ELEMENTARY SOLUTIONS FOR CERTAIN PARABOLIC PARTIAL DIFFERENTIAL EQUATIONS(1)

BY

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1. **Introduction.** Let S be an open interval(2) (s_1, s_2), not necessarily bounded above or below; let $c(x)$ be continuous on S and $\leqq 0$; let $m(dx)$ be a Borel measure, strictly positive on open subsets of S; and let $D(\mathfrak{B})$ be those continuous functions $u(x)$ on S to the real numbers R such that the one-sided derivatives,

$$u^+(x) = \lim_{\epsilon \downarrow 0} \epsilon^{-1}(u(x+\epsilon) - u(x))$$

and

$$u^+(dx)/m(dx) = \lim_{\epsilon \downarrow 0} m(x, x+\epsilon]^{-1}(u^+(x+\epsilon) - u^+(x)),$$

exist, the second being continuous on S, so that the operator,

$$\mathfrak{B}: u \in D(\mathfrak{B}) \to u^+(dx)/m(dx) + c(x)u(x), \tag{1.1}$$

is linear on the domain $D(\mathfrak{B})$ to continuous functions. Such operators have been characterised by W. Feller [1] and are important for the description of certain random processes: see W. Feller [2]. The classical second order operator,

$$\mathfrak{B} = a(x)d^2/dx^2 + b(x)d/dx + c(x), \tag{1.2}$$

is a special case, provided a, b, and c are continuous on S, $a(x) > 0$, and $c(x) \leqq 0$: see [1, p. 95].

The purpose of this paper is to construct (1) the spectral representation of the operator (1.1), acting in a suitable Hilbert space, and (2) the elementary solution for the parabolic partial differential equation,

$$u_t(t, x) = \mathfrak{B}u(t, x), \qquad t > 0, \tag{1.3}$$

subject to certain (classical) side conditions, to be described in §2.

Concerning (1): since (1.1) is a generalized Sturm-Liouville operator, it is not surprising that it can be represented by an eigendifferential expansion like those introduced by H. Weyl [3] in his studies of the classical Sturm-Liouville operator and perfected by M. H. Stone [4], E. C. Titchmarsh [5],

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(2) The convention that 0 belong to S is harmless and will be understood below.

and K. Kodaira [6]. These writers place no restriction on the behaviour of c save that it be continuous. Here, we take it $\leqq 0$, which makes the appropriate contractions of $\mathfrak{B}$ negative semi-definite and permits a simple, completely real construction for the eigendifferential expansions.

Concerning (2): let $C(S)$ be the space of bounded continuous functions $u(x)$ on S to R, provided with the customary norm, $\|u\|_\infty = \sup |u(x)|$, let $C(\mathfrak{B})$ be those $u \in D(\mathfrak{B}) \cap C(S)$ such that $\mathfrak{B}u \in C(S)$ once more, and let B be a *classical side condition*, that is, the collection of those $u \in C(\mathfrak{B})$ which satisfy a pair of *classical boundary conditions*,

$$(1/3)u(s_1) + (2/3)u^+(s_1) = 0 \quad \text{and} \quad u(s_2) = 0,$$

or the like. Given such a side condition, we shall construct the *elementary solution* $p(t, x, s)(t > 0, x \in S, s \in S)$ to (1.3) and show that it has the properties listed below:

E1. $p(t, \cdot, \cdot)$ is positive and symmetric on $S \times S$ for each $t > 0$.

E2. The derivatives $\partial^n/\partial t^n p(t, \cdot, s)$ belong to the side condition B and

$$\partial^n/\partial t^n p(t, \cdot, s) = \mathfrak{B}^n p(t, \cdot, s), \qquad t > 0,\ s \in S,\ n \geqq 0.$$

E3. $\int_S p(t, x, s) m(ds) \leqq 1$, $t > 0$, $x \in S$.

E4. The Chapman-Kolmogorov identity,

$$p(t_1 + t_2, x, s) = \int_S p(t_1, x, \xi) p(t_2, \xi, s) m(d\xi),$$

is satisfied.

E5. The operators,

$$S_t : u \in C(S) \to \int_S p(t, x, s) u(s) m(ds), \qquad t > 0,$$

constitute a semi-group, mapping $C(S)$ into $C(S)$, and one has

$$\partial^n/\partial t^n (S_t u)(x) = \mathfrak{B}^n (S_t u)(x), \quad t > 0,\ u \in C(S),\ n > 0.$$

E6. Given $v \in C(S)$ such that $\mathfrak{B}v$ is continuous near $x \in S$, one has

$$(S_t v)(x) = v(x) + t(\mathfrak{B}v)(x) + o(t), \qquad (t \downarrow 0).$$

REMARK 1.1. Elementary solutions for the classical operator (1.2), subject to a variety of classical side conditions, have been constructed by J. Elliott [7] and E. Hille [8]. Also, W. Feller [9] has constructed the elementary solution for (1.2) with *time dependent* coefficients, subject to the uniqueness condition,

$$\int_{x<0} a(t, x)^{-1/2} dx = \int_{x>0} a(t, x)^{-1/2} dx = +\infty, \qquad t > 0.$$

Our plan of attack is this: let B be a classical side condition, let $p(t, x, s)$ be the corresponding elementary solution, and choose $v \in C(S)$. Keeping E4 and E5 in mind, it is clear that the transform,

$$w = \int_0^{+\infty} e^{-\mu t}(S_t v)(x)dt$$
$$(1.4) \qquad = \int_S \int_0^{+\infty} e^{-\mu t} p(t, x, s) dt v(s) m(ds), \qquad \mu > 0,$$

should belong to B and that we should have $(\mu - \mathfrak{B})w = v$, or, what is just about the same, that the kernel,

$$(1.5) \qquad G(\mu, x, s) = \int_0^{+\infty} e^{-\mu t} p(t, x, s) dt, \qquad \mu > 0,$$

should be the *Green function* for the problem,

$$(1.6) \qquad (\mu - \mathfrak{B})w = v, \qquad w \in \mathrm{B},\ v \in C(S),$$

and this shows that, to *construct* the elementary solution, it is a good bet to construct the Green function for (1.6) and to invert (1.5).

§2 contains the precise description of the classical side conditions, the construction of the Green function, and various properties of the *Green operators*,

$$(1.7) \qquad \mathfrak{G}_\mu : v \in C(S) \to \int_S G(x, s, \mu) v(s) m(ds), \qquad \mu > 0,$$

which we shall need in §§3 and 4.

§3 contains the eigendifferential expansions for the Green operators, the Green function, and the operator $\mathfrak{B}$, viewed in the appropriate Hilbert space. These are constructed by approximating the proper Green operators by suitable compact Green operators. Similar approximations have been made by N. Levinson [10; 11] and K. Yosida [12; 13].

§4 begins with the remark that the eigendifferential expansion for the Green function suggests a simple way to compute the inverse transform,

$$(1.8) \qquad p(t, x, s) = (2\pi i)^{-1} \int_{\epsilon - i\infty}^{\epsilon + i\infty} e^{t\mu} G(x, s, \mu) d\mu, \qquad t, \epsilon > 0,$$

and it is shown that $p(t, x, s)$ is the elementary solution to (1.3), subject to the going side condition and enjoying the properties listed in E1, E2, $\cdots$, E6 above.

REMARK 1.2. S. Karlin and J. McGregor [14] and W. Ledermann and G. Reuter [15] have constructed the elementary solutions for certain parabolic equations, $u_t = \mathfrak{B}u$, in which $\mathfrak{B}$ is a *difference* operator, and have obtained the appropriate eigendifferential expansions. Karlin and McGregor have announced similar results for certain *differential* operators which will be published soon(3).

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2. **Side conditions and Green functions.** Let $\mathfrak{B}$ be the operator (1.1), acting on the domain $D(\mathfrak{B})$, let $m(dx)$ be the measure involved in $\mathfrak{B}$, let $n(dx)$ be the measure $(1-c(x))m(dx)$, set

$$J_i = (-)^i \int_0^{s_i} n(0, x)dx \quad \text{and} \quad K_i = (-)^i \int_0^{s_i} x\, n(dx) \quad (i = 1, 2),$$

and, imitating W. Feller [16, p. 487], let us make the

BOUNDARY CLASSIFICATION. Given $i=1$ or 2, the boundary s_i is said to be

regular	if $J_i < +\infty$	and	$K_i < +\infty$;
exit	if $J_i < +\infty$	but	$K_i = +\infty$;
entrance	if $J_i = +\infty$	but	$K_i < +\infty$;
natural	if $J_i = +\infty$	and	$K_i = +\infty$(4).

REMARK. H. Weyl's Grenzpunkt-Grenzkreis boundary classification [3, pp. 223–228] can be made, but will not be necessary below. The regular boundaries correspond to the Grenzkreisfalle always, but the exit, entrance, and natural boundaries can exhibit both kinds of behaviour: see J. Elliott [7, pp. 408–409].

Keeping the Boundary Classification in mind, choose numbers p_1 and $p_2 \in [0, 1]$, take $i=1$ or 2, and let B_i be those $v \in D(\mathfrak{B})$ such that(5)

$$\begin{aligned} (1 - p_i)v(s_i) + (-)^i p_i v^+(s_i) &= 0, && s_i \text{ regular}, \\ v(s_i) &= 0, && s_i \text{ exit}, \\ v^+(s_i) &= 0, && s_i \text{ entrance}, \\ v(s) \text{ is bounded near } s_i&, && s_i \text{ natural}. \end{aligned}$$

These manifolds are the *classical boundary conditions*, and the intersections,

$$B = B_1 \cap C(\mathfrak{B}) \cap B_2,$$

are the *classical side conditions*. We find it convenient to distinguish the

(3) [Private communication.]

(4) This neat description of the boundary classification was suggested to me by K. Itô [private communication].

(5) Here and below, the numbers, $v(s_i)$ and $v^+(s_i)$, are the boundary values, $\lim_{s\to s_i} v(s)$ and $\lim_{s\to s_i} v^+(s)$ $(i=1, 2)$.

minimal boundary conditions, B_1^0 and B_2^0, which correspond to $p_1=0$ and $p_2=0$.

The next step is to choose a side condition B and to construct the Green function for (1.6). Given an operator (1.2), this is classical, and, since the calculations necessary here differ so little from the classical ones, we shall be content to have a sketch: compare W. Feller [16, pp. 482–493] and E. Hille [8, pp. 104–124].

Given $\mu \in R$, the *homogeneous equation*,

$$\mathfrak{B}v = \mu v, \qquad v \in D(\mathfrak{B}),\ \mu > 0, \tag{2.1}$$

is equivalent to the integral equation,

$$\begin{aligned} v(x) - v(0) - xv^+(0) &= \int_0^x ds \int_{(0,s)} (\mu - c)v\,m(dt) \qquad (x > 0) \\ &= \int_x^0 ds \int_{(s,0]} (\mu - c)v\,m(dt) \qquad (x \leqq 0), \end{aligned} \tag{2.2}$$

and the classical iterative procedure shows that (2.2) has (1) two independent solutions and (2) one and only one solution v with prescribed $v(0)$ and $v^+(0)$. We find it convenient to distinguish the two independent solutions, $e_1(\cdot, \mu)$ and $e_2(\cdot, \mu)$, specified by the conditions,

$$e_1(0, \mu) = 1, \qquad \overset{+}{e_1}(0, \mu) = 0, \qquad e_2(0, \mu) = 0, \qquad \overset{+}{e_2}(0, \mu) = 1,$$

and to remark that $e_1(x, \cdot)$ and $e_2(x, \cdot)$ are continuous on $R(x \in S)$.

Given $\mu > 0$, (2.1) has a positive strictly increasing solution $u_1(\cdot, \mu) \in B_1$, a positive strictly decreasing solution $u_2(\cdot, \mu) \in B_2$, these solutions are independent, and no solution belonging to B_1 is independent of $u_1(\cdot, \mu)$, no solution belonging to B_2 is independent of $u_2(\cdot, \mu)$. Moreover, the Wronskian, $u_1^+ u_2 - u_1 u_2^+$, is constant(6) and positive on S, and we can suppose that it is $=1$. The behaviour of these solutions near the boundary s_2 is exhibited in Table 1. We note that to describe the behaviour of $u_1(\cdot, \mu)$ and $u_2(\cdot, \mu)$ near the boundary s_1, we have merely to interchange the subscripts 1 and 2 and to substitute

$$-u^+ \quad \text{and} \quad \int_{s_1}^0 (\cdot) m(dx) \quad \text{for } u^+ \text{ and} \quad \int_0^{s_2} (\cdot) m(dx).$$

To continue, set the contraction $\mathfrak{B}/B = \mathfrak{G}$ and make up the Green function, according to the classical prescription,

$$\begin{aligned} G(x, s, \mu) &= u_1(x, \mu) u_2(s, \mu) \qquad (x \leqq s) \\ &= u_1(s, \mu) u_2(x, \mu) \qquad (x > s). \end{aligned} \tag{2.3}$$

(6) This fact is due to W. Feller [1, p. 105].

TABLE 1

	Regular	Exit	Entrance	Natural
$u_1(s_2, \mu)$	$<+\infty$	$<+\infty$	$=+\infty$	$=+\infty$
$u_2(s_2, \mu)$	$=0 \quad p_2=0$ $>0 \quad p_2>0$	$=0$	>0	$=0$
$u_1^+(s_2, \mu)$	$<+\infty$	$=+\infty$	$<+\infty$	$=+\infty$
$u_2^+(s_2, \mu)$	$<0 \quad p_2<1$ $=0 \quad p_2=1$	<0	$=0$	$=0$
$\int_0^{s_2} u_1 m(dx)$	$<+\infty$		$=+\infty$	
$\int_0^{s_2} u_2 m(dx)$	$<+\infty$	$<+\infty$	$<+\infty$	$<+\infty$

Then the Green operator,

$$(2.4) \qquad \mathfrak{G}_\mu : v \in C(S) \to \int_S G(x, s, \mu) v(s) m(ds), \qquad \mu > 0,$$

maps $C(S)$ onto B and satisfies $(\mu - \mathfrak{B})\mathfrak{G}_\mu = 1$, being, in short, inverse to the operator $(\mu - \mathfrak{G})$,

$$(2.5) \qquad \mu \|\mathfrak{G}_\mu\|_\infty \leqq 1, \qquad \mu > 0,$$

the resolvent equation,

$$(2.6) \qquad \mathfrak{G}_\mu - \mathfrak{G}_\lambda + (\mu - \lambda)\mathfrak{G}_\mu \mathfrak{G}_\lambda = 0, \qquad \mu, \lambda > 0,$$

is satisfied, and, combining (2.5) and (2.6) and making a short computation, one obtains

$$(2.7) \qquad d^n/d\mu^n \mathfrak{G}_\mu = (-)^n n! \mathfrak{G}_\mu^{n+1}, \qquad n, \mu > 0,$$

the derivatives being taken in the normed $C(S)$ operator topology: compare E. Hille [17, pp. 99 and 110]. Notice that (2.7) is equivalent to the classical Green function identity,

$$(2.8) \qquad \begin{aligned} \partial^n/\partial\mu^n G(x, s, \mu) = (-)^n n! \int_{S \times \cdots \times S} & G(x, \xi_1, \mu) G(\xi_1, \xi_2, \mu) \\ & \cdots G(\xi_n, s, \mu) m(d\xi_1) m(d\xi_2) \cdots m(d\xi_n), \qquad n, \mu > 0. \end{aligned}$$

We shall need to know that the side condition B is not inconveniently small. The necessary information is contained in the

APPROXIMATION THEOREM. *Let $C_0(S)$ be those $v \in C(S)$ vanishing near the boundaries and let $C_0(\mathfrak{B})$ be the intersection $C(\mathfrak{B}) \cap C_0(S)$. Given $\epsilon > 0$ and a positive $v \in C_0(S)$, there is a positive $v_\epsilon \in C_0(\mathfrak{B})$, vanishing outside the support of v and such that*

$$\|v - v_\epsilon\|_\infty < \epsilon.$$

Proof. The construction is straightforward and will not be reproduced.

One more remark and this will be convenient below:

$$(2.9) \qquad (\mathfrak{G}_\mu v)^+(x) = \int_S \partial^+/\partial x G(x, s, \mu) v(s) m(ds) \text{(}^7\text{)}, \qquad v \in C(S),\ \mu > 0.$$

3. **Eigendifferential expansions.** Let $L_2(S)$ be the space of m-measurable functions v on S to R such that $\|v\|_2 = (\int_S v(x)^2 m(dx))^{1/2} < +\infty$, modulo the ideal of null functions, let $(u, v) = \int_S u(x)v(x)m(dx)$ be the inner product, and consider the new Green operators,

$$(3.1) \qquad \mathfrak{E}_\mu : u \in L_2(S) \to \int_S G(x, s, \mu) u(s) m(ds), \qquad \mu > 0.$$

These have a sense because $\|G(x, \cdot, \mu)\|_2 < +\infty$ $(x \in S,\ \mu > 0)$, and, keeping the norm condition (2.5) in mind, we have

$$\begin{aligned} \|\mathfrak{E}_\mu u\|_2^2 &= \int_S m(dx) \left(\int_S G(x, s, \mu)^{1/2} G(x, s, \mu)^{1/2} u(s) m(ds) \right)^2 \\ &\leqq \int_S m(dx) (\mathfrak{G}_\mu 1)(x) \int_S G(x, s, \mu) u(s)^2 m(ds) \\ &\leqq \mu^{-2} \|u\|_2^2, \qquad \mu > 0,\ u \in L_2(S), \end{aligned}$$

so that

$$(3.2) \qquad \mu \|\mathfrak{E}_\mu\|_2 \leqq 1, \qquad \mu > 0.$$

Also, it is clear that these operators are solutions to (2.6), and, as such, have a common range Q and a common null-space N. Choose $u \in N$. Since $G(\cdot, \cdot, \mu)$ is symmetric on $S \times S$, the operator $\mathfrak{E}_\mu$ is symmetric,

$$(\mathfrak{E}_\mu u, v) = (u, \mathfrak{G}_\mu v) = 0, \qquad v \in C_0(S),$$

and, noticing that such $\mathfrak{G}_\mu v$ are dense in $L_2(S)$ by the Approximation Theorem of §2, we see that $u = 0$, that N is trivial, and that our new Green operators are one-one. But more than this: since (2.7) is a consequence of (2.6), we have

(7) Here, $\partial^+/\partial x$ stands for the obvious one-sided partial derivative.

$$(3.3) \qquad (\mathfrak{G}_\mu u, u) = \int_\mu^{+\infty} \|\mathfrak{G}_\lambda u\|_2^2 d\lambda, \qquad \mu > 0, u \in L_2(S),$$

which together with the remark just above, shows that our new Green operators are, in addition, positive definite.

To continue, choose $\mu > 0$ and let $\mathfrak{G}$ be the operator $\mu - \mathfrak{G}_\mu^{-1}$, acting on the common range Q. Simple computations, based on (2.6), show that $\mathfrak{G}$ is independent of μ and satisfies

$$(3.4) \qquad (\mu - \mathfrak{G})\mathfrak{G}_\mu = 1, \qquad \mu > 0,$$

and it is clear that $\mathfrak{G}$ is maximal symmetric and negative semi-definite, mapping Q onto $L_2(S)$.

The subject of this section is the spectral structure of the operator $\mathfrak{G}$ and the new Green operators (3.1).

REMARK 3.1. The precise description of the domain Q will not be necessary below, but we shall sketch it here. Let $D(\mathfrak{Q})$ be those m-measurable u on which the operator,

$$\mathfrak{Q}: u \to \mathfrak{Q}u = u^+(dx)/m(dx) + c(x)u(x),$$

makes sense, the derivatives being taken in the appropriate sense, and let $L_2(\mathfrak{Q})$ be those $u \in D(\mathfrak{Q}) \cap L_2(S)$ such that $\mathfrak{Q}u$ belongs to $L_2(S)$ once more. Given $i = 1$ or 2, let I be the interval (min $(0, s_i)$, max $(0, s_i)$), and let Q_i be those $u \in D(\mathfrak{Q})$ such that

$$\begin{aligned} u &\in L_2(I), & m(I) &= +\infty, \\ u^+(s_i) &= 0, & m(I) &< +\infty, \end{aligned} \qquad s_i \text{ natural},$$

and $\qquad u \in B_i \qquad$ otherwise.

Then, the domain Q is the intersection, $Q_1 \cap L_2(\mathfrak{Q}) \cap Q_2$, and the operator $\mathfrak{G}$ is the contraction $\mathfrak{Q}/Q$.

Getting to work, let us begin with a special case.

THEOREM 3.1. *When no natural boundaries are present, the Green operators (3.1) are compact, and there exist numbers $0 \geqq \mu_1 > \mu_2 > \cdots \downarrow -\infty$ and a basis $(v_i: i > 0)$ in $L_2(S)$ such that*

$$\mathfrak{G}_\mu = \sum_{i>0} (\mu - \mu_i)^{-1} v_i \otimes v_i, \qquad \mu > 0,$$

the operators $v_i \otimes v_i$ being the projections, $v_i \otimes v_i$: $u \in L_2(S) \to v_i(u, v_i)$. The numbers μ_i are the common eigenvalues and the functions v_i the common eigenfunctions of the operators $\mathfrak{S}$ and $\mathfrak{G}$, that is, $v_i \in B \cap Q$ and $\mathfrak{S}v_i = \mu_i v_i = \mathfrak{G}v_i$ for each $i > 0$. Moreover, the Green function is represented by the eigenfunction expansion,

$$G(x, s, \mu) = \sum_{i>0} (\mu - \mu_i)^{-1} v_i(x) v_i(s), \qquad \mu > 0,$$

the sum converging uniformly on compact squares in $S\times S$.

Proof. This has been proved for the classical operator (1.2), subject to the condition that $c(x)$ be continuous on the closed interval $[s_1, s_2]$, by J. Elliott [7, pp. 412–417], and the calculations necessary here are much the same. Actually, she shows that the traces,

$$\operatorname{tr}(\mathfrak{G}_\mu) = \int_S G(s, s, \mu)m(ds) = \sum_{i>0} (\mu - \mu_i)^{-1} \tag{3.5}$$

are $< +\infty$, which is stronger than the compactness statement, but this will not be necessary below.

Dropping the condition that no natural boundaries be present, the Green operators are not necessarily compact, and the eigenfunction expansion for the Green function may not have a sense, but it has a counterpart, namely the eigendifferential expansion advertised in §1, to which we now turn.

THEOREM 3.2. *There is a Borel measure* $\mathfrak{f}(\cdot)$ *on* $(-\infty, 0]$ *to* 2×2, *symmetric, positive semi-definite matrices, such that, if* $\mathfrak{e}(\cdot, \mu)$ *is the* 2-*vector whose entries are the solutions,* $e_1(\cdot, \mu)$ *and* $e_2(\cdot, \mu)$, *to* (2.1) *and if the measure* $e(x, s, d\mu)$ *is the inner product* $\mathfrak{e}(x, \mu)\mathfrak{f}(d\mu)\mathfrak{e}(s, \mu)$ *of the vectors* $\mathfrak{e}(x, \mu)$ *and* $\mathfrak{f}(d\mu)\mathfrak{e}(s, \mu)$, *then*

$$G(x, s, \lambda) = \int_{-\infty}^{0+} (\lambda - \mu)^{-1} e(x, s, d\mu), \qquad \lambda > 0,$$

the integral converging uniformly on compact squares in $S\times S$.

Proof. Let $S_1 \subset S_2 \subset \cdots$ be open intervals in S such that $\bigcup_{n>0} S_n = S$, each containing 0, coinciding with S near the regular, exit, and entrance boundaries, and falling strictly short of the natural boundaries, and set $S_n = (s_{1n}, s_{2n})$. Let $\mathfrak{B}_n$ be the operator $\mathfrak{B}$ cut down to S_n, let $C(\mathfrak{B}_n)$ be those $u \in C(S_n) \cap D(\mathfrak{B}_n)$ such that $\mathfrak{B}_n u$ belongs to $C(S_n)$ once more, let B_{in} be the minimal boundary condition $D(\mathfrak{B}_n)(v\colon v(s_{in}) = 0)$ when s_i is a natural boundary and let B_{in} express the boundary condition expressed in B_i when s_i is not a natural boundary, let B_n be the side condition $B_{1n} \cap C(\mathfrak{B}_n) \cap B_{2n}$, and let $G_n(\cdot, \cdot, \cdot)$, $w_1(\cdot, \cdot)$, and $w_2(\cdot, \cdot)$ be the corresponding Green function and positive solutions to (2.1).

Take $n > 0$. Then S_n has no natural boundaries, and, invoking Theorem 3.1, we can represent the Green function $G_n(\cdot, \cdot, \cdot)$ by the eigenfunction expansion,

$$G_n(x, s, \lambda) = \sum_{i>0} (\lambda - \mu_i)^{-1} v_i(x) v_i(s), \qquad \lambda > 0, \tag{3.6}$$

the numbers μ_i and the functions v_i being the appropriate eigenvalues and eigenfunctions. Since $\mathfrak{B}_n v_i = \mu_i v_i$ and since the entries of $\mathfrak{e}(\cdot, \mu_i)$ are independent solutions of this equation, there exist 2×2 matrices $\mathfrak{f}(\mu_i)$ such that

$$v_i(x)v_i(s) = \mathfrak{e}(x, \mu_i)\mathfrak{f}(\mu_i)\mathfrak{e}(s, u_i) = e(x, s, \mu_i), \qquad i > 0.$$

Choose a Borel set M in $(-\infty, 0]$ and set

$$\mathfrak{f}(M) = \sum_{\mu_i \in M} \mathfrak{f}(\mu_i).$$

Then the entries of $\mathfrak{f}(M)$ are

$$f_{11}(M) = \sum_{\mu_i \in M} v_i(0)^2 \geqq 0,$$

$$f_{12}(M) = \sum_{\mu_i \in M} v_i(0)v_i^+(0) = f_{21}(M),$$

$$f_{22}(M) = \sum_{\mu_i \in M} v_i^+(0)^2 \geqq 0,$$

the inequality,

$$f_{12}(M)^2 = f_{21}(M)^2 \leqq f_{11}(M)f_{22}(M), \tag{3.7}$$

is satisfied, and $\mathfrak{f}(\cdot)$ is a Borel measure on $(-\infty, 0]$ to 2×2, symmetric, positive semi-definite matrices.

The expansion (3.6) can now be written

$$G_n(x, s, \lambda) = \int_{-\infty}^{0+} (\lambda - \mu)^{-1} e(x, s, d\mu), \qquad \lambda > 0. \tag{3.8}$$

We wish to construct the same sort of expansion for the Green function $G(\cdot, \cdot, \cdot)$, but this requires some estimates, namely,

$$\int_{-\infty}^{0+} (\lambda - \mu)^{-2} f_{11}(d\mu) \leqq \lambda^{-1} G_n(0, 0, \lambda), \tag{3.9}$$

$$\int_{-\infty}^{0+} (\lambda - \mu)^{-2} \left| f_{12}(d\mu) \right| = \int_{-\infty}^{0+} (\lambda - \mu)^{-2} \left| f_{21}(d\mu) \right| \leqq \lambda^{-1}, \tag{3.10}$$

and

$$\int_{-\infty}^{0+} (\lambda - \mu)^{-2} f_{22}(d\mu) \leqq \lambda^{-1} G_n(0, 0, \lambda)^{-1}. \tag{3.11}$$

The estimate (3.9) is the simplest to come by. We have

$$\int_{-\infty}^{0+} (\lambda - \mu)^{-2} f_{11}(d\mu) = \sum_{i>0} (\lambda - \mu_i)^{-2} v_i(0)^2,$$

and, observing that

$$(\lambda - \mu_i)^{-1} v_i(x) = \int_{S_n} G_n(x, s, \lambda) v_i(s) m(ds), \qquad i > 0, \tag{3.12}$$

we see that this is

$$= \sum_{i>0}\left(\int_{S_n} G_n(0, s, \lambda)v_i(s)m(ds)\right)^2 = \int_{S_n} G_n(0, s, \lambda)^2 m(ds)$$

$$\leqq G_n(0, 0, \lambda)\int_{S_n} G_n(0, s, \lambda)m(ds) \leqq \lambda^{-1}G_n(0, 0, \lambda).$$

Coming to (3.11), we invoke (2.9) and (3.12) and obtain

$$\begin{aligned}\int_{-\infty}^{0+} (\lambda - \mu)^{-2} f_{22}(d\mu) &= \sum_{i>0} (\lambda - \mu_i)^{-2} v_i^+(0)^2 \\ &= \sum_{i>0}\left(\int_{S_n} \partial^+/\partial x G_n(0, s, \lambda)v_i(s)m(ds)\right)^2 \\ &= \int_{S_n} (\partial^+/\partial x G_n(0, s, \lambda))^2 m(ds) \\ &= w_2^+(0, \lambda)^2 \int_{(s_{1n}, 0]} w_1^2 m(ds) \\ &\quad + w_1^+(0, \lambda)^2 \int_{(0, s_{2n})} w_2^2 m(ds)\end{aligned}$$

$\leqq$ Wronskian (w_1, w_2).

$$\left[w_2(0, \lambda)^{-1}\int_{(s_{1n}, 0]} w_1 m(ds) + w_1(0, \lambda)^{-1}\int_{(0, s_{2n})} w_2 m(ds)\right] \leqq \lambda^{-1}G_n(0, 0, \lambda)^{-1},$$

and now to prove (3.10), we have only to remember (3.7), (3.9), and (3.11), and to notice that, by Schwarz's inequality,

$$\begin{aligned}\int_{-\infty}^{0+} (\lambda - \mu)^{-2} |f_{12}(d\mu)| &= \int_{-\infty}^{0+} (\lambda - \mu)^{-2} |f_{21}(d\mu)| \\ &\leqq \left(\int_{-\infty}^{0+} (\lambda - \mu)^{-2} f_{11}(d\mu)\right)^{1/2}\left(\int_{-\infty}^{0+} (\lambda - \mu)^{-2} f_{22}(d\mu)\right)^{1/2} \leqq \lambda^{-1}.\end{aligned}$$

The next step is to make $n \uparrow +\infty$. To avoid confusion, set $\mathfrak{f}(\cdot) = \mathfrak{f}_n(\cdot)$ and $e(\cdot, \cdot, \cdot) = e_n(\cdot, \cdot, \cdot)$. A simple calculation shows that the Green functions $G_n(\cdot, \cdot, \lambda)$ converge to the original Green function $G(\cdot, \cdot, \lambda)(n \uparrow +\infty)$ uniformly on compact squares in $S \times S$ for each $\lambda > 0$, and, combining this with the estimates (3.9), (3.10), and (3.11) and the weak-star compactness of measures, it is clear that we can choose a Borel measure $\mathfrak{f}(\cdot)$ on $(-\infty, 0]$ to 2×2, symmetric, positive semi-definite matrices, compact intervals $R_1 \subset R_2 \subset \cdots$ such that $\bigcup_{n>0} R_n = (-\infty, 0]$, and open intervals $S_1 \subset S_2 \subset \cdots \subset S$, like those described above, such that $\int_{R_j} v(\mu)\mathfrak{f}_n(d\mu) \to \int_{R_j} v(\mu)\mathfrak{f}(d\mu)$ $(n \uparrow +\infty)$ for each $j>0$ and each $v \in C(R_j)$.

Let us make these choices. Then, keeping the positive semi-definite character of $\mathfrak{f}(\cdot)$ in mind and setting $\mathfrak{e}(x, \mu)\mathfrak{f}(d\mu)\mathfrak{e}(s, \mu) = e(x, s, d\mu)$, we have

$$\begin{aligned}\int_{-\infty}^{0+} (\lambda - \mu)^{-1} e(x, x, d\mu) &= \lim_{j \uparrow +\infty} \lim_{n \uparrow +\infty} \int_{R_j} (\lambda - \mu)^{-1} e_n(x, x, d\mu) \\ &\leqq \lim_{n \uparrow +\infty} G_n(x, x, \lambda) \\ &= G(x, x, \lambda) < +\infty, \qquad \lambda > 0, x \in S,\end{aligned}$$

and, by Schwarz's inequality,

$$\begin{aligned}\int_{\mu \notin R_j} (\lambda - \mu)^{-1} \left| e(x, s, d\mu) \right| &\leqq \left(\int_{\mu \notin R_j} (\lambda - \mu)^{-1} e(x, x, d\mu) \right)^{1/2} G(s, s, \lambda)^{1/2} \\ &= o(1) \ (j \uparrow +\infty), \qquad \lambda > 0, x, s \in S,\end{aligned}$$

the $o(1)$ being uniform in s on compacts. Consequently, the integral,

$$G_\infty(x, s, \lambda) = \int_{-\infty}^{0+} (\lambda - \mu)^{-1} e(x, s, d\mu), \tag{3.13}$$

converges and is continuous in s for each $\lambda > 0$ and $x \in S$. This kernel should coincide with the Green function $G(\cdot, \cdot, \cdot)$ but, to prove this, it is necessary to estimate the tail of the integral, $\int_{-\infty}^{0+} (\lambda - \mu)^{-1} e_n(x, s, d\mu)$.

Choose $u \in C_0(\mathfrak{B})$ and $n > 0$ so large that S_n contains the support of u. Then $u \in B_n$ and

$$\begin{aligned}&\left| \int_S \int_{\mu \notin R_j} (\lambda - \mu)^{-1} e_n(x, s, d\mu) u(s) m(ds) \right| \\ &\qquad = \left| \int_S \sum_{\mu_i \notin R_j} (\lambda - \mu_i)^{-1} v_i(x) v_i(s) u(s) m(ds) \right| \\ &\qquad = \left| \sum_{\mu_i \notin R_j} (\lambda - \mu_i)^{-1} v_i(x) \int_{S_n} u v_i m(ds) \right| \\ &\qquad = \left| \sum_{\mu_i \notin R_j} (\lambda - \mu_i)^{-1} v_i(x) \mu_i^{-1} \int_{S_n} u \mathfrak{B}_n v_i m(ds) \right| \\ &\qquad = \left| \sum_{\mu_i \notin R_j} (\lambda - \mu_i)^{-1} v_i(x) \mu_i^{-1} \int_{S_n} \mathfrak{B}_n u v_i m(ds) \right| \\ &\qquad \leqq \sup_{\mu_i \notin R_j} \left| \mu_i \right|^{-3/2} G_n(x, x, \lambda)^{1/2} \left(\int_S (\mathfrak{B} u)^2 m(ds) \right)^{1/2} \\ &\qquad = o(1) \ (j \uparrow +\infty), \qquad \lambda > 0, x \in S,\end{aligned}$$

by Schwarz's inequality, the $o(1)$ being uniform in n provided n is so large that S_n supports u.

Therefore, keeping the same u, we have

$$\begin{aligned}\int_S G(x, s, \lambda)u(s)m(ds) &= \lim_{n\uparrow+\infty} \int_S G_n(x, s, \lambda)u(s)m(ds) \\ &= \lim_{n\uparrow+\infty} \int_S \int_{-\infty}^{0+} (\lambda - \mu)^{-1} e_n(x, s, d\mu)u(s)m(ds) \\ &= \lim_{n\uparrow+\infty} \int_S \int_{R_j} (\lambda - \mu)^{-1} e_n(x, s, d\mu)u(s)m(ds) + o(1),\end{aligned}$$

where the $o(1) \rightarrow 0$ $(j \uparrow +\infty)$ and is uniform in n in the sense just described,

$$\begin{aligned}&= \int_S \int_{R_j} (\lambda - \mu)^{-1} e(x, s, d\mu)u(s)m(ds) + o(1) \\ &\rightarrow \int_S G_\infty(x, s, \lambda)u(s)m(ds) \qquad (j \uparrow +\infty),\end{aligned}$$

by bounded convergence, that is,

$$\int_S (G_\infty(x, s, \lambda) - G(x, s, \lambda))u(s)m(ds) = 0, \qquad u \in C_0(B).$$

But, $C_0(\mathfrak{B})$ being dense in $C_0(S)$ by the Approximation Theorem of §2 and $G_\infty(x, \cdot, \lambda) - G(x, \cdot, \lambda)$ being continuous on S for each $\lambda > 0$ and $x \in S$, it is clear that

$$G(x, s, \lambda) = G_\infty(x, s, \lambda) = \int_{-\infty}^{0+} (\lambda - \mu)^{-1} e(x, s, d\mu)$$

everywhere, and, in particular, that

$$(3.14) \qquad \int_{R_j} (\lambda - \mu)^{-1} e(x, x, d\mu) \uparrow G(x, x, \lambda) \qquad (j \uparrow +\infty).$$

To complete the proof, we have merely to show that the integral in (3.13) converges uniformly on compact squares in $S \times S$, but, by virtue of (3.14) and the fact that $G(x, x, \lambda)$ is continuous on S, this can be done just as in Mercer's theorem: see [18, pp. 117–118].

Corollary. *The measure* $\mathfrak{f}(\cdot)$ *is unique.*

Proof. This is an immediate consequence of the uniqueness theorem for Stieltjes transforms: see D. V. Widder [19, p. 337].

To continue, remembering that $\mathfrak{G}$ is maximal symmetric and negative semi-definite, it is a simple matter to show that there is a spectral measure $\mathfrak{p}(\cdot)$ on $(-\infty, 0]$ to projections, such that

$$(3.15) \qquad \mathfrak{G} = \int_{-\infty}^{0+} \mu\mathfrak{p}(d\mu) \quad \text{and} \quad \mathfrak{G}_\lambda = \int_{-\infty}^{0+} (\lambda - \mu)^{-1}\mathfrak{p}(d\mu), \qquad \lambda > 0.$$

The projections $\mathfrak{p}(\cdot)$ will now be calculated.

THEOREM 3.3. *Let M be a bounded Borel set in $(-\infty, 0]$ and let $e(x, s, M)$ be the kernel $\int_M e(x, s, d\mu)$. Then*

$$\|e(x, \cdot, M)\|_2 \leqq e(x, x, M)^{1/2}, \qquad x \in S,$$

and the projection $\mathfrak{p}(M)$ is the Carleman operator,

$$\mathfrak{p}(M): u \in L_2(S) \to \int_S e(x, s, M)u(s)m(ds).$$

Proof. Choose v_1 and $v_2 \in C_0(S)$ and a bounded Borel set M in $(-\infty, 0]$. Then

$$\begin{aligned}(\mathfrak{G}_\lambda v_1, v_2) &= \int_S v_2(x)m(dx)\int_S G(x, s, \lambda)v_1(s)m(ds) \\ &= \int_S v_2(x)m(dx)\int_S\int_{-\infty}^{0+} (\lambda - \mu)^{-1}e(x, s, d\mu)v_1(s)m(ds) \\ &= \int_{-\infty}^{0+} (\lambda - \mu)^{-1}\int_S v_2(x)\mathfrak{e}(x, \mu)m(dx)\mathfrak{f}(d\mu)\int_S v_1(s)\mathfrak{e}(s, \mu)m(ds),\end{aligned}$$

the integral being absolutely convergent,

$$\begin{aligned}&= \left(\int_{-\infty}^{0+} (\lambda - \mu)^{-1}\mathfrak{p}(d\mu)v_1, v_2\right) \\ &= \int_{-\infty}^{0+} (\lambda - \mu)^{-1}(\mathfrak{p}(d\mu)v_1, v_2), \qquad \lambda > 0,\end{aligned}$$

and so

$$(\mathfrak{p}(M)v_1, v_2) = \int_M\int_S v_2(x)\mathfrak{e}(x, \mu)m(ds)\mathfrak{f}(d\mu)\int_S v_1(s)\mathfrak{e}(s, \mu)m(ds),$$

by the uniqueness theorem for Stieltjes transforms [19, p. 336]. Since, in addition, $C_0(S)$ is dense in $L_2(S)$, we have

$$\mathfrak{p}(M)v = \int_S e(x, s, M)v(s)m(ds), \qquad v \in C_0(S),$$

and now we shall be content to prove

$$\|e(x,\ \cdot,\ M)\|_2 \leqq e(x,\ x,\ M)^{1/2}, \qquad x \in S, \tag{3.16}$$

leaving the second (obvious) statement to the reader.

Set

$$e(x,\ s,\ v) = \int_{-\infty}^{0+} v(\mu) e(x,\ s,\ d\mu), \qquad v \in C_0(-\infty,\ 0].$$

Then, by bounded convergence and Fatou's lemma, it suffices to show that

$$\|e(x,\ \cdot,\ v)\|_2 \leqq e(x,\ x,\ v^2)^{1/2}, \quad x \in S,\ v \in C_0(-\infty,\ 0],$$

and to prove this, we have merely to observe that, by Fatou's lemma, in the notation of Theorem 3.2,

$$\begin{aligned}
\|e(x,\ \cdot,\ v)\|_2^2 &= \int_S \left(\int_{-\infty}^{0+} v(\mu) e(x,\ s,\ d\mu) \right)^2 m(ds) \\
&\leqq \liminf_{n \uparrow +\infty} \int_{S_n} \left(\int_{-\infty}^{0+} v(\mu) e_n(x,\ s,\ d\mu) \right)^2 m(ds) \\
&= \liminf_{n \uparrow +\infty} \int_{S_n} \left(\sum_{i>0} v(\mu_i) v_i(x) v_i(s) \right)^2 m(ds) \\
&= \liminf_{n \uparrow +\infty} \sum_{i>0} v(\mu_i)^2 v_i(x)^2 \\
&= \lim_{n \uparrow +\infty} \int_{-\infty}^{0+} v(\mu)^2 e_n(x,\ x,\ d\mu) \\
&= e(x,\ x,\ v^2), \qquad x \in S.
\end{aligned}$$

Combining (3.15) and Theorem 3.3, we have the eigendifferential expansions,

$$\mathfrak{G}: u \in L_2(S) \to \int_{-\infty}^{0+} \mu(e(x,\ \cdot,\ d\mu),\ u) \tag{3.17}$$

and

$$\mathfrak{G}_\lambda: u \in L_2(S) \to \int_{-\infty}^{0+} (\lambda - \mu)^{-1}(e(x,\ \cdot,\ d\mu),\ u), \qquad \lambda > 0, \tag{3.18}$$

advertised above: compare M. H. Stone [4, pp. 448–530].

REMARK 3.2. Splitting the positive measure $f_{11}+f_{22}$ into pure jump and continuous parts, the point spectrum of $\mathfrak{G}$ is the support of the pure jump part and has uniform multiplicity 1, the continuous spectrum is the support of the continuous part and has multiplicity 1 or 2.

4. **Elementary solutions.** Remembering that the Green function has the representation,

$$(4.1) \qquad G(x, s, \lambda) = \int_{-\infty}^{0+} (\lambda - \mu)^{-1} e(x, s, d\mu), \qquad \lambda > 0,$$

and that (1.5) is supposed to be true, it is natural to hope that the kernel,

$$(4.2) \qquad p(t, x, s) = \int_{-\infty}^{0+} e^{t\mu} e(x, s, d\mu), \qquad t > 0,$$

will be the elementary solution to (1.3), subject to the side condition B, and will enjoy the properties listed in E1, E2, $\cdots$, and E6 in §1.

Starting from scratch, take compact intervals $R_1 \subset R_2 \subset \cdots$ such that $\bigcup_{j>0} R_j = (-\infty, 0]$. Then each

$$\int_{R_j} e^{t\mu} e(x, s, d\mu), \qquad j > 0,$$

is continuous on $(0, +\infty) \times S \times S$, and, choosing $\epsilon > 0$ at pleasure, we have, by Theorem 3.2,

$$\begin{aligned} \int_{\mu \notin R_j} e^{t\mu} \left| e(x, s, d\mu) \right| &\leqq \int_{\mu \notin R_j} e^{\epsilon\mu} \left| e(x, s, d\mu) \right| \\ &\leqq \text{constant} \int_{\mu \notin R_j} (1 - \mu)^{-1} \left| e(x, s, d\mu) \right| \\ &= o(1) \qquad (j \uparrow +\infty), \qquad t \geqq \epsilon, \end{aligned}$$

the $o(1)$ being uniform on compact squares in $S \times S$. This shows that (4.2) exists and is continuous on $(0, +\infty) \times S \times S$.

To continue,

$$(4.3) \qquad G(x, s, \mu) = \int_0^{+\infty} e^{-\mu t} p(t, x, s) dt, \qquad \mu > 0,$$

by a classical theorem for the Stieltjes transform [17, p. 334]. Since, in addition,

$$(-)^n \partial^n / \partial \mu^n G(x, s, \mu) > 0, \qquad n \geqq 0,$$

by the Green function identity (2.8), $G(x, s, \cdot)$ is completely monotonic, and, combining this with S. Bernstein's theorem [18, pp. 160–162], we see that $p(t, x, s)$ is positive, and now to check E1, we have merely to notice that $p(t, \cdot, \cdot)$ is symmetric on $S \times S$, $e(\cdot, \cdot, d\mu)$ being symmetric there.

Remark 4.1. One can show that $p(t, x, s)$ is actually *strictly* positive.

To show that $p(t, x, s)$ satisfies E2, it is necessary to construct certain auxiliary functions $\phi(\cdot, \cdot, \cdot)$ whose purpose is to connect the behaviour of $p(t, \cdot, s)$ near the boundaries to the behaviour of $p(\cdot, x, s)$ near $t=0$. This is carried out in

THEOREM 4.1. *Given x and $s \in S$, there exists a right continuous increasing function $\phi(x, s, \cdot)$ on $[0, +\infty)$ to R, vanishing at* 0 *and such that*

$$\int_{0-}^{+\infty} e^{-\mu t} d\phi(x, s, t) = u_1(x, \mu)u_1(s, \mu)^{-1} \quad (x < s) \qquad \mu > 0, \tag{1}$$
$$= u_2(x, \mu)u_2(s, \mu)^{-1} \quad (x > s)$$

$$\phi(x, s, +\infty) \leqq 1, \tag{2}$$

(3) $\phi(\cdot, s, t)$ *is convex and increasing on* (s_1, s), *convex and decreasing on* (s, s_2),

and

$$\phi(x, s, t) = o(t^n) \qquad (t \downarrow 0), \qquad n > 0, x \neq s. \tag{4}$$

Proof. Choose $s \in S$, $i=1$ or 2, let I be the open interval $(\min(s, s_i), \max(s, s_i))$, let $\mathfrak{B}^0$ be the operator $\mathfrak{B}$ cut down to I, let B^0 be the side condition corresponding to the going boundary condition at s_i and to the minimal boundary condition at s, let $(\mathfrak{G}_\mu^0: \mu > 0)$ be the corresponding Green operators, and let

$$v(x, \mu) = u_i(x, \mu)u_i(s, \mu)^{-1}, \qquad x \in I, \mu > 0.$$

Choose $\epsilon > 0$ and set $\Delta v = v(\cdot, \mu+\epsilon) - v(\cdot, \mu)$. Then $\Delta v = -\epsilon \mathfrak{G}_\mu^0 v(\cdot, \mu+\epsilon)$ because $\Delta v \in B^0$ and $(\mu - \mathfrak{B}^0)\Delta v = -\epsilon v(\cdot, \mu+\epsilon)$, $\|\Delta v\|_\infty \leqq \epsilon\mu^{-1}$ because $\|v(\cdot, \mu+\epsilon)\|_\infty \leqq 1$, and, making $\epsilon \downarrow 0$ in the equation,

$$\epsilon^{-1}\Delta v = -\mathfrak{G}_\mu^0 v(\cdot, \mu + \epsilon),$$

we have

$$\partial/\partial\mu v = -\mathfrak{G}_\mu^0 v, \tag{4.4}$$

the derivative being taken in the strong $C(I)$ topology.

Now choose $n > 0$ and suppose

$$\partial^n/\partial\mu^n v = (-)^n n!(\mathfrak{G}_\mu^0)^n v, \tag{4.5}$$

the derivative being taken in the same sense. Then (2.7) and (4.4) show that there exists one more strong derivative,

$$\partial^{n+1}/\partial\mu^{n+1} v = (-)^{n+1}(n + 1)!(\mathfrak{G}_\mu^0)^{n+1} v,$$

and, by induction, we see that (4.5) is true for each $n > 0$. Thus,

$$(-)^n \partial^n/\partial\mu^n v(x, \mu) \geqq 0, \qquad n \geqq 0, x \in I,$$

that is, $v(x, \cdot)$ is completely monotonic ($x \in I$), and so, by S. Bernstein's theorem [19, pp. 160–162], there is a right continuous increasing function $\phi(x, s, t)$ on $[0, +\infty)$ to R, vanishing at 0 and such that

$$\int_{0-}^{+\infty} e^{-\mu t} d\phi(x, s, t) = v(x, \mu), \qquad x \in I, \mu > 0,$$

which proves (1)(8).

Statement (2) is obvious, v being bounded by 1, and so we turn to (3). Choose x_1 and $x_2 \in I$ and a number $p \in [0, 1]$, let Δ be the difference operator,

$$\Delta: u \to pu(x_1) + (1 - p)u(x_2) - u(px_1 + (1 - p)x_2),$$

let

$$v_n = v + \mu \overset{0}{\mathfrak{G}}_\mu v + \cdots + \mu^n (\overset{0}{\mathfrak{G}}_\mu)^n v, \qquad n \geqq 0,$$

and consider the transform,

$$\mu^{-1}\Delta v(\cdot, \mu) = \int_{0-}^{+\infty} e^{-\mu t} \Delta\phi(\cdot, s, t) dt. \tag{4.6}$$

Keeping (4.5) in mind and differentiating both sides of (4.6) n times, we obtain

$$\partial^n / \partial \mu^n \mu^{-1} \Delta v = (-)^n n! \mu^{-(n+1)} \Delta v_n, \qquad n \geqq 0. \tag{4.7}$$

Since $v_n^+(dx)/m(dx) = \mu^{n+1}(\mathfrak{G}_\mu^0)^n v - cv_n$ is positive on I, the one-sided derivative v_n^+ increases on I, v_n is convex on I, Δv_n is positive, and, coming back to (4.7), we see that

$$(-)^n \partial^n / \partial \mu^n \mu^{-1} \Delta v(\cdot, \mu) \geqq 0, \qquad n \geqq 0, x \in I.$$

Consequently, by S. Bernstein's theorem [19, pp. 160–162],

$$0 \leqq \int_{t_1}^{t_2} \Delta\phi(\cdot, s, t) dt, \qquad t_2 > t_1 > 0,$$

and, varying t_1 and t_2, it is clear that $\Delta\phi(\cdot, s, t)$ is positive and that $\phi(\cdot, s, t)$ is convex on I, x_1, x_2, and p having been chosen at pleasure.

To show that $\phi(\cdot, s, t)$ increases on I ($i = 1$), choose x and $x + \epsilon \in I$ ($\epsilon > 0$), notice that

$$\phi(x, s, t) = \int_{[0,t]} d\phi(x, x + \epsilon, \sigma)\phi(x + \epsilon, s, t - \sigma),$$

by the convolution rule [19, p. 91], and use (2). Similarly, $\phi(\cdot, s, t)$ decreases on I ($i = 2$), and this completes the proof of (3).

(8) The construction is due to W. Feller [private communication].

To prove (4), take $i=1$ once more, choose $x\in I$, and consider the transform

$$\int_{0-}^{+\infty} e^{-\mu t}d\phi(x, s, t) = u_1(x, \mu)u_1(s, \mu)^{-1}, \qquad \mu > 0. \tag{4.8}$$

We have

$$\begin{aligned} u_1(s, \mu) &= u_1(x, \mu) + (s - x)u_1^+(x, \mu) + \int_x^s d\sigma \int_{(x,\sigma)} (\mu - c)u_1 m(dt) \\ &> u_1(x, \mu)\left(1 + \mu \int_x^s m(x, \sigma)d\sigma\right), \end{aligned}$$

a simple iteration shows that

$$u_1(s, \mu) > u_1(x, \mu)(1 + \alpha_1\mu + \alpha_2\mu^2 + \cdots), \tag{4.9}$$

the numbers $(\alpha_i: i>0)$ being >0, and, putting (4.9) back into (4.8), we see that

$$\int_{0-}^{+\infty} e^{-\mu t}d\phi(x, s, t) < (1 + \alpha_1\mu + \alpha_2\mu^2 + \cdots)^{-1} = o(\mu^{-n}) \qquad (\mu \uparrow +\infty)$$

for each $n>0$, which, combined with the standard Tauberian theorem [19, p. 192], shows that $\phi(x, s, t)=o(t^n)(t\downarrow 0)(n\geqq 0)$. The same ideas work when $i=2$.

Coming to the connection between $\phi(\cdot, \cdot, \cdot)$ and $p(\cdot, \cdot, \cdot)$, we state the simple

LEMMA 4.1. *Let $h(t)$ have two continuous derivatives $(t>0)$, let $h(t)=o(t^n)$ $(t\downarrow 0)$ for each $n>0$, and let $h''(t)=O(t^k)(t\downarrow 0)$ for some, not necessarily positive, k. Then $h'(t)=o(t^n)(t\downarrow 0)$ for each $n>0$.*

We then prove

THEOREM 4.2. (1) *The derivatives,*

$$\partial^n/\partial t^n p(t, x, s) = \int_{-\infty}^{0+} \mu^n e^{t\mu}e(x, s, d\mu), \qquad n > 0,$$

exist and are continuous on $(0, +\infty)\times S\times S$,

$$\partial^n/\partial t^n p(t, x, s) = o(1) \qquad (t \downarrow 0), \qquad n \geqq 0, x \neq s, \tag{2}$$

and

$$\partial^n/\partial t^n p(t, x, s) = \int_0^t d\phi(x, \xi, \sigma)\partial^n/\partial t^n p(t - \sigma, \xi, s), \quad t > 0, n \geqq 0, \tag{3}$$

provided ξ is strictly between x and s.

Proof. Statement (1) should be obvious. To prove (2), the crude estimate,

$$t^{n+1}\partial^n/\partial t^n p(t, x, s) = o(1) \qquad (t \downarrow 0), \qquad n \geqq 0, \tag{4.10}$$

will be convenient. To see that this is true, take $n \geqq 0$, remark that

$$(t^{n+1}\partial^n/\partial t^n p(t, x, s))^2 \leqq t^{2n+2}\int_{-\infty}^{0+} \mu^{2n}e^{t\mu}e(x, x, d\mu)\int_{-\infty}^{0+} e^{t\mu}e(s, s, d\mu),$$

by Schwarz's inequality, and $e^{-t\mu/2}$ being $>(t\mu/2)^{2n}(2n!)^{-1}$ on $(-\infty, 0]$, observe that this expression is

$$\begin{aligned}
&\leqq \text{constant } t\int_{-\infty}^{0+} e^{t\mu/2}e(x, x, d\mu)t\int_{-\infty}^{0+} e^{t\mu}e(s, s, d\mu) \\
&= \text{constant } (t/2)p(t/2, x, x)tp(t, s, s) \\
&\leqq \text{constant } \int_0^{t/2} p(\sigma, x, x)d\sigma\int_0^t p(\sigma, s, s)d\sigma \\
&= o(1), \qquad (t \downarrow 0),
\end{aligned}$$

which is precisely (4.10).

To continue, take x and $s \in S$ and choose some convenient ξ strictly between the two. Then the convolution,

$$\psi(t) = \int_0^t d\phi(x, \xi, \sigma)\int_0^{t-\sigma} p(\tau, \xi, s)d\tau,$$

is continuous in t,

$$\begin{aligned}
\int_0^{+\infty} e^{-\mu t}d\psi(t) &= G(x, s, \mu) \\
&= \int_0^{+\infty} e^{-\mu t}p(t, x, s)dt,
\end{aligned}$$

by the convolution rule [19, p. 91], and we have

$$\int_0^t p(\sigma, x, s)d\sigma = \psi(t) \leqq \phi(x, \xi, t) = o(t^n) \qquad (t \downarrow 0) \tag{4.11}$$

for each $n > 0$, which, combined with Lemma 4.1 and (4.10) proves (2), and now it is a trivial exercise to differentiate the first two entries in (4.11)$(n+1)$ times, proving (3).

Quite a number of interesting facts about $p(\cdot, \cdot, \cdot)$ can now be obtained at little cost, and, in particular, E2 can be checked. The necessary information is contained in

THEOREM 4.3. *Given* $n \geqq 0$, *we have*

$$\partial^+/\partial x\, \partial^n/\partial t^n p(t, x, s) = \int_{-\infty}^{0+} \mu^n e^{t\mu} \mathfrak{e}^+(x, \mu) \mathfrak{f}(d\mu) \mathfrak{e}(s, \mu), \tag{1}$$

$$\begin{aligned} \partial^+/\partial x\, \partial^+/\partial s\, \partial^n/\partial t^n p(t, x, s) &= \int_{-\infty}^{0+} \mu^n e^{t\mu} \mathfrak{e}^+(x, \mu) \mathfrak{f}(d\mu) \mathfrak{e}^+(s, \mu) \\ &= \partial^+/\partial s\, \partial^+/\partial x\, \partial^n/\partial t^n p(t, x, s), \end{aligned} \tag{2}$$

$$\partial^n/\partial t^n p(t, \cdot, s) \in B, \tag{3}$$

$$\partial^n/\partial t^n p(t, \cdot, s) = \mathfrak{G}^n p(t, \cdot, s), \tag{4}$$

identically in the free variables, the integrals in (1) *and* (2) *converging uniformly on compact squares in* $S \times S$.

Proof. Choose $t>0$, x and $x+\epsilon \in S$ $(\epsilon>0)$, and compact intervals $R_1 \subset R_2 \subset \cdots$ such that $\bigcup_{j>0} R_j = (-\infty, 0]$. Then, making $j \uparrow +\infty$, we have

$$\begin{aligned} &\partial^n/\partial t^n p(t, x+\epsilon, s) - \partial^n/\partial t^n p(t, x, s) \\ &= o(1) + \int_{R_j} \mu^n e^{t\mu} [\mathfrak{e}(x+\epsilon, \mu) - \mathfrak{e}(x, \mu)] \mathfrak{f}(d\mu) \mathfrak{e}(s, \mu) \\ &= o(1) + \int_{R_j} \mu^n e^{t\mu} \left[\epsilon e^+(x, \mu) + \int_x^{x+\epsilon} d\sigma \int_{(x,\sigma)} (\mu - c) \mathfrak{e}(\xi, \mu) m(d\xi) \right] \mathfrak{f}(d\mu) \mathfrak{e}(s, \mu) \\ &= o(1) + \epsilon \int_{R_j} \mu^n e^{t\mu} \mathfrak{e}^+(x, \mu) \mathfrak{f}(d\mu) \mathfrak{e}(s, \mu) \\ &\qquad + \int_x^{x+\epsilon} d\sigma \int_{(x,\sigma)} m(d\xi) \int_{R_j} \mu^n e^{t\mu} (\mu - c) e(\xi, s, d\mu), \end{aligned}$$

the interchanges being justified by Fubini's theorem. Since, in addition, the integral,

$$\int_{-\infty}^{0+} \mu^n e^{t\mu} (\mu - c) e(\xi, s, d\mu),$$

converges uniformly on compact squares in $S \times S$, we see that

$$\begin{aligned} &\int_x^{x+\epsilon} d\sigma \int_{(x,\sigma)} m(d\xi) \int_{R_j} \mu^n e^{t\mu} (\mu - c) e(\xi, s, d\mu) \\ &\qquad \to \int_x^{x+\epsilon} d\sigma \int_{(x,\sigma)} m(d\xi) \int_{-\infty}^{0+} \mu^n e^{t\mu} (\mu - c) e(\xi, s, du) \qquad (j \uparrow +\infty) \\ &\qquad = \int_x^{x+\epsilon} d\sigma \int_{(x,\sigma)} (\partial/\partial t - c) \partial^n/\partial t^n p(t, \xi, s) m(d\xi), \end{aligned}$$

that the integral,

$$\int_{-\infty}^{0+} \mu^n e^{t\mu} \mathfrak{e}^+(x, \mu) \mathfrak{f}(d\mu) \mathfrak{e}(s, \mu),$$

converges uniformly on compact squares in $S \times S$, that

$$\begin{aligned} \partial^n/\partial t^n p(t, x+\epsilon, s) - \partial^n/\partial t^n p(t, x, s) \\ = \epsilon \int_{-\infty}^{0+} \mu^n e^{t\mu} \mathfrak{e}^+(x, \mu) \mathfrak{f}(d\mu) \mathfrak{e}(s, \mu) \\ + \int_x^{x+\epsilon} d\sigma \int_{x,\sigma} (\partial/\partial t - c) \partial^n/\partial t^n p(t, \xi, s) m(d\xi), \end{aligned}$$

and this proves not only (1) but also

$$\partial^n/\partial t^n p(t, \cdot, s) \in D(\mathfrak{G}) \tag{4.12}$$

and

$$(\partial/\partial t - \mathfrak{G}) \partial^n/\partial t^n p(t, \cdot, s) = 0, \qquad t > 0, s \in S. \tag{4.13}$$

To prove (2), we have merely to substitute

$$\int_{-\infty}^{0+} \mu^n e^{t\mu} \partial^+/\partial x e(x, s, d\mu) \quad \text{for} \quad \int_{-\infty}^{0+} \mu^n e^{t\mu} e(x, s, d\mu)$$

and to copy the steps, and, to complete the proof, it is sufficient to show that

$$\partial^n/\partial t^n p(t, \cdot, x) \in B_1 \cap B_2, \qquad n \geqq 0, t > 0, x \in S. \tag{4.14}$$

Given $x \in S$, $\phi(s, x, t)$ is positive, bounded by 1, and decreases as s approaches s_2. Moreover, $\phi(\cdot, x, t)$ being convex near s_2,

$$\epsilon^{-1}(\phi(s+\epsilon, x, t) - \phi(s, x, t)) \downarrow \partial^+/\partial s \phi(s, x, t) \qquad (\epsilon \downarrow 0)$$

boundedly in t, and $\partial^+/\partial s\phi(s, x, t)$ is negative and increases as s approaches s_2.

Collecting these remarks, we have, by bounded convergence,

$$\int_0^{+\infty} e^{-\mu t} \lim_{s \uparrow s_2} \phi(s, x, t) dt = \mu^{-1} u_2(s_2, \mu) u_2(x, \mu)^{-1},$$

$$\int_0^{+\infty} e^{-\mu t} \lim_{s \uparrow s_2} \partial^+/\partial s \phi(s, x, t) dt = \mu^{-1} u_2^+(s_2, \mu) u_2(x, \mu)^{-1},$$

and, consequently, $\phi(\cdot, x, t) \in B_2$ for almost all $t > 0$. Similarly, $\phi(\cdot, x, t) \in B_1$ for almost all $t > 0$, and so

$$\phi(\cdot, x, t) \in B_1 \cap B_2, \qquad x \in S, \text{ almost all } t > 0. \tag{4.15}$$

Now take $i=1$ or 2, $n\geqq 0$, $t>0$, $x\in S$, choose s near the boundary s_i, and pick an ϵ such that $s<x+\epsilon<x(i=1)$ and $x<x+\epsilon<s(i=2)$. Then

$$\partial^n/\partial t^n p(t, s, x) = \int_0^t d\phi(s, x+\epsilon, \sigma)\partial^n/\partial t^n p(t-\sigma, x+\epsilon, x),$$

and, integrating this once by parts, we see that

$$(4.16)\quad \partial^n/\partial t^n p(t, s, x) = \int_0^t \phi(s, x+\epsilon, \sigma)\partial^{n+1}/\partial t^{n+1} p(t-\sigma, x+\epsilon, x)d\sigma.$$

Keeping in mind the properties of ϕ invoked above, we have, by bounded convergence in (4.16),

$$(4.17)\quad \lim_{s\to s_i} \partial^n/\partial t^n p(t, s, x) = \int_0^t \lim_{s\to s_i} \phi(s, x+\epsilon, \sigma)\partial^{n+1}/\partial t^{n+1} p(t-\sigma, x+\epsilon, x)d\sigma$$

and

$$(4.18)\quad \lim_{s\to s_i} \partial^+/\partial s\, \partial^n/\partial t^n p(t, s, x)$$
$$= \int_0^t \lim_{s\to s_i} \partial^+/\partial s\, \phi(s, x+\epsilon, \sigma)\partial^{n+1}/\partial t^{n+1} p(t-\sigma, x+\epsilon, x)d\sigma,$$

which, together with (4.15), prove (4.14).

COROLLARY 4.1.

$$\int_{-\infty}^{0+} e^{-t\mu} f(d\mu) = \begin{Vmatrix} 1 & \partial^+/\partial x \\ \partial^+/\partial s & \partial^+/\partial x \partial^+/\partial s \end{Vmatrix} p(t, 0, 0), \qquad t>0.$$

Proof. Set x and $s=0$ in (4.2) and in statements (1) and (2) in Theorem 4.3.

REMARK 4.2. Simple computations show that

	Regular	Exit	Entrance	Natural
$\lim_{s\to s_i} p(t, s, x)$	$=0 \quad p_i=0$ $>0 \quad p_i>0$	$=0$	>0	$=0$
$(-)^i \lim_{s\to s_i} \partial^+/\partial s p(t, s, x)$	$<0 \quad p_i<1$ $=0 \quad p_i=1$	<0	$=0$	$=0$

but a real notion of the *shape* of $p(\cdot, \cdot, \cdot)$ is hard to come by. One appropriate remark is this: suppose $p(t, x, \cdot)$ has a local maximum at x for each $t>0$ and each $x\in S$. Then $s_1=-\infty$, $s_2=+\infty$, $c(x)$ is constant on S, and the operator

$\mathfrak{B}-c$ is a constant multiple of the classical second derivative.

The auxiliary function $\phi(\cdot,\ \cdot,\ \cdot)$ is remarkably smooth, too. To support this contention, we prove

THEOREM 4.4. *Given $s\in S$, $t>0$, and $n>0$, we have*

$$\partial^n/\partial t^n\phi(\cdot,\ s,\ t)\in B_1\cap B_2, \tag{1}$$

$$\partial^n/\partial t^n\phi(\cdot,\ s,\ t)=\mathfrak{B}^n\phi(\cdot,\ s,\ t)\qquad \textit{on } (s_1,\ s)\cup(s,\ s_2), \tag{2}$$

and

$$\partial^n/\partial t^n\phi(x,\ s,\ t)=o(1)\qquad (t\downarrow 0),\qquad x\neq s, \tag{3}$$

the existence of the derivatives being part of the assertion.

Proof. Since no new ideas are involved here, we shall be content to have a sketch. Choose $s\in S$, $i=1$ or 2, let $\mathfrak{B}^0$ be the operator $\mathfrak{B}$ cut down to the interval $I=(\min\ (s,\ s_i),\ \max\ (s,\ s_i))$, let $p^0(t,\ x,\ \xi)(t>0,\ x,\ \xi\in I)$ be the kernel (4.2) corresponding to the going boundary condition at s_i and to the minimal boundary condition at s, let $w_1(\cdot,\ \cdot)$ and $w_2(\cdot,\ \cdot)$ be the corresponding positive solutions of (2.1), and remark that $w_i(\cdot,\ \cdot)$ and $u_i(\cdot,\ \cdot)$ are proportional.

Observing that

$$\begin{aligned} 1 &= w_1^+(s,\ \mu)w_2(s,\ \mu)-w_1(s,\ \mu)w_2^+(s,\ \mu)\\ &= -w_1(s,\ \mu)w_2^+(s,\ \mu) && (i=1)\\ &= w_1^+(s,\ \mu)w_2(s,\ \mu) && (i=2),\end{aligned}$$

we have

$$\begin{aligned}\int_0^{+\infty}e^{-\mu t}\lim_{\xi\to s}\partial^+/\partial\xi\, p^0(t,\ x,\ \xi)dt &= \lim_{\xi\to s}\partial^+/\partial\xi\int_0^{+\infty}e^{-\mu t}p^0(t,\ x,\ \xi)dt\\ &= w_1(x,\ \mu)w_2^+(s,\ \mu)=-w_1(x,\ \mu)w_1(s,\ \mu)^{-1} && (i=1)\\ &= w_1^+(s,\ \mu)w_2(x,\ \mu)=w_2(x,\ \mu)w_2(s,\ \mu)^{-1} && (i=2)\\ &= (-)^i u_i(x,\ \mu)u_i(s,\ \mu)^{-1}\\ &= (-)^i\int_0^{+\infty}e^{-\mu t}d\phi(x,\ s,\ t), && x\in I,\ \mu>0,\end{aligned}$$

the necessary interchanges being justified by (3) in Theorem 4.2, and, taking inverse transforms, it is clear that

$$\phi(x,\ s,\ t)=(-)^i\int_0^t\lim_{\xi\to s}\partial^+/\partial\xi\, p^0(\sigma,\ x,\ \xi)d\sigma,\qquad x\in I, \tag{4.19}$$

which can be combined with Theorems 4.2 and 4.3 to complete the proof.

COROLLARY 4.2. *Given $s\in S$, $t>0$, and $x\in(\min\ (s,\ s_i),\ \max\ (s,\ s_i))$, the flux* *identities,*

$$\partial^{n+1}/\partial t^{n+1}\phi(x, s, t) = (-)^i \lim_{\xi\to s} \partial^+/\partial\xi\, \partial^n/\partial t^n p^0(t, x, \xi), \qquad n \geq 0,$$

are satisfied $(i=1, 2)$.

Proof. One has merely to differentiate (4.19) $(n+1)$ times.

To come back to $p(t, x, s)$, we have still to show that E3, $\cdots$, E6 are satisfied.

Choose $x\in S$, set

$$\phi(t) = \int_S p(t, x, s)m(ds), \qquad t > 0, \tag{4.20}$$

and, keeping (1.4), (2.5), and (2.7) in mind, observe that ϕ is lower semi-continuous, that

$$\int_0^{+\infty} e^{-\mu t}\phi(t)dt = (\mathfrak{G}_\mu 1)(x), \qquad \mu > 0, \tag{4.21}$$

and that

$$\mu^{n+1} \left| \partial^n/\partial\mu^n(\mathfrak{G}_\mu 1)(x) \right| \leq n!, \qquad n \geq 0. \tag{4.22}$$

But (4.22) is precisely the condition that $\phi(t)$ be ≤ 1 almost everywhere (dt) [19, pp. 315–316], which, combined with the fact that ϕ is lower semi-continuous, proves E3.

Before we continue, it is convenient to prove

LEMMA 4.2. *Given* $n\geq 0$, $\partial^n/\partial t^n p(t, x, \cdot)$ *is continuous in* (t, x) *on* $(0, +\infty)\times S$ *in the strong*(9) $L_1(S)$ *topology.*

Proof. Choose $n\geq 0$, a neighborhood $N=(x_1, x_2)$ strictly interior to S, $\epsilon>0$ so small that $s_1<x_1-\epsilon$ and $x_2+\epsilon<s_2$, pick $t_2>t_1>0$, and set $I_1=(s_1, x_1-\epsilon]$ and $I_2=[x_2+\epsilon, s_2)$. Remembering (1.6), statements (1) and (3) in Theorem 4.2, and statement (3) in Theorem 4.4 and setting

$$\begin{aligned} v(s) &= \sup_{t<t_2} \left| \partial^{n+1}/\partial t^{n+1}\phi(x_i, x_i + (-)^i\epsilon, t) \right| \\ &\qquad \int_0^{t_2} p(\sigma, x_i + (-)^i\epsilon, s)d\sigma \qquad (s \in I_i,\ i = 1, 2), \\ &= \sup_{(t,x)\in(t_1,t_2)\times N} \left| \partial^n/\partial t^n p(t, x, s) \right| \qquad (s \notin I_1 \cup I_2), \end{aligned}$$

we see that $v\in L_1(S)$, that

(9) $L_1(S)$ is the space of m-measurable functions v on S to R such that $\|v\|_1=\int_S |v(x)|m(dx)<+\infty$, modulo the ideal of null functions.

$$
\begin{aligned}
v(s) &\geqq \sup_{t<t_2}\left|\int_0^t \partial^{n+1}/\partial t^{n+1}\phi(x_i, x_i+(-)^i\epsilon, t-\sigma)p(\sigma, x_i+(-)^i\epsilon, s)d\sigma\right| \\
&\geqq \sup_{t<t_2}\left|\partial^n/\partial t^n p(t, x_i, s)\right| \\
&\geqq \left|\int_0^t d\phi(x, x_i, \sigma)\partial^n/\partial t^n p(t-\sigma, x_i, s)\right| \\
&= \left|\partial^n/\partial t^n p(t, x, s)\right| \quad s\in I_i,\ i=1, 2,\ (t, x)\in(t_1, t_2)\times N,
\end{aligned}
$$

and that

$$v(s) \geqq \left|\partial^n/\partial t^n p(t, x, s)\right| \quad s\notin I_1\cup I_2, \qquad (t, x)\in(t_1, t_2)\times N,$$

and hence, by dominated convergence, that $\partial^n/\partial t^n p(t, x, \cdot)$ is continuous on $(t_1, t_2)\times N$, which completes the proof, t_1, t_2, and N having been chosen at pleasure.

Coming to E4, remember that $p(t, \cdot, \cdot)$ is symmetric on $S\times S$, choose x and $s\in S$, set

$$\phi(\sigma) = \int_S p(t-\sigma, x, \xi)p(\sigma, \xi, s)m(d\xi), \qquad t>\sigma>0,$$

and, keeping Lemma 4.2 and the table in Remark 4.2 in mind, remark that

$$
\begin{aligned}
\phi'(\sigma) &= o(1) + \int_{x_1}^{x_2} \partial/\partial\sigma p(t-\sigma, x, \xi)p(\sigma, \xi, s)m(d\xi) \\
&= o(1) - \text{Wronskian}\,(p(t-\sigma, x, \cdot), p(\sigma, \cdot, s))\Big|_{x_1}^{x_2} \\
&= o(1) \qquad (x_1\downarrow s_1,\ x_2\uparrow s_2).
\end{aligned}
$$

This shows that

$$
\begin{aligned}
0 &= \int_{t_2<\sigma<t_1+t_2} \phi'(\sigma)d\sigma \qquad (t_2<\sigma<t_1+t_2) \\
&= \int_S p(t-(t_1+t_2), x, \xi)p(t_1+t_2, \xi, s)m(d\xi) \\
&\quad - \int_S p(t-t_2, x, \xi)p(t_2, \xi, s)m(d\xi) \qquad (t>t_1+t_2),
\end{aligned}
$$

and, making $t\downarrow t_1+t_2$, we obtain

$$p(t_1+t_2, x, s) = \int_S p(t, x, \xi)p(s, \xi, s)m(d\xi),$$

this being the Chapman-Kolmogorov identity, listed in E4.

To continue, consider the operators

$$(4.23)\qquad S_t: v \in L_\infty(S) \to \int_S p(t, x, s)v(s)m(ds), \qquad t > 0\,(^{10}),$$

and notice that E1, E3, E4, (4.3), and Lemma 4.2 have the

COROLLARY 4.3. *S_t maps $L_\infty(S)$ into $C(S)$, $\|S_t\|_\infty \leq 1$, S_t is positivity-preserving, $(S_t: t>0)$ is a semi-group, and*

$$(\mathfrak{G}_\mu v)(x) = \int_S G(x, s, \mu)v(s)m(ds) = \int_0^{+\infty} e^{-\mu t}(S_t v)(x)dt, \qquad \mu > 0.$$

This covers the first statement in E5, and, to check the second, we have merely to remark that E2 and Lemma 4.2 have the

COROLLARY 4.4. *Let $D_\infty(\mathfrak{B})$ be those $u \in D(\mathfrak{B})$ such that $\mathfrak{B}u \in D(\mathfrak{B})$, $\mathfrak{B}^2 u \in D(\mathfrak{B})$, and so on, and let v be a member of $L_\infty(S)$. Then $v(t, x) = (S_t v)(x)$ belongs to $D_\infty(\mathfrak{B})$, the derivatives $\partial^n/\partial t^n v(t, x)$ exist and are continuous on $(0, +\infty) \times S$, and we have*

$$\partial^n/\partial t^n v(t, \cdot) = \mathfrak{B}^n v(t, \cdot), \qquad t, n > 0.$$

Branching out in another direction, consider the behavior of the operators (4.23) near $t=0$, and let us prove

THEOREM 4.5. *Given $x \in S$ and a neighborhood N containing x, setting*

$$\alpha(s) = \int_{(x,s)} m(x, \sigma)d\sigma \qquad (s > x)$$

$$= \int_{(s,x)} m(\sigma, x]d\sigma \qquad (s < x),$$

and making $t \downarrow 0$, we have

$$(1)\qquad \int_{S-N} p(t, x, s)m(ds) = o(t^n), \qquad n > 0,$$

$$(2)\qquad \int_N p(t, x, s)m(ds) = 1 + tc(x) + o(t),$$

$$(3)\qquad \int_N p(t, x, s)(s - x)m(ds) = o(t),$$

$$(4)\qquad \int_N p(t, x, s)\alpha(s)m(ds) = t + o(t).$$

---

(10) $L_\infty(S)$ is the space of m-measurable functions v on S to R such that $\|v\|_\infty = \inf(\beta: m(x: |v(x)| > \beta) = 0) < +\infty$, modulo the ideal of null functions.

Proof. Given $x\in S$, $\epsilon>0$, and keeping statement (4) in Theorem 4.1 and statement (3) in Theorem 4.2 in mind, we have

$$\begin{aligned}\int_{|x-s|>2\epsilon} p(t, x, s)m(ds) &= \int_0^t d\phi(x, x-\epsilon, \sigma)\int_{s<x-2\epsilon} p(t-\sigma, x-\epsilon, s)m(ds)\\ &\quad + \int_0^t d\phi(x, x+\epsilon, \sigma)\int_{s\geqq x+2\epsilon} p(t-\sigma, x+\epsilon, s)m(ds)\\ &\leqq \phi(x, x-\epsilon, t)+\phi(x, x+\epsilon, t)\\ &= o(t^n) \qquad (t\downarrow 0), \qquad\qquad n>0,\end{aligned}$$

which proves (1).

Coming to (2), (3), and (4), pick $w_i\in C_0(S)$ $(i=1, 2, 3)$ such that

$$v_i=\int_{\theta<s} d\theta\int_{\sigma<\theta} w_i(\sigma)m(d\sigma)$$

vanishes near the boundaries, and, on the neighborhood N, v_1 coincides with 1, v_2 with constant$+(s-x)$, and v_3 with constant+constant $(s-x)+\alpha(s)$([11]). Then $v_i\in C_0(\mathfrak{B})$, and if we set $u_i=(\mu-\mathfrak{G})v_i$ and keep (1) in mind, we shall have

$$\begin{aligned}\int_N p(t, x, s)v_i(s)m(ds) &= (S_tv_i)(x)+o(t)\\ &= (S_t\mathfrak{G}_\mu u_i)(x)+o(t)\\ &= e^{\mu t}\int_t^{+\infty} e^{-\mu\sigma}(S_\sigma u_i)(x)d\sigma+o(t)\\ &= v_i(x)+(e^{\mu t}-1)\int_t^{+\infty} e^{-\mu\sigma}(S_\sigma u_i)(x)d\sigma\\ &\quad -\int_0^t e^{-\mu\sigma}(S_\sigma u_i)(x)d\sigma+o(t)\\ &= v_i(x)+t(\mu v_i(x)-u_i(x))+o(t)\\ &= v_i(x)+t(\mathfrak{B}v_i)(x)+o(t) \qquad\qquad (t\downarrow 0),\end{aligned}$$

and, setting $i=1, 2, 3$ (in this order) proves, successively, (2), (3), and (4).

REMARK 4.3. Estimates like these are important for the description of random processes with continuous sample functions. The standard procedure is to consider a smooth Markov transition density $p(t, x, s)$, to postulate such conditions (1), (2), (3), and (4), and to show that $p(\cdot, \cdot, s)$ satisfies the *backward equation* (1.3): see A. N. Kolmogorov [20], W. Feller [9], A. Khinchin [21], and, for the modern approach, W. Feller [2].

([11]) The actual construction is straightforward and will not be reproduced here.

This brings us to E6. Given $v \in C(S)$ such that $\mathfrak{B}v$ is continuous near $x \in S$, choose $\epsilon > 0$ and notice that

$$\begin{aligned} v(x) + (s - x)v^+(x) + \alpha(s)\left[(\mathfrak{B}v)(x) - c(x)v(x) - \epsilon\right] & \\ < v(s) & \\ < v(x) + (s - x)v^+(x) + \alpha(s)\left[(\mathfrak{B}v)(x) - c(x)v(x) + \epsilon\right] & \end{aligned}$$

on some neighborhood N containing x. Multiplying this by $p(t, x, s)$, integrating over N, remembering statements (2), (3), and (4) in Theorem 4.5, and making $t \downarrow 0$, we have

$$\begin{aligned} v(x) + t\left[(\mathfrak{B}v)(x) - \epsilon\right] + o(t) &< \int_N p(t, x, s)v(s)m(ds) \\ &< v(x) + t\left[(\mathfrak{B}v)(x) + \epsilon\right] + o(t), \end{aligned}$$

and, combining this with statement (1) in Theorem 4.5, we see that

$$(S_t v)(x) = v(x) + t(\mathfrak{B}v)(x) + o(t) \qquad (t \downarrow 0),$$

which is E6.

E1, E2, $\cdots$, and E6 have now been checked, and, in particular, we can state

THEOREM 4.6. *The kernel $p(t, x, s)$ is the elementary solution to* (1.3), *subject to the side condition* B.

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Page 69, lines 1–2. Delete "linear." Add "$+f''(x_0)(x-x_0)^2$."