \mid v \ll q\} \iff (\exists w \sqsubseteq q) K \subseteq \bigcup \nu^{\mathrm{fs}}(u) \cup \bigcup \nu^{\mathrm{fs}}(w) \iff (\exists w \sqsubseteq q) K \subseteq \bigcup \nu^{\mathrm{fs}}(\widehat{u}\widehat{w}) \iff (\exists w \sqsubseteq q) \widehat{u}\widehat{w} \ll p$ (for $\sqsubseteq$ and $\widehat{u}$ see Section 1). There is a machine M that on input (p, q) prints all words $\widehat{u}$ such that $\widehat{u}\widehat{w} \ll p$ for some $w \sqsubseteq q$ (and writes 11 from time to time). Then f_M realizes intersection of a compact and a closed set. □

By Theorem 11.6, $(K, A) \mapsto A$ for $A \subseteq K$ is (κ, ψ^-, κ)-computable. A number of corollaries can be derived easily in combination with Lemma 10 and Lemma 3, in particular for binary union and intersection. For example, union $(U, V) \mapsto U \cup V$ is $(\nu, \nu, \bigcup \nu^{\mathrm{fs}})$ computable by (7) and $(\bigcup \nu^{\mathrm{fs}}, \nu, \theta)$-computable by Lemma 10.1. By definition, complementation is (θ, ψ^-) computable on the open sets and (ψ^-, θ)-computable on the closed sets. Some negative results are summarized in the following theorem.

Theorem 12.

1. *In general complementation of open sets is not (θ, ψ)-continuous and not (θ^-, ψ^-)-continuous.*
2. *In general complementation of closed sets is not (ψ^-, θ^-)-continuous and not (ψ, θ)-continuous.*
3. *In general for no representations ψ_1, ψ_2 of the closed sets intersection is (ψ_1, ψ_2, ψ)-continuous.*

Proof: 1.,2. This follows from Theorem 7

3. For the computable real line (Example 1.1) there are no (!) representations ψ_1, ψ_2 of the closed sets such that intersection is (ψ_1, ψ_2, ψ)-continuous [Wei00, DWW07]. □

5 Abstract Characterizations of Computability

In Section 3 we have defined explicitly some representations of the points and of spaces of subsets. In this section we characterize their equivalence classes without defining representatives. Since two multi-representations induce the same kind of computability on a set iff they are equivalent, we characterize the computability concepts induced by these representations.

Theorem 13. *Let $\mathbf{X} = (X, \tau, \beta, \nu)$ be a computable topological space.*

1. *For every representation γ of a subset $Y \subseteq X$,* $\{(x, U) \in Y \times \beta \mid x \in U\}$ *is (γ, ν)-r.e.* $\iff \gamma \leq \delta$.
2. *For every representation γ of a subset $\mathcal{T} \subseteq 2^X$,* $\{(x, W) \in X \times \mathcal{T} \mid x \in W\}$ *is (δ, γ)-r.e.* $\iff \gamma \leq \theta$.
3. *For every representation γ of a set $\mathcal{B}$ of closed sets,* *"$A \cap V \neq \emptyset$" is (γ, ν)-r.e.* $\iff \gamma \leq \psi$.
4. *For every multi-representation γ of a subset $\mathcal{S} \subseteq 2^X$,* *"$A \cap V \neq \emptyset$" is (γ, ν)-r.e.* $\iff \gamma \leq \widetilde{\psi}$.
5. *For every multi-representation γ of a subset $\mathcal{L} \subseteq 2^X$,* $\{(K, W) \in \mathcal{L} \times \tau \mid K \subseteq W\}$ *is (γ, θ)-r.e.* $\iff \gamma \leq \kappa$.
6. *For every multi-representation γ of a set of compact sets,* *"$K \subseteq V$" is $(\gamma, \bigcup \nu^{\mathrm{fs}})$-r.e.* $\iff \gamma \leq \kappa$.

The statements remain true for effective topological spaces if "-open" is substituted for "-r.e." and "$\leq_t$" is substituted for "$\leq$" (the topological version of the theorem).

Proof: 1. Suppose "$x \in U$" is (γ, ν)-r.e. Then there is some Type-2 machine M that halts on input $(p, u) \in \mathrm{dom}(\gamma) \times \mathrm{dom}(\nu)$ iff $\gamma(p) \in \nu(u)$. Let N be a Type-2 machine that on input p successively for all $(u, n) \in \mathrm{dom}(\nu) \times \mathbb{N}$ runs the machine M on input (p, u) and writes 11 if the computation does not halt in n steps, and writes $\iota(u)$ otherwise. Then for $p \in \mathrm{dom}(\gamma)$ and $u \in \mathrm{dom}(\nu)$, $f_N(p)$ has the subword $\iota(u)$ iff M halts on input $(p, u) \in \mathrm{dom}(\gamma) \times \mathrm{dom}(\nu)$ iff $\gamma(p) \in \nu(u)$. Therefore, $\gamma(p) = \delta \circ f_N(p)$ for all $p \in \mathrm{dom}(\gamma)$, hence $\gamma \leq \delta$.

On the other hand, suppose that there is a computable function $f :\subseteq \Sigma^\omega \to \Sigma^\omega$ such that $\gamma(p) = \delta \circ f(p)$ for all $p \in \mathrm{dom}(\gamma)$. Let M be a Type-2 machine, which on input $(p, u) \in \Sigma^\omega \times \Sigma^*$ computes $f(p)$ and halts as soon as $\iota(u)$ has been detected as a subword of $f(p)$. Then M halts on input $(p, u) \in \mathrm{dom}(\gamma) \times \mathrm{dom}(\nu)$ iff $\gamma(p) \in \nu(u)$, hence "$x \in U$" is (γ, ν)-r.e.

2. Suppose, "$x \in W$" is (δ, γ)-r.e. Then there is an r.e. set $R \subseteq \Sigma^* \times \Sigma^*$ such that $\delta(p) \in \gamma(q) \iff (p, q) \in \bigcup\{u\Sigma^\omega \times v\Sigma^\omega \mid (u, v) \in R\}$ for all $p \in \mathrm{dom}(\delta)$ and $q \in \mathrm{dom}(\gamma)$. Therefore, $\gamma(q) = \bigcup\{\delta[u\Sigma^\omega] \mid (\exists v)(q \in v\Sigma^\omega \ \wedge \ (u, v) \in R)\}$ for every $q \in \mathrm{dom}(\gamma)$. Since $\delta[u\Sigma^\omega] = \emptyset$ if $u \notin \mathrm{dom}(\nu^{\mathrm{fs}})$ and $\delta[u\Sigma^\omega] = \bigcap\{\nu(v) \mid$

$v \ll u\} = \bigcap \nu^{\rm fs}(u)$ otherwise, $\delta[u\Sigma^\omega] = \theta \circ f(u)$ for some computable function f by Theorem 11.1. Therefore, $\gamma(q) = \bigcup\{\theta \circ f(u) \mid (\exists v)(q \in v\Sigma^\omega \ \wedge \ (u, v) \in R)\}$. There is a machine M that on input q lists all words $w \in \mathrm{dom}(\nu)$ such that $w \ll f(u)$ and $q \in v\Sigma^\omega$ for some $(u, v) \in R$. Then f_M translates γ to θ.

Next, we show that "$x \in W$" is (δ, θ)-r.e. For $p \in \mathrm{dom}(\delta)$ and $q \in \mathrm{dom}(\theta)$,

$$\delta(p) \in \theta(q) \iff (\exists w,\ w \ll q)\delta(p) \in \nu(w) \iff (\exists w)(w \ll q \wedge w \ll p)\,.$$

There is a Type-2 machine M that halts on input (p, q) iff there is some w such that $w \ll q$ and $w \ll p$. This machine halts on input $(p, q) \in \mathrm{dom}(\delta) \times \mathrm{dom}(\theta)$ iff $\delta(p) \in \theta(q)$. Therefore, "$x \in W$" is (δ, θ)-r.e.

Finally for $W \in \mathrm{range}(\gamma)$, "$x \in W$" is (δ, γ)-r.e. if $\gamma \le \theta$, since "$x \in W$" is (δ, θ)-r.e.

3. Let $\mathbf{Z} := (\mathcal{B}, \sigma, \lambda)$ where λ is a notation of a family of subsets of $\mathcal{B}$ such that $\lambda(w) := \{A \in \mathcal{B} \mid A \cap \nu(w) \neq \emptyset\}$. Then $\mathbf{Z}$ is a computable predicate space. Let $T(\mathbf{Z}) := (\mathcal{B}, \tau_\lambda, \beta_\lambda, \nu_\lambda)$ be the associated computable topological space. Then $\delta_{\mathbf{Z}} = \psi$ by (21,13) and $\delta_{\mathbf{Z}} \equiv \delta_\lambda$ by Lemma 9. By Theorem 13.1 applied to $T(\mathbf{Z})$ and Lemma 9.3, $\gamma(p) \cap \nu(u)$ is r.e. $\iff$ $\gamma(p) \in \lambda(u)$ is r.e. $\iff$ $\gamma(p) \in \nu_\lambda(u)$ is r.e. $\iff \gamma \le \delta_\lambda \iff \gamma \le \psi$.

4. Suppose $A \in \widetilde{\psi}(p)$. Then $A \cap \nu(w) \neq \emptyset \iff w \ll p$. There is a machine that halts on input (p, w) iff $w \ll p$. Therefore, "$A \cap V \neq \emptyset$" is $(\widetilde{\psi}, \nu)$-r.e. If $\gamma \le \widetilde{\psi}$, then "$A \cap V \neq \emptyset$" is (γ, ν)-r.e. .

On the other hand assume that "$A \cap V \neq \emptyset$" is (γ, ν)-r.e. . Then there is a machine M that halts on input (p, w) ($p \in \mathrm{dom}(\gamma)$, $w \in \mathrm{dom}(\nu)$) iff $A \cap \nu(w) \neq \emptyset$ for some (and thus for all) $A \in \gamma(p)$. There is another machine N that lists all $w \in \mathrm{dom}(\nu)$ such that M halts on input (p, w). Then f_N translates γ to $\widetilde{\psi}$.

5. Suppose "$K \subseteq V$" is (γ, θ)-r.e. Then there is a machine M that halts on input $(p, q) \in \mathrm{dom}(\gamma) \times \mathrm{dom}(\theta)$ iff $K \subseteq \theta(q)$ for all $K \in \gamma(p)$. Let $K \subseteq \bigcup_{u \in I} \nu(u)$. There is some $q \in \Sigma^\omega$ such that $I = \{u \mid u \ll q\}$. Since $K \subseteq \theta(q)$, the machine M halts on input (p, q) and reads at most a finite prefix u of q. Since $\iota(v)$ has odd length for all $v \in \Sigma^*$, there is some $w \in \Sigma^*11$ such that $\bigcup \nu^{\rm fs}(u) = \bigcup \nu^{\rm fs}(uw)$. Then $q' := uw1^\omega \in \mathrm{dom}(\theta)$ and M that halts on input (p, q'), hence $K \subseteq \theta(q')$. Now, $\theta(q') = \bigcup_{u \in I'} \nu(u)$ for some finite $I' \subseteq I$. Therefore, K is compact. We conclude that γ is a multi-representation of compact sets.

There is a machine N that on input $p \in \mathrm{dom}(\gamma)$ writes a list of all $\iota(u)$ such that $u \in \mathrm{dom}\bigcup \nu^{\rm fs})$ and M on input $(p, u1^\omega)$ halts in at most $|u|$ steps. Then $K \in \kappa \circ f_N(P)$ if $K \in \gamma(p)$. Therefore, $\gamma \le \kappa$.

We show that "$K \subseteq W$" is (κ, θ)-r.e. For $K \in \kappa(p)$ and $W = \theta(q)$, $K \subseteq W$ iff there is some $u \ll p$ such that $v \ll q$ for all $v \ll u$. There is a machine that halts on input (p, q) iff this condition is true. Therefore, "$K \subseteq W$" is (κ, θ)-r.e. If $\gamma \le \kappa$, then "$K \subseteq W$" is (γ, θ)-r.e. (see Section 2).

6. By easy arguments for compact sets K, "$K \subseteq V$" is $(\gamma, \bigcup \nu^{\rm fs})$-r.e. iff "$K \subseteq W$" is (γ, θ)-r.e. Apply Theorem 13.5.

The continuous versions of the statements can be proved similarly using Type-2 machines with an oracle $r \in \Sigma^\omega$ on an additional input tape. $\square$

Let F be a class of multi-representations and let $\gamma \in F$ be such that $\gamma' \leq \gamma$ for all $\gamma' \in F$. Then γ can be called *complete in* F (more precisely, $\leq$-complete). Since translation cannot gain the amount and the quality of information contained in names, a complete multi-representation can also be called the (up to equivalence) "poorest" representation in F.

While a complete representation is unique only up to equivalence, the induced computability concept is the same for all multi-representations complete in F.

Let $\gamma :\subseteq \Sigma^\omega \to X$ be a multi-representation. If $Y \subseteq X$ is γ-r.e., then there is a Type-2 machine M that halts on input $p \in \mathrm{dom}(\gamma)$ iff p is a γ-name of some $y \in Y$. A computation halting on input p can be interpreted as a proof of "$y \in Y$" for all $y \in \gamma(p)$ and the machine M machine can be interpreted as a "proof system" for Y. Correspondingly, if $Y \subseteq X$ is γ-open, then there are a Type-2 machine M and an "oracle" $q \in \Sigma^\omega$ such that for $p \in \mathrm{dom}(\gamma)$, the machine M halts on input (q, p) iff p is a γ-name of some $y \in Y$. In this case, (M, p) is our "oracle proof system". In our context the basic principle underlying the concept of "proof" is *finiteness*. If a Type-2 machine halts, it can read only a finite portion of its input. Therefore, a finite amount of information suffices to obtain a positive answer.

Theorem 13.1 can be formulated as follows. The representation δ is complete in the class of all representations γ of points such that the element relation "$x \in U$" ($x \in X$, $U \in \beta$) becomes provable (precisely, (γ, ν)-r.e.). Provability of the element relation "$x \in U$" does not define δ but only the equivalence class of this representation, that is, the computability concept on X induced by it (remember: two multi-representations are equivalent iff they induce the same concept of computability). The other five statements from Theorem 13. can be interpreted accordingly.

The following observation is noteworthy: while in 1., 3., 4. and 6. computability concepts *on the given sets* (points, closed sets, all sets, compact sets) are characterized by provability, in 2. the open sets and in 5. the compact sets are characterized simultaneously. Thus provability of "$x \in W$" (for points x) is characteristic for the open sets W and provability of "$K \subseteq W$" (for open sets W) is characteristic for the compact sets. By the topological version of the theorem, the open sets are the biggest class of sets W such that "$x \in W$" is δ-open and the compact sets are the biggest class of sets K such that "$K \subseteq W$" is θ-open.

In summary, for a computable topological space by requiring provability we can define in turn computability on the points, the open sets, computability on the open sets, the compact sets and computability on the compact sets. Roughly speaking, merely the idea of *finiteness* suffices to define these concepts for computable topological spaces.

Corollary 14. *For points x, open sets W, closed sets A and compact sets K,*

$$\text{“}x \in W\text{” is } (\delta,\theta)\text{-r.e.,} \tag{22}$$

$$\text{“}K \subseteq W\text{” is } (\kappa,\theta)\text{-r.e.,} \tag{23}$$

$$\text{“}A \cap W \neq \emptyset\text{” is } (\psi,\theta)\text{-r.e.,} \tag{24}$$

$$\text{“}K \cap A = \emptyset\text{” is } (\kappa,\psi^-)\text{-r.e.} \tag{25}$$

Proof: (22) Let $\gamma := \theta$ in Theorem 13.2.
(23) Let $\gamma := \kappa$ in Theorem 13.5.
(24) $\psi(p) \cap \theta(q) \neq \emptyset$ iff $(\exists w \ll q)\psi(p) \cap \nu(w) \neq \emptyset$. Apply Theorem 13.3.
(25)Observe that $\kappa(p) \cap \psi^-(q) = \emptyset$ iff $\kappa(p) \subseteq \theta(q)$. Apply (23). □

Theorem 13.1 generalizes [Wei00, Theorem 3.2.10]. The conclusion “W is open iff W is δ-open” from Theorem 13.2 is the fact that τ is the final topology of the admissible representation δ [KW85, Wei00].

6 Some Additions

Every singleton set $\{x\}$ is compact. The representation κ can be considered as an extension of δ.

Lemma 15. *Define* $\mathrm{ec} : X \to \mathcal{K}$ *by* $\mathrm{ec}(x) := \{x\}$. *Then* ec *is* (δ, κ)*-computable and* ec^{-1} *is* (κ, δ)*-computable.*

Proof: There is a machine that on input $p \in \Sigma^\omega$ lists all $\iota(v)$, $v \in \mathrm{dom}(\nu^{\mathrm{fs}})$, such that $u \ll p$ and $u \ll v$. Then f_M realizes ec. On the other hand there is a machine N that on input q lists all $\iota(u)$ such that $u \in \mathrm{dom}(\nu)$ and $\iota(u) \ll q$. Then f_N realizes ec^{-1}. □

Although in a T_0-space singletons $\{x\}$ may not be closed (see Examples 1.2 and 1.3), the representations ψ and ψ^- of the closed sets can be considered as extensions of the representations δ and δ^- of points, respectively.

Lemma 16. *Define* $\mathrm{cl} : X \to \mathcal{A}$ *by* $\mathrm{cl}(x) := \overline{\{x\}}$. *Then* cl *is injective,* cl *is* (δ, ψ)*-computable and* (δ^-, ψ^-)*-computable, and* cl^{-1} *is* (ψ, δ)*-computable and* (ψ^-, δ^-)*-computable.*

Proof: In the proof of Lemma 6.5 we have shown that cl is injective. For all $w \in \Sigma^*$, $\overline{\{x\}} \cap \nu(w) \neq \emptyset \iff x \in \nu(w)$, hence $\psi(p) = \overline{\{x\}} \iff \delta(p) = x$. Therefore, the identity realizes cl and cl^{-1} w.r.t. δ and ψ.

By definition, $\mathrm{cl} \circ \delta^-(p) = X \setminus \theta(p) = \psi^-(p)$ for $p \in \mathrm{dom}(\mathrm{cl})$. Therefore cl is (δ^-, ψ^-)-computable and cl^{-1} is (ψ^-, δ^-)-computable. □

Notice that (X, τ) is a T_1-space [Eng89] iff all sets $\{x\}$ $(x \in X)$ are closed (that is, $\overline{\{x\}} = \{x\}$). We show that for a T_1-space the multi-representation κ is single-valued. In general, however, κ may be properly multi-valued, but still the sets $\kappa(q)$ have a simple structure. In contrast to T_2-spaces [Eng89], for T_1-spaces compact sets may be not closed and the intersection of compact sets may be not compact.

Theorem 17.

1. *There is a computable topological space such that κ is not single-valued.*
2. *For computable topological spaces that are T_1, the multi-representation κ is single-valued.*
3. *There is a computable topological space that is T_1 with compact subsets, which are not closed, and with two compact sets, the intersection of which is not compact.*

Proof: 1. In Example 1.2, $K \subseteq \mathbb{R}$ is compact iff K has a minimum or $K = \emptyset$ (which is the only closed compact set), and for non-empty sets, $K_1, K_2 \in \kappa(p)$ for some p iff K_1 and K_2 have the same minimum. Therefore, κ is not single-valued.

2. Suppose, $K, L \subseteq X$ are compact such that $K \neq L$. Then w.l.o.g. $x \in K \setminus L$ for some $x \in X$. Since τ is a T_1-topology for every $y \in L$ there is some $U_y \in \beta$ such that $y \in U_y$ and $x \notin U_y$. Since L is compact and $L \subseteq \bigcup_{y \in L} U_y$, there is some finite set $F \subseteq L$ such that $L \subseteq \bigcup_{y \in F} U_y$. Therefore, we have a finite cover of L with base elements that does not cover K, since $(\forall y) x \notin U_y$. Since K and L have different sets of finite covers, they cannot have the same κ-name.

3. In Example 1.4, $\mathbb{N} \cup \{-1\}$ and $\mathbb{N} \cup \{-2\}$ are non-closed compact sets, the intersection $\mathbb{N}$ of which is not compact. In particular, the space is not T_2. $\square$

By Theorem 13.2 the relation $x \in W$ is (δ, θ)-r.e. Therefore, a θ-name of an open set W contains the information how to verify by a machine $\delta(q) \in W$ iff $\delta(q) \in W$. There are various equivalent other ways to encode this information. First, we prove a more general lemma. Let $\rho_<$ and δ_{Si} be the inner representations of the points of the lower real line $\mathbf{R}_<$ and Sierpinski space $\mathbf{Si}$, respectively (Example 1). For $W \subseteq X$ define the characteristic functions $\mathrm{cf}^{\mathbb{R}}_W : X \to \mathbb{R}$ and $\mathrm{cf}^{\mathrm{Si}}_W : X \to \mathbf{Si} = \{\bot, \top\}$ by $\mathrm{cf}^{\mathbb{R}}_W(x) = 0$ and $\mathrm{cf}^{\mathrm{Si}}_W(x) = \bot$ if $x \notin W$ and $\mathrm{cf}^{\mathbb{R}}_W(x) = 1$ and $\mathrm{cf}^{\mathrm{Si}}_W(x) = \top$ if $x \in W$.

Definition 18. Let $\gamma :\subseteq \Sigma^\omega \to X$ be a representation. Define representations $\theta^{\mathrm{dom}}_\gamma$, $\theta^{\mathrm{cf}}_\gamma$ and $\theta^{\mathrm{Si}}_\gamma$ of the γ-open sets as follows:

$$\theta^{\mathrm{dom}}_\gamma(p) = W \iff \gamma^{-1}[W] = \mathrm{dom}(\eta^{\omega *}_p) \cap \mathrm{dom}(\gamma)\,, \tag{26}$$

$$\theta^{\mathrm{cf}}_\gamma(p) = W \iff \eta^{\omega\omega}_p \text{ is a } (\gamma, \rho_<)\text{-realization of } \mathrm{cf}^{\mathbb{R}}_W\,, \tag{27}$$

$$\theta^{\mathrm{Si}}_\gamma(p) = W \iff \eta^{\omega\omega}_p \text{ is a } (\gamma, \delta_{\mathrm{Si}})\text{-realization of } \mathrm{cf}^{\mathrm{Si}}_W\,. \tag{28}$$

Lemma 19. $\theta_\gamma^{\mathrm{dom}} \equiv \theta_\gamma^{\mathrm{cf}} \equiv \theta_\gamma^{\mathrm{Si}}$.

By the utm-theorem for $\eta^{\omega a}$, there is a machine M_a that on input (p,q) computes $\eta_p^{\omega a}(q)$ ($a \in \{\omega, *\}$). By definition: $\theta_\gamma^{\mathrm{dom}}(p) = W$ iff the machine M_* halts on input (p,q) iff $\gamma(q) \in W$ (for all $q \in \mathrm{dom}(\gamma)$); $\theta_\gamma^{\mathrm{cf}}(p) = W$ iff the machine M_ω with input (p,q), $q \in \mathrm{dom}(\gamma)$, computes a list of all rational numbers $c < 0$ if $\gamma(q) \notin W$ and a list of all rational numbers $c < 1$ if $\gamma(q) \in W$; $\theta_\gamma^{\mathrm{Si}}(p) = W$ iff the machine M_ω with input (p,q), $q \in \mathrm{dom}(\gamma)$, writes some $r \in \mathrm{dom}(\delta_{\mathrm{Si}})$ such that $\iota(1) \ll r$ iff $\gamma(q) \in W$. In the first case $\gamma(q) \in W$ is detected by halting, in the second case by writing some (code of a) rational number > 0, and in the last case by writing $\iota(1)$ as a subword.

Proof: $\boldsymbol{\theta_\gamma^{\mathrm{dom}} \leq \theta_\gamma^{\mathrm{cf}}}$: The representation ρ' of the real numbers, defined by $\rho'(q) = x$ iff $q = \iota(u_0)\iota(u_1)\iota(u_2)\ldots$ such that $\nu_{\mathbb{Q}}(u_i) \leq \nu_{\mathbb{Q}}(u_{i+1})$ and $x = \sup_i \nu_{\mathbb{Q}}(u_i)$ is equivalent to $\rho_<$ [Wei00]. By the utm-theorem there is a machine N that on input (p,q) computes $\eta_p^{\omega *}(q)$. There is a machine M that on input (p,q) simulates the machine N on input (p,q). For the ith step of N it writes $\iota(u_0)$ for some u_0 with $\nu_{\mathbb{Q}}(u_0) = 0$. If the machine N halts, the machine M continues writing $\iota(u_1)$ such that $\nu_{\mathbb{Q}}(u_1) = 1$ forever. Then for all (p,q), $\rho' \circ f_M(p,q) \in \{0,1\}$ and $\eta_p^{\omega *}(q)$ exists iff $\rho' \circ f_M(p,q) = 1$. Let $h :\subseteq \Sigma^\omega \to \Sigma^\omega$ be a translation of ρ' to $\rho_<$. Since $(p,q) \mapsto h \circ f_M(p,q)$ is computable, by the smn-theorem there is a computable function $r : \Sigma^\omega \to \Sigma^\omega$ such that $h \circ f_M(p,q) = \eta_{r(p)}^{\omega\omega}(q)$. Suppose, $\theta_\gamma^{\mathrm{dom}}(p) = W$. Then for all $q \in \mathrm{dom}(\gamma)$, $\rho_< \circ \eta_{r(p)}^{\omega\omega}(q) \in \{0,1\}$ and $\rho_< \circ \eta_{r(p)}^{\omega\omega}(q) = 1 \iff \eta_p^{\omega *}(q)$ exists $\iff \gamma(q) \in W \iff \mathrm{cf}_W^{\mathbb{R}} \circ \gamma(q) = 1$. Therefore, $\theta_\gamma^{\mathrm{cf}} \circ r(p) = W$.

$\boldsymbol{\theta_\gamma^{\mathrm{cf}} \leq \theta_\gamma^{\mathrm{Si}}}$: There is a machine that on input (p,q) simulates a machine N computing $\eta_p^{\omega\omega}(q)$ writing $\iota(0)$ after every step of N. As soon as some subword $\iota(v)$ has occurred in the output of N such that $\nu_{\mathbb{Q}}(v) > 0$ M continues writing $\iota(1)$ forever. By the method used above a computable translation from $\theta_\gamma^{\mathrm{cf}}$ to $\theta_\gamma^{\mathrm{Si}}$ can be found.

$\boldsymbol{\theta_\gamma^{\mathrm{Si}} \leq \theta_\gamma^{\mathrm{dom}}}$: There is a machine that on input (p,q) simulates a machine N computing $\eta_p^{\omega\omega}(q)$ and halts as soon as the subword $\iota(1)$ has occurred in the output of N. Continue as in the first case. □

Applied to our admissible representation δ we obtain three representations of the open sets that are equivalent to θ.

Theorem 20. *For the inner representations δ and θ of the points and the open sets, respectively, for a computable topological space* $\mathbf{X}$*,*

$$\theta \equiv \theta_\delta^{\mathrm{dom}} \equiv \theta_\delta^{\mathrm{cf}} \equiv \theta_\delta^{\mathrm{Si}}\,.$$

Proof: By Lemma 19 it suffices to prove $\theta \equiv \theta_\delta^{\mathrm{dom}}$.

$\boldsymbol{\theta} \leq \boldsymbol{\theta_\delta^{\mathrm{dom}}}$: By Theorem 13.2 "$x \in W$" is (δ, θ)-r.e. Therefore, there is a machine M that on input (p, q), $p \in \mathrm{dom}(\theta)$ and $q \in \mathrm{dom}(\delta)$, halts iff $\delta(q) \in \theta(p)$. By the smn-theorem there is a computable function $r : \Sigma^\omega \to \Sigma^\omega$ such that $f_M(p, q) = \eta^{\omega*}_{r(p)}(q)$. Therefore, $\delta(q) \in \theta(p)$ iff $q \in \mathrm{dom}(\eta^{\omega*}_{r(p)})$, hence $\theta(p) = \theta_\delta^{\mathrm{dom}} \circ r(p)$.

$\boldsymbol{\theta_\delta^{\mathrm{dom}}} \leq \boldsymbol{\theta}$: By (26) for $p \in \mathrm{dom}(\theta_\delta^{\mathrm{dom}})$ and $q \in \mathrm{dom}(\delta)$, $\delta(q) \in \theta_\delta^{\mathrm{dom}}(p)$ iff $\eta_p^{\omega*}(q)$ exists. There is a machine M that halts on input (p, q) iff $\eta_p^{\omega*}(q)$ exists. Therefore, "$x \in W$" is $(\delta, \theta_\delta^{\mathrm{dom}})$-r.e. By Theorem 13.2, $\theta_\delta^{\mathrm{dom}} \leq \theta$. $\square$

For the special case of $\mathbb{R}^n$ and for computable metric spaces the theorem has been proved in [BW99] and [BP03], respectively.

7 Inessential Modifications

The Definition 4 of a computable topological space $\mathbf{X} = (X, \tau, \beta, \nu)$ can be modified in various ways without changing the induced computability concepts on points and subsets.

If the T_0-condition is violated, then there are points $x \neq y$ such that $\{U \in \beta \mid x \in U\} = \{U \in \beta \mid y \in U\}$, that is, x and y can not be distinguished by their neighborhoods. By (11), δ becomes a multi-representation. After identifying such points we obtain a representation of a T_0-space. (Formally they are called equivalent and the space must be factorized.) Notice that by (20) in Definition 8 the topology of a computable predicate space is T_0.

The notation ν of the base β induces the various computability concepts on points and open, closed and quasi-compact sets by means of Definitions 5. In applications these concepts should be invariant under "inessential changes" of the base β and its notation ν.

Definition 21. The computable topological spaces $\mathbf{X} = (X, \tau, \beta, \nu)$ and $\mathbf{X}' = (X, \tau, \beta', \nu')$ are *equivalent* iff $\nu \leq \theta'$ and $\nu' \leq \theta$.

As an example of equivalent topological spaces consider the real line and canonical notations ν and ν' of the open intervals the endpoints of which are binary fractions or ternary fractions, respectively. The computability concepts introduced in Definition 5 can be called "computationally robust" since they are the same for equivalent topological spaces. Usually, non-robust concepts [BW99, Wei00, BP03] have only few applications.

Theorem 22 (robustness). *Let $\mathbf{X} = (X, \tau, \beta, \nu)$ and $\mathbf{X}' = (X, \tau, \beta', \nu')$ be computable topological spaces.*

1. $\nu \leq \theta' \iff \delta' \leq \delta \iff \theta \leq \theta'$.
2. $\mathbf{X}$ *and* $\mathbf{X}'$ *are equivalent* $\iff \delta \equiv \delta' \iff \theta \equiv \theta'$.

3. *If $\delta' \le \delta$, then*

$$\theta \le \theta',\ \psi' \le \psi,\ \widetilde{\psi}' \le \widetilde{\psi},\ \kappa' \le \kappa,\ (\theta')^- \le \theta^-, \psi^- \le (\psi')^-,\ \delta^- \le (\delta')^- .$$

4. *If $\mathbf{X}$ and $\mathbf{X}'$ are equivalent, then $\gamma \equiv \gamma'$ for each naming system γ from Definition 5, where γ' is the representation for $\mathbf{X}'$ corresponding to γ.*

Proof: 1. Suppose $\nu \le \theta'$. Then $[\delta', \nu] \le [\delta', \theta']$. Since $\theta' \le \theta'$, by "$\Longleftarrow$" in Theorem 13.2 for $\mathbf{X}'$, $x \in U$ ($U \in \tau$) is $[\delta', \theta']$-r.e. Therefore $x \in U$ ($U \in \beta$) is $[\delta', \nu]$-r.e. By Theorem 13.1, $\delta' \le \delta$. Suppose $\delta' \le \delta$. Since $\theta \equiv \theta$, $x \in W$ is (δ, θ)-r.e. by Theorem 13.2, hence (δ', θ)-r.e. By "$\Longrightarrow$" in Theorem 13.2, $\theta \le \theta'$. Suppose $\theta \le \theta'$. Then $\nu \le \theta'$ since $\nu \le \theta$.

2. Immediately from 1.

3. Suppose $\delta' \le \delta$.
"$\theta \le \theta'$": This follows from 1.
"$\psi' \le \psi$": By Corollary 14, $A \cap V \ne \emptyset$ (A closed and V open) is (ψ', θ')-r.e. Since $\nu \le \theta'$, $A \cap V \ne \emptyset$ (A closed and $V \in \beta$) is (ψ', ν)-r.e. Therefore, $\psi' \le \psi$ by Theorem 13.3
"$\widetilde{\psi}' \le \widetilde{\psi}$": From the case above by Lemma 6.3.
"$\kappa' \le \kappa$": By Theorem 11, $\bigcup \nu^{\mathrm{fs}} \le \theta$, hence $\bigcup \nu^{\mathrm{fs}} \le \theta'$ by 1. By Corollary 14, $K \subseteq W$ (for quasi-compact K and open W) is (κ', θ')-r.e., hence $(\kappa', \bigcup \nu^{\mathrm{fs}})$-r.e. (for quasi-compact K and finite unions of base elements). Therefore, $\kappa' \le \kappa$ by Theorem 13.2.
The remaining statements follow immediately from Definition 5.

4. This follows from 3. □

By Lemma 9, for a computable predicate space $\mathbf{Z} = (X, \sigma, \lambda)$ the space $T(\mathbf{Z}) = (X, \tau, \beta_\lambda, \nu_\lambda)$ where $\nu_\lambda(\iota(u_1) \ldots \iota(u_k)) = \lambda(u_1) \cap \ldots \cap \lambda(u_k)$ and τ is the topology generated by the subbase σ is a computable topological space such that $\delta_{\mathbf{Z}} \equiv \delta_\lambda$. If σ is not only a subbase but a base of τ, $\mathbf{Y} := (X, \tau, \sigma, \lambda)$ is an effective topological space, which may be computable.

For the topology τ we have the basis β_λ with notation ν_λ (defined via formal intersections of subbase elements) and the basis σ with notation λ.

Lemma 23. *Let $\mathbf{Y} = (X, \tau, \sigma, \lambda)$ be an effective topological space such that $\mathbf{Z} = (X, \sigma, \lambda)$ is a computable predicate space. Then $T(\mathbf{Z}) = (X, \tau, \beta_\lambda, \nu_\lambda)$ and $\mathbf{Y}$ are equivalent iff $\mathbf{Y}$ is a computable topological space.*

Proof: Straightforward, apply Theorem 11. □

In this article we start from a computable topological space $\mathbf{X} = (X, \tau, \beta, \nu)$ as the most general space for introducing computability and consider a computable predicate space $\mathbf{Z} = (X, \sigma, \lambda)$ via $T(\mathbf{Z}) = (X, \tau, \beta_\lambda, \nu_\lambda)$ as a special case

(Lemma 9) where $\nu_\lambda(\iota(u_1)\ldots\iota(u_k)) = \lambda(u_1) \cap \ldots \cap \lambda(u_k)$ is the notation of "formal intersections" of subbase elements.

By Lemma 23 we could equivalently start from a computable predicate space $\mathbf{Z} = (X, \sigma, \lambda)$ as the most general space for introducing computability via the space $T(\mathbf{Z}) = (X, \tau, \beta_\lambda, \nu_\lambda)$ and consider a computable topological space $\mathbf{X} = (X, \tau, \beta, \nu)$ as the special predicate space $\mathbf{Z} = (X, \beta, \nu)$ for which β is a base of a topology with computable intersection (10). Roughly speaking, the approach via a base and the approach via a subbase to computable topology are equivalent.

For a computable topological space $\mathbf{X} = (X, \tau, \beta, \nu)$, the notation ν must have a recursive domain. Admitting a recursively enumerable domain is no generalization.

Theorem 24 (r.e. domain). *Let $\mathbf{X}' = (X, \tau, \beta, \nu')$ be an effective topological space such that dom(ν') is r.e. and (10) is true for some r.e. set $S' \subseteq (\mathrm{dom}(\nu'))^3$. Then there is a notation $\nu :\subseteq \Sigma^* \to \beta$ such that $\nu \equiv \nu'$, $\mathbf{X} = (X, \tau, \beta, \nu)$ is a computable topological space and for each representation γ from Definition 5 for $\mathbf{X}$, $\gamma \equiv \gamma'$, where γ' is the naming system defined for $\mathbf{X}'$ correspondingly.*

Proof: If $\mathrm{dom}(\nu')$ is recursive, then define $\nu := \nu'$. Otherwise, there is a computable injective function $h : \Sigma^* \to \Sigma^*$ such that $\mathrm{dom}(\nu') = \mathrm{range}(h)$. Define $\nu(u) := \nu' \circ h(u)$. Then ν has recursive domain and $\nu \equiv \nu'$ since $h^{-1} :\subseteq \Sigma^* \to \Sigma^*$ is computable. Let $(u, v, w) \in S \iff (h(u), h(v), h(w)) \in S'$. Then S is an r.e. set such that (10) is true. Therefore, $\mathbf{X}$ is a computable topological space.

For the representations in Definition 5 the unprimed versions can be translated easily to equivalent primed ones by means of the function h and primed versions can be translated to equivalent unprimed ones by means of the function h^{-1}. $\square$

If the set of non-empty base elements is r.e., the empty base elements can be ignored.

Lemma 25. *Let $\mathbf{X} = (X, \tau, \beta, \nu)$ be a computable topological space such that $\beta' := \{U \in \beta \mid U \neq \emptyset\}$ is ν-r.e. Then there is a notation ν' of β' such that $\mathbf{X}' = (X, \tau, \beta', \nu')$ is a computable topological space equivalent to $\mathbf{X}$.*

Proof: There is a computable function $g :\subseteq \Sigma^* \to \Sigma^*$ with recursive domain B such that $g[B] = \{u \mid \nu(u) \neq \emptyset\}$. Define $\nu' := \nu \circ g$. With S from (10) let

$$S' := \{(u', v', w') \mid (g(u'), g(v'), g(w')) \in S)\}\,.$$

Then $\nu'(u') \cap \nu'(v') = \bigcup\{\nu'(w') \mid (u', v', w') \in S'\}$, hence $\mathbf{X}'$ is a computable topological space.

Since $\nu'(u') = \nu \circ g(u')$ for all $u' \in B = \mathrm{dom}(\nu')$, $\nu' \leq \theta$. There is a machine M that on input u writes all $u' \in B$ such that $g(u') = u$ (and writes 11 from time to time). If $\nu(u) = \emptyset$, then M writes 11 repeatedly, hence $\theta' \circ f_M(u) = \emptyset$. If $\nu(u) \neq \emptyset$, then M writes words u' such that $g(u') = u$, hence $\nu'(u') = \nu \circ g(u') = \nu(u)$. Therefore, $\theta' \circ f_M(u) = \nu'(u') = \nu(u)$. We obtain $\nu \leq \theta'$. In summary, $\mathbf{X}$ and $\mathbf{X}'$ are equivalent. □

8 Subspaces and Products

We consider restrictions and products of effective topological spaces. Let $\mathbf{X} = (X, \tau, \beta, \nu)$ be an effective topological space. For $B \subseteq X$ define the restriction $\mathbf{X}_B = (B, \tau_B, \beta_B, \nu_B)$ of $\mathbf{X}$ to B by $\mathrm{dom}(\nu_B) := \mathrm{dom}(\nu)$, $\nu_B(w) := \nu(w) \cap B$, $\beta_B := \mathrm{range}(\nu_B)$ and $\tau_B := \{W \cap B \mid W \in \tau\}$. Let δ_B, θ_B, ... , ψ_B^- be the representations for $\mathbf{X}_B$ from Definition 5. For a multi-function $f : X \rightrightarrows Y$ and $Z \subseteq Y$ define $f|^Z : X \rightrightarrows Z$ by $f|^Z(x) := f(x) \cap Z$ for all $x \in X$.

Lemma 26.

1. *$\mathbf{X}_B$ is an effective topological space, which is computable if $\mathbf{X}$ is computable,*
2. *$\delta_B = \delta|^B$,*
3. *$\theta_B(p) = \theta(p) \cap B$ for all $p \in \mathrm{dom}(\theta_B) = \mathrm{dom}(\theta)$,*
4. *$\psi_B^-(p) = \psi^-(p) \cap B$ for all $p \in \mathrm{dom}(\psi_B^-) = \mathrm{dom}(\psi^-)$,*
5. *$\psi_B|^{\mathcal{C}} = \psi|^{\mathcal{C}}$ for $\mathcal{C} := \{C \subseteq B \mid C$ closed in $\mathbf{X}\}$.*
6. *$\kappa_B|^{\mathcal{L}} = \kappa|^{\mathcal{L}}$ for $\mathcal{L} := \{K \subseteq B \mid K$ compact in $\mathbf{X}\}$.*

Proof: 1. Straightforward.

2. For $x \in B$, $x \in \nu(w) \iff x \in \nu_B(w)$, see (11).

3. Straightforward.

4. $\psi_B^-(p) = B \setminus \theta_B(p) = B \setminus (\theta(p) \cap B) = B \setminus \theta(p) = B \cap \psi^-(p)$.

5. For $C \subseteq B$, $\nu_B(w) \cap C \neq \emptyset \iff \nu(w) \cap B \cap C \neq \emptyset \iff \nu(w) \cap C \neq \emptyset$, see (13).

6. Similar to 5., see (15). □

For $i = 1, 2$ let $\mathbf{X}_i = (X_i, \tau_i, \beta_i, \nu_i)$ be effective topological spaces with representations δ_i, θ_i, ... ,ψ_i^- from Definition 5. Define the product $\overline{\mathbf{X}} = (X_1 \times X_2, \overline{\tau}, \overline{\beta}, \overline{\nu})$ of $\mathbf{X}_1$ and $\mathbf{X}_2$ such that $\mathrm{dom}(\overline{\nu}) = (\langle u_1, u_2 \rangle \mid u_1 \in \mathrm{dom}(\nu_1),\ u_2 \in \mathrm{dom}(\nu_2)\}$, $\overline{\nu}\langle u_1, u_2 \rangle = \nu_1(u_1) \times \nu_2(u_2)$, $\overline{\beta} = \mathrm{range}(\overline{\nu})$ and $\overline{\tau}$ is the product topology generated by $\overline{\beta}$. Let $\overline{\delta}$, $\overline{\theta}$, ... , $\overline{\psi}^-$ be the representations for $\overline{\mathbf{X}}$ from Definition 5.

Lemma 27.

1. *$\overline{\mathbf{X}}$ is an effective topological space, which is computable if $\mathbf{X}_1$ and $\mathbf{X}_2$ are computable.*
2. *$\overline{\delta} \equiv [\delta_1, \delta_2]$*

3. *$(x_1, x_2) \mapsto (x_1, x_2)$ is $(\delta_1, \delta_2, \overline{\delta})$-computable, the projections $(x_1, x_2) \mapsto x_i$ are $(\overline{\delta}, \delta_i)$-computable.*
4. *For open sets, the product $(W_1, W_2) \mapsto W_1 \times W_2$ is $(\theta_1, \theta_2, \overline{\theta})$-computable. Furthermore, the product is $(\theta_1^-, \theta_2^-, \overline{\theta}^-)$-computable if "$U_1 \neq \emptyset$" is ν_1-r.e. and "$U_2 \neq \emptyset$" is ν_2-r.e.*
5. *For open sets, the projection $W \mapsto \mathrm{pr}_1[W]$ is $(\overline{\theta}, \theta_1)$-computable if "$U_2 \neq \emptyset$" is ν_2-r.e.*
6. *For closed sets, the product $(A_1, A_2) \mapsto A_1 \times A_2$ is $(\psi_1, \psi_2, \overline{\psi})$-computable and $(\psi_1^-, \psi_2^-, \overline{\psi}^-)$-computable.*
7. *For compact sets, $(K_1, K_2) \mapsto K_1 \times K_1$ is $(\kappa_1, \kappa_2, \overline{\kappa})$-computable and the projection $\overline{K} \mapsto \mathrm{pr}_1[\overline{K}]$ is $(\overline{\kappa}, \kappa_1)$-computable.*

Proof: 1. There are sets S_1 and S_2 such that (10) for $\mathbf{X_1}$ and $\mathbf{X}_2$, respectively. Let $\overline{S} := \{(\langle u_1, u_2\rangle, \langle v_1, v_2\rangle, \langle w_1, w_2\rangle) \mid (u_1, v_1, w_1) \in S_1,\ (u_2, v_2, w_2) \in S_2\}$. The set $\overline{S}$ satisfies (10) for $\overline{\mathbf{X}}$. It is r.e. if S_1 and S_2 are r.e.

2. From a list of all pairs $\langle u_1, u_2\rangle$ such that $(x_1, x_2) \in \nu_1(u_1) \times \nu_2(u_2)$ a list of all u_1 such that $x_1 \in \nu_1(u_1)$ and a list of all u_2 such that $x_2 \in \nu_2(u_2)$ can be computed and vice versa.

3. Immediate or from 2.

4. Straightforward.

5. Straightforward.

6. Use: $A_1 \times A_2 \cap \overline{\nu}\langle u_1, u_2\rangle \neq \emptyset \iff A_1 \cap \nu_1(u_1) \neq \emptyset \ \wedge\ A_2 \cap \nu_2(u_2) \neq \emptyset$.

7. First, we show that $K_1 \times K_2$ is compact. Suppose, $K_1 \times K_2 \subseteq \bigcup_{i\in I} U_i \times V_i$ ($U_i \in \beta_1$, $V_i \in \beta_2$). Suppose, $x \in K_1$. For $I_x := \{i \in I \mid x \in U_i\}$, $K_2 \subseteq \bigcup_{i\in I_x} V_i$, hence $K_2 \subseteq \bigcup_{i\in J_x} V_i$ for some finite set $J_x \subseteq I_x \subseteq I$, since K_2 is compact. For $W_x := \bigcap_{i\in J_x} U_i$, $K_1 \subseteq \bigcup_{x\in K_1} W_x$, hence $K_1 \subseteq \bigcup_{x\in F} W_x$ for some finite set $F \subseteq K_1$ since K_1 is compact. Therefore,

$$K_1 \times K_2 \subseteq \bigcup_{x\in F} W_x \times K_2 \subseteq \bigcup_{x\in F} \bigcup_{i\in J_x} W_x \times V_i \subseteq \bigcup_{x\in F} \bigcup_{i\in J_x} U_i \times V_i,$$

which is a finite subcover. If in the above consideration the finitely many finite sets J_x are called $F_1, \dots, F_n$, then

$$\begin{array}{l} K_1 \times K_2 \subseteq \bigcup_{i\in I} U_i \times V_i, \text{ iff there are finite sets } F_1, \dots, F_n \subseteq I \text{ such that} \\ (\forall j \leq n)\, K_2 \subseteq \bigcup_{i\in F_j} V_i \ \text{ and } \ K_1 \subseteq \bigcup_{j\leq n} W_j \quad (\text{where } W_j := \bigcap_{i\in F_j} U_i)\,. \end{array} \tag{29}$$

(The condition $W_j \neq \emptyset$ is not necessary and has been omitted). Define a multi-representation γ of a set of compact subsets of $X_1 \times X_2$ by $\gamma\langle p_1, p_2\rangle := \{K_1 \times K_2 \mid K_1 \in \kappa_1(p_1),\ K_2 \in \kappa_2(p_2)\}$. Theorem 13.6 it suffices to show that $\overline{K} \subseteq \overline{U}$ is $(\gamma, \bigcup \overline{\nu}^{\mathrm{fs}})$-r.e.

Therefore, we need a machine that halts on input $(\langle p_1, p_2\rangle, w)$ iff for all $K_1 \in \kappa_1(p_1)$ and $K_2 \in \kappa_2(p_2)$, $K_1 \times K_2 \subseteq \bigcup_{i\in I} U_i \times V_i$, where $\nu^{\mathrm{fs}}(w) = \{U_1 \times V_1, \dots, U_m \times V_m\}$. By (29), we need a machine M that halts on input $(\langle p_1, p_2\rangle, w)$ iff there are sets $F_1, \dots, F_n \subseteq \{1, \dots, m\}$ such that

$(\forall j \leq n)\, K_2 \subseteq \bigcup_{i\in F_j} V_i$ and $K_1 \subseteq \bigcup_{j\leq n} \bigcap_{i\in F_j} U_i$.
There is such a machine, since from $F_1,\ldots,F_n \subseteq \{1,\ldots,m\}$ for each $j \leq n$ a $\bigcup \nu_2^{\mathrm{fs}}$-name of $\bigcup_{i\in F_j} V_i$ and a θ_1-name of $\bigcup_{j\leq n} \bigcap_{i\in F_j} U_i$ can be computed (Theorem 11) and since $K_2 \subseteq W$ is $(\kappa_2, \bigcup \nu_2^{\mathrm{fs}})$-r.e. (apply Theorem 13.6 to $\gamma := \kappa := \kappa_2$) and $K_1 \subseteq W$ is (κ_1, θ_1)-r.e. (apply Theorem 13.5 to $\gamma := \kappa := \kappa_1$).

Computability of the projection on compact sets follows from the more general Theorem 38.6 below. □

The generalization to finite products is straightforward.

9 The Space of Continuous Functions

In this section let $\mathbf{X}_i = (X_i, \tau_i, \beta_i, \nu)_i$ $(i = 1, 2)$ be effective topological spaces with representations δ_i, θ_i, ... ,ψ_i^- from Definition 5. A partial function $f :\subseteq X_1 \to X_2$ is continuous iff for every $W \in \tau_2$, $f^{-1}[W]$ is open in $\mathrm{dom}(f)$, that is, $f^{-1}[W] = V \cap \mathrm{dom}(f)$ for some $V \in \tau_1$. The following conditions are equivalent:

$$f \text{ is continuous,} \tag{30}$$

$$(\forall x \in \mathrm{dom}(f), W \in \tau_2)(f(x) \in W \Longrightarrow (\exists V \in \tau_1)(x \in V \;\wedge\; f[V] \subseteq W)), \tag{31}$$

$$f[\mathrm{cls}_{\mathrm{dom}(f)}(C)] \subseteq \overline{f[C]} \text{ for every } C \subseteq \mathrm{dom}(f), \tag{32}$$

$$f \text{ has a continuous } (\delta_1, \delta_2)\text{-realization.} \tag{33}$$

The equivalences of (30), (31) and (32) are well-known [Eng89]. The equivalence of (30) and (33) is the "main theorem" for admissible representations [Wei00, Theorem 3.2.11], since for an effective topological space $\mathbf{X} = (X, \tau, \beta, \nu)$ the representation δ is *admissible* w.r.t. the topology τ. We use these characterizations to define a number of multi-representations of the set of partial continuous functions $f :\subseteq X_1 \to X_2$. We use properly multi-valued representations since in many applications specifying the domains of functions explicitly is difficult or unnecessary and many computability results can already be proved without explicit information about the domains of the functions. The names of our multi-representations do not fix the domains (Theorem 29.2). The applications, for example in Section 11, strongly justify using multi-representations.

Definition 28. Define multi-representations of the set $\mathrm{CP}(X_1, X_2)$ of all partial continuous functions $f :\subseteq X_1 \to X_2$ as follows:

1. $f \in \overrightarrow{\delta_1}(p) :\Longleftrightarrow\; f \circ \delta_1(q) = \delta_2 \circ \eta_p^{\omega\omega}(q)$ for all $q \in \mathrm{dom}(f \circ \delta_1)$,
2. $f \in \overrightarrow{\delta_2}(p) :\Longleftrightarrow\; f^{-1}[\theta_2(q)] = \theta_1 \circ \eta_p^{\omega\omega}(q) \cap \mathrm{dom}(f)$ for all $q \in \mathrm{dom}(\theta_2)$,
3. $f \in \overrightarrow{\delta_3}(p) :\Longleftrightarrow\; f^{-1}[\nu_2(v)] = \theta_1 \circ \eta_p^{*\omega}(v) \cap \mathrm{dom}(f)$ for all $v \in \mathrm{dom}(\nu_2)$,
4. $f \in \overrightarrow{\delta_4}(p) :\Longleftrightarrow \begin{cases} (w \ll p \Longrightarrow (\exists u \in \mathrm{dom}(\nu_1),\, v \in \mathrm{dom}(\nu_2)) w = \langle u, v\rangle) \\ \text{and } f^{-1}[\nu_2(v)] = \bigcup_{\langle u,v\rangle \ll p} \nu_1(u) \cap \mathrm{dom}(f), \end{cases}$

5. $f \in \overrightarrow{\delta_5}(p) :\iff \overline{f[C]} = \psi_2 \circ \eta_p^{\omega\omega}(q)$ if $C \subseteq \mathrm{dom}(f)$ and $\overline{C} = \psi_1(q)$,
6. $f \in \overrightarrow{\delta_6}(p) :\iff f[K] \in \kappa_2 \circ \eta_p^{\omega\omega}(q)$ if $K \subseteq \mathrm{dom}(f)$ and $K \in \kappa_1(q)$,
7. $f \in \overrightarrow{\delta_7}(p) :\iff \begin{cases} U := \nu_1 \circ \eta_p^{\omega *}\langle q_1, q_2\rangle \text{ exists, } x \in U \text{ and } f[U] \subseteq W \\ \text{if } x = \delta_1(q_1),\ W = \theta_2(q_2) \text{ and } f(x) \in W, \end{cases}$
8. $f \in \overrightarrow{\delta_8}(p) :\iff \begin{cases} U := \nu_1 \circ \eta_p^{\omega *}\langle q_1, v\rangle \text{ exists, } x \in U \text{ and } f[U] \subseteq V \\ \text{if } x = \delta_1(q_1),\ V = \nu_2(v) \text{ and } f(x) \in V. \end{cases}$

For the multi-representation $\overrightarrow{\delta_1}$ we will use the name $[\delta_1 \to_p \delta_2]$ from [Wei00, Wei08], see Section 2. If we call p a *program* of η_p^{ab}, then in 1. a name p is a program for computing f w.r.t. (δ_1, δ_2), in 2. a name p is a program for computing $W \mapsto f^{-1}[W]$ for open W w.r.t. (θ_2, θ_1), etc. By Lemma 6.3, $f \in \overrightarrow{\delta_5}(p)$ iff p is a program for computing $C \mapsto f[C]$ for $C \subseteq \mathrm{dom}(f)$ w.r.t. $(\widetilde{\psi}_1, \widetilde{\psi}_2)$.

Theorem 29. *Let $\mathbf{X_1}, \mathbf{X_2}$ be computable topological spaces.*

1. *The multi-functions $\overrightarrow{\delta_i}$, $i = 1, \dots, 8$, are multi-representations of the set* $\mathrm{CP}(X_1, X_2)$ *of all partial continuous functions $f :\subseteq X_1 \to X_2$ such that*
$$\overrightarrow{\delta_1} \equiv \overrightarrow{\delta_2} \equiv \overrightarrow{\delta_3} \equiv \overrightarrow{\delta_4} \equiv \overrightarrow{\delta_5} \equiv \overrightarrow{\delta_6} \le \overrightarrow{\delta_7} \le \overrightarrow{\delta_8}\,. \tag{34}$$
2. *For $1 \le i \le 8$ and every $p \in \mathrm{dom}(\overrightarrow{\delta_i})$, $g \in \overrightarrow{\delta_i}(p)$ if $f \in \overrightarrow{\delta_i}(p)$ and g is a restriction of f.*
3. *For $1 \le i \le 6$ and every $p \in \mathrm{dom}(\overrightarrow{\delta_i})$, $f(x) = g(x)$ if $f, g \in \overrightarrow{\delta_i}(p)$ and $x \in \mathrm{dom}(f) \cap \mathrm{dom}(g)$*
4. *For $1 \le i \le 4$ and every $p \in \mathrm{dom}(\overrightarrow{\delta_i})$ there is some $f_{ip} \in \overrightarrow{\delta_i}(p)$ such that for every $f :\subseteq X_1 \to X_2$, $f \in \overrightarrow{\delta_i}(p) \iff f$ is a restriction of f_{ip}.*
5. *In general, Theorem 29.3 is not true for $\overrightarrow{\delta_7}$ and $\overrightarrow{\delta_8}$, $\overrightarrow{\delta_7} \not\le \overrightarrow{\delta_1}$, $\overrightarrow{\delta_8} \not\le \overrightarrow{\delta_1}$, $\overrightarrow{\delta_8} \not\le \overrightarrow{\delta_7}$ and Theorem 29.4 is not true for $\overrightarrow{\delta_5}$ and $\overrightarrow{\delta_6}$.*

By Theorem 29.2 the classes $\overrightarrow{\delta_i}(p)$ are closed under restriction. By Theorem 29.4 for $1 \le i \le 4$ every non-empty class $\overrightarrow{\delta_i}(p)$ contains a function with maximal domain. By Theorem 29.3 for $1 \le i \le 6$ the restriction of $\overrightarrow{\delta_i}$ to a class of continuous functions with fixed domain is single-valued. Single-valued representations of classes of partial functions can also be obtained by adding information about the domains to names, for example $\overrightarrow{\delta_1} \wedge \theta$ and $\overrightarrow{\delta_1} \wedge \psi$ (or equivalent ones) as a representation of the continuous functions with open domains and with closed domains, respectively, [Her99, Wei01, WZ07]. The separation of evaluation information from domain information is in particular meaningful if (names of) the domains of the considered functions are not known. Theorem 38 shows that for many results the domain information of continuous functions is not needed.

For the representations $\overrightarrow{\delta_7}$ and $\overrightarrow{\delta_8}$, which are derived from the continuity characterization (31), function values are no longer defined uniquely (if they exist) by names, see Example 3.

Proof: 1. By the main theorem for admissible representations [Wei00, Theorem 3.2.11] a function $f :\subseteq X_1 \to X_2$ is continuous iff it has a continuous (δ_1, δ_2)-realization $h :\subseteq \Sigma^\omega \to \Sigma^\omega$. Since every continuous function $h :\subseteq \Sigma^\omega \to \Sigma^\omega$ has an extension $\eta_p^{\omega\omega}$ (for some $p \in \Sigma^\omega$) and since every extension of a realization is a realization, the function $f :\subseteq X_1 \to X_2$ is continuous iff, for some p, $\eta_p^{\omega\omega}$ is a (δ_1, δ_2)-realization, that is, iff $f \in \overrightarrow{\delta_1}(p)$. Therefore, $\overrightarrow{\delta_1}$ is a multi-representation of $\mathrm{CP}(X_1, X_2)$.

If $f \in \overrightarrow{\delta_8}(p)$, then for all $x \in \mathrm{dom}(f)$ and all $V \in \beta_2$ there is some $T \in \tau_1$ such that $x \in T$ and $f[T] \subseteq V$, hence, f is continuous. Since in general $\mathrm{range}(\gamma) \subseteq \mathrm{range}(\delta)$ if $\gamma \leq \delta$, from (34) we can conclude that the multi-functions $\overrightarrow{\delta_i}$, $i = 1, \ldots, 8$, are multi-representations of the set $\mathrm{CP}(X_1, X_2)$. We prove (34).

$\boldsymbol{\overrightarrow{\delta_1} \leq \overrightarrow{\delta_4}}$: By Theorem 11 and Lemma 10 there is a computable function $h_1 :\subseteq \Sigma^* \to \Sigma^\omega$ such that $\delta_1[w\Sigma^\omega] = \bigcap \nu_1^{\mathrm{fs}}(w) = \theta_1 \circ h_1(w)$ for all $w \in \mathrm{dom}(\nu_1^{\mathrm{fs}})$. By the utm-theorem for $\eta^{\omega\omega}$ there is a machine M that computes the function $(p, q) \mapsto \eta_p^{\omega\omega}(q)$.

There is a machine N that on input $p \in \mathrm{dom}(\delta_1)$ writes all $\iota\langle u, v\rangle$ for which there are $w_1 \in \mathrm{dom}(\nu_1^{\mathrm{fs}})$, $w_2 \in \mathrm{dom}(\nu_2^{\mathrm{fs}})$ such that M on input $(p, w_1 1^\omega)$ writes w_2 in at most $|w_1|$ steps, $u \ll h_1(w_1)$, and $v \ll w_2$. (We also force N to write 11 from time to time in order to produce a result in Σ^ω .)

Let $f \in \overrightarrow{\delta_1}(p)$. Then for $v \in \mathrm{dom}(\nu_2)$ and $x \in \mathrm{dom}(f)$,

$$
\begin{aligned}
& x \in f^{-1}[\nu_2(v)] \\
\iff & f(x) \in \nu_2(v) \\
\iff & (\exists w_1 \in \mathrm{dom}(\nu_1^{\mathrm{fs}}))(\exists w_2 \in \mathrm{dom}(\nu_2^{\mathrm{fs}}))(x \in \delta_1[w_1\Sigma^\omega],\ \text{on input } (p, w_1 1^\omega) \\
& \quad \text{the machine } M \text{ writes } w_2 \text{ in at most } |w_1| \text{ steps and } v \ll w_2) \\
\iff & (\exists w_1 \in \mathrm{dom}(\nu_1^{\mathrm{fs}}))(\exists w_2 \in \mathrm{dom}(\nu_2^{\mathrm{fs}}))(\exists u \ll h_1(w_1)) \\
& (x \in \nu_1(u),\ \text{on input } (p, w_1 1^\omega) \text{ the machine } M \text{ writes } w_2 \\
& \quad \text{in at most } |w_1| \text{ steps and } v \ll w_2) \\
\iff & (\exists u)(x \in \nu_1(u) \ \wedge\ \iota\langle u, v\rangle \ll f_N(p)) \\
\iff & x \in \bigcup\{\nu_1(u) \mid \langle u, v\rangle \ll f_N(p)\}.
\end{aligned}
$$

Therefore, $f^{-1}[\nu_2(v)] = \bigcup\{\nu_1(u) \mid \langle u, v\rangle \ll f_N(p)\} \cap \mathrm{dom}(f)$, hence $f \in \overrightarrow{\delta_4} \circ f_N(p)$. This shows that f_N translates $\overrightarrow{\delta_1}$ to $\overrightarrow{\delta_4}$.

$\boldsymbol{\overrightarrow{\delta_4} \leq \overrightarrow{\delta_2}}$: There is a machine M that on input (p, q), $p, q \in \Sigma^\omega$, lists all $u \in \Sigma^*$ such that $v \ll q$ and $\langle u, v\rangle \ll p$ for some $v \in \Sigma^*$. Let $f \in \overrightarrow{\delta_4}(p)$. Then for $x \in \mathrm{dom}(f)$ and $q \in \mathrm{dom}(\theta_2)$,

$$
\begin{aligned}
& x \in f^{-1}[\theta_2(q)] \\
\iff & (\exists v \ll q)\, x \in f^{-1}[\nu_2(v)] \\
\iff & (\exists v \ll q)(\exists u)(\langle u, v\rangle \ll p \ \wedge\ x \in \nu_1(u)) \\
\iff & (\exists u)(x \in \nu_1(u) \ \wedge\ u \ll f_M(p, q)) \\
\iff & x \in \theta_1 \circ f_M(p, q))\,.
\end{aligned}
$$

By the smn-theorem there is some computable function $g : \Sigma^\omega \to \Sigma^\omega$ such that $f_M(p, q) = \eta_{g(p)}^{\omega\omega}(q)$. Then $f^{-1}[\theta_2(q)] = \theta_1 \circ \eta_{g(p)}^{\omega\omega}(q) \cap \mathrm{dom}(f)$, hence $f \in$

$\overrightarrow{\delta_2} \circ g(p)$. Therefore, g translates $\overrightarrow{\delta_4}$ to $\overrightarrow{\delta_2}$.

$\overrightarrow{\boldsymbol{\delta_2}} \leq \overrightarrow{\boldsymbol{\delta_3}}$: By the utm-theorem for $\eta^{\omega\omega}$ and the smn-theorem for $\eta^{*\omega}$, there is a computable function $g : \Sigma^\omega \to \Sigma^\omega$ such that $\eta_p^{\omega\omega}(\iota(v)1^\omega) = \eta^{*\omega}_{g(p)}(v)$.

Let $f \in \overrightarrow{\delta_2}(p)$. Since $\nu_2(v) = \theta_2(\iota(v)1^\omega)$, $f^{-1}[\nu_2(v)] = f^{-1}[\theta_2(\iota(v)1^\omega)] = \theta_1 \circ \eta_p^{\omega\omega}(\iota(v)1^\omega) \cap \mathrm{dom}(f) = \theta_1 \circ \eta^{*\omega}_{g(p)}(v) \cap \mathrm{dom}(f)$, hence $f \in \overrightarrow{\delta_3} \circ g(p)$. Therefore, g translates $\overrightarrow{\delta_2}$ to $\overrightarrow{\delta_3}$.

$\overrightarrow{\boldsymbol{\delta_3}} \leq \overrightarrow{\boldsymbol{\delta_1}}$: There is a machine M that on input $(p,q) \in (\Sigma^\omega)^2$ prints all $\iota(v)$, $v \in \mathrm{dom}(\nu_2)$, such that for some $u \in \mathrm{dom}(\nu_1)$, $u \ll q$ and $u \ll \eta_p^{*\omega}(v)$ (apply the utm-theorem).

Let $f \in \overrightarrow{\delta_3}(p)$ and $x = \delta_1(q) \in \mathrm{dom}(f)$. Then

$$\begin{aligned}
& f(x) \in \nu_2(v) \\
\iff & x \in f^{-1}[\nu_2(v)] \\
\iff & x \in \theta_1 \circ \eta_p^{*\omega}(v) \\
\iff & (\exists u)(x \in \nu_1(u) \ \wedge \ u \ll \eta_p^{*\omega}(v) \\
\iff & (\exists u)(u \ll q \ \wedge \ u \ll \eta_p^{*\omega}(v) \\
\iff & v \ll f_M(p,q),
\end{aligned}$$

hence $f(x) = \delta_2 \circ f_M(p,q)$. By the smn-theorem there is a computable function $g : \Sigma^\omega \to \Sigma^\omega$ such that $f_M(p,q) = \eta^{\omega\omega}_{g(p)}(q)$. Since $f \circ \delta_1(q) = \delta_2 \circ \eta^{\omega\omega}_{g(p)}(q)$, $f \in \overrightarrow{\delta_1} \circ g(p)$. Therefore, $\overrightarrow{\delta_3} \leq \overrightarrow{\delta_1}$.

$\overrightarrow{\boldsymbol{\delta_4}} \leq \overrightarrow{\boldsymbol{\delta_5}}$: Suppose $f \in \overrightarrow{\delta_4}(p)$. Let $C \subseteq \mathrm{dom}(f)$ and $\overline{C} = \psi_1(q)$. Then

$$\begin{aligned}
& \nu_2(v) \cap \overline{f[C]} \neq \emptyset \\
\iff & \nu_2(v) \cap f[C] \neq \emptyset \\
\iff & (\exists x \in C)\, f(x) \in \nu_2(v) \\
\iff & (\exists x \in C)\, x \in f^{-1}[\nu_2(v)] = \textstyle\bigcup_{\langle u,v\rangle \ll p} \nu_1(u) \cap \mathrm{dom}(f) \\
\iff & (\exists x \in C)(\exists u)\, (\langle u,v\rangle \ll p \ \wedge \ x \in \nu_1(u)) \\
\iff & (\exists u)\, (\langle u,v\rangle \ll p \ \wedge \ C \cap \nu_1(u) \neq \emptyset) \\
\iff & (\exists u)\, (\langle u,v\rangle \ll p \ \wedge \ u \ll q).
\end{aligned}$$

There is a machine M that on input (p,q) lists all $\iota(v)$ such that for some u, $(\langle u,v\rangle \ll p \ \wedge \ u \ll q)$. By the smn-theorem there is a computable function $r : \Sigma^\omega \to \Sigma^\omega$ such that $f_M(p,q) = \eta^{\omega\omega}_{r(p)}(q)$. Then $\overline{f[C]} = \psi_2 \circ \eta^{\omega\omega}_{r(p)}(q)$, hence, $f \in \overrightarrow{\delta_5} \circ r(p)$. Therefore, r translates $\overrightarrow{\delta_4}$ to $\overrightarrow{\delta_5}$.

$\overrightarrow{\boldsymbol{\delta_5}} \leq \overrightarrow{\boldsymbol{\delta_1}}$: Suppose $f \in \overrightarrow{\delta_5}(p)$. Let $x \in \mathrm{dom}(f)$ and $x = \delta_1(q)$. Then $\{x\} \subseteq \mathrm{dom}(f)$ and $\overline{\{x\}} = \psi_1(q)$, hence $\overline{f[\{x\}]} = \psi_2 \eta_p^{\omega\omega}(q)$ and $v \ll \eta_p^{\omega\omega}(q) \iff \nu_2(v) \cap \overline{f[\{x\}]} \neq \emptyset \iff \nu_2(v) \cap f[\{x\}] \neq \emptyset \iff f(x) \in \nu_2(v)$. Therefore, $f(x) = \delta_2 \circ \eta_p^{\omega\omega}(q)$. We conclude, $f \in \overrightarrow{\delta_1}(p)$, hence the identity translates $\overrightarrow{\delta_5}$ to $\overrightarrow{\delta_1}$.

$\overrightarrow{\boldsymbol{\delta_2}} \leq \overrightarrow{\boldsymbol{\delta_6}}$: Suppose $f \in \overrightarrow{\delta_2}(p)$ and $K \in \kappa_1(q)$. Then $f[K] \subseteq \bigcup \nu_2^{fs}(v) \iff K \subseteq f^{-1}[\bigcup \nu_2^{fs}(v)]$. By Lemma 10, $\bigcup \nu_2^{fs}(v) = \theta_2 \circ h(v)$ for some computable function h, hence $f^{-1}[\bigcup \nu_2^{fs}(v)] = \theta_1 \circ \eta_p^{\omega\omega} \circ h(v)$. There is a machine that on input (p,q) lists all $\iota(v)$, $v \in \mathrm{dom}(\nu_2^{fs})$, such that there is some $w \ll q$ such that

$u \ll \eta_p^{\omega\omega} \circ h(v)$ for all $u \ll w$. Then f_M realizes $(f, K) \mapsto f[K]$.

$\boldsymbol{\overrightarrow{\delta_6} \leq \overrightarrow{\delta_1}}$: In Lemma 15 let h_1 realize ec_1 and let h_2 realize ec_2^{-1}. If $f \in \overrightarrow{\delta_6}(p)$, then $\mathrm{ec}_2^{-1} \circ \eta_p^{\omega\omega} \circ \mathrm{ec}_1$ realizes $x \mapsto f(x)$. There is a computable function g such that $\eta_{g(p)}^{\omega\omega}(q) = \mathrm{ec}_2^{-1} \circ \eta_p^{\omega\omega} \circ \mathrm{ec}_1(q)$. Then g translates $\overrightarrow{\delta_6}$ to $\overrightarrow{\delta_1}$.

$\boldsymbol{\overrightarrow{\delta_2} \leq \overrightarrow{\delta_7}}$: There is a machine M that on input $(p, \langle q_1, q_2 \rangle) \in (\Sigma^\omega)^2$ searches for some $u \in \mathrm{dom}(\nu_1)$ such that $u \ll q_1$, $u \ll \eta_p^{\omega\omega}(q_2)$ (apply the utm-theorem), and writes u if the search was successful.

Let $f \in \overrightarrow{\delta_2}(p)$. Let $x = \delta_1(q_1) \in \mathrm{dom}(f)$ and $f(x) \in W = \theta_2(q_2)$. Then $x \in f^{-1}[\theta_2(q_2)] = \theta_1 \circ \eta_p^{\omega\omega}(q_2) \cap \mathrm{dom}(f)$, hence $u \ll q_1$ and $u \ll \eta_p^{\omega\omega}(q_2)$ for some $u \in \mathrm{dom}(\nu_1)$. Therefore, $u := f_M(p, \langle q_1, q_2 \rangle)$ exists. Since $u \ll q_1$ and $u \ll \eta_p^{\omega\omega}(q_2)$, $x \in \nu_1(u)$ and $\nu_1(u) \subseteq \theta_1 \circ \eta_p^{\omega\omega}(q_2))$, hence $f[\nu_1(u)] \subseteq \theta_2(q_2)$.

By the smn-theorem for $\eta^{\omega *}$ there is a computable function $r : \Sigma^\omega \to \Sigma^\omega$ such that $f_M(p, \langle q_1, q_2 \rangle) = \eta_{r(p)}^{\omega *}\langle q_1, q_2 \rangle$. Then $f \in \overrightarrow{\delta_7} \circ r(p)$. Therefore, r translates $\overrightarrow{\delta_2}$ to $\overrightarrow{\delta_7}$.

$\boldsymbol{\overrightarrow{\delta_7} \leq \overrightarrow{\delta_8}}$: This follows from $\nu_2 \leq \theta_2$.

2. Let $f \in \overrightarrow{\delta_1}(p)$ and let g be a restriction of f. Then $g \circ \delta_1(q) = \delta_2 \circ \eta_p^{\omega\omega}(q)$ for all for all $q \in \mathrm{dom}(g \circ \delta_1)$, hence $g \in \overrightarrow{\delta_1}(p)$. Therefore for $i = 1, \ldots, 8$, $g \in \overrightarrow{\delta_i}(p)$ if $f \in \overrightarrow{\delta_i}(p)$ and g restricts f, since $\overrightarrow{\delta_1} \leq \overrightarrow{\delta_i}$.

3. Suppose $f, g \in \overrightarrow{\delta_1}(p)$ and $x = \delta_1(q) \in \mathrm{dom}(f) \cap \mathrm{dom}(g)$. Then $f(x) = g(x)$ since $f \circ \delta_1(q) = \delta_2 \circ \eta_p^{\omega\omega}(q) = g \circ \delta_1(q)$. Therefore for $i = 1, \ldots, 6$, $f(x) = g(x)$ if $f, g \in \overrightarrow{\delta_i}(p)$ and $x \in \mathrm{dom}(f) \cap \mathrm{dom}(g)$ since $\overrightarrow{\delta_i} \leq \overrightarrow{\delta_1}$.

4. Define f_{ip} by $\mathrm{graph}(f_{ip}) = \bigcup\{\mathrm{graph}(f) \mid f \in \overrightarrow{\delta_i}(p)\}$. By 3. the function f_{ip} is well-defined.

$i = 1$: Suppose $x = \delta_1(q) \in \mathrm{dom}(f_{1p})$. Then $x \in \mathrm{dom}(f)$ for some $f \in \overrightarrow{\delta_1}(p)$. Since $f_{1p} \circ \delta_1(q) = f \circ \delta_1(q) = \delta_2 \circ \eta_p^{\omega\omega}(q)$, $f_{1p} \in \overrightarrow{\delta_1}(p)$.

$i = 2$: Suppose $x = \delta_1(q) \in \mathrm{dom}(f_{2p})$. Then $x \in \mathrm{dom}(f)$ for some $f \in \overrightarrow{\delta_2}(p)$. We obtain $x \in f_{2p}^{-1}[\theta_2(q)] \iff f_{2p}(x) \in \theta_2(q) \iff f(x) \in \theta_2(q)$
$\iff x \in \theta_1 \circ \eta_p^{\omega\omega}(q) \cap \mathrm{dom}(f) \iff x \in \theta_1 \circ \eta_p^{\omega\omega}(q) \cap \mathrm{dom}(f_{2p})$.
Therefore, $f_{2p} \in \overrightarrow{\delta_2}(p)$.

$i = 3$: Replace $\theta_2(q)$ by $\nu_2(q)$ and $\eta^{\omega\omega}$ by $\eta^{*\omega}$ in "$i = 2$".

$i = 4$: Similar to the case "$i = 2$".

5. See Example 3. □

Example 3. 1. Let $\mathbf{X_1} = \mathbf{X_2} = (X, \tau, \beta, \nu)$ be a computable topological space such that X has at least two elements. There is a machine M that on input $\langle q_1, q_2 \rangle$ writes some u such that $u \ll q_1$ and diverges if no such word u exists. Then $f_M = \eta_p^{\omega *}$ for some $p \in \Sigma^\omega$.

Let $c \in X$ and $f_c(x) := c$ for all $x \in X$. Suppose (see Definition 28.7) $x = \delta(q_1)$, $W = \theta(q_2)$ and $f_c(x) \in W$. Then $\eta_p^{\omega *}\langle q_1, q_2 \rangle = u$ for some u such that $x \in \nu(u)$ and $f_c[\nu(u)] = \{c\} = \{f_c(x)\} \subseteq W$. Therefore, $f_c \in \overrightarrow{\delta_7}(p)$.

For $c, d \in X$, $c \neq d$, $f_c, f_d \in \overrightarrow{\delta_7}(p)$ but $f_c(c) = c \neq d = f_d(c)$. Therefore Theorem 29.3 is false for $\overrightarrow{\delta_7}$.

Since $\overrightarrow{\delta_7} \leq \overrightarrow{\delta_8}$, Theorem 29.3 is false also for $\overrightarrow{\delta_8}$. If $\overrightarrow{\delta_7} \leq \overrightarrow{\delta_1}$ or $\overrightarrow{\delta_8} \leq \overrightarrow{\delta_1}$, then Theorem 29.3 must be false also for $\overrightarrow{\delta_1}$. Therefore $\overrightarrow{\delta_7} \not\leq \overrightarrow{\delta_1}$ and $\overrightarrow{\delta_8} \not\leq \overrightarrow{\delta_1}$.

2. For showing $\overrightarrow{\delta_8} \not\leq \overrightarrow{\delta_7}$ consider the computable topological space $\mathbf{R}$ for the real line from Example 1. We may assume that the basis β contains only intervals of length < 1. For $i \in \mathbb{N}$ define $f_i :\subseteq \mathbb{R} \to \mathbb{R}$ by $\mathrm{dom}(f_i) := (-1; 1)$ and $f_i(x) := 3i + x$. Let $(a; b) + c := (a + c; b + c)$. There is a machine M that on input (q, v) such that $-1 < \rho(q) < 1$ and $v \in \mathrm{dom}(\nu)$ searches some $i \in \mathbb{N}$ such that $\nu(v) \cap (3i - 1; 3i + 1) \neq \emptyset$ and writes some u such that $u \ll q$ and $\nu(u) \subseteq (-1; 1) \cap (\nu(v) - 3i)$. Notice that there is at most one number i. Suppose $f_j \circ \rho(q) \in \nu(v)$. Then on input (q, v) the machine M finds $i = j$ and some u such that $\rho(q) \in \nu(u) \subseteq (-1; 1) \cap (\nu(v) - 3j)$, hence $f_j[\nu(u)] \subseteq f_j[\nu(v) - 3j] = \nu(v)$. There is some $p \in \Sigma^\omega$ such that $f_M(q, v) = \eta_p^{\omega *}\langle q, v\rangle$ if $f_M(q, v)$ exists. Therefore, $f_i \in \overrightarrow{\delta_8}(p)$ for all $i \in \mathbb{N}$.

   Suppose $\overrightarrow{\delta_8} \leq \overrightarrow{\delta_7}$. Then there is some $p \in \Sigma^\omega$ such that $f_i \in \overrightarrow{\delta_7}(p)$ for all $i \in \mathbb{N}$. Let $\rho(q_1) = x = 0$ and $\theta_2(q_2) = W = \bigcup_{i \in \mathbb{N}} \{(3i - 2^{-i}; 3i + 2^{-i})\}$. Let $i \in \mathbb{N}$. Since $f_i \in \overrightarrow{\delta_7}(p)$ and $f_i(x) \in W$. Then $u := \eta_p^{\omega *}\langle q_1, q_2\rangle$ exists such that $0 \in \nu_1(u)$ and $f_i[\nu(u)] \subseteq W$. Therefore, the length of $\nu(u) \cap (-1; 1)$ is less than $2 \cdot 2^{-i}$. Since $0 \in \nu_1(u)$, $\nu_1(u)$ must have length 0. Contradiction.

3. Let $\mathbf{X} = (X, \tau, \beta, \nu)$ be a computable topological space such that there are $u, v \in \mathrm{dom}(\nu)$ and $c, d \in X$ such that $c \in \nu(u)$, $d \in \nu(v)$ and $\nu(u) \cap \nu(v) = \emptyset$. Let f_c, f_d and f_{cd} be the restriction of the identity on X to $\{c\}$, $\{d\}$ and $\{c, d\}$, respectively. There is a machine that on input $q \in \Sigma^\omega$ copies q to the output tape but halts as soon as it has detected $\iota(u)$ and $\iota(v)$ as subwords of q. There is some $p \in \Sigma^\omega$ such that $f_M = \eta_p^{\omega\omega}$. Then $f_c, f_d \in \overrightarrow{\delta_5}(p)$ but $f_{cd} \notin \overrightarrow{\delta_5}(p)$. Therefore, the condition in Theorem 29.4 is violated for $\overrightarrow{\delta_5}$.

   A similar example violates the condition in Theorem 29.4 for $\overrightarrow{\delta_6}$.

Some of the equivalences in Theorem 29.1 have been proved in [Sch03] for slightly less general spaces. For the case of semi-computable metric spaces the equivalence of $\overrightarrow{\delta_1}$ and $\overrightarrow{\delta_4}$ has been proved in [Wei93]. The function $\mathrm{eval} : (f, x) \mapsto f(x)$ is $(\overrightarrow{\delta_1}, \delta_1, \delta_2)$-computable, even more, we can characterize the equivalence class of $\overrightarrow{\delta_1} = [\delta_1 \to_p \delta_2]$ as follows:

Theorem 30. *For every multi-representation $\gamma : \Sigma^\omega \rightrightarrows F$ of a set F of partial functions $f :\subseteq X_1 \to X_2$,*

$$\mathrm{eval} : (f, x) \mapsto f(x) \text{ is } (\gamma, \delta_1, \delta_2)\text{-computable} \iff \gamma \leq [\delta_1 \to_p \delta_2]\,.$$

Proof: We apply the more general result [Wei08, Corollary 34]. For every multi-representation γ' of a class F' of multi-functions $f : X_1 \rightrightarrows X_2$, the apply multi-function is $(\gamma', \delta_1, \delta_2)$-computable iff $\gamma' \leq [\delta_1 \rightrightarrows \delta_2]$. Since γ is such a multi-representation, the apply function is $(\gamma, \delta_1, \delta_2)$-computable iff $\gamma \leq [\delta_1 \rightrightarrows \delta_2]$. Since γ is a multi-representation of partial functions and $[\delta_1 \to_p \delta_2]$ is the restriction of $[\delta_1 \rightrightarrows \delta_2]$ to the (single-valued) partial functions [Wei08], $\gamma \leq [\delta_1 \rightrightarrows \delta_2] \iff \gamma \leq [\delta_1 \to_p \delta_2]$. □

Compare this result with Theorem 13, which is of similar type. In accordance with Theorem 29.5, in general evaluation $(f, x) \mapsto f(x)$ is not computable for $\overrightarrow{\delta_7}$ and $\overrightarrow{\delta_8}$. The reason is shown in Example 3: in general for $x \in \mathrm{dom}(f)$ and $f \in \overrightarrow{\delta_8}(p)$, the value $f(x)$ is not defined uniquely by the name p of f (Theorem 29.3). We only mention that some of the above statements of Theorem 29 remain true for effective topological spaces and that all the statements remain true for effective topological spaces if "$\leq$" (reducible) is replaced by "$\leq_t$" (continuously reducible), (use oracles for $\mathrm{dom}(\nu)$ and S in Definition 4).

10 Where are the Points?

In our approach we have started from a computable topological space $\mathbf{X} := (X, \tau, \beta, \nu)$, that is, a set of points, a topology on it and a notation of a base such that for some r.e. set S, $\nu(u) \cap \nu(v) = \bigcup\{\nu(w) \mid (u, v, w) \in S\}$ (Definition 4). However, the only information about $\mathbf{X}$ we have used is the set S, which contains some, but not all, Boolean information about the sets $\nu(v)$. Similarly we can a consider an effective predicate space $\mathbf{Z} := (X, \sigma, \lambda)$ where the only concrete information is $\mathrm{dom}(\lambda)$. Finally, we can assume that the full Boolean information on the topology is given.

In all these cases the elements $U \in \beta$ can be interpreted as "regions for points" (see pointless topology, locales [Joh82, Joh83]). Given an abstract notation $\nu :\subseteq \Sigma^* \to \beta$ we can ask in which way the regions can be filled by points. The answer depends on the axioms for β and ν.

Definition 31. Let $\mathbf{Z} := (X, \sigma, \lambda)$ be an effective predicate space and let $\mathbf{X} := (X, \tau, \beta, \nu)$ be an effective topological space.

1. $L \subseteq \Sigma^*$ *realizes* $\mathbf{Z}$ if $L = \mathrm{dom}(\lambda)$.
2. $S \subseteq \mathrm{dom}(\nu)^3$ *realizes intersection* for $\mathbf{X}$ if
   $$\nu(u) \cap \nu(v) = \bigcup\{\nu(w) \mid (u, v, w) \in S\} \text{ for all } u, v \in \mathrm{dom}(\nu),$$
3. Q *realizes inclusion* (on the topology) for $\mathbf{X}$ if
   $$Q = \{(u, D) \mid u \in \mathrm{dom}(\nu),\ D \subseteq \mathrm{dom}(\nu),\ \nu(u) \subseteq \bigcup \nu[D]\}.$$

Lemma 32. *Let Q realize inclusion for the spaces $\mathbf{X} := (X, \tau, \beta, \nu)$ and $\mathbf{X}' := (X', \tau', \beta', \nu')$. If S realizes intersection for $\mathbf{X}$, then S realizes intersection for $\mathbf{X}'$.*

Proof: Obviously, $\mathrm{dom}(\nu) = \mathrm{dom}(\nu')$ and for all $u, v \in \mathrm{dom}(\nu)$, $\nu(u)\subseteq\nu(v) \iff \nu'(u)\subseteq\nu'(v)$ (choose $D = \{v\}$). Then $\nu(w)\subseteq\nu(u) \cap \nu(v) \Longrightarrow \nu'(w)\subseteq\nu'(u) \cap \nu'(v)$, hence $\bigcup\{\nu'(w) \mid (u, v, w) \in S\}\subseteq\nu'(u) \cap \nu'(v)$. On the other hand,
$\nu'(t)\subseteq\nu'(u) \cap \nu'(v) \Longrightarrow \nu(t)\subseteq\nu(u) \cap \nu(v) \Longrightarrow \nu(t)\subseteq \bigcup\{\nu(w) \mid (u, v, w) \in S\} \Longrightarrow \nu'(t)\subseteq \bigcup\{\nu'(w) \mid (u, v, w) \in S\}$, hence $\nu'(u)\cap\nu'(v)\subseteq \bigcup\{\nu'(w) \mid (u, v, w) \in S\}$. □

From the set S inclusion on the base cannot be defined, since after deleting some points in $\mathbf{X}$ the set S still realizes intersection but the inclusion order on the base may have changed. For filling the regions $\nu(w)$ with points we will consider three cases:
(1) only $\mathrm{dom}(\lambda)$ is fixed in $T(\mathbf{Z})$ (Definition 8),
(2) a fixed set S for realizing intersection, and
(3) a fixed set Q for realizing inclusion.

Definition 33.
1. For effective topological spaces $\mathbf{X} := (X, \tau, \beta, \nu)$ and $\mathbf{X}' := (X', \tau', \beta', \nu')$ a function $f : X \to X'$ *embeds* $\mathbf{X}$ into $\mathbf{X}'$ if $\mathrm{dom}(\nu') = \mathrm{dom}(\nu)$, f is injective and $\nu(w) = f^{-1}[\nu'(w)]$ for all $w \in \mathrm{dom}(\nu)$. If some f embeds $\mathbf{X}$ into $\mathbf{X}'$, we write $\mathbf{X} \preceq \mathbf{X}'$.
2. An effective topological space $\mathbf{X}'$ is called *complete* in a class $\mathcal{T}$ of effective topological spaces if $\mathbf{X}' \in \mathcal{T}$ and $\mathbf{X} \preceq \mathbf{X}'$ for all $\mathbf{X} \in \mathcal{T}$.

Roughly speaking, $\mathbf{X} \preceq \mathbf{X}'$ means that $\mathbf{X}$ can be obtained from $\mathbf{X}'$ by deleting some points from the regions and renaming the remaining points. In 1. we refrain from further generalizations, for example, from changing $\mathrm{dom}(\nu)$. Obviously, $\preceq$ is a preorder on the class of all effective topological spaces.

Proposition 34.
1. *In Definition 33 the embedding $f : X \to X'$ is (δ, δ')-computable and its inverse is (δ', δ)-computable.*
2. *If S realizes intersection for $\mathbf{X}'$ and $\mathbf{X} \preceq \mathbf{X}'$, then S realizes intersection for $\mathbf{X}$.*

Proof: 1. For every $x \in X$, $x \in \nu(w) \iff f(x) \in \nu'(w)$. By (11) the identity on Σ^ω realizes f as well as f^{-1}.
2. By assumption, $\nu'(u) \cap \nu'(v) = \bigcup\{\nu'(w) \mid (u, v, w) \in S'\}$. Apply f^{-1}. □

First we consider the spaces $T(\mathbf{Z})$ where $\mathbf{Z}$ is an effective predicate space (Definition 8).

Theorem 35. *For $L\subseteq\Sigma^*$ define the effective predicate space $\mathbf{Z}_L = (Z_L, \sigma_L, \lambda_L)$ by $Z_L := 2^L$, $\mathrm{dom}(\lambda_L) := L$ and $\lambda_L(u) := \{A\subseteq L \mid u \in A\}$, Then $T(\mathbf{Z}_L)$ is complete in the class $\mathcal{T}_L$ of all spaces $T_L(\mathbf{Z})$ such that $\mathbf{Z} = (Z, \sigma, \lambda)$ is an effective predicate space with $L = \mathrm{dom}(\lambda)$.*

Proof: First we show (20) for $\mathbf{Z}_L$. Let $A, B \in Z_L$ such that $\{U \in \sigma_L \mid A \in U\} = \{U \in \sigma_L \mid B \in U\}$. Then $\{u \in L \mid A \in \nu_L(u)\} = \{u \in L \mid B \in \nu_L(u)\}$, hence $\{u \in L \mid u \in A\} = \{u \in L \mid u \in B\}$, that is, $A = B$. Therefore, $\mathbf{Z_L}$ is an effective predicate space.

Let $\mathbf{Z} = (Z, \sigma, \lambda)$ be an effective predicate space with $L = \mathrm{dom}(\lambda)$. Define $f : Z \to Z_L$ by $f(x) := \{u \in L \mid x \in \lambda(u)\}$. Then $f(x) = f(y) \Longrightarrow \{u \in L \mid x \in \lambda(u)\} = \{u \in L \mid y \in \lambda(u)\} \Longrightarrow \{U \in \sigma \mid x \in U\} = \{U \in \sigma \mid y \in U\} \Longrightarrow x = y$, hence f is injective. Since $x \in f^{-1}[\lambda_L(u)] \iff f(x) \in \lambda_L(u) \iff u \in f(x) \iff x \in \lambda(u)$, $\lambda(u) = f^{-1}[\lambda_L(u)]$ and hence $\nu_\lambda(v) = f^{-1}[\nu_{\lambda_L}(v)]$ for $v \in \mathrm{dom}(\nu^{\mathrm{fs}})$. Therefore, $\mathcal{T}(\mathbf{Z}) \preceq \mathcal{T}(\mathbf{Z}_L)$. □

Notice that (X_L, σ_L) is a Scott domain [Sco76]. For a computable topological space $\mathbf{X}$ a set S that realizes intersection was the only concrete information we have used so far.

Theorem 36. *For $S \subseteq (\Sigma^*)^3$ let $\mathcal{T}_S$ be the class of all effective topological spaces $\mathbf{X} := (X, \tau, \beta, \nu)$ such that S realizes intersection for $\mathbf{X}$. If $\mathcal{T}_S \neq \emptyset$, then there is some space $\mathbf{X}_S$ that is complete in $\mathcal{T}_S$.*

Proof: Let $S \subseteq (\Sigma^*)^3$ such that $\mathcal{T}_S \neq \emptyset$. Since an effective topological space is a T_0-space with countable base, its cardinality is at most $2^{\aleph_0}$, the cardinality of the real numbers. Therefore, every effective topological space $\mathbf{X}$ can be obtained from an effective topological space on a subset of the real numbers $X' \subseteq \mathbb{R}$ by bijective renaming. Formally,

$$(\forall \mathbf{X} \in \mathcal{T}_S)(\exists \mathbf{X}' \in \mathcal{T}_S)\,(X' \subseteq \mathbb{R} \ \wedge \ \mathbf{X} \preceq \mathbf{X}' \ \wedge \ \mathbf{X}' \preceq \mathbf{X})\,. \tag{35}$$

Let $D := \mathrm{pr}_1(S)$ be the first projection of S and let $\{\mathbf{X}_i \mid i \in I\}$, $\mathbf{X}_i := (X_i, \tau_i, \beta_i, \nu_i)$, be the set of all effective topological spaces on the set of real numbers in $\mathcal{T}_S$. By assumption $I \neq \emptyset$. Define the disjoint union X of the sets X_i and a notation ν of subsets of it by $\mathrm{dom}(\nu) := D$ and

$$X := \{(i, x) \mid i \in I, x \in X_i\}, \quad \nu(w) := \{(i, x) \in X \mid x \in \nu_i(w)\}\,.$$

Since S realizes intersection for every $\mathbf{X}_i$, $\nu(u) \cap \nu(v) = \{(i, x) \mid x \in \nu_i(u) \cap \nu_i(v)\} = \{(i, x) \mid (\exists\, w)((u, v, w) \in S \ \wedge \ x \in \nu_i(w))\} = \bigcup\{\nu(w) \mid (u, v, w) \in S\}$, ν is a notation of a base of a topology τ on X, which, however, may not be T_0. We define an equivalence relation $\equiv$ on X by

$$(i, x) \equiv (j, y) \iff \{u \in D \mid (i, x) \in \nu(u)\} = \{u \in D \mid (j, y) \in \nu(u)\}$$

and factorize.

Define $X_S := X/_\equiv$. Since $(j, y) \in \nu(u)$ if $(i, x) \in \nu(u)$ and $(i, x) \equiv (j, y)$, $\nu(u)$ is a union of full equivalence classes. For $u \in D$ define $\nu_S(u) := \nu(u)/_\equiv =$

$\{(i,x)/_{\equiv} \mid (i,x) \in \nu(u)\}$. Finally, let $\beta_S := \mathrm{range}(\nu_S)$ and let τ_S be the topology generated by β_S. By the definitions, $(i,x)/_{\equiv} \in \nu_S(u) \iff x \in \nu_i(u)$. We show that $\mathbf{X}_S = (X_S, \tau_S, \beta_S, \nu_S)$ is complete in $\mathcal{T}_S$.

Since $\{u \mid (i,x)/_{\equiv} \in \nu_S(u)\} = \{u \mid (j,y)/_{\equiv} \in \nu_S(u)\} \iff \{u \mid (i,x) \in \nu(u)\} = \{u \mid (j,y) \in \nu(u)\} \iff (i,x) \equiv (j,y) \iff (i,x)/_{\equiv} = (j,y)/_{\equiv}$, the smallest topology τ_S containing β_S, is a T_0-topology.

Since $(i,x)/_{\equiv} \in \nu_S(u) \cap \nu_S(v) \iff x \in \nu_i(u) \cap \nu_i(v) \iff (\exists w)((u,v,w) \in S \ \wedge\ x \in \nu_i(w)) \iff (\exists w)((u,v,w) \in S \ \wedge\ (i,x)/_{\equiv} \in \nu_S(w)) \iff (i,x)/_{\equiv} \in \bigcup\{\nu_S(w) \mid (u,v,w) \in S\}$, $\mathbf{X}_S \in \mathcal{T}_S$.

It remains to show $\mathbf{X} \preceq \mathbf{X}_S$ for every $\mathbf{X} = (X, \tau, \beta, \nu) \in \mathcal{T}_S$. Since $\mathbf{X} \preceq \mathbf{X}_i$ for some $i \in I$ by (35), it suffices to show $\mathbf{X}_i \preceq \mathbf{X}_S$.

Define $f : X_i \to X_S$ by $f(x) := (i,x)/_{\equiv}$. For $x, y \in X_i$, $f(x) = f(y) \implies (i,x)/_{\equiv} = (i,y)/_{\equiv} \implies (i,x) \equiv (i,y) \implies (\forall u)((i,x) \in \nu(u) \iff (i,y) \in \nu(u)) \implies (\forall u)(x \in \nu_i(u) \iff y \in \nu_i(u)) \implies x = y$, therefore, the function f is injective. Finally, $x \in \nu_i(w) \iff (i,x) \in \nu(w) \iff (i,x)/_{\equiv} \in \nu_S(w) \iff f(x) \in \nu_S(w)$, hence $\nu_i(w) = f^{-1}[\nu_S(w)]$. Therefore, $\mathbf{X}_i \preceq \mathbf{X}_S$. □

For a topology τ, $F \subseteq \tau$ is a *filter of open sets* if $F \neq \emptyset$, $\emptyset \notin F$, $U \cap V \in F$ if $U, V \in F$ and $V \in F$ if $U \in F$, and $U \subseteq V \in \tau$ [Eng89]. The filter F is *completely prime* if for every $\alpha \subseteq \tau$ with $\bigcup \alpha \in F$, $U \in F$ for some $U \in \alpha$. The space is *sober* if every completely prime filter is the set of open neighborhoods of a unique point [Sün00]. Sobriety of X is precisely a condition that forces the lattice of open subsets of X to determine X up to homeomorphism.

For a point $x \in X$ of an effective topological space $\mathbf{X} = (X, \tau, \beta, \nu)$ by (11), $p \in \Sigma^\omega$ is a δ-name of x if it is a list of the set $H := \{u \in \mathrm{dom}(\nu) \mid x \in \nu(u)\}$. The set H has the properties

$$H \neq \emptyset, \quad \emptyset \notin \nu[H]\,, \tag{36}$$

$$(\forall u, v \in H)(\exists w \in H)\, \nu(w) \subseteq \nu(u) \cap \nu(v)\,, \tag{37}$$

$$(\forall D \subseteq \mathrm{dom}(\nu))\,((\exists u \in H)\nu(u) \subseteq \bigcup \nu[D] \implies (\exists w \in D)\, w \in H)\,, \tag{38}$$

$$(\forall u, v \in \mathrm{dom}(\nu))\,((u \in H \ \wedge\ \nu(u) \subseteq \nu(v)) \implies v \in H)\,. \tag{39}$$

By (36) and (37), $F_H := \{U \in \tau \mid (\exists v \in H)\, \nu(v) \subseteq U\}$ is a filter and $\nu[H]$ is a *filter base* [Eng89]. (39), which follows already from (38), is a normalization axiom for the filter base and induces $H = \nu^{-1}\nu[H]$. By (38), F_H is a completely prime filter, which is the set of open neighborhoods of the point x. The point x is defined uniquely by the set H since $\mathbf{X}$ is a T_0-space. There may be, however, completely prime filters that are not neighborhood filters of a point. *Sobrification* adds points for all these completely prime filters such that the inclusion relation on the open sets remains unchanged. We will consider the class of all effective topological spaces $\mathbf{X}$ which have a common set Q realizing inclusion.

Theorem 37. *For $Q \subseteq \Sigma^* \times 2^{\Sigma^*}$ let $\mathcal{T}_Q$ be the class of all effective topological spaces $\mathbf{X} := (X, \tau, \beta, \nu)$ such that $X \neq \emptyset$ and Q realizes inclusion for $\mathbf{X}$. If $\mathcal{T}_Q \neq \emptyset$, then there is some effective topological space $\mathbf{X}_0$ that is complete in $\mathcal{T}_Q$.*

Proof: First, observe that for every effective topological spaces $\mathbf{X} := (X, \tau, \beta, \nu)$ such that $\mathbf{X} \in \mathcal{T}_Q$, $\mathrm{dom}(\nu)$ is the first projection of Q. Since $\mathcal{T}_Q \neq \emptyset$ there is some effective topological space $\mathbf{X} := (X, \tau, \beta, \nu) \in \mathcal{T}_Q$ such that $X \neq \emptyset$. Let

$$\begin{aligned} X_0 &:= \{H \subseteq \mathrm{dom}(\nu) \mid (36)\text{-}(39)\}, \\ \mathrm{dom}(\nu_0) &:= \mathrm{dom}(\nu), \\ \nu_0(u) &:= \{H \in X_0 \mid u \in H\} \text{ for all } u \in \mathrm{dom}(\nu)\,. \end{aligned}$$

Let $\beta_0 := \mathrm{range}(\nu_0)$, let τ_0 be the topology generated by β_0 and let $\mathbf{X}_0 := (X_0, \tau_0, \beta_0, \nu_0)$. Notice that $H \in \nu_0(u) \iff u \in H \iff \nu(u) \in \nu[H]$ (the last "$\Longleftarrow$" by (39)).

$\mathbf{X}_0$ is a T_0-space: For $H, H' \in X_0$, $\{u \mid H \in \nu_0(u)\} = \{u \mid H' \in \nu_0(u)\} \iff \{u \mid u \in H\} = \{u \mid u \in H'\} \iff H = H'$. Therefore, $\mathbf{X}_0$ is an effective topological space.

We show that for all $u \in \mathrm{dom}(\nu)$ and $D \subseteq \mathrm{dom}(\nu)$,

$$\nu(u) \subseteq \bigcup \nu[D] \iff \nu_0(u) \subseteq \bigcup \nu_0[D]\,. \tag{40}$$

Suppose $\nu(u) \subseteq \bigcup \nu[D]$ and $H \in \nu_0(u)$. Since $u \in H$, $w \in H$ for some $w \in D$ by (38). Therefore, $H \in \nu_0(w)$ for some $w \in D$, hence $H \in \bigcup \nu_0[D]$. This proves "$\Longrightarrow$". On the other hand, suppose $\nu_0(u) \subseteq \bigcup \nu_0[D]$ and $x \in X$. Let $H_x := \{u \in \mathrm{dom}(\nu) \mid x \in \nu(u)\}$. Then $H_x \in X_0$ and $x \in \nu(u) \Longrightarrow u \in H_x \Longrightarrow H_x \in \nu_0(u) \Longrightarrow (\exists w \in D)\, H_x \in \nu_0(w) \Longrightarrow (\exists w \in D)\, w \in H_x \Longrightarrow (\exists w \in D)\, x \in \nu(w) \Longrightarrow x \in \bigcup \nu[D]$. This proves "$\Longleftarrow$".

Since Q realizes inclusion for $\mathbf{X}$, by (40) Q realizes inclusion for $\mathbf{X}_0$. Since there is some $x \in X$, $H_x \neq \emptyset$ and $H_x \in X_0$, hence $X_0 \neq \emptyset$. Therefore, $\mathbf{X}_0 \in \mathcal{T}_Q$.

We show $\mathbf{X} \preceq \mathbf{X}_0$. Define $f : X \to X_0$ by $f(x) := H_x = \{u \mid x \in \nu(u)\}$. Since $\mathbf{X}$ is a T_0-space, f is injective. Since $x \in f^{-1}[\nu_0(w)] \iff f(x) \in \nu_0(w) \iff H_x \in \nu_0(w) \iff w \in H_x \iff x \in \nu(w)$, $f^{-1}\nu_0(w) = \nu(w)$ for all $w \in \mathrm{dom}(\nu)$. Therefore, $\mathbf{X} \preceq \mathbf{X}_0$.

We show that $\mathbf{X}_0$ is complete in $\mathcal{T}_Q$. Above we have constructed $\mathbf{X}_0$ from some arbitrary $\mathbf{X} \in \mathcal{T}_Q$ via (36 - 39). Since from $\mathbf{X}$ only the set Q was needed to define $\mathbf{X}_0$, $\mathbf{X}_0$ remains unchanged if in the above proof $\mathbf{X}$ is replaced by any other $\mathbf{X}' \in \mathcal{T}_Q$. Therefore, $\mathbf{X}' \preceq \mathbf{X}_0$ for all $\mathbf{X}' \in \mathcal{T}_Q$. $\square$

The space $\mathbf{X}_0$ constructed above is sober. The constructions in Theorems 35, 36 and 37 are various kinds of *completion*, where Theorem 37 presents "effective" sobrification.

11 Examples of Computable Operators

A variety of operations on points, sets and functions are computable w.r.t. the representations from Definition 5. We give some additional examples.

Theorem 38.

1. $\mathrm{eval} : (f, x) \mapsto f(x)$ *is* $([\delta_1 \to_p \delta_2], \delta_1, \delta_2)$*-computable.*
2. *For* $f :\subseteq X_1 \to X_2$ *and* $g :\subseteq X_2 \to X_3$, $(f, g) \mapsto g \circ f$ *is* $([\delta_1 \to_p \delta_2], [\delta_2 \to_p \delta_3], [\delta_1 \to_p \delta_3])$*-computable.*
3. *The multi-function* $(f, W) \rightrightarrows T$ *mapping every continuous function* $f :\subseteq X_1 \to X_2$ *and every open set* $W \subseteq X_2$ *to some open set* $T \subseteq X_1$ *such that* $f^{-1}[W] = T \cap \mathrm{dom}(f)$ *is* $([\delta_1 \to_p \delta_2], \theta_2, \theta_1)$*-computable.*
4. *The function* $(f, C) \mapsto f[C]$ *for* $C \subseteq \mathrm{dom}(f)$ *is* $([\delta_1 \to_p \delta_2], \widetilde{\psi}_1, \widetilde{\psi}_2)$ *- computable.*
5. *The function* $(f, A) \mapsto \overline{f[A]}$ *for closed* $A \subseteq \mathrm{dom}(f)$ *is* $([\delta_1 \to_p \delta_2], \psi_1, \psi_2)$ *- computable.*
6. *The function* $(f, K) \mapsto f[K]$ *for compact* $K \subseteq \mathrm{dom}(f)$ *is* $([\delta_1 \to_p \delta_2], \kappa_1, \kappa_2)$*-computable.*

Proof: 1. By Theorem 30 or as follows: The function h, $h(p, q) := \eta^{\omega\omega}_p(q)$, is computable. Suppose $(f, x) \in \mathrm{dom(eval)}$, that is, $x \in \mathrm{dom}(f)$, $f \in \overrightarrow{\delta_1}(p)$ and $x = \delta_1(q)$. By Definition 281, $\mathrm{eval}(f, x) = \delta_2 \circ h(p, q)$. Since $\overrightarrow{\delta_1} = [\delta_1 \to_p \delta_2]$, eval is $([\delta_1 \to_p \delta_2], \delta_1, \delta_2)$-computable.

2. Suppose, $f \in [\delta_1 \to_p \delta_2](p_1)$, $g \in [\delta_2 \to_p \delta_3](p_2)$, $x = \delta_1(q)$ and $y = \delta_2(r)$. By 1. there are computable functions f_{12} and f_{23} such that $f(x) = \delta_2 \circ f_{12}(p_1, q)$ and $g(y) = \delta_3 \circ f_{23}(p_2, r)$. Setting $y := f(x)$ and $r := f_{12}(p_1, q)$ we obtain $(g \circ f)(x) = \delta_3 \circ f_{23}(p_2, f_{12}(p_1, q)) = \delta_3 \circ \eta^{\omega\omega}_{h(p_1,p_2)}(q)$ for some computable function h. Therefore, composition is computable.

3. Suppose, $f \in \overrightarrow{\delta_1}(p)$ and $W = \theta_2(q)$. By Theorem 29 there is a computable function h such that $f \in \overrightarrow{\delta_2} \circ h(p)$. By the definition of $\overrightarrow{\delta_2}$, $f^{-1}[W] = \theta_1 \circ \eta^{\omega\omega}_{h(p)}(q) \cap \mathrm{dom}(f)$. By the utm-theorem for $\eta^{\omega\omega}$ there is a computable function H such that $H(p, q) = \eta^{\omega\omega}_{h(p)}(q)$. Then H realizes $(f, W) \mapsto f^{-1}[W]$.

4. Suppose, $f \in \overrightarrow{\delta_1}(p)$ and $C \in \widetilde{\psi}_1(q)$. By Theorem 29 there is a computable function h such that $f \in \overrightarrow{\delta_3} \circ h(p)$. By the definition of $\overrightarrow{\delta_3}$ for $v \in \mathrm{dom}(\nu_2)$, $\nu_2(v) \cap f[C] \neq \emptyset \iff f^{-1}[\nu_2(v)] \cap C \neq \emptyset \iff \theta_1 \circ \eta^{*\omega}_{h(p)}(v) \cap C \neq \emptyset \iff (\exists u)(u \ll \eta^{*\omega}_{h(p)}(v) \ \wedge \ u \ll q)$. There is a computable function H such that $H(p, q)$ is a list of all $\iota(v)$, $v \in \mathrm{dom}(\nu_2)$, such that $u \ll q$ and $u \ll \eta^{*\omega}_{h(p)}(v)$ for some u. Then $f[C] \in \widetilde{\psi}_2 \circ H(p, q)$, therefore, H realizes $(f, C) \mapsto f[C]$.

5. This follows from 4. above.

6. By Definition 28.6, $f \in \delta_6(p)$, $K \in \kappa_1(q)$ and $K \subseteq \mathrm{dom}(f)$ implies $f[K] \in \kappa_2 \circ \eta_p^{\omega\omega}(q)$. By the utm-theorem for $\eta^{\omega\omega}$ the function h, $h(p,q) = \eta_p^{\omega\omega}(q)$, is computable. Therefore, $(f, K) \mapsto f[K]$ is $(\delta_6, \kappa_1, \kappa_2)$-computable. By Theorem 28, the operation is $([\delta_1 \to_p \delta_2], \kappa_1, \kappa_2)$-computable. □

As a simple application let $\gamma := \widetilde{\psi} \wedge \kappa$. Then γ generalizes the (equivalence class of the) minimal cover representation κ_{mc} of the compact subsets of $\mathbb{R}^n$ in [KW87, BW99][Wei00, Definition 5.2.4] and of a computable metric space in [BP03] . From Theorem 38.4 and 6 we conclude that $(f, K) \mapsto f[K]$ for compact $K \subseteq \mathrm{dom}(f)$ is $([\delta_1 \to_p \delta_2], \gamma_1, \gamma_2)$-computable. As shown in [Wei08] the relatively computable functions are not only closed under simple composition but more generally under flowchart programming. Thus Theorem 38.2 can be generalized to operators defined by flowcharts. As another example we consider Dini's Theorem.

Theorem 39 (Dini). *Let $(f_n)_{n\in\mathbb{N}}$ be a monotonically increasing sequence of real-valued functions on a compact space that converges pointwise to a continuous function. Then the convergence is uniform.*

The first effective version of Dini's theorem has been proved by Kamo [Kam05]. He works in the terminology introduced by Pour-El and Richards [PER89] using the definitions of an "effectively compact metric space" and of "computable sequence of computable functions" introduced by Mori, Tsujii and Yasugi [MTY97, YMT99]. His theorem is formulated as follows.

Theorem 40 (effective Dini, Kamo's version). *Let $(M, d, \mathcal{S})$ be an effectively compact metric space. Let (g_n) be a computable sequence of real-valued functions on M and f a computable real-valued function on M. If g_n converges pointwise monotonically to f as $n \to \infty$, then g_n converges effectively uniformly to f.*

Roughly speaking, on a compact metric space with natural computability assumptions on compactness and sequences of real-valued functions, *there is* a computable modulus of convergence. Here we prove as a more effective version that for the more general computable topological spaces the modulus of convergence *can be computed* from the functions and from the compact subset, see [GW05] for a preliminary version.

Let $\rho_<$ be the lower representation of the real numbers (Example 1). It suffices to prove the theorem for an increasing sequence of real functions converging to 0 pointwise. It suffices to consider only $(\delta, \rho_<)$-continuous (lower semi-continuous) functions.

Theorem 41 (computable Dini). *The multi-valued operator D mapping each triple $((f_n)_{n\in\mathbb{N}}, K, k)$ such that*

$$\begin{aligned}
&(\forall n)\ f_n :\subseteq X \to \mathbb{R} \quad \textit{is } (\delta, \rho_<)\textit{-continuous},\\
&K \quad \textit{is compact and} \quad k \in \mathbb{N},\\
&(\forall n)\ K \subseteq \mathrm{dom}(f_n),\\
&(\forall n)(\forall x \in K)\ f_n(x) \le f_{n+1}(x),\\
&(\forall x \in K) \sup_n f_n(x) = 0
\end{aligned}$$

to some $j \in \mathbb{N}$ such that $(\forall x \in K)\ 2^{-k} < f_j(x)$ is $([\delta \to_p \rho_<]^\omega, \kappa, \nu_\mathbb{N}, \nu_\mathbb{N})$-computable.

Proof: Let $\rho_<$ be the inner representation of points let $\kappa_<$ be the representation of the compact subsets for the lower real line $\mathrm{R}_< = (\mathbb{R}, \tau_<, \beta_<, \nu_<)$ from Example 1. By Dini's theorem, $\sup_{n\in\mathbb{N}} \inf_{x\in K} f_n(x) = 0$. Since $\inf_{x\in K} f_n(x) = \inf f_n[K]$,

$$\sup_{n\in\mathbb{N}} \inf f_n[K] = 0 \ \wedge\ (\forall n) \inf f_n[K] \le \inf f_{n+1}[K]\,.$$

By Theorem 38, $(f, K) \mapsto f[K]$ is $[\delta \to_p \rho_<], \kappa, \kappa_<)$-computable. Since $L \mapsto \inf L$ is $(\kappa_<, \rho_<)$-computable, $(f, K) \mapsto \inf f[K]$ is $[\delta \to_p \rho_<], \kappa, \rho_<)$-computable. Therefore, $((f_n)_{n\in\mathbb{N}}, K) \mapsto (\inf f_n[K])_{n\in\mathbb{N}}$ is $([\delta \to_p \rho_<]^\omega, \kappa, \rho_<^\omega)$-computable. Finally, the multi-function $h : ((x_n)_n, k) \rightrightarrows j$ such that $2^{-k} < x_j$ for nondecreasing sequences $(x_n)_n$ with $\sup x_n = 0$ is $(\rho_<^\omega, \nu_\mathbb{N}, \nu_\mathbb{N})$-computable. The multi-function D is obtained by composition from the above computable multi-functions. Therefore, D is computable. $\square$

By type conversion [Wei08, Theorem 35] the multi-function $((f_n)_{n\in\mathbb{N}}, K) \rightrightarrows m$ where $m : \mathbb{N} \to \mathbb{N}$ is a modulus of uniform convergence is $([\delta \to_p \rho_<]^\omega, \kappa, [\nu_\mathbb{N} \to \nu_\mathbb{N}])$-computable. The upper bound 0 can be replaced by an upper semi-continuous function g, which then is a further argument of D.

12 Final Remarks

In this article we have laid merely a basis for a general computable topology. From here an immense number of further fields can be studied. A next step could be the search for δ-computable points in a computable topological space. In Example 2 there are a recursive set $L \subseteq \Sigma^*$ realizing $\mathbf{Z}$, an r.e. set S realizing intersection and an r.e. set Q realizing inclusion for $T(\mathbf{Z})$. If we delete the computable real numbers from $\mathbb{R}$, still L, S and Q are realizers. Therefore it is meaningful to search for computable points in the complete spaces for recursive sets L and for r.e. sets S and Q found in Theorems 35–37, see also [GSW07].

For the complete space $T_L(\mathbf{Z})$ from in Theorems 35 the answer is simple: an element $A \in Z_L = 2^L$ is δ-computable iff it is an r.e. subset of L. Other next steps could be the study of computable separation and the investigation of computably locally compact spaces started already in [XG07].

In applications usually there are fixed representation for the sets. Of course, in this case one can simply say "r.e." or "computable" and omit the unwieldy prefixes such as "$([\gamma \to_p \delta], \psi)$-".

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