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Elementary formal systems[1)]

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§1. **Introduction.** Like the canonical languages of Post [**2**], [**3**], elementary formal systems (as they are to be defined) provide a direct characterization of recursive enumerability for sets (and relations) of formal expressions, without recourse to Gödel numbering.[3)] In this paper we develop just enough of the theory of these systems to construct a "universal" system and to prove its recursive unsolvability. This proof is of unusual brevity; no number theory is employed, and the Post normal form theorem for canonical systems is circumvented.

§2. **Elementary formal systems.** For any finite alphabet K we define an *elementary formal system* (E) *over* K as a collection of the following items: (i) the alphabet K; (ii) A new alphabet of symbols called *variables*; (iii) another alphabet of symbols called *predicates*, each of which is assigned a unique positive integer called its*degree*; (iv) two more symbols $\rightarrow$ and ,; (v) A finite set $A_1, \cdots, A_z$ of expressions which are (well formed) *formulas*, according to the definition given below; these strings are called the *axioms* of the system (E).

By a *term* of (E) we mean any string composed of symbols of K and variables (or either one alone). By an *atomic* formula of (E) we mean an expression of the form $Pt_1, \cdots, t_n$, where $t_1, \cdots, t_n$ are terms and P is a predicate of degree n. By a (well formed) *formula* of (E) we mean either an atomic formula

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2) This paper is a condensation of Chapter 1 of the author's forthcoming monograph [**1**]. We are publishing it separately, since it forms a completely self contained unit, designed to give the general reader a quick insight into the essential nature of undecidability arguments.

3) Our systems offer certain technical advantages over those of Post, in that their structure is simpler to describe and their techniques are particularly simple to apply. [cf. [**1**] for a complete development of recursive function theory from the viewpoint of elementary formal systems.] The "productions" of Post (whose definition involves a cumbersome metamathematical notation) are replaced by substitution and modus ponens, as sole rules of inference. Thus these 2 familiar logistic rules suffice for the construction of *all* formal mathematical systems.

F_1 or an expression of the form $F_1 \to F_2 \to \cdots \to F_n$, where each F_i is atomic; in such an expression, F_n is called the *conclusion* and $F_1, \cdots, F_{n-1}$ are called the *premisses*. [In the intended interpretation, the symbol "$\to$" stands for implication with association to the right; e. g. $F_1 \to F_2 \to F_3$ is to be read "F_1 and F_2 together imply F_3" or "F_1 implies that F_2 implies F_3"].

A (well formed) formula X of (E) is said to be *provable* in (E) or a *theorem* of (E) iff X is either an axiom of (E) or else is obtainable from the axioms of (E) by finitely many applications of the 2 rules: (i) to substitute any (non-empty) string in K for (all occurrences of) any variable; (ii) to infer X_2 from X_1 and $X_1 \to X_2$, providing X_1 is atomic. [The reason for this proviso is that implication is associated to the right; a formula $X_1 \to X_2$ can be read "X_1 implies X_2" only if X_1 is atomic].

A predicate P of degree n is said to *represent* the set of all n-tuples $(X_1, \cdots, X_n)$ (of strings in K) such that $PX_1, \cdots, X_n$ is provable in (E). A set, or relation, W of strings in K is called *formally representable* over K iff there exists an elementary formal system (E) over K in which W is represented by some predicate.[4] A mathematical system (M) in an alphabet K can now be called *formal* or *finitary* iff its set of theorems is formally representable over K[5].

§ 3. **Recursive enumerability.** Just as any non-negative integer is uniquely expressible as a polynomial in powers of 2 with coefficients 0 and 1, so is any *positive* integer uniquely expressible as a polynomial in powers of 2 with coefficients 1 and 2. We let D be the 2-sign alphabet $\{1, 2\}$; these 2 symbols we call *dyadic digits*; any string $a_n a_{n-1} \cdots a_1 a_0$ of dyadic digits is called a *dyadic numeral*. This numeral is identified with the positive integer $a_0 + 2a_1 + 4a_2 + \cdots + 2^n a_n$.[6] Any set (or relation) A of positive integers shall be identified with the corresponding set (or relation) of dyadic numerals. We call A *recursively enumerable* (abbreviated r. e.) iff A is formally representable over D.[7]

4) Formal representability of W is equivalent to representability in a Post canonical system, which in turn is equivalent to recursive enumerability (under any of the standard Gödel numberings). The equivalence of formal representability to Post's canonical representability is substantiated in [**1**]. The following interesting question arises: If K is a sub-alphabet of L and if W is a set (or relation) of strings in K and if W is formally representable over L, is W necessarily formally representable over K? We answer this question affirmatively in [**1**].

5) Other characterizations of formal systems, also basically along the lines of Post, have been provided by Markov [**4**] and Lorenzen [**5**].

6) Our choice of 1 and 2, rather than 0 and 1, is made for certain technical reasons. Our program could (with minor modifications) also be carried out using 0 and 1 (cf. [**7**]).

7) Our choice of 2 as a base is one of convenience rather than necessity. Post's definition of recursive enumerability used 1 as a base. In [**1**] we show that our definition of recursive enumerability is invariant under change of base.

And A is called *recursive* iff both A and its complement $\tilde{A}$ are r. e. [If A is a relation of degree n, then $\tilde{A}$ is understood to be the complement of A relative to the set of all n-tuples of numbers (positive integers).]

§ **4. A universal system.** We now wish to construct a so-called *universal* system U in which we can, so to speak, express all propositions of the form that such and such a number is in such and such a recursively enumerable set.

Preparatory to the construction of U, we need a device for "transcribing" all elementary formal systems over D into one finite alphabet. We take 3 symbols: $v, ', p$. By a *transcribed* variable we mean any of the strings v', v'', v''', etc.; by a *transcribed* predicate we mean a string of p's followed by a string of accents; the number of p's is to indicate the *degree* of the predicate. We now define a *transcribed* system to be a system like an elementary formal system over D, except that we use transcribed variables and transcribed predicates in place of individual symbols for variables and predicates. It is obvious that representability in a transcribed system is equivalent to representability in a system which is not transcribed. We use the terms "transcribed term", "transcribed (well formed) formula" in their obvious contexts. For any transcribed formulas $X_1, X_2, \cdots, X_n, Y$, we say that Y is *derivable* from $X_1, \cdots, X_n$ iff Y is provable in that transcribed system whose axioms are $X_1, \cdots, X_n$.

We now construct the system U.[8] The *alphabet* K_9 of U shall consist of the nine symbols: $1\,2\,v\,'\,p\,,\rightarrow * \vdash$. We refer to these signs as the 1st, 2nd, $\cdots$, 9th symbols of U respectively. [The first seven are used for constructing all transcribed systems.] The *sentences* of U shall be all expressions of the form $X_1 * X_2 * \cdots * X_n \vdash Y$, where $X_1, X_2, \cdots, X_n, Y$ are transcribed well formed formulas. [For $n=1$ the expression is of the form $X_1 \vdash Y$.] Such a sentence shall be called *true* in U iff Y is derivable from $X_1, \cdots, X_n$. We let T be the set of all true sentences. By a *predicate* H of U (not to be confused with a transcribed predicate) we mean an expression of the form $X_1 * \cdots * X_n \vdash P$, where $X_1, \cdots, X_n$ are transcribed formulas and P is a transcribed predicate; the degree n of H is, by definition, that of P. This predicate H is said to *represent* (in U) the set of all n-tuples $(a_1, \cdots, a_n)$ of numbers such that $P\ a_1, a_2, \cdots, a_n$ is derivable from $X_1, \cdots, X_n$ —alternatively H represents in U the relation (or set) represented by P in that transcribed systems whose axioms are $X_1, \cdots, X_n$. Thus the number sets (and relations) representable in U are precisely those which are recursively enumerable— it is in this sense that U is called a "universal" system for all r. e. sets and relations.

§ **5. The recursive unsolvability of U.** As we have remarked, we are

8) The details of this construction differ a bit from those in [1]. We believe the present version to be a slight improvement.

identifying the positive integers with their corresponding dyadic numerals. We let $g_1, g_2, \cdots, g_9$ be the respective numbers (numerals) 12, 122, $\cdots$, 1222222222. For any string X in the alphabet K_9, we define its *Gödel number* X_0 to be the number (numeral) obtained from X by substituting $g_1, g_2, \cdots, g_9$ respectively for the 1st, 2nd, $\cdots$, 9th symbols of U. [E. g. the Gödel number of v' 2 is 122212222122.] Our Gödel numbering has the technical advantage of being an isomorphism with respect to concatenation—i. e. for any strings X, Y in K_9, the Gödel number (XY) of XY is simply $X_0 Y_0$ (i. e. X_0 *followed by* Y_0). For any set W of strings in K_9 we let W_0 be the corresponding set of Gödel numbers. Thus, e. g., T_0 is the set of Gödel numbers of all *true* sentences of U.

The set T is formally representable over K_9 and the set T_0 is r. e. (we prove this in the appendix). We now wish to show that the system U is recursively unsolvable, in the sense that T_0 is not a *recursive* set. To show that the complement $\tilde{T}_0$ of T_0 is not recursively enumerable, we employ the following modification of Gödel's well known diagonalization agreement.

For any string X in K_9 we define its *norm* to be the string XX_0—i. e. X followed by its own Gödel number (numeral). [We might note that if X is a predicate of U of degree 1, then the norm of X is a sentence which is true in U iff the Gödel number of X lies in the set represented by X.][9] Every number n (looked at as a dyadic numeral) itself has a Gödel number n_0 (e. g. the Gödel number of 121 is 1212212), and hence also a norm $n\,n_0$. For any number set A we define A^* to be the set of all numbers whose norm is in A. Thus $n \in A^*$ iff $n\,n_0 \in A$.

Lemma. *If A is r. e., so is A^*.*

Proof. Let (E) be an elementary formal system over D in which the predicate P represents A. We add a new unary predicate Q and a new binary predicate G and the new axioms:

$G1, 12$

$G2, 122$

$Gx, y \rightarrow Gz, w \rightarrow Gxz, yw$

$Gx, y \rightarrow Pxy \rightarrow Qx$

[x, y, z, w are variables]

Then G represents the set of all ordered pairs (x, y) such that $x_0 = y$, and Q represents A^*.

Proposition 1. *For every r. e. set A there is a sentence X such that X is true (in U) iff its Godel number X_0 is in A. (and hence also: $X_0 \in T_0 \leftrightarrow X_0 \in A$).*

Proof. Let A be r. e. Then so is A^* by the above lemma. Then A^* is represented in U by some predicate H. Then for every number n,

9) This notion of "norm" plays here a role quite analogous to that in [4].

$$Hn \text{ is true} \leftrightarrow n \in A^*$$
$$\leftrightarrow nn_0 \in A.$$

Setting $n=h$, where h is the Gödel number of H, we have:

$$Hh \text{ is true} \leftrightarrow hh_0 \in A.$$

However hh_0 is the Gödel number of the sentence Hh. Therefore Hh is true iff its own Gödel number is in A.

Proposition 1 says, in effect, that every r. e. set A either contains some element of T_0 or lacks some element outside T_0. Since the complement $\tilde{T}_0$ of T_0 cannot possibly have this property, it immediately follows that $\tilde{T}_0$ is not r. e. We hence have

THEOREM 1. [After Post's form of Church's theorem]. *The system U is not recursively solvable.*

REMARKS. Theorem 1 means (intuitively) that there exists no "mechanical" procedure to determine which numbers are in which recursively enumerable sets. Given any stronger system (S)—i. e., one in which it is possible to "effectively" translate all sentences of U into sentences of (S) in such a manner as to preserve both truth and falsity—it can be shown that (S) in turn must be "undecidable". This approach has been utilized (for other formulations of U) to establish Gödel's incompleteness theorem.

The sentence Hh constructed in our proof of Proposition 1 is a highly simplified variant of Gödel's famous sentence which "refers" to its own Gödel number; it can be thought of as expressing the proposition that its own Gödel number is in the set A. It was our purpose to capture the crucial ideal behind Gödel's construction utilizing a bare minimum of formal machinery.

Appendix

Formal representability of T and T_0

In this appendix we prove that the set T of true sentences of U is formally representable over K_9 and that T_0 is r. e. We shall represent T in an elementary formal system (E) over K_9. We first note that the implication sign of (E) is to be distinct from the implication sign of transcribed systems. We could continue to use "$\rightarrow$" for the latter; we prefer however to use "$\rightarrow$" to denote the implication sign of (E) (since it will occur so frequently), and we shall now denote the implication sign of transcribed systems by "imp". Similarly we shall now use the ordinary comma for our punctuation sign of (E), and "com" for the punctuation sign of transcribed systems. Variables of (E) (not to be confused with transcribed variables) will be denoted by "x", "y", "z", "w", with or without subscripts. Predicates of (E) (not to be confused either with predicates of U or transcribed predicates) will be introduced as needed.

We now introduce the axioms of (E) in groups, first explaining what each newly introduced predicate of (E) is to represent.

N represents the set of numbers (dyadic numerals).

$N1$
$N2$
$Nx \rightarrow Ny \rightarrow Nxy$

Acc represents the set of strings of accents

Acc'
$Accx \rightarrow Accx'$

V represents the set of transcribed variables.

$Accx \rightarrow Vvx$

P represents the set of transcribed predicates.

$Accx \rightarrow Ppx$
$Px \rightarrow Ppx$

t represents the set of trancribed terms.

$Nx \rightarrow tx$
$Vx \rightarrow tx$
$tx \rightarrow ty \rightarrow txy$

F_0 represents the set of transcribed atomic formulas.

$Accx \rightarrow ty \rightarrow F_0pxy$
$F_0x \rightarrow ty \rightarrow F_0px \,\mathrm{com}\, y$

F represents the set of transcribed formulas.

$F_0x \rightarrow Fx$
$F_0x \rightarrow Fy \rightarrow Fx \,\mathrm{imp}\, y$

dv represents the relation "x and y are distinct transcribed variables".

$vx \rightarrow Accy \rightarrow dvx, xy$
$dvx, y \rightarrow dvy, x$

S represents the set of all quadruples (x, y, z, w) such that x is any string (well formed or not) which is compounded from numerals, transcribed variables, transcribed predicates, com, imp; y is a transcribed variable, z is a numeral, and w is the result of substituting z for all occurrences of y in x (that is, all occurrences which are not immediately followed by more accents).

$Nx \rightarrow Vy \rightarrow Nz \rightarrow Sx, y, z, x$
$Vx \rightarrow Nz \rightarrow Sx, x, z, z$
$dv\, x, y \rightarrow Nz \rightarrow Sx, y, z, x$
$Px \rightarrow Vy \rightarrow Nz \rightarrow Sx, y, z, x$
$Vy \rightarrow Nz \rightarrow S\,\mathrm{com}, y, z, \mathrm{com}$
$Vy \rightarrow Nz \rightarrow S\,\mathrm{imp}, y, z, \mathrm{imp}$
$Sx, y, z, w \rightarrow Sx_1, y, z, w_1 \rightarrow Sxx_1, y, z, ww_1$

T represents the set of true sentences of U

$$Fx \to Tx \vdash x$$
$$Tx \vdash y \to Fz \to Tx{*}z \vdash y$$
$$Tx \vdash y \to Fz \to Tz{*}x \vdash y$$
$$Tx \vdash y \to Sy, z_1, z_2, w \to tx \vdash w$$
$$Tx \vdash y \to Tx \vdash y \text{ imp } z \to F_0 y \to Tx \vdash z$$

This completes the construction of the system (E) in which T is represented. To represent T_0 over $\{1, 2\}$, just take all the above axioms and replace each symbol of K_9 by its Gödel number.

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