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Elementary examples of Loewner chains generated by densities

Abstract. We study explicit examples of Loewner chains generated by absolutely continuous driving measures, and discuss how properties of driving measures are reflected in the shapes of the growing Loewner hulls.

1. Introduction. Loewner's differential equation has served as an important tool in one-dimensional complex analysis since its discovery by C. Loewner in the 1920's. This equation, in its classical form, parametrizes families of conformal mappings of reference domains, such as the unit disk or the upper half-plane, in terms of unimodular, or real-valued, driving functions respectively.

The Schramm–Loewner evolutions (SLE's) of O. Schramm (see the textbook [17]) have generated renewed interest in the problem of understanding geometric properties of deterministic solutions of the Loewner equation. The work of J. Lind, D. Marshall, and S. Rohde (see [18], [20], [19]) has shed light on how curves generated by the Loewner equation depend on the driving function. L. Kadanoff, W. Kager, and B. Nienhuis (see [15]) have derived and studied explicit solutions to the Loewner equation, and G. Ivanov, D. Prokhorov, and A. Vasil'ev [13] have considered the case of singular solutions. Recently, F. Bracci, M. D. Contreras, S. Díaz-Madrigal, P.

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Gumenyuk, and others, have worked towards a unified Loewner theory covering, in particular, the radial (disk) and chordal (half-plane) cases. (See, for example, [2] and [6].) In this note, we deal with the radial (disk) case.

When dealing with processes where growth takes place simultaneously at multiple points, it is natural to turn to the more general version of the Loewner equation, sometimes referred to as the Loewner–Kufarev equation. Here, driving functions are replaced by driving measures on the unit circle or the real line. For example, L. Carleson and N. Makarov (see [4] and [5]) and G. Selander (see [25]), have used this version of the Loewner equation to study deterministic variants of Diffusion-limited Aggregation (DLA), and related Laplacian growth models. The Hastings–Levitov growth models (see [12]) can also be described within the framework of measure-driven Loewner chains (see [24] and also [14]). This line of investigation has recently been continued by physicists; for instance, in [11] fingered growth in a channel is studied, and in [9] a general framework for approaching Laplacian growth via the Loewner equation is proposed. However, Loewner chains that are generated by prescribed families of non-atomic measures seem to have received somewhat less attention than the case of moving point masses (see [17] and [21] for some background material). The special case of time-independent driving measures is intimately connected with the theory of one-parameter semigroups of analytic self-maps pioneered by E. Berkson and H. Porta, and developed by numerous subsequent authors (see for instance the book [10] and the references therein, and the survey [26]). The recent papers (see [16] and [27]) of A. Kuznetsov and Vasil'ev address boundary properties of general Loewner chains. For instance, Kuznetsov derives Hölder continuity properties for the Loewner maps under certain size and smoothness assumptions on the driving terms (see the next section for a definition). Very recently, after the first version of this note was submitted, Bracci, Contreras, and Díaz-Madrigal (see [3]) noted the connection between β-points of Loewner maps and singularities of driving measures in the time-independent case.

The purpose of this note is to exhibit some very simple but explicit examples of deterministic Loewner chains driven by measures absolutely continuous with respect to arc length. Some of the chains we consider appeared as (deterministic) scaling limits in an anisotropic random growth model studied in [14]; this originally led to our interest in finding closed expressions for these particular mappings. Our examples, obtained by elementary means, illustrate how certain properties of the density of the measure (smoothness, symmetries, zeros) are reflected in the shapes of the growing sets of the evolutions. In certain cases, we find explicit solutions to the Loewner equations in question by applying classical transforms to simpler conformal maps. A rescaling procedure then allows us to analyze the asymptotic shapes of the Loewner hulls as $t \to \infty$. Some of the examples and observations in this note are folklore and known to specialists, but we have not found them conveniently gathered in one place; other examples provide concrete illustrations of more general theorems (especially regarding boundary fixed points). Our point of view is perhaps slightly different in that we work in the exterior disk with decreasing domains, and in that we start out by prescribing the driving measures, an approach natural in the context of growth processes, and then analyze the geometric properties of the resulting mappings.

2. Background material. We work in the setting of the complement of the unit disk $\mathbb{D}$, that is, the exterior disk $\Delta = \{z \in \mathbb{C}_\infty : |z| > 1\}$. In its general form, the (radial) Loewner (or Loewner–Kufarev–Pommerenke) equation reads

$$\partial_t f(z,t) = zf'(z,t) \int_{\mathbb{T}} \frac{z+\zeta}{z-\zeta} d\mu_t(\zeta), \quad z \in \Delta, \tag{2.1}$$

where $\{\mu_t\}_{t\geq 0}$ is a family of positive measures on the unit circle $\mathbb{T}$. We usually identify the unit circle with the interval $[0,1)$ for computational purposes. Given an initial condition $f(z,0) = \phi(z)$, with ϕ conformal, the Loewner equation admits a unique solution if, for instance, $\|\mu_t\| = 1$ and the integral appearing the right-hand side is measurable in t. We shall choose $\phi(z) = z$ for simplicity. Solving the Loewner equation, we obtain a Loewner chain, that is, a family $\{f(\cdot,t)\}_{t\geq 0}$ of conformal maps

$$f(\cdot,t) \colon \Delta \to \Omega(t) = \mathbb{C} \setminus K(t)$$

where the sets $K(t)$ are compact and connected; and they satisfy $K(0) = \overline{\mathbb{D}}$ and $K(t) \supsetneq K(s)$, $t > s$. The maps are normalized so that $f(\infty,t) = \infty$ and $f'(\infty,t) = \operatorname{cap}(K(t)) = e^t$. We shall usually refer to the sets $K(t)$ as the hulls associated with the Loewner chain. In this note, we are working with the decreasing version of Loewner chains; in many applications, the increasing version, where $\{f(\Delta,t)\}$ forms a sequence of growing domains, is more natural. The increasing and decreasing theories are similar, but not completely equivalent (see [7]). In both cases, the classical form of the Loewner equation is recovered by setting $\mu_t = \delta_{\lambda(t)}$, where λ is a unimodular driving function.

The Herglotz theorem sets up a correspondence between probability measures on the unit circle and analytic functions $p\colon \Delta \to \mathbb{C}$ with $\operatorname{Re}(p(z)) > 0$ and $p(\infty) = 1$; any such function p is of the form

$$p(z) = \int_{\mathbb{T}} \frac{z+\zeta}{z-\zeta} d\mu(\zeta).$$

An equivalent (and more classical) form of the Loewner equation reads

$$\partial_t f(z,t) = zf'(z,t)\, p(z,t). \tag{2.2}$$

(See [21] and [27] for in-depth discussions and historical remarks.) When we wish to emphasize that the function $p(z,t)$ arises as the integral of the measure μ_t against the kernel

$$S(z, e^{2\pi ix}) = \frac{z + e^{2\pi ix}}{z - e^{2\pi ix}} = 1 + 2\sum_{j=1}^{\infty} e^{2\pi ijx} z^{-j}, \tag{2.3}$$

we shall write $p = \mathcal{S}[\mu_t]$. The real part of S is the standard Poisson kernel

$$P_\Delta(z, e^{2\pi ix}) = \frac{r^2 - 1}{r^2 - 2r\cos(2\pi(x - \xi)) + 1}, \quad z = re^{2\pi i\xi} \in \Delta,$$

of the exterior disk Δ. In terms of the Fourier coefficients

$$\hat{\mu}_t(j) = \int_0^1 e^{-2\pi ijx} d\mu_t(x), \quad j \in \mathbb{Z},$$

of the driving measures, we have

$$p(z, t) = \mathcal{S}[\mu_t](z) = 1 + 2\sum_{j=1}^{\infty} \hat{\mu}_t(-j) z^{-j}, \quad z \in \Delta. \tag{2.4}$$

In the case of time-independent measures, the solutions to the Loewner equation

$$\partial_t f(z, t) = z f'(z, t) p(z) \tag{2.5}$$

can be obtained by conjugating a fixed starlike (conformal) map ψ with the exponential scaling $z \mapsto \exp(t)z$. Recall that a function f holomorphic in Δ is said to be starlike if its omitted set $E = \mathbb{C} \setminus f(\Delta)$ is starlike with respect to the point $0 \in E$. (See, for instance, [21, p. 47].) Families of conformal mappings of this type have been studied in the context of iteration theory and in connection with composition operators (usually in the unit disk $\mathbb{D}$, as in [1]). In dynamics terms, we are dealing with one-parameter semigroups with interior Denjoy–Wolff point (placed at ∞ in the setting of Δ), and the mapping ψ acts as a Koenigs function for $\{f(\cdot, t)\}$. A systematic discussion from the dynamical viewpoint can be found, for instance, in [10] and in the survey [26] – in fact, Siskakis's examples in some cases contain unit disk analogs of the Loewner chains we obtain starting with explicit choices of growth measures.

For the reader's convenience we describe the conjugacy in detail. Assume, for a moment, that f is of the form

$$f(z, t) = \varphi(e^t \psi(z)), \quad z \in \Delta, \ t \geq 0, \tag{2.6}$$

where $\psi \colon \Delta \to \Omega$ is a conformal map of the exterior disk, with expansion at infinity of the form $\psi(z) = z + c_0 + c_1 z^{-1} + \cdots$, and $\varphi = \psi^{-1}$ is the inverse map. In order for everything to be well-defined, we require $e^t\Omega \subset \Omega$. Clearly $f(z, 0) = z$. Differentiating with respect to t and z respectively, we obtain

$$\partial_t f(z, t) = e^t \psi(z) \varphi'(e^t \psi(z)) \quad \text{and} \quad f'(z, t) = e^t \psi'(z) \varphi'(e^t \psi(z)). \tag{2.7}$$

In view of (2.5), we then have

$$\psi'(z) = \frac{1}{zp(z)}\psi(z), \quad z \in \Delta. \tag{2.8}$$

The equation (2.8) can be integrated, and we obtain the following formula for ψ:

$$\log \psi(z) = \int \frac{dz}{zp(z)} \quad \text{or} \quad \psi(z) = \exp\left(\int \frac{dz}{zp(z)}\right). \tag{2.9}$$

Here, the antiderivative is chosen so as to ensure that ψ has the desired expansion at ∞. The latter formula can be reformulated in terms of a standard integral representation of the function $q = 1/p$. Namely, writing $q = \mathcal{S}[\nu]$ for a probability measure ν on the unit circle (which in general is not so easy to determine explicitly), expanding $S(z, e^{2\pi ix}) = 2/(z - e^{2\pi ix}) - 1/z$, and reversing the order of integration, we recover the standard formula

$$\psi(z) = z \exp\left[2\int_0^1 \log\left(1 - \frac{e^{2\pi ix}}{z}\right) d\nu(x)\right].$$

(See, for instance, [22, Chapter 3].) Now conversely, given p, the formula (2.9) defines an analytic function that, in view of (2.8), satisfies

$$\operatorname{Re}\left(\frac{z\psi'(z)}{\psi(z)}\right) = \operatorname{Re}\left(\frac{1}{p(z)}\right) > 0, \quad z \in \Delta.$$

It then follows that ψ is a starlike univalent function, and we can run our construction backwards to obtain solutions to (2.5) by setting $f(z,t) = \varphi(\exp(t)\psi(z))$. In particular, solutions to (2.5) exist for all $t > 0$.

The representation (2.6) can also be derived by applying the method of characteristics to (2.2) (cf. [4, Section 1.1] and [16]), in the special case when p does not depend on the variable t. In the general time-dependent case, this approach leads to the initial value problem

$$\dot{u}(t) = -u(t)p(u(t), t), \quad u(T) = z, \tag{2.10}$$

whose solutions satisfy $u(0) = f(z, T)$ for $T > 0$ fixed. Writing $u(t) = r(t)\exp(i\theta(t))$ and taking real and imaginary parts in (2.10), we obtain

$$\dot{r}(t) = -r(t)\operatorname{Re}\{p(u(t), t)\}, \quad r(T) = |z|, \tag{2.11}$$

and

$$\dot{\theta}(t) = -\operatorname{Im}\{p(u(t), t)\}, \quad \theta(T) = \arg z. \tag{2.12}$$

Suppose now that the non-tangential extension of the driving term, $p(\zeta, t) = \angle \lim_{z \to \zeta} p(z, t)$, vanishes at $\zeta \in \mathbb{T}$ (by uniqueness, this cannot happen on any set of positive measure), or more precisely, that

$$\angle \lim_{z \to \zeta} \frac{p(z,t)}{(z - \zeta)} = L$$

is finite. In view of (2.10), we expect the point ζ to be a boundary fixed point of the Loewner mappings, that is, $f(\zeta, t) = \zeta$ for all t (see [8] for the time-independent case). In the case of sufficiently smooth absolutely continuous measures, a necessary, but not sufficient, condition for $p(\zeta, t) = 0$ is that the density ρ_t is zero at the point in question. This can be seen by writing

$$p(z, t) = \mathcal{S}[\rho_t](z) = P[\rho_t](z) + iQ[\rho_t](z),$$

where $P[\rho_t]$ denotes the Poisson integral and $Q[\rho_t]$ is the usual conjugate function. We have $\angle \lim P[\rho_t] = \rho_t$, and since the corresponding boundary values of $Q[\rho_t]$ are given by the Hilbert transform, p vanishes at ζ if

$$\rho_t(x) = \mathcal{H}[\rho_t](x) = 0, \quad \zeta = e^{2\pi i x}.$$

For a detailed discussion of boundary fixed points in the time-independent setting, we refer the reader to [8] and [10]; we shall return to this topic later.

3. Examples with time-independent driving measures.

3.1. Disks and slits. We begin by considering two extremes: normalized Lebesgue measure on $\mathbb{T}$, and a point mass located at the point $\zeta = 1$. The associated functions p in these cases are

$$p(z) = 1 \quad \text{and} \quad p(z) = \frac{z+1}{z-1},$$

respectively. In the first case, we obtain the partial differential equation

$$\partial_t f(z, t) = z f'(z, t),$$

which is solved by $f(z, t) = e^t z$, $z \in \Delta$. That is, the Loewner hulls are simply growing closed disks.

The Loewner equation associated with the constant point mass is

$$\partial_t f(z, t) = z f'(z, t) \frac{z+1}{z-1}, \tag{3.1}$$

and it is well known that the solutions to this problem are conformal maps mapping the exterior disk to the disk minus a straight slit emanating from the point 1. An explicit formula for this map can be found by first solving the associated equation

$$\psi'(z) = \frac{z-1}{z(z+1)} \psi(z).$$

It is not difficult to see that

$$\psi(z) = z + \frac{1}{z} + 2 \quad \text{and} \quad \varphi(z) = \frac{1}{2}\left(z - 2 + \sqrt{z^2 - 4z}\right),$$

and, letting $s(z, t) = \varphi(e^t \psi(z))$, we obtain (cf. [20, p. 772])

$$s(z, t) = \frac{e^t}{2z}\left[z^2 + 2(1 - e^{-t})z + 1 + (z+1)\sqrt{z^2 + 2(1 - 2e^{-t})z + 1}\right]. \tag{3.2}$$

Note that $s(-1, t) = -1$, and so we have a fixed point on the boundary.

3.2. m-fold symmetry. We next consider driving measures of the form

$$d\mu_m\big(e^{2\pi ix}\big) = \rho_m(x)dx = 2\sin^2(m\pi x)\,dx, \tag{3.3}$$

for $m = 1, 2, 3, \ldots$. Measures of this form are used to grow random clusters in a simplified model of anisotropic DLA studied in [23], and in [14]. The smooth densities ρ_m have m maxima and m zeros (of order 2) on the circle; we shall see shortly how this is reflected in the hulls of the evolutions.

A short computation shows that the analytic function p appearing in the Loewner equation is

$$p_m(z) = 1 - \frac{1}{z^m}, \quad z \in \Delta,$$

and hence we are led to the differential equation

$$\partial_t f(z,t) = zf'(z,t)\left(1 - \frac{1}{z^m}\right). \tag{3.4}$$

This equation can be solved using (2.9), but we prefer to give a more geometrically enlightening construction here.

We begin with the simple case $m = 1$, that is, we consider the equation

$$\partial_t f(z,t) = (z-1)f'(z,t). \tag{3.5}$$

First, we verify that

$$F(z,t) = e^t\big(z - 1 + e^{-t}\big)$$

satisfies (3.5) with initial condition $f(z,0) = z$. We next define, for $m \geq 2$, the functions

$$\phi_m(z,t) = \left[F(z^m, mt)\right]^{1/m}, \quad z \in \Delta. \tag{3.6}$$

Note that the univalence of $F(\cdot,t)$, and the fact that $0 \notin F(\Delta,t)$, guarantee that this yields a family of well-defined conformal maps. A computation shows that

$$\partial_t\phi_m(z,t) = [F(z^m,mt)]^{\frac{1}{m}-1}\partial_t F(z^m,mt),$$

and similarly, we find that

$$z\phi_m'(z,t) = z^m[F(z^m,mt)]^{\frac{1}{m}-1}F'(z^m,mt).$$

The fact that F solves (3.5) then implies that ϕ_m satisfies (3.4).

Lemma 3.1. *The solution to* (3.4) *is given by the conformal maps*

$$\phi_m(z,t) = e^t(z^m - 1 + e^{-mt})^{1/m}, \quad z \in \Delta. \tag{3.7}$$

Applying the inversion map $z \mapsto 1/z$, we recover the maps of [26, Example 4, Table 1].

We note that the angular limits of ϕ_m fix the points on the unit circle corresponding to the m-th roots of unity, that is, the m zeros of the density ρ_m, and that the growing hulls are smooth. In the case $m = 1$, the hulls are simply disks tangent to $\mathbb{T}$ at $\zeta = 1$.

FIGURE 1. Hulls associated with the equation (3.4) with $m = 1$ and $m = 4$, for $0 \leq t \leq 1$.

However, we have $|\phi_m'(1,t)| = \exp(mt)$ and so the angular derivative of ϕ_m at the fixed points grows rather fast. This has to be so by [8, Theorem 1] as the roots of unity are non-superrepelling fixed points under the dynamics induced by $\phi_m(\cdot, t)$.

The m-th root transform can be applied to any normalized conformal map of the exterior disk, and in particular, to any function in a Loewner chain. The key feature of the particular Loewner equation (3.4) that allows us to obtain m-fold solutions from the 1-fold solution is that $p_m(z) = p_1(z^m)$. We record this observation as a proposition.

Proposition 3.2. *Suppose that two functions p and $\tilde{p}$ appearing in the right-hand side of the Loewner equation* (2.2) *satisfy*

$$\tilde{p}(z) = p(z^m), \quad z \in \Delta. \tag{3.8}$$

Then the corresponding conformal mappings satisfy $\tilde{f}(z,t) = [f(z^m, mt)]^{1/m}$.

A sufficient condition on the two families of driving measures for this to occur to hold is provided by the next lemma.

Lemma 3.3. *Let $\{\mu\}_{t\geq 0}$ be a family of absolutely continuous probability measures with densities $\{\rho_t\}_{t\geq 0}$, and let $\{\mu^m\}_{t\geq 0}$ be the family of measures whose densities are given by $\rho_t^m(e^{2\pi i x}) = \rho_t(e^{2\pi i m x})$.*

Then, if $\mathcal{S}[\mu_t](z) = p(z,t)$, we have $\mathcal{S}[\mu_t^m](z) = p(z^m, t)$.

Proof. This amounts to a simple exercise in elementary Fourier Analysis: if ρ_t has expansion $\sum_{j\in\mathbb{Z}} \hat{\rho}_t(j) e^{2\pi i j x}$, then the Fourier series of ρ_t^m is $\sum_{j\in\mathbb{Z}} \hat{\rho}_t(j) e^{2\pi i j m x}$. We then have

$$\mathcal{S}[\mu_t] = \sum_{j=0}^{\infty} \hat{\rho}_t(-j) c_j(z) = p(z,t),$$

with $c_j(z) = z^j$ and we read off that

$$\mathcal{S}[\mu_t^m](z) = \sum_{j=0}^{\infty} \hat{\rho}_t(-j) c_{mj}(z) = p(z^m, t). \qquad \square$$

While we have formulated the lemma for absolutely continuous driving measures, an analogous statement holds for general probability measures satisfying

$$\mu_m = \frac{1}{m}(\varphi_m)_* \mu = \frac{1}{m}(\mu \circ \varphi_m^{-1}), \tag{3.9}$$

with $\varphi_m(\zeta) = \zeta^{1/m}$. For instance, the observations of this subsection apply to driving measures of the form

$$\mu_m = \frac{1}{m} \sum_{j=0}^{m-1} \delta_{\zeta(j,m)}, \tag{3.10}$$

where $\zeta(0,m), \zeta(1,m), \ldots$ denote the m-th roots of unity. In this example, a direct computation confirms that

$$\frac{1}{m} \sum_{j=0}^{m-1} \frac{z + \zeta(j,m)}{z - \zeta(j,m)} = \frac{z^m + 1}{z^m - 1},$$

and hence that (3.8) holds. Hence, the conformal maps generated by (3.10) map the exterior disk onto the disk slit by m symmetrically placed straight slits; the mappings can be obtained by taking the mth root transform of the slit map (3.2). These functions appear as limit configurations in a geodesic growth model in the exterior disk studied by Selander (see [25, p. 68]). More generally, (2.9) can be used together with partial fractions expansions to find solutions corresponding to more general configurations of point masses of the form

$$\mu = \sum_{j=0}^{n} \alpha_j \delta_{\zeta_j}; \quad \sum_{j=0}^{n} \alpha_j = 1, \quad \zeta_j \in \mathbb{T}.$$

3.3. Sine densities. For an interesting comparison to the simplest driving measure in the previous example, namely the one with density $2\sin^2(\pi x)$, we turn our attention to the driving measure μ with

$$d\mu(e^{2\pi i x}) = |\sin(\pi x)| dx$$

In this case, the density has a zero of order 1 at the point 0, and a discontinuous derivative at the same point. We compute that

$$p(z) = \mathcal{S}[\mu](z) = \frac{z-1}{\sqrt{z}} \operatorname{arcoth}\sqrt{z}, \quad z \in \Delta,$$

which leads to the Loewner equation

$$\partial_t f(z,t) = zf'(z,t)\frac{z-1}{\sqrt{z}}\ \mathrm{arcoth}\sqrt{z}. \tag{3.11}$$

Here we have used the expansion

$$\mathrm{arcoth}(w) = \sum_{j=0}^{\infty} \frac{w^{-1-2j}}{2j+1}, \quad |w| > 1;$$

recall that $\mathrm{arcoth}(w) = (1/2)\log[(w+1)/(w-1)]$. By solving

$$\psi'(z) = \frac{1}{\sqrt{z}(z-1)\mathrm{arcoth}\sqrt{z}}\,\psi(z)$$

and then inverting, we obtain that the solution to (3.3) is given by

$$f(z,t) = \coth^2(e^{-\frac{t}{2}}\mathrm{arcoth}\sqrt{z}), \quad z \in \Delta. \tag{3.12}$$

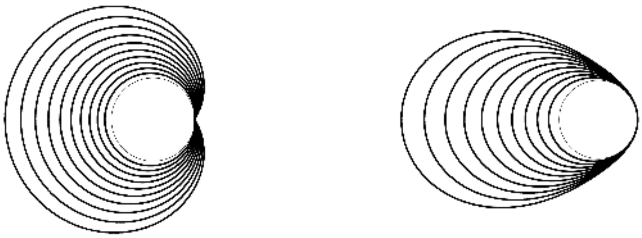

FIGURE 2. Left: Hulls associated with the equation (3.11), for $0 \leq t \leq 1$. Right: Hulls associated with $d\mu = (8/3)\sin^4(\pi x)dx$, for $0 \leq t \leq 1$.

Note that, due to symmetry, the composed maps are not sensitive to an initial choice of branch in the square root, and hence are well-defined. We again find that the boundary extension of the conformal maps in (3.12) fix the point on the circle where the density is zero. This time, and we shall return to this later, the hulls are singular at this point, and the angular derivative has $|f'(1,t)| = \infty$. In dynamics terms, the point $\zeta = 1$ is a super-repelling fixed point.

We next turn to a density with a zero of higher order at $x = 0$, namely

$$d\mu\big(e^{2\pi ix}\big) = \frac{8}{3}\sin^4(\pi x)dx.$$

Straight-forward calculations show that, for this choice of measure,

$$p(z) = 1 - \frac{4}{3z} + \frac{1}{3z^2},$$

and, by (2.9), we have

$$\psi(z) = (z-1)\sqrt{\frac{z-1}{z-\frac{1}{3}}}.$$

An algebraic expression for φ can be found by solving the equation

$$z^3 - 3z^2 + (3-w^2)z - 1 + \frac{w^2}{3} = 0$$

for w. The hulls generated by this density now exhibit a higher order tangency with the unit circle at $\zeta = 1$.

More generally, we could consider densitites

$$\rho_n(x) = \frac{2^{2n}(n!)^2}{(2n)!}\sin^{2n}(\pi x), \quad n = 1, 2, \ldots$$

(as in [23]) in which case

$$p_n(z) = 1 + 2\sum_{k=1}^{n}\frac{(-1)^k (n!)^2 z^{-k}}{(n+k)!(n-k)!} = 1 - \frac{2n}{(n+1)z}\,{}_2F_1\left(1, 1-n; 2+n; \frac{1}{z}\right);$$

${}_2F_1(a,b;c;x)$ denotes the usual Gauss hypergeometric function. The computations for ψ become more involved as n increases; one verifies, however, that $p_n'(1) = n/(2n-1)$.

3.4. Discontinuous and unbounded densities. We make a few remarks regarding discontinuous and unbounded densities. Let χ_E denote the characteristic function of a set $E \subset [0,1]$. For a number $\eta \in (0,1)$, we consider the measures μ_η with

$$d\mu_\eta\big(e^{2\pi i x}\big) = \frac{1}{\eta}\chi_{[0,\eta]}(x)dx.$$

Again computing with Fourier series (see [14] for details), we find that

$$p_\eta(z) = \mathcal{S}[\mu_\eta](z) = 1 + \frac{2}{\eta\pi}\arctan\left(\frac{e^{i\pi\eta}\sin(\pi\eta)}{z - e^{i\pi\eta}\cos(\pi\eta)}\right), \quad z \in \Delta. \tag{3.13}$$

Having a closed expression for the function p allows us to generate pictures of the growing hulls for different choices η easily (see [14] for simulations). Unfortunately, the solutions to the Loewner equation for this choice of measure do not seem to admit any obvious description in terms of elementary functions.

We observe that the hulls do not exhibit growth outside the parts of the unit circle corresponding to the interval $[0,\eta]$, as is to be expected in view of our initial discussion of boundary fixed points and (2.11). Instead, the boundary extensions of the conformal maps $f(\cdot,t)$ to this subarc induce a flow on the circle (that breaks down when the moving points hit the protruding part of the hulls). This phenomenon is discussed at greater length in the paper [14]. The differential equation governing the circle flows in [14] is obtained by applying a logarithmic transformation to the inverse mappings $g(\cdot,t) = f^{-1}(\cdot,t)$, restricted to $\partial K(t) \cap \mathbb{T}$. In the notation of this note, we are simply considering the inverses of solutions to the equation (2.12) for the angular part of our conformal maps, restricted to the boundary.

A rather natural example of an unbounded density on the circle would be $\rho(x) = 1 - \log(2\sin(\pi x))$, giving rise to

$$p(z) = \mathcal{S}[\rho](z) = 1 + \log[z/(z-1)].$$

We have unfortunately not been able to find a closed expression for the conformal maps generated by this density, but we do note that

$$\angle \lim_{z\to 1}(z-1)p(z) = 0,$$

and so, by [3], the point $\zeta = 1$ is not a β-point for $f(\cdot,t)$.

3.5. The Poisson kernel and the slit map revisited. In order to further explore the relationship between the growth of hulls and the local size of the measure, we look for a family of measures that interpolate between uniform density, that is Lebesgue measure, and support concentrated at a single point. We should expect the corresponding conformal maps to exhibit similar interpolating properties, with the map $f(z) = e^t z$ and the slit map (3.2) as extremes.

We are able to observe this directly if we select driving measures ν_r with

$$d\nu_r(e^{2\pi i x}) = P_\Delta(r, e^{2\pi i x})dx, \quad r > 1. \tag{3.14}$$

That is, we choose the Poisson kernel itself, at the point $z = r$, as the density. We quickly find that

$$p_r(z) = \mathcal{S}[\nu_r](z) = \frac{rz+1}{rz-1},$$

and in the corresponding Loewner equation,

$$\partial_t f(z,t) = zf'(z,t)\frac{rz+1}{rz-1}, \tag{3.15}$$

we recover the differential equation for the slit map as $r \to 1$, and the equation generating growing disks as $r \to \infty$.

Now let $s(z,t)$ again denote the slit map (3.2), and define the dilated maps

$$\sigma_r(z,t) = \frac{1}{r}s(rz,t). \tag{3.16}$$

Then, since $\sigma_r'(z,t) = s'(rz,t)$, and $\partial_t\sigma_r(z,t) = \frac{1}{r}\partial_t s(rz,t)$ we see that σ_r satisfies the equation (3.15). The Loewner hulls evolve along level lines of the slit maps (3.2), scaled back so as to have the correct capacity.

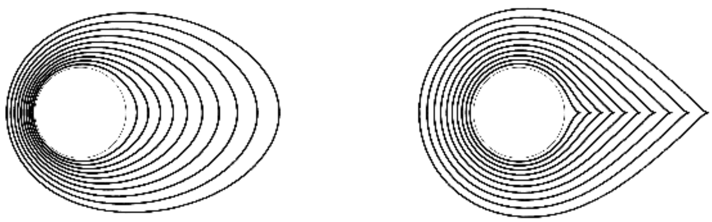

FIGURE 3. Left: Hulls associated with the equation (3.15) with $r = 2.2$, at time $0 \le t \le 1$. Right: Hulls associated with $d\mu_{3/4} = 3/4dx + 1/4\delta_0$, for $0 \le t \le 1$.

Lemma 3.4. *The solutions to* (3.15) *are given by the conformal mappings*

$$\sigma_r(z,t) = \frac{e^t}{2z}\left[r^2z^2+2(1-e^{-t})rz+1 + (rz+1)\sqrt{r^2z^2+2(1-2e^{-t})rz+1}\right]$$

To see what the dilation of the function $p = \mathcal{S}[\mu_t]$ means in terms of general driving measures we look at the Fourier representation

$$p_r(z,t) = p(rz,t) = 1 + 2\sum_{j=1}^{\infty}\hat{\mu}_t(-j)(rz)^{-j} = 1 + 2\sum_{j=1}^{\infty}\hat{\mu}_t^r(-j)z^{-j},$$

where the coefficients $\hat{\mu}_t^r(-j)$ are given as products

$$\hat{\mu}_t^r(-j) = r^{-j}\hat{\mu}_t(-j). \tag{3.17}$$

We recall that multiplication of Fourier coefficients of measures corresponds to computing the Fourier coefficients of the convolution measure and we recognize r^{-j} as the Fourier coefficients of the absolutely continuous measure ν_r whose density is the Poisson kernel $P_\Delta\big(r, e^{2\pi ix}\big)$.

Proposition 3.5. *Let $\{f(\cdot,t)\}_{t\ge 0}$ denote the conformal mappings that solve the Loewner equation driven by the family of probability measures $\{\mu_t\}_{t\ge 0}$. Then the solution to the Loewner equation driven by the convolution measures with*

$$d\mu_t^r = d(\mu_t * \nu_r), \quad r > 1, \tag{3.18}$$

are given by $\{f^r(\cdot,t)\}_{t\ge 0}$, where $f^r(z,t) = r^{-1}f(rz,t)$.

This means that solving the Loewner equation for a measure smoothed by convolution with the Poisson kernel amounts to considering the dilated versions of the conformal mappings associated with the measure itself; just as convolutions with absolutely continuous measures are always absolutely continuous, the dilation of a conformal mapping is always smooth up to the boundary. In both settings, we recover the original object as $r \to 1$. (The Poisson kernel converges to a point mass, which is the unit for the convolution product of measures.) This proposition also serves as an illustration of the continuity result [14, Proposition 1]; there it is shown that weak* convergent measures (on a product space) yield conformal maps converging locally uniformly.

4. Sums of driving measures. In this section, we consider Loewner mappings that arise by driving the Loewner equation by a weighted sum of two probability measures.

We begin with a driving measure with singular component; the result is yet another way to "spread out" growth in the slit map. We introduce the family

$$d\mu_a(e^{2\pi i x}) = a dx + (1-a)\delta_0, \quad a \in [0,1]. \tag{4.1}$$

That is, we add uniform measure to a point mass of strength $1-a$ at the point $\zeta = 1$, and we obtain the functions

$$p_a(z) = \mathcal{S}[\mu_a](z) = \frac{z+1-2a}{z-1}. \tag{4.2}$$

The growing hulls that arise from the Loewner equation with this choice of driving measures again interpolate between the singular slit map ($a = 0$) and the very smooth growing disks ($a = 1$), with cusp-like features for intermediate a.

For instance, the choice $a = 3/4$ results in particularly simple computations for $\psi(z) = 2z^2/(2z-1)$ and φ. In this case, we find that the solution to the corresponding Loewner equation is given by

$$f(z,t) = \frac{e^t z}{2z-1}\left(z + \sqrt{z^2 + e^{-t}(1-2z)}\right).$$

These functions map Δ conformally onto the complement of tear-shaped domains exhibiting cusps, as we shall see in the next section. At this point, we simply note that $\zeta = 1$ is a regular pole (in the sense of [3]) for p_a, of mass $\angle \lim_{z\to 1}(z-1)p_a(z) = 2-2a$.

Next, we return to our previous example with m-fold symmetry (with an integer $m \geq 1$) and introduce two families of time-dependent measures $\{\mu_m(\cdot,t)\}_t$ and $\{\nu_m(\cdot,t)\}_t$:

$$d\mu_m(e^{2\pi i x}, t) = (1-t)dx + 2t\sin^2(m\pi x)dx, \quad t \in (0,1), \tag{4.3}$$

and

$$d\mu_m^*(e^{2\pi i x}, t) = t dx + 2(1-t)\sin^2(m\pi x)dx, \quad t \in (0,2). \tag{4.4}$$

We have

$$p_m(z,t) = \mathcal{S}[\mu_t](z) = 1 - \frac{t}{z^m} \quad \text{and} \quad p_m^*(z,t) = \mathcal{S}[\mu_t^*](z) = 1 - \frac{1-t}{z^m},\ z \in \Delta,$$

and we wish to solve $\partial_t f(z,t) = zf'(z,t)p_m(z,t)$. Because of the time-dependence, our previous analysis using (2.9) does not apply. Instead, we rely on (2.10) to solve the Loewner equation. In the first case, this leads to the ordinary differential equation

$$\dot{u}(s) = -u(s) + \frac{s}{[u(s)]^{m-1}}, \quad u(t) = z,$$

and we verify that $u(s) = [e^{m(t-s)}(z^m - t + 1/m) + s - 1/m]^{1/m}$ is the unique solution to this problem. Similar arguments apply to the equation involving p_m^*, and after evaluating at $s = 0$, we deduce the following.

Lemma 4.1. *The solutions to the Loewner equation driven by* (4.3) *and* (4.4) *are given by*

$$\Phi_m(z,t) = e^t \left[z^m - t + \frac{1}{m}(1 - e^{-mt}) \right]^{1/m}, \quad z \in \Delta, \ t \in [0,1],$$

and

$$\Phi_m^*(z,t) = e^t \left[z^m + t - \frac{m+1}{m}(1 - e^{-mt}) \right]^{1/m}, \quad z \in \Delta, \ t \in [0,2].$$

The exterior disk is mapped onto smooth domains exhibiting m-fold symmetry that becomes more, respectively less pronounced. In the simplest case $m = 1$, we have $\Phi_m(z,t) = e^t(z + 1 - t) - 1$ and $\Phi_m^*(z,t) = e^t(z + t - 2) + 2$, and the exterior disk is mapped onto the complement of a closed disk of increasing radius, and moving center. It should be noted that while $\Phi_m(\cdot,t)$ and $\Phi_m^*(\cdot,t)$ are well-defined conformal maps for all $t \geq 0$, the evolution cannot be continued as a Loewner chain past $t = 1$ and $t = 2$ respectively. To see this, note that the function $p_1^*(z,t) = 1 + (t-1)/z$ has an initial zero at $\zeta = 1$, no zeros when $0 < t < 2$, a zero at $\zeta = -1$ when $t = 2$, and fails to have positive real part when $t > 2$. This is reflected in the fact that the domain inclusion required of a Loewner chains is not satisfied once $t > 2$.

5. Asymptotic shapes and boundary fixed points. There is a natural way of associating limit shapes with a families of Loewner hulls in Δ. Given a Loewner chain $\{f(\cdot,t)\}_{t\geq 0}$, we define a family of rescaled maps $\{\tilde{f}(\cdot,t)\}_{t\geq 0}$ by setting

$$\tilde{f}(z,t) = \frac{1}{\operatorname{cap}(K(t))} f(z,t) = e^{-t} f(z,t), \quad z \in \Delta. \tag{5.1}$$

Since the sets $K(t)$ are connected, dividing by logarithmic capacity leads to sets $\tilde{K}(t)$ having diameter comparable to 1. We then say that the hull associated with $\tilde{f}(\cdot,\infty) = \lim_{t\to\infty} \tilde{f}(\cdot,t)$, if it exists, is the asymptotic hull.

In general, this limit (taken in the sense of uniform convergence on compacts) need not exist. Consider the driving measure with

$$d\mu_c(e^{2\pi i x}, t) = 2\sin^2(\pi x - ct)dx, \quad c \in \mathbb{R},$$

which gives rise to $p_c(z,t) = 1 - \exp(2ict)/z$. A short computation based on (2.10) shows that the corresponding Loewner equation is solved by

$$f_c(z,t) = e^t z + \frac{1-2ic}{1+4c^2}\left(1 - e^{(1+2ic)t}\right) \quad z \in \Delta,\ t > 0.$$

These are conformal maps of the exterior disk onto the complement of expanding and rotating disks. We see that rescaled maps $\tilde{f}_c(\cdot, t)$ do not converge as $t \to \infty$. However, we can let $c \to \infty$ in the formula above, and obtain $f_\infty(z,t) = e^t z$; this corresponds to the fact that the measures on the product space $\mathbb{T} \times [0,t]$ given by $d\mu_c dt$ converge weak* (by the Riemann–Lebesgue lemma).

For time-independent driving measures, however, $\tilde{f}(\cdot, \infty)$ is well defined; taking the limit entails computing

$$\tilde{f}(z,\infty) = \lim_{t\to\infty} e^{-t}\varphi(e^t\psi(z)) = \psi(z),$$

so that the limit map is simply the starlike function ψ.

Applying this procedure to our examples, we immediately find that the limit map in the case of the uniform measure is the identity and the limit hull is the closed unit disk.

In the case of the density $\rho = 2\sin^2(\pi x)$, we see that

$$\tilde{f}(z,t) = z - 1 + e^{-t} \to \tilde{f}(z,\infty) = z - 1, \quad t \to \infty,$$

so that the limit hull is again a disk, this time centered at $z = -1$. A similar analysis of the m-fold maps shows that

$$\tilde{f}(z,\infty) = (z^m - 1)^{1/m} = [(z-1)(z-\zeta(1,m))\cdots(z-\zeta(m-1,m))]^{\frac{1}{m}}.$$

In this case we obtain limit hulls in the form of m-fold lemniscates: the domains $\tilde{f}(\Delta,\infty)$ exhibit m corners of inner angle $\alpha = \pi/m$, with symmetrically placed vertices. For $\rho_2 = 8/3\sin^4(\pi x)$ we obtain

$$\tilde{f}(z,\infty) = (z-1)^{3/2}\left(z - \frac{1}{3}\right)^{-1/2},$$

a conformal map onto the complement of a tear-shaped set, with a corner of interior angle $3\pi/2$ at $z = 0$. These observations essentially follow from a general result regarding semigroups (see for instance [10]): if $p(\zeta) = 0$ and $p'(\zeta) = a$, then the map ψ exhibits a corner of opening $1/a\pi$ at ζ. From this we deduce that the maps ψ that arise from the sine densities ρ_n exhibit corners of inner angles $(2 - 1/n)\pi$, with the slit map as limiting degenerate case. Thus, precomposing the squared-sine density with $\zeta \mapsto \zeta^m$ decreased the angle at the fixed points; whereas raising the density to an integer power (and renormalizing) increased the angle.

We next return to the density $\rho = |\sin(\pi x)|$. First, we study the boundary of the hulls at the fixed point in finite time. We set $a(t) = \exp(-t/2)$ and write things out in terms of the functions

$$\coth(z) = \frac{e^{2z}+1}{e^{2z}-1} \quad \text{and} \quad \operatorname{arcoth}(w) = \frac{1}{2}\log\left(\frac{w+1}{w-1}\right),$$

and considering the quantity

$$\lim_{z\to 1} \arg\left(\frac{f(z,t)-1}{(z-1)^{a(t)}}\right)$$

we find that the hulls arising from the sine density exhibit a corner of opening $\alpha(t) = a(t)\pi = \exp(-t/2)\pi$ at the point 1 (see [22, Chapter 3]). Next, we again set $\tilde{f}(z,t) = e^{-t}f(z,t)$, and pass to the limit. We then see that the domain associated with the limit map

$$\tilde{f}(z,\infty) = \frac{1}{\operatorname{arcoth}^2(\sqrt{z})} \asymp \log^{-2}(z-1), \quad z \to 1,$$

exhibits an outward pointing cusp at the origin.

Turning to the slit map s in (3.2) and its relatives σ_r, rescaling, and passing to the limit, we find that

$$\tilde{s}(z,\infty) = z + \frac{1}{z} + 2 \quad \text{and} \quad \tilde{\sigma}_r(z,\infty) = z + \frac{1}{r^2 z} + \frac{2}{r}.$$

Each $\tilde{\sigma}_r(\cdot,\infty)$ maps the exterior of the unit disk to the complement of an ellipse centered at $z = 2/r$, with axis ratio $(1-r^{-2})/(1+r^{-2})$. As is to be expected, we recover the line segment $[0,4]$ when $r = 1$ (the point mass) and the closed unit disk when $r = \infty$ (uniform measure). Finally, the limit map associated with the measure (4.1) with $a = 3/4$ is

$$\tilde{f}(z,\infty) = \frac{2z^2}{2z-1}$$

and maps the exterior disk onto a domain with an inward pointing cusp. In this case, an analysis of the maps $f(z,t)$ confirms that this cusp, or expressed differently, this β-point, is present for every $t > 0$. This is also guaranteed by the results in [3] since p_a has a regular pole at $\zeta = 1$ for all $0 \le a < 1$.

6. Final remarks. As we have seen, the local behavior of a density near a zero determines whether the growing hulls are smooth, or exhibit outward corners or cusps, near a fixed point. These kinds of questions have been studied in depth in the time-independent (semigroup) framework (see [10] and references therein). Our example of a measure with a singular component shows that measures that are very concentrated at certain points can lead to inward corners and cusps; a precise analysis in the time-independent case is carried out in [3]. Presumably, very irregular driving measures (e.g. random measures) should lead to hulls with complicated and perhaps interesting structures (cf. [26, Section 8]).

In Kuznetsov's paper [16] size assumptions are imposed on the function p, for instance, that $a < \mathrm{Re}(p(z,t)) < b$ (or $a < \rho_t(x) < b$), with $a > 0$ and $b < \infty$. Kuznetsov then asserts that the conformal mappings arising from the Loewner equation are Hölder continuous in the closed unit disk; if the density is also assumed to be Hölder continuous, the unit disk $\mathbb{D}$ is mapped onto rectifiable quasidisks (cf. [16, Theorem 4, Section 5]). Our examples illustrate why these size assumptions are in some sense necessary; in the case of the sine density the angle at the fixed point gets arbitrarily small as t increases.

The case of time-dependent driving measures is of course more interesting from the point of view of applications to growth processes in Physics then the time-independent cases we have have mostly focused on here. However, explicit solutions are more difficult to obtain, and we believe that our examples do capture some of the flavor of the evolutions we should expect in that case. The case of random or irregular, but not purely singular, driving measures does not seem to have been studied extensively; we suspect that a better understanding of Loewner chains generated by such measures might be useful when studying certain scaling limits of random planar growth models (cf. [14]).

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