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ELEMENTARY SPINOR ALGEBRA FOR POLARIZED BEAMS IN STORAGE RINGS

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Two-component spinors and Pauli matrices can usefully be applied to classical problems of rotations in three dimensions. In particular, they constitute a concise method of analyzing problems of spin motion in storage rings and accelerators. An elementary treatment of spinor algebra is developed and applied to some fundamental kinematics of polarized beams in storage rings and to the basic configuration of the Siberian Snake scheme. Further possibilities of the formalism are indicated.

1. INTRODUCTION

Polarized beams in accelerators and storage rings can be studied largely on the basis of classical equations of motion. Much of the analysis then consists of the rotation of ordinary real vectors in three-dimensional space, exactly as in the kinematics of rigid body motion. Such rotations are commonly described by linear transformations using real orthogonal 3×3 matrices; although this method has a direct physical appeal, it is rather cumbersome for detailed analysis since the nine matrix elements contain only three independent parameters to describe an arbitrary rotation.

The algebra of two-component spinors and Pauli matrices, which form part of the SU(2) group, provides a much more compact and elegant formalism for describing classical rotations in real three-dimensional space. Although spinor notation is normally associated with quantum mechanics, in particular to describe the internal degree of freedom of the electron known as the spin, it is very closely related to the quaternions (hypercomplex numbers) used by Hamilton over a century ago.[1] Although spinors might at first sight appear to be somewhat abstract entities to use in classical physics, they are hardly more so than complex numbers used in electrical engineering. Furthermore, the transformations of spinors retain to a considerable extent the physical picture, in much the same way as the operators of quantum mechanics are closely analogous to the dynamical variables of classical mechanics. The algebra of spinors can readily be related to other ways of describing rotations, such as the Euler angles and the Cayley-Klein parameters.[2]

The purpose of this report is to describe the basic properties of two-component spinors and their transformations, and to illustrate their application to the study of classical spin motion of particles in storage rings and accelerators. To this end, a few special examples of an elementary nature are invoked, but no attempt is made here to treat spin motion generally. A considerable amount of literature exists on this subject, much of which is cited in review papers, for example, those of Baier[3] and of Derbenev et al.[4]

2. BASIC SPINOR ALGEBRA

2.1 *The Pauli Matrices*

A conventional definition of the Pauli matrices is

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}; \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}; \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix};$$

they are used together with the identity (unit) matrix

$$I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

The Pauli matrices have zero trace, minus-unity determinant, and the simple commutation properties

$$\left.\begin{aligned} \sigma_x\sigma_x &= \sigma_y\sigma_y = \sigma_z\sigma_z = I \\ \sigma_x\sigma_y &= -\sigma_y\sigma_x = i\sigma_z \\ \sigma_y\sigma_z &= -\sigma_z\sigma_y = i\sigma_x \\ \sigma_z\sigma_x &= -\sigma_x\sigma_z = i\sigma_y \end{aligned}\right\} \qquad (2.1)$$

We use the notation * for the complex conjugate (c.c.) and † for the Hermitian conjugate (c.c. of the transpose).

It is easy to show that the Pauli matrices are

$$\left.\begin{aligned} &\text{i) Unitary;} \quad \sigma_i\sigma_i^\dagger = I \\ &\text{ii) Hermitian;} \quad \sigma_i^\dagger = \sigma_i \end{aligned}\right\} \quad i = x, y, z.$$

2.2 *Matrix vectors*

Let **x**, **y**, **z** be orthogonal unit vectors in Cartesian three-dimensional space. Then we can define a matrix vector (or vector matrix), i.e., a vector $\boldsymbol{\sigma}$ whose components are Pauli matrices,

$$\boldsymbol{\sigma} = \mathbf{x}\sigma_x + \mathbf{y}\sigma_y + \mathbf{z}\sigma_z, \qquad (2.2)$$

and apply standard vector operations. The scalar product becomes

$$\begin{aligned} \boldsymbol{\sigma} \cdot \boldsymbol{\sigma} &= \mathbf{x} \cdot \mathbf{x}\sigma_x\sigma_x + \mathbf{y} \cdot \mathbf{y}\sigma_y\sigma_y + \mathbf{z} \cdot \mathbf{z}\sigma_z\sigma_z \\ &= 3I, \end{aligned} \qquad (2.3)$$

where we have used the obvious property that the **x**, **y**, **z** commute with the σ_i.

It is not really necessary to write the unit vectors **x**, **y**, **z** explicitly, and the σ_i can be formally manipulated as if they were components of a three-vector. In fact the σ_i are often called the unit rotators for their respective axes.

Since the σ_i are Hermitian we have also

$$\boldsymbol{\sigma}^\dagger = \boldsymbol{\sigma}$$

and hence

$$\begin{aligned} \boldsymbol{\sigma}^\dagger \cdot \boldsymbol{\sigma} &= \boldsymbol{\sigma} \cdot \boldsymbol{\sigma}^\dagger = \boldsymbol{\sigma}^\dagger \cdot \boldsymbol{\sigma}^\dagger \\ &= \boldsymbol{\sigma} \cdot \boldsymbol{\sigma} = 3I. \end{aligned} \qquad (2.4)$$

A vector product can also be formed according to the normal rules of vector algebra, taking account of the commutation properties of Eq. (2.1). Thus

$$\begin{aligned} \boldsymbol{\sigma} \times \boldsymbol{\sigma} &= [\mathbf{x}\sigma_x + \mathbf{y}\sigma_y + \mathbf{z}\sigma_z] \times [\mathbf{x}\sigma_x + \mathbf{y}\sigma_y + \mathbf{z}\sigma_z] \\ &= \mathbf{x}(\sigma_y\sigma_z - \sigma_z\sigma_y) + \mathbf{y}(\sigma_z\sigma_x - \sigma_x\sigma_z) \\ &\quad + \mathbf{z}(\sigma_x\sigma_y - \sigma_y\sigma_x) \\ &= 2i[\mathbf{x}\sigma_x + \mathbf{y}\sigma_y + \mathbf{z}\sigma_z]; \end{aligned}$$

so

$$\boldsymbol{\sigma} \times \boldsymbol{\sigma} = 2i\boldsymbol{\sigma}. \qquad (2.5)$$

The scalar product may be formed also with any arbitrary real three-vector **b**:

$$\left.\begin{aligned} &S = \boldsymbol{\sigma} \cdot \mathbf{b} = \mathbf{b} \cdot \boldsymbol{\sigma} = b_x\sigma_x + b_y\sigma_y + b_z\sigma_z, \\ &\text{or in full:} \\ &S = \begin{pmatrix} b_z & b_x - ib_y \\ b_x + ib_y & -b_z \end{pmatrix} \end{aligned}\right\} \qquad (2.6)$$

using the definition of the Pauli matrices; S is Hermitian, traceless, and has determinant

$$|S| = -[b_x^2 + b_y^2 + b_z^2] = -b^2. \qquad (2.7)$$

Powers of S can be formed:

$$S^2 = SS = (b_x^2 + b_y^2 + b_z^2)I = b^2 I, \qquad (2.8)$$

and in general

$$\left.\begin{aligned} S^n &= b^n I, & n \text{ even} \\ &= b^{n-1}S, & n \text{ odd} \end{aligned}\right\}. \qquad (2.9)$$

The product of two matrices $S_1 = \boldsymbol{\sigma} \cdot \mathbf{b}_1$ and $S_2 = \boldsymbol{\sigma} \cdot \mathbf{b}_2$ is given by

$$\begin{aligned} S_1S_2 &= (\boldsymbol{\sigma} \cdot \mathbf{b}_1)(\boldsymbol{\sigma} \cdot \mathbf{b}_2) \\ &= I(\mathbf{b}_1 \cdot \mathbf{b}_2) + i\sum_j \sigma_j \sum_{k,m} \epsilon_{jkm} b_{1k} b_{2m}, \\ &\qquad j, k, m = x, y, z, \end{aligned} \qquad (2.10)$$

where ϵ_{jkm} is the Levi-Civita density, sometimes called the antisymmetric Kronecker symbol, and is defined by:

$$\epsilon_{jkm} = \begin{cases} 0 & \text{if any two indices are identical,} \\ +1 & \text{for even permutations of the indices,} \\ -1 & \text{for odd permutations of the indices.} \end{cases}$$

Then, since

$$\sum_{k,m} \epsilon_{jkm} b_{1k} b_{2m} = (\mathbf{b}_1 \times \mathbf{b}_2)_j,$$

Eq. (2.10) can be written

$$S_1 S_2 = I(\mathbf{b}_1 \cdot \mathbf{b}_2) + i\boldsymbol{\sigma} \cdot (\mathbf{b}_1 \times \mathbf{b}_2). \quad (2.11)$$

The commutator of S_1 and S_2 follows immediately from Eq. (2.11);

$$[S_1, S_2] = S_1 S_2 - S_2 S_1 = 2i\boldsymbol{\sigma} \cdot (\mathbf{b}_1 \times \mathbf{b}_2). \quad (2.12)$$

2.3 *Spinors*

We consider a two-component column vector

$$\psi = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix},$$

where ψ_1, ψ_2 are in general complex numbers. Such a state vector is called a spinor in the context of quantum mechanics for spin one-half particles. The Hermitian conjugate

$$\psi^\dagger = (\psi_1^*, \psi_2^*)$$

is formed consistently with the concepts of matrix algebra, i.e., by taking the complex conjugate of the transpose.

A scalar P_x may be formed as follows:

$$P_x = \psi^\dagger \sigma_x \psi = (\psi_1^*, \psi_2^*) \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} = (\psi_1^*, \psi_2^*) \begin{pmatrix} \psi_2 \\ \psi_1 \end{pmatrix} = \psi_1^* \psi_2 + \psi_2^* \psi_1. \quad (2.13)$$

Clearly $P_x^* = P_x$; hence P_x is real.

Similarly,

$$P_y = \psi^\dagger \sigma_y \psi = -i(\psi_1^* \psi_2 - \psi_2^* \psi_1) \quad (2.14)$$

and

$$P_z = \psi^\dagger \sigma_z \psi = |\psi_1|^2 - |\psi_2|^2 \quad (2.15)$$

are both real. We make the hypothesis that P_x, P_y, P_z are components of a real three-vector:

$$\mathbf{P} = \mathbf{x}P_x + \mathbf{y}P_y + \mathbf{z}P_z = \mathbf{x}\psi^\dagger\sigma_x\psi + \mathbf{y}\psi^\dagger\sigma_y\psi + \mathbf{z}\psi^\dagger\sigma_z\psi,$$

$$\mathbf{P} = \psi^\dagger \boldsymbol{\sigma} \psi. \quad (2.16)$$

The square of the modulus of **P** can be expressed in terms of the components of ψ, using Eqs. (2.13), (2.14) and (2.15), as

$$\begin{aligned} |\mathbf{P}|^2 = \mathbf{P} \cdot \mathbf{P} &= P_x^2 + P_y^2 + P_z^2 \\ &= (\psi_1^*\psi_2 + \psi_2^*\psi_1)^2 - (\psi_1^*\psi_2 - \psi_2^*\psi_1)^2 \\ &\quad + (|\psi_1|^2 - |\psi_2|^2)^2 \\ &= 4|\psi_1|^2|\psi_2|^2 + (|\psi_1|^2 - |\psi_2|^2)^2 \\ &= (|\psi_1|^2 + |\psi_2|^2)^2, \end{aligned}$$

whence

$$|\mathbf{P}| = |\psi_1|^2 + |\psi_2|^2 = \psi^\dagger\psi. \quad (2.17)$$

The hypothesis is confirmed if Eq. (2.17) is invariant under transformation, which will be established in Sections 2.4 and 2.5.

We shall interpret **P** as a spin vector (or polarization vector) of unit length, which implies a normalization condition

$$|\psi_1|^2 + |\psi_2|^2 = (\psi_1^*, \psi_2^*) \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} = 1. \quad (2.18)$$

From Eq. (2.15) we then have

$$P_z = |\psi_1|^2 - |\psi_2|^2 = 2|\psi_1|^2 - 1 = 1 - 2|\psi_2|^2 \quad (2.19)$$

and note that

$$\left.\begin{array}{ll} \text{if } |\psi_1|^2 = 1 & P_z = +1 \\ \text{if } |\psi_1|^2 = 0 & P_z = -1 \end{array}\right\}. \tag{2.20}$$

Furthermore,

$$\begin{aligned} P_x^2 + P_y^2 &= 1 - P_z^2 = 1 - (2|\psi_1|^2 - 1)^2 \\ &= 4|\psi_1|^2(1 - |\psi_1|^2) \\ &= 4|\psi_1|^2|\psi_2|^2, \end{aligned}$$

whence

$$\sqrt{P_x^2 + P_y^2} = \pm 2|\psi_1||\psi_2|. \tag{2.21}$$

2.4 *Differential equation of spinor transformation*

We examine the properties of the equation

$$\frac{d\psi}{d\theta} = -\frac{i}{2}(\boldsymbol{\sigma}\cdot\mathbf{b})\psi \tag{2.22}$$

where **b** is an arbitrary real three-vector and θ is an independent variable which, for the moment, we do not need to define further. For constant **b** we can write a formal solution satisfying Eq. (2.22)

$$\psi(\theta) = \exp\left[-\frac{i}{2}(\boldsymbol{\sigma}\cdot\mathbf{b})\theta\right]\psi(0). \tag{2.23}$$

The complex matrix exponential may be interpreted by a generalization of the normal expansion. We write $S = \boldsymbol{\sigma}\cdot\mathbf{b}$ as in Eq. (2.6), put $b = |\mathbf{b}| = (b_x^2 + b_y^2 + b_z^2)^{1/2}$, and use Eq. (2.9). Then

$$\begin{aligned} \exp\left[-iS\frac{\theta}{2}\right] &= I + \left(-iS\frac{\theta}{2}\right) + \frac{1}{2!}\left(-iS\frac{\theta}{2}\right)^2 \\ &\quad + \frac{1}{3!}\left(-iS\frac{\theta}{2}\right)^3 + \dots \\ &= I\left[1 - \frac{1}{2!}\left(\frac{b\theta}{2}\right)^2 + \frac{1}{4!}\left(\frac{b\theta}{2}\right)^4 \dots\right] \\ &\quad - \frac{iS}{b}\left[\left(\frac{b\theta}{2}\right) - \frac{1}{3!}\left(\frac{b\theta}{2}\right)^3 + \frac{1}{5!}\left(\frac{b\theta}{2}\right)^5 \dots\right], \end{aligned}$$

or

$$\begin{aligned} \exp\left[-\frac{i}{2}(\boldsymbol{\sigma}\cdot\mathbf{b})\theta\right] &= I\cos\left(\frac{b\theta}{2}\right) - \frac{i}{b}(\boldsymbol{\sigma}\cdot\mathbf{b})\sin\left(\frac{b\theta}{2}\right). \end{aligned} \tag{2.24}$$

It can readily be verified by putting Eq. (2.24) into Eq. (2.23), differentiating, and using Eq. (2.8), that Eq. (2.22) is satisfied by this form.

If ψ satisfies Eq. (2.22), $\psi^\dagger\psi$ is invariant, for

$$\begin{aligned} \frac{d}{d\theta}(\psi^\dagger\psi) &= \frac{d\psi^\dagger}{d\theta}\psi + \psi^\dagger\frac{d\psi}{d\theta} \\ &= \frac{i}{2}\psi^\dagger(\boldsymbol{\sigma}\cdot\mathbf{b})\psi - \frac{i}{2}\psi^\dagger(\boldsymbol{\sigma}\cdot\mathbf{b})\psi = 0. \end{aligned}$$

2.5 *Equation of three-vector* **P**

Differentiating Eq. (2.16) yields

$$\frac{d\mathbf{P}}{d\theta} = \frac{d\psi^\dagger}{d\theta}\boldsymbol{\sigma}\psi + \psi^\dagger\boldsymbol{\sigma}\frac{d\psi}{d\theta}. \tag{2.25}$$

With **b** real, the Hermitian conjugate of Eq. (2.22) is

$$\frac{d\psi^\dagger}{d\theta} = \frac{i}{2}\psi^\dagger(\boldsymbol{\sigma}\cdot\mathbf{b}), \tag{2.26}$$

since $\boldsymbol{\sigma}$ is Hermitian, and the Hermitian conjugate of a product is the product of the Hermitian conjugates taken in reverse order. Then Eq. (2.25) becomes

$$\begin{aligned} \frac{d\mathbf{P}}{d\theta} &= \frac{i}{2}\psi^\dagger(\boldsymbol{\sigma}\cdot\mathbf{b})\boldsymbol{\sigma}\psi - \frac{i}{2}\psi^\dagger\boldsymbol{\sigma}(\boldsymbol{\sigma}\cdot\mathbf{b})\psi \\ &= \frac{i}{2}\psi^\dagger[(\boldsymbol{\sigma}\cdot\mathbf{b})\boldsymbol{\sigma} - \boldsymbol{\sigma}(\boldsymbol{\sigma}\cdot\mathbf{b})]\psi. \end{aligned} \tag{2.27}$$

From the vector triple product identity

$$(\mathbf{A}\times\mathbf{B})\times\mathbf{C} = (\mathbf{C}\cdot\mathbf{A})\mathbf{B} - \mathbf{A}(\mathbf{C}\cdot\mathbf{B}),$$

we have

$$(\boldsymbol{\sigma}\times\boldsymbol{\sigma})\times\mathbf{b} = (\mathbf{b}\cdot\boldsymbol{\sigma})\boldsymbol{\sigma} - \boldsymbol{\sigma}(\mathbf{b}\cdot\boldsymbol{\sigma}),$$

whence

$$\frac{d\mathbf{P}}{d\theta} = -\frac{i}{2}\psi^\dagger[\mathbf{b}\times(\boldsymbol{\sigma}\times\boldsymbol{\sigma})]\psi$$

$= \psi^\dagger(\mathbf{b} \times \boldsymbol{\sigma})\psi$, from Eq. (2.5)

$= \mathbf{b} \times (\psi^\dagger \boldsymbol{\sigma} \psi)$, since $\mathbf{b}$ and $\psi^\dagger$ commute.

Hence,

$$\frac{d\mathbf{P}}{d\theta} = \mathbf{b} \times \mathbf{P}, \qquad (2.28)$$

which is the familiar three-vector precession equation.

The invariance of $|\mathbf{P}|^2 = \mathbf{P} \cdot \mathbf{P}$ follows from

$$\frac{d}{d\theta}(\mathbf{P} \cdot \mathbf{P}) = 2\mathbf{P} \cdot \frac{d\mathbf{P}}{d\theta} = 2\mathbf{P} \cdot (\mathbf{b} \times \mathbf{P}) = 0,$$

which establishes that $\mathbf{P}$ is a vector and that Eqs. (2.22) and (2.28) are equivalent representations.

3. SPIN MOTION IN AN ELECTROMAGNETIC FIELD

The kinematics of classical spin motion in an electromagnetic field are governed by the equation

$$\frac{d\mathbf{P}}{dt} = \boldsymbol{\Omega} \times \mathbf{P}, \qquad (3.1)$$

where the axial vector $\boldsymbol{\Omega}$ is given by

$$\boldsymbol{\Omega} = -\frac{e}{m\gamma}\left[(1 + \gamma a)\mathbf{B} - (\gamma - 1)a\frac{\mathbf{v}(\mathbf{v} \cdot \mathbf{B})}{v^2} + \gamma\left(a + \frac{1}{\gamma + 1}\right)\frac{\mathbf{E} \times \mathbf{v}}{c^2}\right]. \qquad (3.2)$$

Here e, m are the charge and mass of the particle, γ is the Lorentz energy factor, $\mathbf{v}$ is the velocity, c the speed of light, $\mathbf{B}$ and $\mathbf{E}$ the magnetic and electric fields, and $a = (g - 2)/2$ is the gyromagnetic anomaly. The vector $\mathbf{P}$ can be considered either as the polarization of an ensemble of particles or as a classical representation of the spin of an individual particle. Equation (3.2) is frequently called the BMT equation[5] although its essentials are due to L. H. Thomas. A particularly clear derivation is given by J. S. Bell[6]:

For $\mathbf{E} = 0$, Eq. (3.2) becomes, for a longitudinal magnetic field $\mathbf{B}_\parallel$ ($\mathbf{B} \times \mathbf{v} = 0$):

$$\boldsymbol{\Omega} = -\frac{e}{m\gamma}(1 + a)\mathbf{B}_\parallel = -\frac{e}{m\gamma} \cdot \frac{g}{2}\mathbf{B}_\parallel, \qquad (3.3)$$

and for a transverse field $\mathbf{B}_\perp$ ($\mathbf{B} \cdot \mathbf{v} = 0$)

$$\boldsymbol{\Omega} = -\frac{e}{m\gamma}(1 + \gamma a)\mathbf{B}_\perp. \qquad (3.4)$$

In cyclic accelerators and storage rings it is convenient to use a Cartesian coordinate system moving with the particle, the y coordinate lying along the direction of the ideal orbit. In this system, which rotates at the local relativistic cyclotron frequency $\boldsymbol{\Omega}_c = -e\mathbf{B}_\perp/m\gamma$, Eq. (3.4) becomes

$$\boldsymbol{\Omega} = -\frac{e\mathbf{B}_\perp}{m\gamma}(\gamma a) = \boldsymbol{\Omega}_c(\gamma a). \qquad (3.5)$$

For a planar orbit $\mathbf{B}_\perp = (0, 0, B_z)$ is the bending field normal to the plane of the orbit. If in Eq. (2.28) we take the independent variable θ to be the bending angle, comparison with Eqs. (3.1) and (3.5) shows that $\mathbf{b}$ has the direction of $-B_z$ and a magnitude (γa), since $|\Omega_c| = d\theta/dt$. Over a length l of a magnet, a bending angle θ of the orbit is accompanied by a precession angle ϕ of $\mathbf{P}$ (around B_z), given by

$$\phi = -b\theta = (\gamma a)\theta = -(\gamma a) \cdot \frac{eB_z l}{mc\beta\gamma}. \qquad (3.6)$$

The corresponding spinor transformation of Eqs. (2.23) and (2.24) contains only the z component, and becomes

$$\psi(\theta) = \left[I \cos\frac{\phi}{2} - i\sigma_z \sin\frac{\phi}{2}\right]\psi(0). \qquad (3.7)$$

The parameter (γa) is very important in spin motion and will occur repeatedly in subsequent sections.

4. TRANSFORMATION THROUGH PIECEWISE-CONSTANT ELEMENTS

4.1 *Basic properties*

A solution (2.23) of the spinor equation (2.19) through a constant element may be written in the form

$$\psi(\theta) = M\psi(0), \qquad (4.1)$$

where the matrix M may be written in an expanded form from Eq. (2.24)

$$M = IC_0 - i\sigma_x C_x - i\sigma_y C_y - i\sigma_z C_z. \qquad (4.2)$$

An important property of the coefficients C_i is obtained in multiplying Eq. (4.1) by its Hermitian conjugate:

$$\psi^\dagger(\theta)\psi(\theta) = \psi^\dagger(0)M^\dagger M\psi(0).$$

Since the normalization condition of Eq. (2.18) is invariant, it follows that

$$M^\dagger M = I, \tag{4.3}$$

and that M is therefore unitary. By forming from Eq. (4.2),

$$M^\dagger M = [IC_0^* + i\sigma_x C_x^* + i\sigma_y C_y^* + i\sigma_z C_z^*] \times [IC_0 - i\sigma_x C_x - i\sigma_y C_y - i\sigma_z C_z]$$

(since the Pauli matrices are Hermitian), and using the commutation rules of Eq. (2.1), it is easily shown that the necessary and sufficient conditions for Eq. (4.3) to be satisfied are that

$$C_0^2 + C_x^2 + C_y^2 + C_z^2 = 1 \tag{4.4}$$

and that the C_i be real. This is also evident if Eq. (4.2) is written with the explicit form of the Pauli matrices:

$$M = \begin{pmatrix} C_0 - iC_z & -C_y - iC_x \\ C_y - iC_x & C_0 + iC_z \end{pmatrix} \tag{4.5}$$

and multiplied by its Hermitian conjugate $M^\dagger$. It is also clear that M is not Hermitian, i.e., $M^\dagger \neq M$, in contrast to the matrix S of Eq. (2.6).

4.2 *Interpretation of the coefficients*

The transformation matrix of Eq. (4.2) can be written in yet another form

$$M = I\cos\frac{\phi}{2} - i\sin\frac{\phi}{2}(\sigma_x\cos\alpha_x + \sigma_y\cos\alpha_y + \sigma_z\cos\alpha_z), \tag{4.6}$$

where

$$C_0 = \cos\frac{\phi}{2}$$

$$C_i = \sin\frac{\phi}{2}\cos\alpha_i, \quad i = x, y, z.$$

In order that Eq. (4.4) be satisfied,

$$\cos^2\alpha_x + \cos^2\alpha_y + \cos^2\alpha_z = 1, \tag{4.7}$$

which is the case if the $\cos\alpha_i$ are the direction cosines of some vector with respect to the appropriate axes. This form is implicit in Eq. (2.6), where the components b_i are proportional to the corresponding direction cosines and in fact equal to these if $b^2 = 1$.

The spinor matrix M therefore represents a rotation (or precession) by an angle ϕ around the axis defined by the direction cosines $\cos\alpha_i$. It is characteristic of spinor algebra that the precession angles always appear as half angles in the arguments. This is closely connected with the physical properties of spin one-half particles, and results in the M matrix being a two-valued function of the corresponding real orthogonal matrix in three-dimensional space (see Ref. 2, Section 4–5). The two sets of matrices are isomorphic however, i.e., to a transformation in one set corresponds one in the other set.

4.3 *Transformation through several elements*

Equation (3.7) is a simple example of a rotation around the z-axis by an angle ϕ; here $\cos\alpha_z = 1$ and $\cos\alpha_x = \cos\alpha_y = 0$. The form of a simple rotation around the x or the y axis is obvious.

Successive transformations with different axes are readily generated, as in the example

$$\psi(\theta_1) = \left[I\cos\frac{\phi_1}{2} - i\sigma_z\sin\frac{\phi_1}{2}\right]\psi(0)$$

$$\begin{aligned}\psi(\theta_2) &= \left[I\cos\frac{\phi_2}{2} - i\sigma_x\sin\frac{\phi_2}{2}\right]\psi(\theta_1) \\ &= \left[I\cos\frac{\phi_2}{2} - i\sigma_x\sin\frac{\phi_2}{2}\right] \times \left[I\cos\frac{\phi_1}{2} - i\sigma_z\sin\frac{\phi_1}{2}\right]\psi(0) \\ &= \left[I\cos\frac{\phi_2}{2}\cos\frac{\phi_1}{2} - i\sigma_z\cos\frac{\phi_2}{2}\sin\frac{\phi_1}{2} - i\sigma_x\sin\frac{\phi_2}{2}\cos\frac{\phi_1}{2} - \sigma_x\sigma_z\sin\frac{\phi_2}{2}\sin\frac{\phi_1}{2}\right]\psi(0)\end{aligned}$$

and, using the commutation properties of the Pauli matrices in Eq. (2.1),

$$\psi(\theta_2) = [IA_0 - i\sigma_x A_x - i\sigma_y A_y - i\sigma_z A_z]\psi(0)$$

where:

$$A_0 = \cos\frac{\phi_2}{2}\cos\frac{\phi_1}{2}$$

$$A_x = \sin\frac{\phi_2}{2}\cos\frac{\phi_1}{2}$$

$$A_y = -\sin\frac{\phi_2}{2}\sin\frac{\phi_1}{2}$$

$$A_z = \cos\frac{\phi_2}{2}\sin\frac{\phi_1}{2}.$$

It is evident that any succession of transformations can be reduced to the canonical form of Eq. (4.2) by suitable substitution of coefficients, and that the resulting transformation can be represented by an equivalent rotation around a specified axis as in Eq. (4.6). Four parameters are present, namely ϕ, α_x, α_y, α_z, and the constraint of Eq. (4.7) leaves three free parameters to describe the rotation. The condition in Eq. (4.4) is automatically satisfied.

4.4 *Polarization projection in real space*

The components of a real three-vector **P** in terms of the spinor notation have been given in Eqs. (2.13) to (2.15) and in the three-vector form in Eq. (2.16). If the spinor has been transformed between two points of a system by a matrix M, one can write

$$\psi(\theta) = M\psi(0)$$

and the Hermitian conjugate

$$\psi^\dagger(\theta) = \psi^\dagger(0)M^\dagger.$$

The transformed expression of Eq. (2.16) is therefore

$$\begin{aligned}\mathbf{P}(\theta) &= \psi^\dagger(\theta)\boldsymbol{\sigma}\psi(\theta) \\ &= \psi^\dagger(0)M^\dagger\boldsymbol{\sigma}M\psi(0)\end{aligned} \qquad (4.8)$$

Although the three-vector form is perfectly well defined, the reduction is rather cumbersome and it is easier to evaluate the components separately.

For the x component,

$$P_x(\theta) = \psi^\dagger(0)M^\dagger\sigma_x M\psi(0). \qquad (4.9)$$

The matrix $M^\dagger\sigma_x M$ is Hermitian, since

$$(M^\dagger\sigma_x M)^\dagger = M^\dagger\sigma_x^\dagger(M^\dagger)^\dagger = M^\dagger\sigma_x M.$$

Expanding, one has

$$\begin{aligned}M^\dagger\sigma_x M = &[IC_0 + i\sigma_x C_x + i\sigma_y C_y + i\sigma_z C_z] \\ &\sigma_x[IC_0 - i\sigma_x C_x - i\sigma_y C_y - i\sigma_z C_z]\end{aligned}$$

which, after a little algebra, reduces to

$$\begin{aligned}M^\dagger\sigma_x M = &\sigma_x\{C_0^2 + C_x^2 - C_y^2 - C_z^2\} \\ &+ \sigma_y\{2(C_xC_y - C_0C_z)\} \\ &+ \sigma_z\{2(C_0C_y + C_xC_z)\}.\end{aligned} \qquad (4.10)$$

Similarly, one obtains for the other components

$$\begin{aligned}M^\dagger\sigma_y M = &\sigma_x\{2(C_0C_z + C_xC_y)\} \\ &+ \sigma_y\{C_0^2 + C_y^2 - C_x^2 - C_z^2\} \\ &+ \sigma_z\{2(C_yC_z - C_0C_x)\}\end{aligned} \qquad (4.11)$$

$$\begin{aligned}M^\dagger\sigma_z M = &\sigma_x\{2(C_zC_x - C_0C_y)\} \\ &+ \sigma_y\{2(C_0C_x + C_yC_z)\} \\ &+ \sigma_z\{C_0^2 + C_z^2 - C_x^2 - C_y^2\}.\end{aligned} \qquad (4.12)$$

Thus, in the development of Eq. (4.8), the final coefficients of the unit matrix I vanish for all three components, which reflects the Hermitian property. Using a compacted notation for the coefficients of the Pauli matrices in Eq. (4.10), the x component of Eq. (4.9) becomes

$$\begin{aligned}P_x(\theta) &= \psi^\dagger(0)\,[T_{xx}\sigma_x + T_{xy}\sigma_y + T_{xz}\sigma_z]\psi(0) \\ &= T_{xx}\psi^\dagger(0)\sigma_x\psi(0) + T_{xy}\psi^\dagger(0)\sigma_y\psi(0) \\ &\quad + T_{xz}\psi^\dagger(0)\sigma_z\psi(0),\end{aligned}$$

or

$$P_x(\theta) = T_{xx}P_x(0) + T_{xy}P_y(0) + T_{xz}P_z(0). \qquad (4.13)$$

The other components can be similarly expressed and Eq. (4.8) can then be written as a 3×3

matrix transformation

$$\begin{pmatrix} P_x(\theta) \\ P_y(\theta) \\ P_z(\theta) \end{pmatrix} = \begin{pmatrix} T_{xx} & T_{xy} & T_{xz} \\ T_{yx} & T_{yy} & T_{yz} \\ T_{zx} & T_{zy} & T_{zz} \end{pmatrix} \begin{pmatrix} P_x(0) \\ P_y(0) \\ P_z(0) \end{pmatrix}, \quad (4.14)$$

where the elements of the T matrix are given in full in the table below.

The T matrix is real and orthogonal, i.e.,

$$\sum_i T_{ij} T_{ik} = \delta_{jk}; \quad i, j, k = x, y, z,$$

where δ_{jk} is the Kronecker δ-symbol. This is easily verified using the normalization condition of the spinor-matrix coefficients in Eq. (4.4). These coefficients appear in quadratic form in the T_{ij} elements because the characteristic half-angles of spinors must transform to full angles in the real three-space of the T matrix.

The elements of the T matrix can be used to impose conditions on the coefficients C_i of the M matrix in order to obtain a desired relation between $\mathbf{P}(\theta)$ and $\mathbf{P}(0)$. For example, if $P_z(0) = +1$, and one requires $P_x(\theta) = -1$, the minimum constraint is

$$T_{xz} = 2(C_x C_z + C_0 C_y) = -1.$$

It follows from the orthogonal property that T_{yz} and T_{zz} must then both vanish, i.e.,

$$2(C_y C_z - C_0 C_x) = 0$$

and

$$C_0^2 + C_z^2 - C_x^2 - C_y^2 = 0.$$

Simple examples like this can be readily inferred directly from the spinor matrix M, but in more complicated cases the use of the T-matrix elements may be more direct.

5. SPIN MOTION IN A STORAGE RING

Particles moving along a given closed orbit in a storage ring (or accelerator) experience magnetic fields which are periodic at revolution frequency, and the corresponding axial vector $\mathbf{\Omega}$ in Eqs. (3.1) and (3.2) is therefore also periodic. One then expects to find a periodic solution of the spin motion corresponding to the given closed orbit. Because the representations in real three-space and in the complex two-space of spinors are isomorphic, there must also be periodic spinor solutions.

5.1 *Eigenvalues and eigenvectors*

We now consider the M matrix, in the form given by Eq. (4.5), to represent the spinor transformation around one full revolution at some arbitrary azimuth of the machine. Instead of ψ we use k_μ ($\mu = a, b$) to represent the two eigenvectors (closed spinor solutions) of the equation

$$(M - \lambda_\mu I) k_\mu = 0. \quad (5.1)$$

Since M is unitary, i.e., $M^\dagger M = I$, the eigenvalues λ_μ lie on the unit circle. The secular equation is

$$|M - \lambda I| = \lambda^2 - \lambda \operatorname{Tr} M + |M| = 0,$$

whence, from Eq. (4.5),

$$\lambda^2 - 2\lambda C_0 + 1 = 0$$

with eigenvalues

$$\lambda_\mu = C_0 \pm i\sqrt{1 - C_0^2} \quad (5.2)$$

which can be written

$$\lambda_\mu = \cos\frac{\phi}{2} \pm i \sin\frac{\phi}{2} = e^{\pm i\phi/2}. \quad (5.3)$$

Here, as in Section 4.2, the coefficient $C_0 = 1/2 \operatorname{Tr} M$ is associated with a rotation ϕ, which

$C_0^2 + C_x^2 - C_y^2 - C_z^2$	$2(C_x C_y - C_0 C_z)$	$2(C_x C_z + C_0 C_y)$
$2(C_x C_y + C_0 C_z)$	$C_0^2 + C_y^2 - C_x^2 - C_z^2$	$2(C_y C_z - C_0 C_x)$
$2(C_z C_x - C_0 C_y)$	$2(C_y C_z + C_0 C_x)$	$C_0^2 + C_z^2 - C_x^2 - C_y^2$

is the precession angle of a polarization vector in one revolution.

If $C_0^2 = 1$ the two eigenvalues are equal and real, $\lambda_\mu = \pm 1$, $M = \pm I$, and Eq. (5.1) is satisfied by any arbitrary spinor k. This degenerate situation corresponds to an integral spin resonance; any arbitrary solution is periodic but there is no stability against perturbations.

For $C_0^2 \neq 1$ the eigenvectors k_μ can be found from the cofactors of the first row of the matrix

$$M - \lambda_\mu I = \begin{pmatrix} C_0 - iC_z - \lambda_\mu & -C_y - iC_x \\ C_y - iC_x & C_0 + iC_z - \lambda_\mu \end{pmatrix}, \quad (5.4)$$

and are given in unnormalized form by

$$k_\mu = \begin{pmatrix} C_0 + iC_z - \lambda_\mu \\ -C_y + iC_x \end{pmatrix} = \begin{pmatrix} iC_z \mp i\sqrt{1 - C_0^2} \\ -C_y + iC_x \end{pmatrix} \quad (5.5)$$

where the two eigenvectors k_a, k_b are distinguished by the sign of the square root. For normalization the eigenvectors are divided by $(k_\mu \dagger k_\mu)^{1/2}$ where, from Eq. (5.5),

$$k_\mu \dagger k_\mu = 2(1 - C_0^2) \mp 2C_z\sqrt{1 - C_0^2}. \quad (5.6)$$

It is also easily verified that the eigenvectors are orthogonal, i.e., $k_\mu \dagger k_\nu = 0$ for $\mu \neq \nu$.

5.2 *Periodic solution* **n** *in real space*

To the periodic solutions k_μ in spinor space correspond periodic spin solutions in real three-space, which can be evaluated by the methods of Sections 2.3 and 4.4. It is convenient to distinguish these real periodic solutions from arbitrary spin (polarization) vectors by using the notation **n** instead of **P**. For the x component, one first forms the un-normalized product:

$$k_\mu \dagger \sigma_x k_\mu = \{-i(C_z \mp \sqrt{1 - C_0^2}),\ -(C_y + iC_x)\} \times \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} i(C_z \mp \sqrt{1 - C_0^2}) \\ -(C_y - iC_x) \end{pmatrix} = 2C_x(C_z \mp \sqrt{1 - C_0^2}).$$

This can also be obtained directly from the spinor components using Eq. (2.13). For normalization we must divide by Eq. (5.6), whence

$$n_x = \frac{k_\mu \dagger \sigma_x k_\mu}{k_\mu \dagger k_\mu} = \frac{2C_x(C_z \mp \sqrt{1 - C_0^2})}{2(1 - C_0^2) \mp 2C_z\sqrt{1 - C_0^2}},$$

which reduces to

$$n_x = \frac{\mp C_x}{\sqrt{1 - C_0^2}}. \quad (5.7)$$

Similarly one can obtain

$$n_y = \frac{\mp C_y}{\sqrt{1 - C_0^2}} \quad (5.8)$$

and

$$n_z = \frac{\mp C_z}{\sqrt{1 - C_0^2}}. \quad (5.9)$$

It is evident that $\mathbf{n} \cdot \mathbf{n} = n_x^2 + n_y^2 + n_z^2 = 1$. Furthermore, in the case of degeneracy when $C_0^2 = 1$, the components of **n** are not defined.

The periodic solution **n** plays an important role in the calculation of spin kinetics, since it is the (local) direction about which other spin solutions precess. Corresponding to each closed orbit in a storage ring there is a periodic spin solution **n** which is, in general, a function both of the azimuth θ and of the energy γ defining the closed orbit, i.e., $\mathbf{n} = \mathbf{n}(\theta, \gamma)$. It is evident that **n** is the spin analogue of the closed orbit itself, through which it is determined. Pursuing the analogy further, it is clear that the local direction of **n** is the polarization projection of an ensemble of particles whose individual spins are precessing around **n**, in the same way that a closed orbit represents the average trajectory of an ensemble of particles with betatron oscillations.

The vector **n** provides a suitable basis for the detailed analysis of spin motion in storage rings using perturbation theory; it acts as a reference vector[7] for calculating perturbed motion arising from betatron and synchrotron oscillations and, in the case of electron storage rings, from the kinetics of quantum emission and synchrotron-radiation damping. In the present context we restrict ourselves to the behaviour of the closed solution **n**, for which the use of two-component spinor algebra is particularly convenient.

In an ideal storage ring in which the closed orbit lies everywhere in the median plane, the only magnetic field component at the orbit is B_z. The spinor transformation around one revolution is then given by Eq. (3.7), with $\theta = 2\pi$ in Eq. (3.6), and can be written

$$M_0 = I\cos(\pi\gamma a) - i\sigma_z \sin(\pi\gamma a). \quad (5.10)$$

Comparison with Eqs. (5.7) to (5.9) shows that $C_x = C_y = 0$, $C_0 = \cos(\pi\gamma a)$, $C_z = \sin(\pi\gamma a)$, and $n_z = \pm 1$, independent of γ. The closed solution is thus either parallel or antiparallel to the magnetic field at all energies for which $C_0^2 \neq 1$. If (γa) is integer, $C_0 = \pm 1$ and n_z is not defined; this is the case of an integer (or imperfection) spin resonance as previously noted.

If the closed orbit deviates from the median plane, either by accident or design, radial fields are present and the coefficients C_x, C_y do not generally vanish. It is then instructive to examine the characteristics of a simple model consisting of a perfect machine of the form of Eq. (5.10) containing a single local perturbation represented quite generally by the spinor transformation

$$M_a = IA_o - i\sigma_x A_x - i\sigma_y A_y - i\sigma_z A_z. \quad (5.11)$$

In order to evaluate the parameters at an arbitrary azimuth in the unperturbed part of the machine, the latter is divided into two parts with matrices

$$M_1 = I\cos\left(\frac{\lambda\chi}{2}\right) - i\sigma_z \sin\left(\frac{\lambda\chi}{2}\right) \quad (5.12)$$

and

$$M_2 = I\cos\left\{\frac{(1-\lambda)\chi}{2}\right\} - i\sigma_z \sin\left\{\frac{(1-\lambda)\chi}{2}\right\} \quad (5.13)$$

respectively. Here $\chi = 2\pi\gamma a$ and λ is a parameter $(0 \leq \lambda \leq 1)$ defining the azimuth (in bending angle) at which the matrix for one revolution is evaluated. This matrix is

$$\begin{aligned} M &= M_1 M_a M_2 \\ &= \left[I\cos\left(\frac{\lambda\chi}{2}\right) - i\sigma_z \sin\left(\frac{\lambda\chi}{2}\right)\right] \\ &\quad \times \left[IA_o - i\sigma_x A_x - i\sigma_y A_y - i\sigma_z A_z\right] \\ &\quad \times \left[I\cos\left\{\frac{(1-\lambda)\chi}{2}\right\} - i\sigma_z \sin\left\{\frac{(1-\lambda)\chi}{2}\right\}\right] \end{aligned}$$

which, after multiplying out, can be expressed in canonical form as

$$M = IC_o - i\sigma_x C_x - i\sigma_y C_y - i\sigma_z C_x,$$

with

$$\left.\begin{aligned} C_o &= A_o \cos\frac{\chi}{2} - A_z \sin\frac{\chi}{2} \\ C_x &= A_x \cos(2\lambda - 1)\frac{\chi}{2} + A_y \sin(2\lambda - 1)\frac{\chi}{2}. \\ C_y &= A_y \cos(2\lambda - 1)\frac{\chi}{2} - A_x \sin(2\lambda - 1)\frac{\chi}{2} \\ C_z &= A_o \sin\frac{\chi}{2} + A_z \cos\frac{\chi}{2}. \end{aligned}\right\} \quad (5.14)$$

For a vanishing perturbation $A_o = 1$, $A_x = A_y = A_z = 0$ and Eqs. (5.14) and (5.10) become identical.

As a simple example of a local perturbation, we can consider a solenoid field parallel to the beam direction; the corresponding spinor matrix is

$$\begin{aligned} M_a &= IA_o - i\sigma_y A_y \\ &= I\cos\frac{\phi}{2} - i\sigma_y \sin\frac{\phi}{2}, \quad (5.15) \end{aligned}$$

where ϕ is the precession angle around the field due to the solenoid. An extreme case of this arises if $\phi = \pi$, which makes $A_o = 0$, $A_y = 1$; Eqs. (5.14) then become

$$\left.\begin{aligned} C_0 &= 0 \\ C_x &= \sin(2\lambda - 1)\frac{\chi}{2} \\ C_y &= \cos(2\lambda - 1)\frac{\chi}{2} \\ C_z &= 0. \end{aligned}\right\} \quad (5.16)$$

This is the basis of the proposal, by Derbenev and Kondratenko[4], now known as the Siberian

Snake; it is characterized by the vanishing of C_0, independently of energy, provided the condition $\phi = \pi$ is maintained. Since the Pauli matrices are traceless, the only contribution to the trace of M comes from the coefficient C_0 of the unit matrix I, and we can write quite generally

$$\frac{1}{2}\operatorname{Tr} M = C_0 = \cos(\pi\nu), \tag{5.17}$$

where ν is the effective precession wave number (or spin tune). For the unperturbed machine of Eq. (5.10), $\nu = \gamma a$ and is therefore proportional to energy. If $C_0 = 0$ however, $\cos(\pi\nu) = 0$ and ν is half integer. This scheme makes it possible, in principle, to accelerate polarized beams over a wide energy range at a fixed spin tune, thus avoiding the crossing of resonances.

It can also be seen from Eqs. (5.16) that diametrically opposite the Snake, where $\lambda = 0.5$, $C_x = 0$ and $C_y = 1$. At this position the closed solution $\mathbf{n}$ has only the y-component ± 1 of Eq. (5.8). Elsewhere around the machine, $\mathbf{n}$ lies in the median plane with an orientation depending on the azimuth.

Transformations of the type indicated by Eq. (5.15) can be obtained using only transverse-field magnets[4,9] and variants using two Snakes of different types[10] can avoid some of the drawbacks of the simplest version. Such schemes can readily be analysed by the same methods as are discussed here. In a similar way, we can examine the effects of unintentional perturbations, such as closed-orbit deviations or incompletely compensated solenoid fields, which can in general modify the projections of $\mathbf{n}$ on the three axes as well as change the spin tune.

5.3 *Variation of* n *with energy*

The energy dependence of $\mathbf{n}$ is most clearly presented in the form $\gamma(\partial\mathbf{n}/\partial\gamma)$, which represents the fractional change in $\mathbf{n}$ for a given fractional change in energy, $\mathbf{n}$ being of unit magnitude. In terms of the components, Eqs. (5.7) to (5.9) yield

$$\begin{aligned}\gamma\frac{\partial n_i}{\partial\gamma} &= \mp\gamma\frac{\partial}{\partial\gamma}\left[\frac{C_i}{\sqrt{1-C_0^2}}\right] \quad (i = x, y, z)\\ &= \mp\frac{2\pi\gamma a}{(1-C_0^2)^{3/2}}\left[(1-C_0^2)\frac{\partial C_i}{\partial\chi} + C_0C_i\frac{\partial C_0}{\partial\chi}\right],\end{aligned} \tag{5.18}$$

since $\chi = 2\pi\gamma a$. In the above it is assumed that the local perturbation contributes only a small fraction of the total precession around the machine, and that the variation of the coefficients A_0, A_x, A_y, A_z with energy may therefore be neglected. From Eq. (5.14) we can then write

$$\left.\begin{aligned}\frac{\partial C_0}{\partial\chi} &= -\frac{1}{2}\left[A_0\sin\frac{\chi}{2} + A_z\cos\frac{\chi}{2}\right]\\ &= -\frac{C_z}{2}\\ \frac{\partial C_x}{\partial\chi} &= -\frac{(2\lambda-1)}{2}\left[A_x\sin(2\lambda-1)\frac{\chi}{2}\right.\\ &\qquad\left. - A_y\cos(2\lambda-1)\frac{\chi}{2}\right]\\ &= (2\lambda-1)\frac{C_y}{2}\\ \frac{\partial C_y}{\partial\chi} &= -\frac{(2\lambda-1)}{2}\left[A_y\sin(2\lambda-1)\frac{\chi}{2}\right.\\ &\qquad\left. + A_x\cos(2\lambda-1)\frac{\chi}{2}\right]\\ &= -(2\lambda-1)\frac{C_x}{2}\\ \frac{\partial C_z}{\partial\chi} &= \frac{1}{2}\left[A_0\cos\frac{\chi}{2} - A_z\sin\frac{\chi}{2}\right] = \frac{C_0}{2}.\end{aligned}\right\} \tag{5.19}$$

Using these expressions in Eq. (5.18) for the corresponding components results in

$$\left.\begin{aligned}\mp\gamma\frac{\partial n_x}{\partial\gamma} &= \frac{\pi\gamma a}{(1-C_0^2)^{3/2}}\\ &\quad\times[(2\lambda-1)(1-C_0^2)\,C_y - C_0C_xC_z]\\ \mp\gamma\frac{\partial n_y}{\partial\gamma} &= -\frac{\pi\gamma a}{(1-C_0^2)^{3/2}}\\ &\quad\times[(2\lambda-1)(1-C_0^2)\,C_x + C_0C_yC_z]\\ \mp\gamma\frac{\partial n_z}{\partial\gamma} &= \frac{\pi\gamma a C_0}{(1-C_0^2)^{3/2}}[C_x^2 + C_y^2].\end{aligned}\right\} \tag{5.20}$$

The term $(1 - C_0^2)^{3/2}$ common to all three components constitutes a resonance denominator since, from Eq. (5.17), it can be written

$$(1 - C_0^2)^{3/2} = \sin^3(\pi\nu). \tag{5.21}$$

Depending on the nature and strength of the perturbation, $\gamma(\partial\mathbf{n}/\partial\gamma)$ can become very large in the vicinity of a spin resonance, where ν approaches integer values. If, for example, in the third of Eqs. (5.20) we replace C_x, C_y from Eqs. (5.14), the z component becomes

$$\mp\gamma\frac{\partial n_z}{\partial\gamma} = \frac{\pi\gamma a\cos(\pi\nu)}{\sin^3(\pi\nu)}[A_x^2 + A_y^2], \tag{5.22}$$

from which it follows that the presence of appreciable x or y components in the perturbation matrix can result in a large energy dependence of $\mathbf{n}$ near a resonance. For an unperturbed machine, $A_x = A_y = 0$, hence $C_x = C_y = 0$ and $\gamma(\partial\mathbf{n}/\partial\gamma)$ vanishes.

The quantity $\gamma(\partial\mathbf{n}/\partial\gamma)$, which has been given the name "spin chromaticity" by Buon,[11] is of particular importance in electron storage rings. It can give rise to both depolarizing and polarizing effects[8] in conjunction with synchrotron-radiation energy loss, and is an important ingredient in a possible method of polarizing electrons by collision with soft photon beams.[12] It should be noted that the simple Siberian Snake represented by Eq. (5.16) has intrinsically a large value of spin chromaticity, which makes it unsuitable for electron machines in the high-energy range.

The depolarizing effects[8] of spin chromaticity appear in the form $(\gamma\partial\mathbf{n}/\partial\gamma)^2$, which can be evaluated for a local perturbation from Eqs. (5.20). Using also Eqs. (5.14) one obtains

$$\left(\gamma\frac{\partial\mathbf{n}}{\partial\gamma}\right)^2 = \frac{\pi^2(\gamma a)^2}{1 - C_0^2}(A_x^2 + A_y^2) \times\left[(2\lambda - 1)^2 + \frac{C_0^2}{1 - C_0^2}\right]. \tag{5.23}$$

If ν is half-integer, $C_0 = \cos(\pi\nu) = 0$; this is the case for a normal machine mid-way between two integer resonances, or for a Siberian Snake scheme with nominal parameters. Then, at a position $\lambda = \frac{1}{2}$ diametrically opposite the perturbation, the spin chromaticity vanishes. This feature is of importance in determining the position of high-field magnetic elements with strong quantum excitation, such as wigglers, in a machine with a known systematic local perturbation.

For a storage ring with isomagnetic bending fields, the average of Eq. (5.23) around the circumference is a measure of the depolarizing effect of a single perturbation in the presence of quantum excitation. Integrating with respect to λ from 0 to 1 yields

$$\left\langle\left(\gamma\frac{\partial\mathbf{n}}{\partial\gamma}\right)^2\right\rangle = \frac{\pi^2(\gamma a)^2}{3(1 - C_0^2)^2} \times (A_x^2 + A_y^2)(1 + 2C_0^2). \tag{5.24}$$

A low depolarization rate requires that Eq. (5.24) be small compared with unity, which puts upper limits on acceptable values of $(A_x^2 + A_y^2)$ as a function of the energy and the exact value of the spin tune.

6. CONCLUDING NOTES

This paper has been written as an elementary introduction to the use of spinor algebra for treating problems of beam polarization in storage rings. In the interests of simplicity, the scope has been restricted to quasi-static situations and piecewise-constant transformations, but the formalism is equally applicable to more complicated problems where the parameters may vary in time, or may have a less simple space variation. The dynamic crossing of spin resonances was first investigated by Froissart and Stora[13] using spinor notation.

The spinor transformations used here correspond to the solutions of pairs of linear, first-order differential equations in two complex variables, the components ψ_1, ψ_2 of the spinor ψ, but problems can also be formulated and solved in terms of one linear, second-order differential equation of a single spinor component, as in Ref. 13.

It will have been noticed that the spinor ψ has a rather ephemeral existence in this treatment, like the wave function or state vector in quantum mechanics. It does not appear to be useful to seek a relation between the spinor components and the polarization vector, although in Ref. 1, Appendix IV, some geometric properties of quarternions can be discerned.

Apart from the formal elegance of spinor algebra there is the practical advantage that problems can be treated with a near-minimum redundancy of parameters whilst still retaining a large degree of physical insight. Despite the use of complex numbers, only four real coefficients are required to describe classical spin motion in an element. For numerical computation, the transformation properties of the Pauli matrices can be represented simply by real operations on these coefficients, and the invariant normalization property can be used for checking.

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