ELEMENTS FOR A THEORY OF RELATIVISTIC COORDINATE SYSTEMS. FORMAL AND PHYSICAL ASPECTS.[a]

Bartolomé Coll
Systèmes de Référence Relativistes, DANOF-CNRS
Observatoire de Paris, 61 Avenue de l'Observatoire,
75014 Paris, France
E-mail: bartolome.coll@obspm.fr

Basic elements for a relativistic theory of coordinate systems are introduced. The main purposes of such a theory are to precise the physical and geometrical status of coordinate systems in general relativity, to structure those presently known, to offer a convenient scheme to incorporate new ones, to reveal voids in our knowledge of their mutual relations, and to incitate their study. Relativistic operational criteria to construct coordinate systems are given, with particular attention to satellite positioning systems, to which the current GPS could be related.

1 Introduction

This talk concerns coordinate systems. They appear in mathematics as well as in physics. And frequently, in theoretical studies, it is not clear when we are using them as mathematical tools or as physical objects. In fact, their status is rather ambiguous, even in a purely geometric context. But the rather intensive use that physicists and mathematicians make of them, the need of better understanding the most part of them and the conviction that they will play increasing roles in physics, justify to tackle the construction of a theory of coordinate systems. Taking into account the progress in experimental precision and the already present relativistic effects in earth positioning, this theory cannot be but a Relativistic Theory of Coordinate Systems. I believe that things and thoughts are ripe to begin this construction.

The structure of such a relativistic theory of coordinate systems is better that of a theory in the sense of Number Theory than that of a theory in the sense of a pyramid-structured deductive theorie, like Set Theory. Like in Number Theory, where numbers are considered as elements of classes (odd, prime, Fibonacci, Bernouilly, etc.), in the theory of relativistic coordinate systems, coordinate systems are not considered individually, but as elements of classes (harmonic, Born, Fermi, co-moving, symmetric, light-like, etc). Of these classes, their existence, internal group structure, intrinsic and extrinsic ways of definition and of generation of their elements, as well as their mutual

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relations, or nature of their intersections, generically creating new classes, are taken into account. These unavoidable aspects constitute the formal part of the theory.

But, as physicists, we are interested in coordinate systems not only as mathematical tools to obtain tensorial results, but also as a true way to reference the space-time in our neighbourhood. So, we are specially interested in their physical realization. This is the physical and operational part of the theory.

The purpose of this talk cannot be to develop this vast program, but only to present a miscellany of some of these aspects. A good part of the results presented here have been obtained in cheerful collaboration with my friends Lluís Bel, Joan Josep Ferrando, Juan Antonio Morales and Albert Tarantola, to which I am very indebted.

2 Aims of a Relativistic Theory of Coordinate Systems

Without exhaustive character, the aims of a relativistic theory of coordinate systems are:

- to specify the geometrical and physical status of coordinate systems,
- to determine the internal groups associated to some classes of coordinate systems (or eventually, their corresponding algebras),
- to find the compatibility relations between the usual classes of coordinate systems (harmonics, Born, Fermi, of evolution, co-moving, adapted to symmetries, etc.),
- to study the geometrical (existence in any space of given dimension) and dimensional (existence in spaces of arbitrary dimension) genericity of many other coordinate systems of physical interest,
- to sound out the geometry of space-times by means of specific coordinate systems,

in order to give an adequate framework to collect all known results, to incorporate the new ones and specially to reveal the absence of results in some usual situations, but also...

- to classify coordinate systems by their causal character, related to their physical construction (there exist 199 causal classes[1], that is to say, roughly speaking, 197 unexplored ones).
- to introduce appropriate notions (extrinsic and intrinsic, integral and differential definitions of generic, primary, immediate, retarded coordinate systems),

allowing to better delimit the physically operational characteristics of coordinate systems.

3 Status of Coordinate Systems in Theoretical Physics

Today, more than 80 years after the creation of the General Theory of Relativity, we are met here to talk about reference frames. Why? Because, in spite of their apparent simplicity, there remain many obscurities on this subject.

I believe that one of the main reasons for this situation is the ambiguity of the status of coordinate systems in physics. Compare, for example, the status of a Cartesian coordinate system in this hall, with origin at that corner and with axis the intersections of the walls and the floor, and that of the Global Positioning System (GPS); the first one seems to consist essentially of a *convention*, the last one stands on the *physical* signals cast by the constellation of satellites and pick up by the GPS receptor.

This ambiguity of possible status is correlated with an almost total absence of epistemological analysis concerning coordinate systems (role of the coordinate systems in obtaining physical laws, in diffusion of experimental protocols, in experimental determination of the tensorial character of physical quantities, etc.).

In geometry, coordinate systems are used to define the structure of differentiable manifolds. In this sense they are *mathematical* objects, but they cannot be *geometrical* objects (roughly speaking, because a geometrical object is a mathematical object that does not change when the coordinate system changes). Coordinates are used in geometry as an scaffolding for the construction of geometrical results; but, once these results obtained, the scaffolding is taken down and the results are presented "clean", free from this "annoying" piece.

Perhaps this is why in theoretical physics almost all physicists consider coordinates as a mathematical tool having no physical interest. This point of view is reinforced by the fact that in the geometrical structure of relativistic theories, physical quantities are represented by tensorial objects.

In fact, this point of view may be compatible with theoretical works on already established formalized theories (because there coordinate systems appear essentially as formal scaffolds allowing mathematical handlings), but in other domains of physics, coordinate systems are basic physical pieces that allow:

- to detect experimentally the tensorial character of the significant physical quantities of an experiment,

- to find experimentally the appropriate tensorial laws relating different physical quantities,

- to ameliorate the experimental definition and determination of physical events, which carries the same type of physical progress than that associate to ameliorations in the definition and the determination of the physical units.

3.1 Physical Status of Coordinate Systems

A qualitative description of physics, as one finds in dictionaries, is the following one: "scientific study of properties of matter and energy". But, because today the devices used for this task play an essential role, they may be explicitly mentioned: "physics is the scientific study of properties of matter and energy by means of ad hoc devices". In fact, I propose the following slightly different qualitative description of physics:

scientific study of interactions between matter
and energy with ad hoc devices.

If the *interaction* may be reduced to a simple *action* of matter and energy on the ad hoc devices, the phenomenon would be classic, otherwise it would be quantum.

The interest of this last description is that devices, and in particular coordinate systems, when used in physics, appear on an equal footing with matter and energy, that is to say, they are physical objects:

coordinate systems are **physical objects.**

Thus, from now on, we shall take as definition of a coordinate system for classical, not quantum, purposes, the following one:

> *A coordinate system in a region of the space-time is a set of real or virtual, passive (test) physical fields, controled and parametrized in such a way to localize every one of the points of the region from the values of the parameters.*

3.2 Mathematical Status of Coordinate Systems

Because physical objects of space-time correspond to geometrical objects on a four-dimensional manifold, and because coordinate systems are generically of local character, coordinate systems on the four-dimensional manifold must be described by local geometric objects. Fortunately, such a description is trivial.

Mathematically, a coordinate system is a set of n (local) coordinate functions on a n-dimensional manifold. These n coordinate functions biunivocally define $\binom{n}{p}$, $0 < p < n$, families of coordinate p-surfaces, namely those obtained when $n - p$ coordinate functions take fixed values. The more usual of these p-surfaces being the n congruences of coordinate lines ($p = 1$) and the n families of coordinate hypersurfaces ($p = n - 1$) .

These families of surfaces are local geometric objects, so that conversely:

> *Physical systems of coordinates on the space-time may be represented by local geometric objects on four-dimensional manifolds, namely by a sufficient and compatible number of local p-surfaces, for some p's such that* $4 < p < 0$.

4 Some Notions on Coordinate Systems

A *definition* of a coordinate system is said *extrinsic* if it refers to another given system. Example: polar coordinates $\{r, \theta\}$ in terms of Cartesian ones, $r^2 = x^2 + y^2$, $\theta = \arctan y/x$.

A *definition* of a coordinate system is said *intrinsic* if it does not refer to a given system. Example: $\{r, \theta\}$, r being the affine parameter of a focal congruence of straight lines, and θ the angle that every line forms with respect to one of them.

An *intrinsic definition* may be *integral*, as in the example above, or *differential* as in the following example: $\{r, \theta\}$ such that $\nabla d\, r = (1/r)(g - dr \otimes dr)$, $dr \cdot d\theta = 0$, $r|d\theta| = \cos 2\theta$. The conditions $\Delta x^\alpha = 0$ for harmonic coordinates constitute another example of a differential intrinsic definition.

A coordinate system is said *generic for a class of space-times* if it may be constructed on any space-time of this class.

An interesting notion is that of *Yano* (or centrifugal) *vector field* associated to a coordinate system, which allows to classify coordinate systems and to describe some of them.

For every coordinate system $\{x^i\}$, the Yano or centrifugal vector field associated to it is the vector field ξ whose components are $\xi^i = x^i$, i.e. the analogue of the position vector for Cartesian coordinates.

Yano showed the converse[2], that to every vector field ξ , there are associated a set of *Yano coordinate systems* $\{x^i\}$, i.e. those in which the vector field has components x^i . Normal coordinates and homothetic vector fields are particular examples of the usefulness of this notion.

5 Harmonic Coordinates

There is no room here to collect the potpourri of questions and answers of the talking version. This section will only be devoted to the open problem of harmonic exterior coframes.

In flat metric spaces it is possible to find specific coordinate systems : those called Cartesian coordinate systems. Is it possible to find, in every metric space, *similar* specific coordinate systems? No general answer is known for this problem.

In particular curved space-times, harmonic coordinates were assimilated by Fock as the analogs of Cartesian coordinates, but under addditional global conditions. Nevertheless, the intrinsic structure of differential geometry being local, and the existence of Cartesian coordinates being local for locally flat spaces, the correct answer to this problem seems to be a *local* answer.

On the other hand, if we want these coordinates to be specific of the given metric, the differential operators determining the coordinates must be concomitants of the metric. The first differential operator concomitant of the metric is the Laplacian. This suggests the old Focks' idea that harmonic coordinates are candidates to be considered as analogues to Cartesian coordinates for non flat spaces. But local harmonic coordinates are excessively abundant, as compared with Cartesian ones, so that *additional* restrictions are to be imposed. To this end, let us observe that the natural frame for covectors (= coframe) dx^α of a system of harmonic coordinates $\{x^\alpha\}$, $\Delta x^\alpha = 0$, is also harmonic, $\Delta(dx^\alpha) = 0$, but that the corresponding natural frame for cotensors of order p , $p \neq 1$, $\{x^\alpha\}$, $\Delta x^\alpha = 0$, is *not* harmonic in general.

Now, the way to impose additional conditions is clear; we give the following definition:

> *A system of coordinates is said total covariant harmonic if, for any p , $0 < p \leq n$, the natural basis for p-cotensors,*
>
> $$dx^{\alpha_1} \otimes ... \otimes dx^{\alpha_n}$$
>
> *is harmonic:*
>
> $$\Delta(dx^{\alpha_1} \otimes ... \otimes dx^{\alpha_n}) = 0 \ .$$

The following two questions are open: does any space-time admit total covariant harmonic coordinates? if not, what are the space-times that admit them?

This problem is mathematically difficult to analyze, essentially due to the complicate properties of Laplacians over the tensorial algebra of symmetric tensors. We may restrict it to the exterior algebra, and give the following definition:

A system of coordinates is said exterior covariant harmonic if, for any p , $0 < p \leq n$, the natural basis for p-forms,

$$dx^{\alpha_1} \wedge ... \wedge dx^{\alpha_n}$$

is harmonic:

$$\Delta(dx^{\alpha_1} \wedge ... \wedge dx^{\alpha_n}) = 0 \ .$$

The analog of the above two questions for this exterior version are also open, but we have an intermediate result. Denoting by δ the divergence operator on the exterior forms α, β of the exterior algebra Λ , the Schouten algebra $(\Lambda, \{\})$ is defined by the braket

$$\{\alpha, \beta\} = \delta\alpha \wedge \beta + (-1)^a \alpha \wedge \delta\beta - \delta(\alpha \wedge \beta) \ ,$$

where a is the degree of α , and we have[3]:

Proposition (Coll-Ferrando): *The De Rham Laplacian over the exterior forms,*

$$\Delta \equiv [d, \delta] = d\delta + \delta d \ ,$$

verifies

$$\Delta\alpha \wedge \beta + \alpha \wedge \Delta\beta - \Delta(\alpha \wedge \beta) =$$

$$\{d\alpha, \beta\} + (-1)^a \{\alpha, d\beta\} + d\{\alpha, \beta\}$$

for any two forms α, β .

By iteration, this proposition allows to show the following one.

Proposition (Coll-Ferrando): *The Laplacian of an exterior product of p closed exterior forms α^i , $i = 1, ..., p$, $d\alpha^i = 0$, admits the expression*

$$\Delta(\alpha^1 \wedge ... \wedge \alpha^p) =$$

$$\sum_{\substack{i=1 \\ j>1}}^{p-1} (-1)^{\pi(i,j;p)} \alpha^1 \wedge ... \hat{i} ... \hat{j} ... \wedge \alpha^p \wedge L^{ij} \ ,$$

where the forms L^{ij} associated to the pairs α^i, α^j are given by

$$L^{ij} \equiv \Delta(\alpha^i \wedge \alpha^j) - \frac{p-2}{p-1}(\Delta\alpha^i \wedge \alpha^j + \alpha^i \wedge \Delta\alpha^j) \quad ,$$

$\pi(i, j; p)$ is the parity associated to α^i and α^j , and the a_i's are the degrees of the α^i .

So we have the following

Theorem (Coll-Ferrando): *A harmonic coordinate system* $\{x^i\}$, $\Delta x^i = 0$ *, is an exterior covariant harmonic system if, and only if, it verifies*

$$\Delta(dx^i \wedge dx^j) = 0 \quad , \quad \forall i,j = 1,...,n \ .$$

6 Physical notions Related to Coordinate Systems

A *definition* of a coordinate system is said *operational* if it contains a detailed protocol for its construction. According to the general definition of a physical coordinate system given in Section 3, the detailed protocol must indicate the nature of the constituent physical fields, their parameterisation and their interrelation.

Like a formal one, an operational definition may be extrinsic or intrinsic.

A *coordinate system* is said *primary* if it is given by an intrinsic operational definition.

In Newtonian physics, the effective construction of coordinate systems, irrespective of their simple or sophisticated character, seems always related to constructions of *parametrized straight lines*. In relativistic physics, straight lines have to be substituted by *geodesics*; and the easiest way to produce generically parametrized geodesics are *light rays*.

But, because today the operational definitions of meters and seconds are obtained from light rays, the metric parametrization of any other type of geodesics (timelike or spacelike) involve also them. For this reason, one can say that in relativistic physics, the effective construction of coordinate systems has to be related to constructions of parametrized lightlike rays.

In fact, light rays may be considered as geometrical lines only for high frequency parametrizations. For low frequency the effective construction of coordinate systems has to be related to the duals of lightlike rays, i.e., to front waves.

Of course, light rays are usually taken into account in the effective construction of coordinate systems. Synges Chronometry is one of the best examples. In it, the scheme of the determination of the radial coordinate r (distance) of a particle P with respect to an inertial observer I , is represented in figure 1. In it, we see that, neither the observer, nor the particle, knows the value of this coordinate in real time. We are led to give the following definition:

A *primary coordinate system* is said *immediate* if the values of the coordinates of any event may be obtained without delay. Otherwise, the primary coordinate system is said *retarded*.

> *The main purpose of a Relativistic Theory of Coordinate Systems is the construction of generic immediate primary coordinate systems for the class of all gravitational fields.*

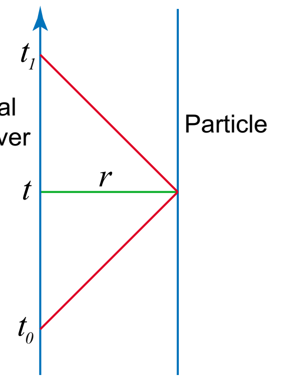

Figure 1: An inertial observer determines the position r of a point particle by means of a light ray. This method gives rise to a *retarded* coordinate system

7 Coordinate Systems of GPS type

Coordinate systems whose congruence of lines consist of light rays were considered many years ago[4]. They may be interesting in some physical situations and also as candidates to measure the gravitational field itself, but here their interest is stopped by the fact that we do not know if they are or not generic for all gravitational fields.

A *dual* version of them (the congruence of light rays being substituted by families of front waves) is the one constituted by four focal points emitting parametrized front waves. Specifically, we shall suppose here that the focal points are four satellites in geodetic motion, and that the parameterized front waves send their proper time.

Due to the similarities of such system with the current well known GPS (Global Positioning System) we shall call them *coordinate systems of GPS type* (see below for the differences).

It is easy to see that coordinate systems of GPS type are relativistic immediate coordinate systems, generic for *all* gravitational fields.

7.1 The current GPS System

The current GPS system consists of a *spatial segment*, constellation of 24 satellites around the earth, a *control segment*, of five earth stations, and a *user* *segment*, which has to allow military and civilian users to find their position by means of appropriate receivers.

Satellites broadcast over the earth their identification code, a shifted TAI (International Atomic Time) called the GPST and their WGS-84 (World Geodetic System) position, calculated and controled by the control segment.

Undoubtedly, the engineering work involved by the GPS, with its rigorous schedule of conditions, is admirable. But, once more GPS shows that, if it is important for technology and science to form an harmonic couple, it is equally important to clearly distinguish one of the other, their spirits being very different. If apart from maintenance and updating, the engineering work may be considered as achieved, the corresponding scientific work still has to be done. Not only the basic theory is not relativistically correct but the fundamental importance of the spatial segment has not been understood.

From the conceptual point of view, there are many inadequacies in theory and practise of the current GPS: *i)* the starting point of the theory must not be the control segment and the model of the earth, but the space segment and its light like coordinate systems; *ii)* satellites must not emit the GPST indicated by the control segment but every satellite must emit its proper time; *iii)* satellites must not emit their position with respect to the earth, but referred to the ICRS (Internation Celestial Reference System; *iv)* the control segment, which individually controls every satellite, must be substituted by a survey system, with the sole mission of surveying the satellites constellation as a whole; *v)* individual satellites must survey their proper trajectories with respect to the others (to fix ideas: if the coordinate system were a Cartesian one, an exterior control station would control the correct motion of the origin and the correct orientation of the system in space, the internal properties of the system, namely orthogonal axes and metric parameterization, would correspond to internal controls); *vi)* users must be able to survey the correct functioning of the system at any moment.

7.2 GPS as a Coordinate System

As compared with the current GPS system, our interest is the construction of a complete relativistic theory of coordinate systems of GPS type exempt of the above criticisms, with special interest in the geometric study of the spatial segment

In a 2D space-time representation, a coordinate system of GPS type is constituted by two (geodesic) satellites emitting their proper time:

According to our definition (subsection 3.1) of a coordinate system the real proper time parametric radiations in Ω define the immediate coordinate system associated to the GPS. Note that no observer drags it (the coordinate system is not co-moving).

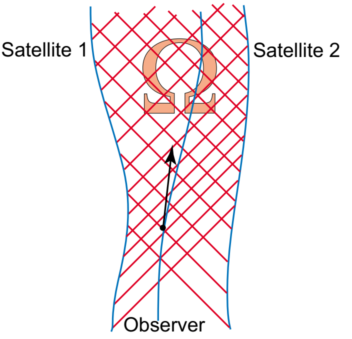

Figure 2: A 2D relativistic primary immediate generic coordinate system for all gravitational fields.

7.3 A Users's Control Result

Concerning the control of the system by users, we have obtained a result that we present in a simplified version:

Theorem: *In 2D Minkowski space-time, let S_1 and S_2 be two geodesic satellites emitting their proper times. Let s_1 and s_2 be respectively their values measured by an user U , and s_1' and s_2' the values respectively measured by S_2 and S_1 at the instants s_2 and s_1 . Then, the space-time metric is given by*

$$ds^2 = \sqrt{\frac{s_1\, s_2}{s_1'\, s_2'}}\; ds_1\, ds_2 \ .$$

Observe that, in terms of these proper times, the expression is independent not only of the velocities of the satellites in their constellation (relative velocities), but also of the velocity of the user with respect to the constellation.

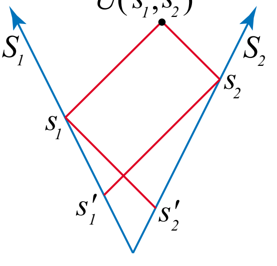

Figure 3: In 2D Minkowski space-time, the coordinates of an event and that of the two satellites allow to completely know the metric at the event.

7.4 Other points of interest of the system of type GPS

i) In the absence of gravitational fields, the theory may be developed in exact and complete form.

ii) By perturbation methods, the theory may be developed exact to first order and complete for the cases:

- perturbations dues to satellite trajectories or to emission frequencies.
- perturbations due to a field of refraction indices taking into account atmospheric influence.
- perturbations dues to weak gravitational fields.

iii) The theory may be generalized to different situations (solar system, post-Newtonian gravitational fields, etc)

iv) These coordinates naturally offer a natural frame for physical vectors and tensors of interest (velocity, acceleration, deformation tensor, electromagnetic field, etc).

v) The current treatment of the spatial segment may be assimilated to a treatment of action at a distance; meanwhile our approach is that of a field theory. If one thinks to the enriching points of view that gravitational (Poisson) or electromagnetic (Maxwell) field theories have introduced on the corresponding action at a distance previous theories, one may hope that our field theoretical approach will allow to see GPS systems under new interesting lights.

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