ELEMENTS OF STOCHASTIC CALCULUS VIA REGULARISATION

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Abstract.

This paper first summarizes the foundations of stochastic calculus via regularization and constructs through this procedure Itô and Stratonovich integrals. In the second part, a survey and new results are presented in relation with finite quadratic variation processes, Dirichlet and weak Dirichlet processes.

Key words: Integration via regularization, weak Dirichlet processes, covariation, Itô formulae.

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1 Introduction

Stochastic integration via regularization is a technique of integration developed in a series of papers by the authors starting from [45], continued in [46, 47, 48, 49, 44] and later carried out by other authors, among them [50, 11, 12, 54, 53, 55, 57, 16, 15, 17, 18, 23]. Among some recent applications to finance, we refer for instance to [31, 4].

This approach constitutes a counterpart of a discretization approach initiated by Föllmer ([19]) and continued by many authors, see for instance [2, 21, 14, 13, 10, 22].

The two theories run parallel and, at the axiomatic level, almost all the results we obtained via regularization can essentially be translated in the language of discretization.

The advantage of using regularization lies in the fact that this approach is natural and relatively simple, and can easily be connected with other approaches. We list now some typical features of the stochastic calculus via regularization.

- Two fundamental notions are the quadratic variation of a process, see Definition 2.3 and forward integral, see Definition 2.1. Calculus via regularization is first of all a calculus related to finite quadratic variation processes, see section 4. A novelty of the paper is a new construction of Itô integral with respect to continuous semimartingales based on forward integrals, see Section 3. Classical calculus will appear as a particular case of calculus via regularization. Let the integrator be a classical Brownian motion W and the integrand an adapted process H such that $\int_0^T H_t^2 dt < \infty$ a.s., where a.s. means almost surely. We will show that the forward integral $\int_0^\cdot H d^- W$ coincides with Itô integral $\int_0^\cdot H dW$, see section 3.5. On the other hand, the discretization approach constitues a sort of Riemann-Stieltjes type integral and only allows integration of processes that are not too irregular, see Remark 3.34. This fact is slightly alleviated using McShane type integrators, see the lines below Remark 3.34.

- The calculus via regularization constitutes a bridge between non causal and causal calculus operating through substitution formulae, see subsection 3.6. A precise link between our forward integration and the one given by the theory of enlargement of filtrations may be given, see [46]. Our integrals can be connected to the well-known Skorohod type integrals, see again [46].

- With the help of symmetric integrals a calculus with respect to processes having a higher variation than 2 may be developed. For instance the fractional Brownian motion is the prototype of such processes.

- This stochastic calculus constitutes some kind of barrier separating the pure pathwise calculus in the sense of T. Lyons and coauthors, see e.g. [35, 34, 30, 27], and any stochastic calculus taking into account an underlying probability. see Section Section 6.

This paper will essentially focuse on the first point.

The paper is organized as follows. First, in Section 2, we recall the basic definitions and properties of forward, backward, symmetric integrals and covariations. The related definitions and properties do not need a particular effort for justification. A significant example is Young integral, see [56]. In Section 3 we redefine Itô integrals in the spirit of integrals via regularization and we prove some typical properties. We essentially define Itô integrals as forward integrals in a subclass and we prolongate through functional analysis methods. Section 4 is devoted to finite quadratic variation processes. In particular we establish C^1-stability properties and Itô type formula of C^2-type. Section 5 provides some survey material with new results related to the class of weak Dirichlet processes introduced by [11] with later developments discussed by [23, 7]. Considerations about Itô formulae under C^1-conditions are discussed as well.

2 Stochastic integration via regularization

2.1 Definitions and fundamental properties

In this paper T will be a fixed positive real number. Let f be a real continuous function defined either on $[0,T]$ or $\mathbb{R}_+$. We will convene that it will be prolongated using the same symbol to the real line, setting

$$f(t) = \begin{cases} f(0) & \text{if} \quad t \le 0 \\ f(T) & \text{if} \quad t > T. \end{cases} \tag{2.1}$$

Let $(X_t)_{t\ge 0}$ be a continuous process and $(Y_t)_{t\ge 0}$ be a process with paths in $L^1_{loc}(\mathbb{R}_+)$, i.e. for any $a > 0$, $\int_0^a |Y_t| dt < \infty$ a.s.

Our generalized stochastic integrals and covariations will be defined through a regularization procedure. More precisely, let $I^-(\varepsilon, Y, dX)$ (resp. $I^+(\varepsilon, Y, dX), I^0(\varepsilon, Y, dX)$ and $C(\varepsilon, Y, X)$) be the ε-forward integral (resp. ε-backward integral, ε-symmetric integral and ε-covariation).

$$I^-(\varepsilon, Y, dX)(t) = \int_0^t Y(s) \frac{X(s+\varepsilon) - X(s)}{\varepsilon} ds; \quad t \ge 0, \tag{2.2}$$

$$I^+(\varepsilon, Y, dX)(t) = \int_0^t Y(s) \frac{X(s) - X((s-\varepsilon)_+)}{\varepsilon} ds; \quad t \ge 0, \tag{2.3}$$

$$I^0(\varepsilon, Y, dX)(t) = \int_0^t Y(s) \frac{X(s+\varepsilon) - X((s-\varepsilon)_+)}{2\varepsilon} ds; \quad t \ge 0, \tag{2.4}$$

$$C(\varepsilon, X, Y)(t) = \int_0^t \frac{\big(X(s+\varepsilon) - X(s)\big)\big(Y(s+\varepsilon) - Y(s)\big)}{\varepsilon} ds; \quad t \geq 0. \qquad (2.5)$$

Observe that previous integral processes are all continuous.

Definition 2.1 *1) A family of processes* $(H_t^{(\varepsilon)})_{t\in[0,T]}$ *is said to converge to* $(H_t)_{t\in[0,T]}$ *in the* **ucp sense**, *if* $\sup_{0\le t\le T} |H_t^{(\varepsilon)} - H_t|$ *goes to* 0 *in probability, as* $\varepsilon \to 0$.

2) Provided that the corresponding limits exist in the ucp sense, we introduce the following integrals and covariations by the following formulae

a) **Forward integral** *:* $\int_0^t Y d^- X = \lim_{\varepsilon\to 0^+} I^-(\varepsilon, Y, dX)(t)$.

b) **Backward integral** *:* $\int_0^t Y d^+ X = \lim_{\varepsilon\to 0^+} I^+(\varepsilon, Y, dX)(t)$.

c) **Symmetric integral** *:* $\int_0^t Y d^\circ X = \lim_{\varepsilon\to 0+} I^\circ(\varepsilon, Y, dX)(t)$.

d) **Covariation** *:* $[X,Y]_t = \lim_{\varepsilon\to 0^+} C(\varepsilon, X, Y)(t)$. *When* $X = Y$ *we often denote* $[X] = [X,X]$.

Remark 2.2 *Let* X, X', Y, Y' *be some processes with* X, X' *being continuous and* Y, Y' *with paths in* $L^1_{loc}(\mathbb{R}_+)$. $\star$ *will be a symbol in* $\{-, +\circ\}$.

1. $(X,Y) \mapsto \int_0^\cdot Y d^\star X$ *and* $(X,Y) \mapsto [X,Y]$ *are bilinear operations.*
2. *The covariation of continuous processes is a symmetric operation.*
3. *When it exists,* $[X]$ *is an increasing process.*
4. *Let* τ *a random time. Then* $[X^\tau, X^\tau]_t = [X,X]_{t\wedge\tau}$ *and*

$$\int_0^t Y 1_{[0,\tau]} d^\star X = \int_0^t Y d^\star X^\tau = \int_0^t Y^\tau d^\star X^\tau = \int_0^{t\wedge\tau} Y d^\star X,$$

   where X^τ *is the process* X *stopped at time* τ *defined by* $X_t^\tau = X_{t\wedge\tau}$.

5. *Given* ξ, η *be two fixed r.v., we have* $\int_0^\cdot (\xi Y_s) d^\star (\eta X_s) = \xi\eta \int_0^\cdot Y_s d^\star X_s$.
6. *Integrals via regularization also have the following localization property. Suppose that* $X_t = X'_t, Y_t = Y'_t, \forall t \in [0,T]$ *on some subset* Ω_0 *of* Ω. *Then*

$$1_{\Omega_0} \int_0^t Y_s d^\star X_s = 1_{\Omega_0} \int_0^t Y'_s d^\star X'_s, \quad t \in [0,T].$$

7. *If Y is an elementary process of the type $Y_t = \sum_{i=1}^{N} A_i 1_{I_i}$, where A_i are random variables and (I_i) a family of real intervals with end-points $a_i < b_i$, then*

$$\int_0^t Y_s d^\star X_s = \sum_{i=1}^{N} A_i (X_{b_i \wedge t} - X_{a_i \wedge t}).$$

Definition 2.3 *1) If $[X]$ exists, X is said to be* **finite quadratic variation process**. *$[X]$ is called* **quadratic variation** *of X. .*

2) If $[X] = 0$, X is called **zero quadratic variation process**.

3) A vector $(X^1, \dots, X^n)$ of continuous processes is said to have all its **mutual covariations** *if $[X^i, X^j]$ exists for every $1 \le i,\, j \le n$.*

We will also use the terminology **bracket** *instead of covariation.*

Remark 2.4 *1) If $(X^1, \dots, X^n)$ has all its mutual covariations, then we have*

$$[X^i + X^j, X^i + X^j] = [X^i, X^i] + 2[X^i, X^j] + [X^j, X^j]. \tag{2.6}$$

From the previous equality, it follows that $[X^i, X^j]$ is the difference of two increasing processes therefore it has bounded variation; consequently the bracket is a classical integrator in the Lebesgue-Stieltjes sense.

2) Relation (2.6) holds as soon as three brackets among the four exist. More generally an identity of the type $I_1 + \dots + I_n = 0$ has the following meaning: if $n-1$ terms among the I_j exist, the remaining one also makes sense and the identity holds true.

3) We will see later, in Remark 5.19, that there exist processes X and Y such that $[X, Y]$ exist but has no finite variation process; in particular (X, Y) does not have all its mutual brackets.

The properties below can be established in a elementary way exploiting the definition of integrals via regularization.

Proposition 2.5 *Let $X = (X_t)_{t \ge 0}$ be a continuous process and $Y = (Y_t)_{t \ge 0}$ be a process with paths in $L^1_{loc}(\mathbb{R}_+)$. Then*

1) $[X, Y]_t = \int_0^t Y d^+ X - \int_0^t Y d^- X.$

2) $\int_0^t Y d^\circ X = \frac{1}{2} \left(\int_0^t Y d^+ X + \int_0^t Y d^- X \right).$

3) **Time reversal**. *We set* $\hat{X}_t = X_{(T-t)}, t \in [0,T]$. *Then we have*

1. $\int_0^t Y d^{\pm} X = -\int_{T-t}^{T} \hat{Y} d^{\mp} \hat{X}, \quad 0 \le t \le T$;
2. $\int_0^t Y d^{\circ} X = -\int_{T-t}^{T} \hat{Y} d^{\circ} \hat{X}, \quad 0 \le t \le T$;
3. $[\hat{X}, \hat{Y}]_t = [X,Y]_T - [X,Y]_{T-t}, \quad 0 \le t \le T$.

4) **Integration by parts**. *If* Y *is continuous we have*

$$\begin{aligned} X_t Y_t &= X_0 Y_0 + \int_0^t X d^- Y + \int_0^t Y d^+ X \\ &= X_0 Y_0 + \int_0^t X d^- Y + \int_0^t Y d^- X + [X,Y]_t. \end{aligned}$$

5) **Kunita-Watanabe inequality**. *If* X *and* Y *are finite quadratic variation processes we have*

$$|[X,Y]| \le \left\{[X]\ [Y]\right\}^{1/2}$$

6) If X *is a finite quadratic variation process and* Y *is a zero quadratic variation process then* (X,Y) *has all its mutual brackets and* $[X,Y] = 0$.

7) Let X *be a bounded variation process and* Y *be a process with locally bounded paths, and at most countable discontinuities. Then*

a) $\int_0^t Y d^+ X = \int_0^t Y d^- X = \int_0^t Y dX$, *where* $\int_0^t Y dX$ *denotes Lebesgue-Stieltjes integral.*

b) $[X,Y] = 0$. *In particular a bounded variation and continuous process is a zero quadratic variation process.*

8) Let X *be an absolutely continuous process and* Y *be a process with locally bounded paths. Then*

$$\int_0^t Y d^+ X = \int_0^t Y d^- X = \int_0^t Y X' ds.$$

Remark 2.6 *If* Y *has more than countable discontinuities then previous point 7) may fail. Take for instance* $Y = 1_{\mathrm{suppdV}}$, *where* V *is a strictly increasing continuous function such that* $V'(t) = 0$ *a.e. (almost everywhere) with respect to Lebesgue measure. Then* $Y = 0$ *Lebesgue a.e., and* $Y = 1, dV$ *a.e. Consequently*

$$\int_0^t Y dV = t, \quad I^-(\varepsilon, Y, dV)(t) = 0 \quad \int_0^t Y d^- V = 0.$$

Remark 2.7 *Point 2) of Proposition 2.5 states that the symmetric integral is the average of the forward and backward integrals.*

Proof (of Proposition 2.5). Points 1), 2), 3), 4) follow immediately from the definition. For illustration, we only prove 3); operating a change of variable $u = T - s$, we obtain

$$\int_0^t Y_s \frac{X_s - X_{(s-\varepsilon)_+}}{\varepsilon} ds = -\int_{T-t}^T \hat{Y}_u \frac{\hat{X}_{u+\varepsilon} - \hat{X}_u}{\varepsilon} du, \quad 0 \le t \le T.$$

Since X is continuous, we can take the limit of both members and the result follows.

5) follows by Cauchy-Schwarz inequality which says that

$$\begin{aligned}
&\frac{1}{\varepsilon}\left|\int_0^t (X_{s+\varepsilon} - X_s)(Y_{s+\varepsilon} - Y_s) ds\right| \\
&\quad \le \left\{\frac{1}{\varepsilon}\int_0^t (X_{s+\varepsilon} - X_s)^2 ds \ \frac{1}{\varepsilon}\int_0^t (Y_{s+\varepsilon} - Y_s)^2 ds\right\}^{\frac{1}{2}}.
\end{aligned}$$

6) is a consequence of 5).

7) Using Fubini, we have

$$\begin{aligned}
\frac{1}{\varepsilon}\int_0^t Y_s (X_{s+\varepsilon} - X_s) ds &= \frac{1}{\varepsilon}\int_0^t ds\, Y_s \int_s^{s+\varepsilon} dX_u \\
&= \int_0^{t+\varepsilon} dX_u \frac{1}{\varepsilon}\int_{(u-\varepsilon)\vee 0}^{u\wedge t} Y_s ds.
\end{aligned}$$

Since Y has at most countable jumps, $\frac{1}{\varepsilon}\int_{(u-\varepsilon)\vee 0}^{u} Y_s ds \to Y_u, d|X|$ a.e. where $|X|$ denotes the total variation of X. Since $t \to Y_t$ is locally bounded, then Lebesgue convergence theorem implies that $\int_0^t Y d^- X = \int_0^t Y dX$.

The fact that $\int_0^t Y d^+ X = \int_0^t Y dX$ follows similarly.

b) is a consequence of point 1).

8) can be reached using similar elementary integration properties. ■

2.2 Young integral in a simplified framework

In this section we will consider the integral defined by Young ([56]) in 1936, and implemented in the stochastic framework by Bertoin, see [3]. Here we will restrict ourselves to the case that integrand and integrator are Hölder continuous processes. As a result, that integral will be shown to coincide with the forward but also with backward and symmetric integral.

Definition 2.8 *1. Let C^α be the set of Hölder continuous functions defined on $[0,T]$ with index $\alpha > 0$. Recall that $f : [0,T] \mapsto \mathbb{R}$ belongs to C^α if*

$$N_\alpha(f) := \sup_{0\le s,t\le T} \frac{|f(t)-f(s)|}{|t-s|^\alpha} < \infty.$$

2. Let $X, Y : [0,T] \mapsto \mathbb{R}$ be two functions of class C^1, then the Young integral of Y on $[a,b] \subset [0,T]$, with respect to X is defined as :

$$\int_a^b Y d^{(y)} X := \int_a^b Y(t)X'(t)dt, \quad 0 \le a \le b \le T.$$

To extend Young integral to Hölder functions we need some estimate of $\int_0^T Y d^{(y)} X$ in terms of Hölder norms of X and Y. More precisely let X and Y as in Definition 2.8 above; then in [14], it is proved:

$$\Big| \int_a^T (Y - Y(a)) d^{(y)} dX \Big| \le C_\rho T^{1+\rho} N_\alpha(X) N_\beta(Y), \quad 0 \le a \le T. \tag{2.7}$$

where $\alpha, \beta > 0,\ \alpha+\beta > 1,\ \rho \in]0, \alpha+\beta-1[$, and C_ρ is a universal constant.

Proposition 2.9 *1. The map $(X,Y) \in C^1([0,T]) \times C^1([0,T]) \mapsto \int_0^\cdot Y d^{(y)} X$ taking its values in C^α, can be continuously extended to a bilinear map from $C^\alpha \times C^\beta$ to $\mathcal{C}^\alpha$. The value of this extension at point $(X,Y) \in \mathcal{C}^\alpha \times C^\beta$ will still be denoted $\int_0^\cdot Y d^{(y)} X$ and is called the* **Young integral** *of Y with respect to X.*

2. Inequality (2.7) is still valid for any $X \in C^\alpha$ and $Y \in C^\beta$.

Proof. 1. Let X, Y be of class $C^1([0,T])$ and

$$F(t) = \int_0^t Y d^{(y)} X = \int_0^t Y(s)X'(s)ds, \quad t \in [0,T].$$

For any $a, b \in [0,T],\ a < b$, we have

$$F(b) - F(a) = \int_a^b \big(Y(t) - Y(a)\big) d^{(y)} X + Y(a)\big(X(b) - X(a)\big).$$

Then (2.7) implies

$$|F(b)-F(a)| \le C_\rho (b-a)^{1+\rho} N_\alpha(X) N_\beta(Y) + \sup_{0\le t\le T} |Y(t)| \; N_\alpha(X)(b-a)^\alpha. \quad (2.8)$$

Consequently $F \in C^\alpha$.

Then the map $(X,Y) \in C^1([0,T]) \times C^1([0,T]) \mapsto \int_0^\cdot Y d^{(y)} X$ being bilinear, may be extended to a continuous bilinear map from $C^\alpha \times \mathcal{C}^\beta$ to C^α.

2. is a consequence of point 1.

■

Before discussing the relation between Young integral and integrals via regularization, we provide an useful technical result.

Lemma 2.10 *Let $0 < \gamma' < \gamma \le 1, \varepsilon > 0$. With Z in C^γ we associate*

$$Z_\varepsilon(t) = \frac{1}{\varepsilon} \int_0^t \big(Z(u+\varepsilon) - Z(u) \big) du, \; t \in [0,T].$$

Then Z_ε converges to Z in $C^{\gamma'}$, as $\varepsilon \to 0$.

Proof of Lemma 2.10.

$$Z_\varepsilon(t) = \frac{1}{\varepsilon} \int_0^t \big(Z(u+\varepsilon) - Z(u) \big) du = \frac{1}{\varepsilon} \int_t^{t+\varepsilon} Z(u) du - \frac{1}{\varepsilon} \int_0^\varepsilon Z(u) du,$$

for any $0 \le t \le T$.

Setting $\Delta_\varepsilon(t) = Z_\varepsilon(t) - Z(t)$, we get

$$\begin{aligned} \Delta_\varepsilon(t) - \Delta_\varepsilon(s) &= \frac{1}{\varepsilon} \int_t^{t+\varepsilon} Z(u) du - Z(t) - \frac{1}{\varepsilon} \int_s^{s+\varepsilon} Z(u) du + Z(s) \\ &= \frac{1}{\varepsilon} \int_t^{t+\varepsilon} \big(Z(u) - Z(t) \big) du - \frac{1}{\varepsilon} \int_s^{s+\varepsilon} \big(Z(u) - Z(s) \big) du, \end{aligned}$$

where $0 \le s \le t \le T$.

a) Suppose $0 \le s < s + \varepsilon < t$. Previous inequality implies

$$|\Delta_\varepsilon(t) - \Delta_\varepsilon(s)| \le \frac{1}{\varepsilon} \int_t^{t+\varepsilon} \big| Z(u) - Z(t) \big| du + \frac{1}{\varepsilon} \int_s^{s+\varepsilon} \big| Z(u) - Z(s) \big| du.$$

Since $Z \in C^\gamma$, then

$$\begin{aligned} |\Delta_\varepsilon(t) - \Delta_\varepsilon(s)| &\le \frac{N_\gamma(Z)}{\varepsilon} \Big(\int_t^{t+\varepsilon} (u-t)^\gamma du + \int_s^{s+\varepsilon} (u-s)^\gamma du \Big) \\ &\le \frac{2 N_\gamma(Z)}{\gamma + 1} \varepsilon^\gamma. \end{aligned}$$

But $\varepsilon < t - s$, consequently

$$|\Delta_\varepsilon(t) - \Delta_\varepsilon(s)| \le \frac{2N_\gamma(Z)}{\gamma+1})\varepsilon^{\gamma-\gamma'}|t-s|^{\gamma'}. \tag{2.9}$$

b) We now investigate the case $0 \le s < t < s + \varepsilon$. The difference $\Delta_\varepsilon(t) - \Delta_\varepsilon(s)$ may be decomposed as follows :

$$\begin{aligned}\Delta_\varepsilon(t) - \Delta_\varepsilon(s) &= \frac{1}{\varepsilon}\int_{s+\varepsilon}^{t+\varepsilon} \big(Z(u) - Z(s+\varepsilon)\big)du - \frac{1}{\varepsilon}\int_s^t \big(Z(u) - Z(s)\big)du \\ &+ \frac{t-s}{\varepsilon}\big(Z(s+\varepsilon) - Z(s)\big) + Z(s) - Z(t).\end{aligned}$$

Proceeding as in previous step and using the inequality $0 < t - s < \varepsilon$, we obtain

$$\begin{aligned}|\Delta_\varepsilon(t) - \Delta_\varepsilon(s)| &\le N_\gamma(Z)\Big(\frac{2}{\gamma+1}\frac{(t-s)^{\gamma+1}}{\varepsilon} + \frac{t-s}{\varepsilon^{1-\gamma}} + (t-s)^\gamma\Big) \\ &\le 2N_\gamma(Z)\frac{\gamma+2}{\gamma+1}\varepsilon^{\gamma-\gamma'}|t-s|^{\gamma'}.\end{aligned}$$

At this point, the above inequality and (2.9) directly imply that $N_{\gamma'}(Z_\varepsilon - Z) \le C\varepsilon^{\gamma-\gamma'}$ and the claim is finally established.

■

In the sequel of this section X and Y will denote stochastic processes.

Remark 2.11 *If X and Y have a.s. Hölder continuous paths respectively of order α and β with $\alpha > 0, \beta > 0$ and $\alpha+\beta > 1$. Then one can easily prove that $[X,Y] = 0$.*

Proposition 2.12 *Let X, Y be two real processes indexed by $[0,T]$ whose paths are respectively a.s. in C^α and C^β, with $\alpha > 0, \beta > 0$ and $\alpha + \beta > 1$. Then for any symbol $\star \in \{+,-,\circ\}$ the integral $\int_0^\cdot Y d^\star X$ coincides with the Young integral $\int_0^\cdot Y d^{(y)} X$.*

Proof of Proposition 2.12.

We establish that the forward integral coincides with the Young integral. The equality concerning the two other integrals is a consequence of Proposition 2.5 1., 2. and Remark 2.11.

By additivity we can suppose, without lost generality, that $Y(0) = 0$.

We set

$$\Delta_\varepsilon(t) := \int_0^t Y d^{(y)} X - \int_0^t Y dX_\varepsilon.$$

where

$$X_\varepsilon(t) = \frac{1}{\varepsilon}\int_0^t \big(X(u+\varepsilon) - X(u)\big)du, \ t \in [0,T].$$

Since $t \mapsto X_\varepsilon(t)$ is of class $C^1([0,T])$, then $\int_0^t Y dX_\varepsilon$ is equal to the Young integral $\int_0^t Y d^{(y)} X_\varepsilon$ and therefore

$$\Delta_\varepsilon(t) = \int_0^t Y d^{(y)}\big(X - X_\varepsilon\big).$$

Let α' such that : $0 < \alpha' < \alpha$ and $\alpha' + \beta > 1$. Applying inequality (2.7) we obtain

$$\sup_{0 \le t \le T} |\Delta_\varepsilon(t)| \le C_\rho T^{1+\rho} N_{\alpha'}(X - X_\varepsilon) N_\beta(Y), \quad \rho \in]0, \alpha' + \beta - 1[.$$

Lemma 2.10 with $Z = X$ and $\gamma = \alpha$ directly implies that $\Delta_\varepsilon(t)$ goes to 0, uniformly a.s. on $[0,T]$, as $\varepsilon \to 0$, concluding the proof of the Proposition.

■

3 Itô integrals and related topics

In this section we propose an alternative construction of Itô integral with respect to a local martingale, based on calculus via regularization. Our approach is inspired by McKean ([36]), section 2.1.

3.1 Some reminders on martingales theory

In this subsection, we recall basic notions related to martingale theory, essentially without proofs, except when they help the reader. For detailed complements, see [29], chap. 1., in particular for definition of adapted and progressively measurable processes.

Let $(\mathcal{F}_t)_{t\ge 0}$ be a filtration on the probability space $(\Omega, \mathcal{F}, P)$ satisfying the usual conditions, see Definition 2.25, chap. 1 in [29].

An adapted process (M_t) of integrable random variables, i.e. verifying $E(|M_t|) < \infty$, $\forall t \ge 0$ is:

- an $(\mathcal{F}_t)$-martingale if $E(M_t|\mathcal{F}_s) = M_s, \quad \forall t \ge s$;
- a $(\mathcal{F}_t)$- submartingale if $E(M_t|\mathcal{F}_s) \ge M_s, \quad \forall t \ge s$

In this paper, all the submartingales (and therefore martingales) will be supposed to be continuous.

Remark 3.1 *From that definition, we can deduce that if $(M_t)_{t\geq 0}$ is a martingale, then $E(M_t) = E(M_0)$, $\forall t \geq 0$. If $(M_t)_{t\geq 0}$ is a supermartingale (resp. submartingale) then $t \longrightarrow E(M_t)$ is decreasing (resp. increasing).*

Definition 3.2 *A process X is said to be* **square integrable** *if $E(X_t^2) < \infty$, for any $t \geq 0$.*

When one speaks of a martingale, without σ field specification, one refers to the natural filtration.

Definition 3.3 1. *A (continuous) process $(X_t)_{t\geq 0}$, is called $(\mathcal{F}_t)$-***local martingale** *(resp.* **local submartingale***) if there is an increasing sequence (τ_n) of stopping times such that $X^{\tau_n} 1_{\tau_n > 0}$ is an $(\mathcal{F}_t)$-martingale (resp. submartingale) and $\lim_{n\to\infty} \tau_n = +\infty$ a.s.*

Remark 3.4

- *A martingale is a local martingale. A bounded local martingale is a martingale.*
- *The set of local martingales is a vector algebra.*
- *If M is an $(\mathcal{F}_t)$- local martingale, τ is a stopping time, then M^τ is again an $(\mathcal{F}_t)$- local martingale.*
- *If M_0 is bounded, it is possible to choose a localizing sequence (τ_n) such that M^{τ_n} is bounded.*

Definition 3.5 *A process S is called (continuous) $(\mathcal{F}_t)$-***semimartingale** *if it is the sum of an $(\mathcal{F}_t)$- local martingale and an $(\mathcal{F}_t)$-adapted continuous finite variation process.*

A basic decomposition in stochastic analysis is the following.

Theorem 3.6 *(***Doob decomposition of a submartingale***)*

Let X be a $(\mathcal{F}_t)$-local submartingale. Then, there is an $(\mathcal{F}_t)$-local martingale M and an adapted, continuous, and finite variation process V (such that $V_0 = 0$) with $X = M + V$. The decomposition is unique.

Definition 3.7 *Let M be an $(\mathcal{F}_t)$-local martingale. We denote by $< M >$ the bounded variation process intervening in the Doob decomposition of local submartingale M^2. In particular $M^2 - < M >$ is an $(\mathcal{F}_t)$-local martingale.*

In Corollary 3.20, we will prove that $< M >$ coincides with $[M, M]$, so that the oblique bracket $< M >$ does not depend on the underlying filtration.

Corollary 3.8 *Let M be an $(\mathcal{F}_t)-$local martingale vanishing at zero, with $< M >= 0$. Then M is identically zero.*

Proof. Due to stopping properties, we may suppose that M is bounded. Hence Remark 3.4 implies that M^2 is a bounded martingale and so $E[M_t^2] = E[M_0^2] = 0$. Consequently, for any $t \geq 0$, $M_t = 0$ a.s. Since M is a continuous process, then a.s., $M_t = 0, \ \forall t \geq 0$. ■

The following result will be needed in section 3.2.

Lemma 3.9 *Let $(M^n_{t\in[0,T]})$ be a sequence of $(\mathcal{F}_t)$ local martingales such that $M_0^n = 0$ and $< M^n >_T$ converges to $< M >_T$ in probability as $n \to \infty$. Then $M^n \to 0$ ucp, when $n \to \infty$..*

Proof. The proof is based on the following inequality stated in [29], Problem 5.25 Chap. 1, which holds for any $(\mathcal{F}_t)$-local martingale (M_t) such that $M_0 = 0$:

$$P\big(\sup_{0\leq u\leq t} |M_u| \geq \lambda\big) \leq P\big(< M >_t \geq \delta\big) + \frac{1}{\lambda^2} E\big[\delta \wedge < M >_t \big], \tag{3.10}$$

for any $t \geq 0, \ \lambda, \delta > 0$. ■

3.2 The Itô integral

Let M be an $(\mathcal{F}_t)$-local martingale. We construct here the Itô integral with respect to M using stochastic calculus via regularization. We will proceed in two steps. First we define the Itô integral $\int_0^\cdot HdM$ for a smooth integrand process H as the forward integral $\int_0^\cdot Hd^-M$. Secondly, we extend $H \mapsto \int_0^\cdot HdM$ with the help of functional analysis arguments. We remark that the classical theory of Itô integrals first defines the integral of simple step processes H, see Remark 3.14, for details.

We first observe that the forward integral of a process H of class C^1 is well defined because Proposition 2.5 4), 7) imply that

$$\int_0^t Hd^-M = H_t M_t - H_0 M_0 - \int_0^t Md^+H = H_t M_t - H_0 M_0 - \int_0^t M_s H_s' ds. \tag{3.11}$$

We denote $\mathcal{C}$ the vector algebra of adapted processes whose paths are of class C^0. This linear space, equipped with the metrizable topology which governs the ucp convergence, is an $F-$space. For the definition and properties of $F-$spaces, see [9], chapter 2.1. We remark that the set $\mathcal{M}_{\mathrm{loc}}$ of continuous $(\mathcal{F}_t)$-local martingales is a closed linear subspace of $\mathcal{C}$, see for instance [23].

We denote by $\mathcal{C}^1$ the subspace of $\mathcal{C}$ of processes whose paths are a.s. of class C^1. The next crucial observation is the following.

Lemma 3.10 *If H is an adapted process in $\mathcal{C}^1$ then $\left(\int_0^\cdot Hd^-M\right)$ is an $(\mathcal{F}_t)$-local martingale whose quadratic variation is given by*

$$< \int_0^\cdot Hd^-M >_t = < \int_0^\cdot H_s^2 d < M >_s .$$

Proof. We only sketch the proof. We restrict ourselves to prove that if M is a local martingale then $Y = \int_0^\cdot Hd^-M$ is a local martingale.

Using localization, we can suppose that H, H' and M are bounded processes.

Let $0 \le s < t$. Since H is of class C^1, then $H_t = H_s + \int_s^t H'_u du$. Therefore (3.11) implies

$$Y_t = H_s M_t - H_0 M_0 - \int_0^s M_u H'_u du + \int_s^t (M_t - M_u) H'_u du. \tag{3.12}$$

Let $u \in [s,t]$, then

$$E\big[(M_t - M_u)H'_u | \mathcal{F}_s\big] = E\Big[E\big(M_t - M_u)H'_u|\mathcal{F}_u\big] | \mathcal{F}_s\Big] = 0.$$

Consequently, taking the conditional expectation with respect to $\mathcal{F}_s$ in (3.12) yields to

$$E\big[Y_t|\mathcal{F}_s\big] = H_s M_s - H_0 M_0 - \int_0^s M_u H'_u du = Y_s.$$

Using similar arguments we can check that $Y^2 - \int_0^\cdot H^2 d < M >$ is a martingale.

■

The previous lemma allows to extend the map $H \mapsto \int_0^t Hd^-M$. Let $\mathcal{L}^2(d < M >)$ be the set of progressively measurable processes such that

$$\int_0^T H^2 d < M >< \infty \text{ a.s.} \tag{3.13}$$

$\mathcal{L}^2(d < M >)$ is an F-space with respect to the metrizable topology d_2: (H^n) converges to H when $n \to \infty$ if $\int_0^T (H_s^n - H_s)^2 d < M >_s \to 0$ in probability, when $n \to \infty$.

Let $\Lambda : \mathcal{C}^1 \to \mathcal{M}_{\mathrm{loc}}$ be the map defined by $\Lambda H = \int_0^\cdot Hd^-M$.

Lemma 3.11 *If $\mathcal{C}^1$ (resp. $\mathcal{M}_{\rm loc}$) is equipped with d_2 (resp. the ucp topology) then Λ is continuous.*

Proof. Let H^k be a sequence of processes in $\mathcal{C}^1$, converging to 0 when $k \to \infty$, according to d_2. We set $N^k = \int_0^\cdot H^k d^- M$. Lemma 3.10 implies that $< N^k >_T$ converges to 0 in probability. Finally Lemma 3.9 concludes the proof. ■

We can now easily define the Itô integral. Since $\mathcal{C}^1$ is dense in $\mathcal{L}^2(d < M >)$, with respect to d^2, Lemma 3.11 and standard functional analysis arguments imply that Λ can be uniquely and continuously extended to $\mathcal{L}^2(d < M >)$.

Definition 3.12 *If H belongs to $\mathcal{L}^2(d < M >)$, then we denote by $\int_0^\cdot HdM := \Lambda H$ and we call it the* **Itô integral of H with respect to M**.

Proposition 3.13 *If H in $\mathcal{L}^2(d < M >)$, then $(\int_0^\cdot HdM)$ is an $(\mathcal{F}_t)$-local martingale with bracket*

$$< \int_0^\cdot HdM >= \int_0^\cdot H^2 d < M > . \tag{3.14}$$

Proof. Let $H \in \mathcal{L}^2(d < M >)$. From Definition 3.12, $(\int_0^\cdot HdM)$ is an $(\mathcal{F}_t)$-local martingale. It remains to prove that (3.14).

Since H belongs to $\mathcal{L}^2(d < M >)$, then there exists a sequence (H_n) of elements in $\mathcal{C}^1$, such that $H_n \to H$ in $\mathcal{L}^2(d < M >)$.

Let us introduce $N_n = \displaystyle\int_0^\cdot H^n dM$ and $N'_n = N_n^2 - < N_n >$.

Therefore $N_n \to N$, ucp, $n \to \infty$ and $< N_n >= \displaystyle\int_0^\cdot H_n^2 d < M >$.

The stochastic Dini lemma (see Lemma 3.1 in [49]) implies that $\displaystyle\int_0^\cdot H_n^2 d < M >$ goes to $\displaystyle\int_0^\cdot H^2 d < M >$ in the ucp sense, as $n \to \infty$. Therefore N'_n converges with respect to the ucp topology, to the local martingale $N^2 - \displaystyle\int_0^\cdot H^2 d < M >$, $n \to \infty$. This actually proves (3.14).

■

Remark 3.14 1. *We recall that whenever $H \in \mathcal{C}^1$*

$$\int_0^\cdot HdM = \int_0^\cdot Hd^- M.$$

This property will be generalized in Propositions 3.16 and 3.33.

2. *We emphasize that Itô stochastic integration based on adapted simple step processes and our method, finally, lead to the same object.*

   If H is of the type $Y1_{]a,b]}$ where Y is an $\mathcal{F}_a$ measurable random variable, it is possible to show that $\int_0^t HdM = Y(M_{t\wedge b} - M_{t\wedge a})$ Since the class of elementary processes obtained by linear combination of previous processes is dense in $\mathcal{L}^2(d < M >)$ and the map Λ is continuous, then $\int_0^\cdot HdM$ equals the classical Itô integral.

In Proposition 3.15 below we state the chain rule property.

Proposition 3.15 *Let $(M_t, t \geq 0)$ be an $(\mathcal{F}_t)$-local martingale, $(H_t, t \geq 0)$ be in $\mathcal{L}^2(d < M >)$, $N := \displaystyle\int_0^\cdot H_s dM_s$ and $(K_t, t \geq 0)$ be a $(\mathcal{F}_t)$-progressively measurable process such that $\displaystyle\int_0^T (H_s K_s)^2 d < M >_s < \infty$ a.s.. Then*

$$\int_0^t K_s dN_s = \int_0^t H_s K_s dM_s, \quad 0 \leq t \leq T. \tag{3.15}$$

Proof. Since the map $\Lambda : H \in \mathcal{L}^2(d < M >) \mapsto \int_0^\cdot HdM$ is continuous, it is sufficient to prove (3.15) for H and K of class C^1.

For simplicity we suppose $M_0 = H_0 = K_0 = 0$.

We have

$$\int_0^t KdN = \int_0^t (N_t - N_u) K'_u du,$$

and

$$\begin{aligned} N_t - N_u &= \int_0^t (M_t - M_v) H'_v dv - \int_0^u (M_u - M_v) H'_v dv \\ &= (M_t - M_u) H_u + \int_u^t (M_t - M_v) H'_v dv, \end{aligned}$$

where $0 \leq u \leq t$.

Using Fubini theorem we get

$$\int_0^t KdN = \int_0^t (M_t - M_u)(K'_u H_u + K_u H'_u) du = \int_0^t (M_t - M_u)(HK)'_u du = \int_0^t HKdM.$$

■

3.3 Connections with calculus via regularizations

Next Proposition will show that, under suitable conditions, the Itô integral corresponds to forward integral.

Proposition 3.16 *Let X be an $(\mathcal{F}_t)$-local martingale and suppose that (H_t) is adapted and has a left limit at each point. Then $\int_0^{\cdot} H_s d^- X_s = \int_0^{\cdot} H_{s-} dX_s$. In particular if H is càdlàg then $\int_0^{\cdot} H_s d^- X_s = \int_0^{\cdot} H_s dX_s$.*

Proof. For simplicity we suppose that H is continuous. Since $s \mapsto \int_{s-\varepsilon}^{s} H_u du$ is of class C^1, then

$$\begin{aligned}
\int_0^t \Big(\frac{1}{\varepsilon}\int_{s-\varepsilon}^{s} H_u du\Big) dX_s &= \int_0^t \Big(\frac{1}{\varepsilon}\int_{s-\varepsilon}^{s} H_u du\Big) d^- X_s \\
&= X_t\Big(\frac{1}{\varepsilon}\int_{t-\varepsilon}^{t} H_u du\Big) - H_0 X_0 - \frac{1}{\varepsilon}\int_0^t (H_s - H_{s-\varepsilon}) X_s ds.
\end{aligned}$$

We modify the second integral in the right-hand side as follows

$$\begin{aligned}
-\int_0^t (H_s - H_{s-\varepsilon}) X_s ds &= \int_0^t H_s (X_{s+\varepsilon} - X_s) ds - \int_{t-\varepsilon}^{t} H_s X_{s+\varepsilon} ds \\
&+ H_0 \int_0^{\varepsilon} X_s ds.
\end{aligned}$$

Consequently

$$\int_0^t \Big(\frac{1}{\varepsilon}\int_{s-\varepsilon}^{s} H_u du\Big) dX_s = \frac{1}{\varepsilon}\int_0^t H_s (X_{s+\varepsilon} - X_s) ds + R_\varepsilon(t), \tag{3.16}$$

where

$$\begin{aligned}
R_\varepsilon(t) &= X_t\Big(\frac{1}{\varepsilon}\int_{t-\varepsilon}^{t} H_s ds\Big) - \frac{1}{\varepsilon}\int_{t-\varepsilon}^{t} H_s X_{s+\varepsilon} ds + H_0\left(\frac{1}{\varepsilon}\int_0^{\varepsilon} X_s ds - X_0\right) \\
&= \frac{1}{\varepsilon}\int_{t-\varepsilon}^{t} H_s (X_t - X_{s+\varepsilon}) ds + H_0\left(\frac{1}{\varepsilon}\int_0^{\varepsilon} X_s ds - X_0\right)
\end{aligned} \tag{3.17}$$

Since $R_\varepsilon \to 0$, ucp, as $\varepsilon \to 0$ and the map $H \mapsto \frac{1}{\varepsilon}\int_{\cdot-\varepsilon}^{\cdot} H_s ds$ is a continuous operator from $\mathcal{L}^2(d<M>)$ to itself, therefore $\int_0^{\cdot} H_s d^- X_s$ and $\int_0^{\cdot} H_s dX_s$ coincide. ■

Remark 3.17 *Let H be a progressively measurable process. As the proof of Proposition 3.16 shows, the conclusion of this proposition is still valid as soon as it is only supposed that, a.s. $H_t = H_{t-}$, $d<M>$. a.e.*

When the integrator is a Brownian motion W, we will see in Theorem 3.33 below that forward integral coincides with the Itô integral for any integrand in $\mathcal{L}^2(d <$

$W>$). This is no longer true when the integrator is a general semimartingale. The following example provides a martingale (M_t) and a deterministic integrand h such that both Itô integral $\int_0^t hdM$ and forward integral $\int_0^t hd^-M$ exist, but are different.

Example 3.18 *Let $\psi : [0,\infty[\mapsto \mathbb{R}$ such that $\psi(0) = 0$, ψ is strictly increasing, and $\psi'(t) = 0$ a.e. (with respect to the Lebesgue measure). Let (M_t) be the process: $M_t = W_{\psi(t)},\ t \geq 0$, and h be the indicator function of the support of the positive measure $d\psi$. Since $W_t^2 - t$ is a martingale, $<W>_t = t$. Clearly (M_t) is a martingale and $<M>_t = \psi(t),\ t \geq 0$. Observe that $h = 0$ a.e. with respect to Lebesgue measure. Then $\int_0^{\cdot} h(s)\frac{M(s+\varepsilon) - M(s)}{\varepsilon}ds = 0$ and so $\int_0^{\cdot} hd^-M = 0$.*

On the other hand, $h = 1,\ d\psi$ a.e., implies $\int_0^t hdM = M_t,\ t \geq 0$.

Remark 3.19 *A significant result of classical stochastic calculus is Bichteler-Dellacherie theorem, see [42] Th. 22, Section III.7. In the regularization approach, an analogous property occurs: if the forward integral exists for a rich class of adapted integrands, then the integrator is forced to be a semimartingale. More precisely we recall the significant statement of [46], Proposition 1.2.*

Let $(X_t, t \geq 0)$ be an $(\mathcal{F}_t)$-adapted and continuous process such that for any càdlàg, bounded and adapted process (H_t), the forward integral $\int_0^{\cdot} Hd^-X$ exists. Then (X_t) is a $(\mathcal{F}_t)$-semimartingale.

From Proposition 3.16 we deduce the relation between oblique and square bracket.

Corollary 3.20 *Let M be an $(\mathcal{F}_t)$-local martingale. Then $<M> = [M]$ and*

$$M_t^2 = M_0^2 + 2\int_0^t Md^-M + <M>_t\ . \tag{3.18}$$

Proof. The proof of (3.18) is very simple and is based on the following identity

$$(M_{s+\varepsilon} - M_s)^2 = M_{s+\varepsilon}^2 - M_s^2 - 2M_s(M_{s+\varepsilon} - M_s).$$

Integrating on $[0, t]$ leads to

$$\begin{aligned} \frac{1}{\varepsilon}\int_0^t (M_{s+\varepsilon} - M_s)^2 ds &= \frac{1}{\varepsilon}\int_0^t M_{s+\varepsilon}^2 ds - \frac{1}{\varepsilon}\int_0^t M_s^2 ds - \frac{2}{\varepsilon}\int_0^t M_s(M_{s+\varepsilon} - M_s)ds \\ &= \frac{1}{\varepsilon}\int_t^{t+\varepsilon} M_s^2 ds - \frac{1}{\varepsilon}\int_0^{\varepsilon} M_s^2 ds - \frac{2}{\varepsilon}\int_0^t M_s(M_{s+\varepsilon} - M_s)ds. \end{aligned}$$

Therefore if we take the limit $\varepsilon \to 0$, we obtain

$$[M]_t = M_t^2 - M_0^2 - 2\int_0^t M_s d^- M_s.$$

Since $t \mapsto M_t$ is continuous, the forward integral $\int_0^\cdot M_d^- M$ coincides with the corresponding Itô integral. Consequently $M_t^2 - M_0^2 - [M]_t$ is a local martingale. This proves both $[M] =< M >$ and (3.18). ■

Corollary 3.21 *Let M, M' be two $(\mathcal{F}_t)$-local martingales. Then (M, M') has all its mutual covariations.*

Proof. Since M, M' and $M + M'$ are continuous local martingales, Corollary 3.20 directly implies that they are finite quadratic variation processes. The bilinearity property of the covariation implies directly that $[M, M']$ exists and equals

$$\frac{1}{2}([M + M'] - [M] - [M']).$$

■

Proposition 3.22 *Let M and M' be two $(\mathcal{F}_t)$-local martingales, H and H' be two progressively measurable processes such that*

$$\int_0^\cdot H^2 d < M >< \infty, \quad \int_0^\cdot H^2 d < M' >< \infty.$$

Then

$$[\int_0^\cdot HdM, \int_0^\cdot H'dM']_t = \int_0^t HH'd[M, M']_t.$$

Next proposition provides a simple example of two processes (M_t) and (Y_t) such that $[M, Y]$ exists even though the vector (M, Y) has no mutual covariation.

Proposition 3.23 *Let (M_t) be an continuous $(\mathcal{F}_t)$-local martingale, (Y_t) a càdlàg and an $(\mathcal{F}_t)$-adapted process. If M and Y are independent then $[M, Y] = 0$.*

Proof. Let $\mathcal{Y}$ be the σ-field generated by (Y_t). We denote $(\tilde{\mathcal{M}}_t)$ the smallest filtration satisfying the usual conditions and containing $(\mathcal{F}_t)$ and $\mathcal{Y}$ (i.e. $\sigma(M_s, s \leq t) \vee \mathcal{Y} \subset \tilde{\mathcal{M}}_t, \forall t \geq 0$).

It is not difficult to show that (M_t) is also an $(\tilde{\mathcal{M}}_t)$-martingale.

Thanks to Proposition 2.5 1., it is sufficient to prove that

$$\int_0^t Y d^- M = \int_0^t Y d^+ M. \tag{3.19}$$

Proposition 3.16 implies that the left member coincides with the $(\mathcal{M}_t)$- Itô integral $\int_0^t YdM$.

For simplicity we suppose in the sequel that Y is continuous, and $M_0 = 0$. We proceed as in the proof of Proposition 3.16. Since $s \mapsto \int_s^{s+\varepsilon} Y_u du$ is of class C^1, then

$$\int_0^t \Big(\frac{1}{\varepsilon}\int_s^{s+\varepsilon} Y_u du\Big) d^- M_s = M_t \Big(\frac{1}{\varepsilon}\int_s^{s+\varepsilon} Y_u du\Big) - \frac{1}{\varepsilon}\int_0^t (Y_{s+\varepsilon} - Y_s) M_s ds.$$

Since process Y is independent of (M_t), then the forward integral in the left-hand side above is actually an Itô integral. Therefore, taking the limit $\varepsilon \to 0$ and using Proposition 3.16, we get

$$\int_0^t YdM = \int_0^t Yd^- M = Y_t M_t - \int_0^t M d^- Y.$$

According to point 4) of Proposition 2.5, the right member of the previous identity is equal to $\int_0^t Yd^+M$. This proves (3.19). ■

3.4 The semimartingale case

We begin this section by proving a technical lemma which implies that the decomposition of a semimartingale is unique.

Lemma 3.24 *Let $(M_t, t \geq 0)$ be a $(\mathcal{F}_t)$-local martingale with bounded variation. Then (M_t) is constant.*

Proof. Since M has bounded variation, then Proposition 2.5, 7) implies that $[M] = 0$. Consequently Corollaries 3.8 and 3.20 imply that $M_t = M_0, \ t \geq 0$.

■

It is now easy to define stochastic integration with respect to continuous semimartingales.

Definition 3.25 *Let $(X_t, t \geq 0)$ be a $(\mathcal{F}_t)$-semimartingale with canonical decomposition $X = M + V$, where M (resp. V) is a continuous, $(\mathcal{F}_t)$-local martingale (resp. bounded variation), continuous and $(\mathcal{F}_t)$-adapted process vanishing at 0. Let $(H_t, t \geq 0)$ be an $(\mathcal{F}_t)$-progressively measurable process, satisfying*

$$\int_0^T H_s^2 d[M,M]_s < \infty, \quad \textit{and} \quad \int_0^T |H_s| d\|V\|_s < \infty, \tag{3.20}$$

where $\|V\|_t$ *is the total variation of* V *over* $[0,t]$.

We set

$$\int_0^t H_s dX_s = \int_0^t H_s dM_s + \int_0^t H_s dV_s, \quad 0 \le t \le T.$$

Remark 3.26 1. *In previous definition, integral with respect to* M *(resp.* V*) is an Itô-type (resp. Stieltjes-type) integral.*

2. *It is clear that* $\int_0^\cdot H_s dX_s$ *is again a continuous* $(\mathcal{F}_t)$*-semimartingale, with martingale part* $\int_0^\cdot H_s dM_s$ *and bounded variation component* $\int_0^\cdot H_s dV_s$.

Once we have introduced stochastic integrals with respect to continuous semimartingales, it is easy to define Stratonovich integrals.

Definition 3.27 *Let* $(X_t, t \ge 0)$ *be a* $(\mathcal{F}_t)$*-semimartingale and* $(Y_t, t \ge 0)$ *be a* $(\mathcal{F}_t)$*-progressively measurable process. The* **Stratonovich** *integral of* Y *with respect to* X *is defined as follows*

$$\int_0^t Y_s \circ dX_s = \int_0^t Y_s dX_s + \frac{1}{2}[Y,X]_t; \quad t \ge 0, \tag{3.21}$$

if $[Y,X]$ *and* $\int_0^\cdot Y_s dX_s$ *exist.*

Remark 3.28 1. *Recall that conditions of type (3.20) ensure existence of the stochastic integral with respect to* X.

2. *If* (X_t) *and* (Y_t) *are* $(\mathcal{F}_t)$*-semimartingales, then* $\int_0^\cdot Y_s \circ dX_s$ *exists and is called* **Fisk-Stratonovich** *integral.*

3. *Suppose that* (X_t) *is an* $(\mathcal{F}_t)$*-semimartingale and* (Y_t) *is a left continuous and* $(\mathcal{F}_t)$*-adapted process such that* $[Y,X]$ *exists. We already have observed (see Proposition 3.16) that* $\int_0^\cdot Y_s dX_s$ *coincides with* $\int_0^\cdot Y_s d^- X_s$*. Proposition 2.5 1) and 2) imply that the Stratonovich integral* $\int_0^\cdot Y_s \circ dX_s$ *is equal to the symmetric integral* $\int_0^\cdot Y_s d^\circ X_s$.

At this point we can easily identify the covariation of two semimartingales.

Proposition 3.29 *Let* $S^i = M^i + V^i$ *two* $(\mathcal{F}_t)$*-semimartingales,* $i = 1, 2$*, where* M^i *are local martingales and* V^i *bounded variation processes. We have* $[S^1, S^2] = [M^1, M^2]$.

Proof. The result follows directly from Corollary 3.21, Proposition 2.5 7), and the bilinearity of the covariation. ■

Corollary 3.30 *Let S^1, S^2 be two $(\mathcal{F}_t)$- semimartingales such that their martingale parts are independent. Then $[S^1, S^2] = 0$.*

Proof. It follows from Proposition 3.23. ■

Proposition 3.16 can be generalized as follows.

Proposition 3.31 *Let X be an $(\mathcal{F}_t)$-semimartingale and suppose that (H_t) is adapted, with left limits at each point. Then $\int_0^{\cdot} H_s d^- X_s = \int_0^{\cdot} H_{s-} dX_s$. If H is càdlàg then $\int_0^{\cdot} H d^- X = \int_0^{\cdot} H dX$.*

Remark 3.32

1. *Forward integral generalizes not only the classical Itô integral but also the integral issued from the enlargement of filtrations theory, see e.g. [28]. Let $(\mathcal{F}_t)$ and $(\mathcal{G}_t)$ be two filtrations fulfilling usual conditions with $\mathcal{F}_t \subset \mathcal{G}_t$, for any t. Let X be a $(\mathcal{F}_t)$-semimartingale with decomposition $M + V$, M being a continuous $(\mathcal{F}_t)$-local martingale and V a continuous with bounded variation $(\mathcal{F}_t)$-adapted process. Let H be a càdlàg bounded $(\mathcal{F}_t)$-adapted process. According to Proposition 3.31, the $(\mathcal{F}_t)$-Itô integral $\int_0^{\cdot} H dX$ equals the $(\mathcal{G}_t)$-Itô integral and it coincides with the forward integral $\int_0^{\cdot} H d^- X$.*

2. *The result stated above is wrong when H has no left limits at each point. Using a tricky example in [41], it is possible to exhibit two filtrations $(\mathcal{F}_t^X)$ and $(\mathcal{G}_t)$ with $\mathcal{F}_t^X \subset \mathcal{G}_t$ for each $t \geq 0$, a bounded and $(\mathcal{F}_t^X)$-progressively measurable process H, such that $\int_0^{\cdot} H d^- X$ equals the $(\mathcal{F}_t^X)$-Itô integral but differs from the $(\mathcal{G}_t)$-Itô integral. More precisely we have.*

   (a) *X is a 3-dimensional Bessel process with natural filtration $(\mathcal{F}_t^X)$ and decomposition*

   $$X_t = W_t + \int_0^t \frac{1}{X_s} ds, \tag{3.22}$$

   where W is an $(\mathcal{F}_t^X)$-Brownian motion,

   (b) *X is an $(\mathcal{G}_t)$-semimartingale with decomposition $M + V$,*

   (c) *H is $(\mathcal{F}_t^X)$-progressively measurable process,*

(d) $H_t(\omega) = 1$ *for almost all* $dt \otimes dP \quad (t, \omega) \in [0, T] \times \Omega$,

(e) $\beta_t = \int_0^t HdX$ *is an* $(\mathcal{G}_t)$*-Brownian motion.*

Property (d) implies that $I^-(\varepsilon, H, dX) = I^-(\varepsilon, 1, dX)$ *so that* $\int_0^t Hd^-X = X_t$. *The* $(\mathcal{F}_t^X)$*-Itô integral* $\int_0^t HdX$ *equals* $\int_0^t HdW + \int_0^t \frac{H_s}{X_s} ds$. *Theorem 3.33 below and Proposition 2.5 8) imply that the previous integral coincides with* $\int_0^t Hd^-X$. *Since a Bessel process cannot be equal to a Brownian motion, the* $(\mathcal{G}_t)$*-Itô integral* $\int_0^t HdX$ *differs from the* $(\mathcal{F}_t^X)$*-Itô integral* $\int_0^t HdX$.

According to ii), $[X]_t = [W]_t = t$*; therefore* M *is an* $(\mathcal{G}_t)$*-Brownian motion. Theorem 3.33 below say that* $\int_0^{\cdot} Hd^-M = \int_0^{\cdot} HdM$*; the additivity of forward integral and Itô integral imply that* $\int_0^{\cdot} Hd^-V \neq \int_0^{\cdot} HdV$. *Consequently it can be deduced from Proposition 2.5 7) a) that* H *is not a.s. with countable discontinuities. This explains why the* $(\mathcal{G}_t)$*-Itô integral* $\int_0^t HdX$ *is different from the* $(\mathcal{F}_t^X)$*-Itô integral* $\int_0^t HdX$.

3.5 The Brownian case

In this section we will investigate the link between forward and Itô integration with respect to a Brownian motion. In this section (W_t) will denote a $(\mathcal{F}_t)$-Brownian motion.

The main result of this subsection is the following.

Theorem 3.33 *Let* $(H_t, t \geq 0)$ *be an* $(\mathcal{F}_t)$*-progressively measurable process satisfying* $\displaystyle\int_0^T H_s^2 ds < \infty$ *a.s. Then the Itô integral* $\displaystyle\int_0^{\cdot} H_s dW_s$ *coincides with the forward integral* $\displaystyle\int_0^{\cdot} H_s d^-W_s$.

Remark 3.34 1. *We would like to illustrate the advantage of using regularization instead of discretization ([19]) through the following example.*

Let g *be the indicator function of* $\mathbb{Q} \cap \mathbb{R}_+$.

Let $\Pi = \{t_0 = 0, t_1, \cdots, t_N = T\}$ *be a subdivision of* $[0, T]$ *and*

$$I(\Pi, g, dW)_t := \sum_i g(t_i)\big(W(t_{i+1} \wedge t) - W(t_i \wedge t)\big); \quad 0 \leq t \leq T.$$

We remark that

$$I(\Pi, g, dW)_t = \begin{cases} 0 & \textit{if } \Pi \subset \mathbb{R} \setminus \mathbb{Q} \\ W_t & \textit{if } \Pi \subset \mathbb{Q}. \end{cases}$$

Therefore there is no canonical definition of $\int_0^t gdW$ *through discretization. This is not surprising since* g *is not everywhere continuous and so is not Riemann integrable. On the contrary, integration via regularization seems drastically more adapted to define* $\int_0^t gd^-W$, *for any* $g \in L^2([0,T])$, *since this integral coincides with the classical Itô-Wiener integral.*

2. *In fact, the discretization approach admits several sophistications in the literature; a significant one is McShane stochastic integration, see [37] chap. 2 and 3. McShane makes use of the so called belated partition in the framework of discretization approach. In such a case a class of non Riemann integrable functions* g *can be integrated versus Brownian motion.*

Proof (of Theorem 3.33) 1) Suppose first that H is moreover a continuous process. Replacing X by W in (3.16) we get

$$\int_0^t \Big(\frac{1}{\varepsilon}\int_{s-\varepsilon}^s H_u du\Big) dW_s = \frac{1}{\varepsilon}\int_0^t H_s(W_{s+\varepsilon} - W_s)ds + R_\varepsilon(t), \tag{3.23}$$

where the reminder term $R_\varepsilon(t)$ is given by (3.17).

Recall the maximal inequality ([51], chap. I.1): there exists a constant C such that for any $\phi \in L^2([0,T])$,

$$\int_0^T \Big(\sup_{0<\eta<1}\Big\{\frac{1}{\eta}\int_{(v-\eta)_+}^v \phi_v dv\Big\}\Big)^2 du \le C\int_0^T \phi_v^2 dv. \tag{3.24}$$

2) We claim that (3.23) may be extended to progressively measurable processes (H_t) satisfying $\int_0^\cdot H_s^2 ds < \infty$.

Let $H_t^n = n\int_{t-1/n}^t H_u du, t \ge 0$.

It is clear that as $n \to \infty$

- for a.e. t, H_t^n converges to H_t,
- (H_t^n) converges to (H_t) in $\mathcal{L}^2(d<W>)$ (i.e. $\int_0^\cdot (H_s^n - H_s)^2 ds$ goes to 0 in the ucp sense).

Since

$$<\int_0^\cdot \Big(\frac{1}{\varepsilon}\int_{s-\varepsilon}^s H_u du\Big)dW_s>_t = \int_0^\cdot \Big(\frac{1}{\varepsilon}\int_{s-\varepsilon}^s H_u du\Big)^2 ds,$$

then (3.24) and Lemma 3.9 imply that (3.23) and (3.17) are valid.

Taking $\varepsilon \to 0$ in (3.23), and using once more (3.24) Lemma 3.9 prove Theorem 3.33.

■

3.6 Substitution formulae

We conclude Section 3, observing that our approach allows us to integrate non adapted integrands in a context which is covered neither by Skorohod integration theory nor by enlargement of filtrations. A class of examples is the following.

Let $(X(t,x), t \geq 0, x \in \mathbb{R}^d)$, $(Y(t,x),\ t \geq 0,\ x \in \mathbb{R}^d)$ be two families of continuous $(\mathcal{F}_t)$ semimartingales depending on a parameter x and $(H(t,x), t \geq 0, x \in \mathbb{R}^d)$ $(\mathcal{F}_t)$ progressively measurable processes depending on x. Let Z be a $\mathcal{F}_T$-measurable r.v., taking its values in $\mathbb{R}^d$.

Under some minimal conditions of Garsia-Rodemich-Rumsey type, see for instance [48, 49], we have

$$\int_0^t H(s,Z)\, d^- X(s,Z) = \int_0^t H(s,x)\, dX(s,x)\Big|_{x=Z},$$

$$[X(\cdot,Z), Y(\cdot,Z)] = [X(\cdot,x), Y(\cdot,x)]\Big|_{x=Z}.$$

The first result is useful to prove existence results for SDE's driven by a semimartingale, with an anticipating initial condition.

It is significant to remark that previous substitution formulae create anticipating calculus in a setting which is not covered by Malliavin non-causal calculus since our integrators may be general semimartingales, while Skorohod integral applies essentially for Gaussian integrators or eventually Poisson type processes. Note that the usual causal Itô calculus may not be applied since $(X(s,Z))_s$ is not a semimartingale, take for instance a r.v. Z such that $\mathcal{F}_T = \sigma(Z)$.

4 Calculus for finite quadratic variation processes

4.1 Stability of the covariation

One basic tool of calculus via regularization states that the family of finite quadratic variation processes is stable through C^1 transformations.

Proposition 4.1 *Let (X^1, X^2) be a vector of processes having all its mutual covariations, $f, g \in C^1(\mathbb{R})$. Then, $[f(X^1), g(X^2)]$ exists and it is given by*

$$[f(X^1), g(X^2)]_t = \int_0^t f'(X^1_s) g'(X^2_s) d[X^1, X^2]_s$$

Proof. Using polarization techniques (bilinearity arguments), it is enough to consider the case $X = X^1 = X^2$ and $f = g$.

Using Taylor's formula, we expand as follows

$$f(X_{s+\varepsilon}) - f(X_s) = f'(X_s)(X_{s+\varepsilon} - X_s) + R(s,\varepsilon)(X_{s+\varepsilon} - X_s), \quad s \geq 0, \varepsilon > 0,$$

where $R(s,\varepsilon)$ denotes here a generic process which converges in the ucp sense to 0, when $\varepsilon \to 0$.

Since f' is unifomly continuous on each compact, this implies that

$$(f(X_{s+\varepsilon}) - f(X_s))^2 = f'(X_s)^2(X_{s+\varepsilon} - X_s)^2 + R(s,\varepsilon)(X_{s+\varepsilon} - X_s)^2$$

Integrating from 0 to t, we get

$$\frac{1}{\varepsilon}\int_0^t (f(X_{s+\varepsilon}) - f(X_s))^2 ds = I_1(t,\varepsilon) + I_2(t,\varepsilon)$$

where

$$\begin{aligned} I_1(t,\varepsilon) &= \int_0^t f'(X_s)^2 \frac{(X_{s+\varepsilon} - X_s)^2}{\varepsilon} ds, \\ I_2(t,\varepsilon) &= \frac{1}{\varepsilon}\int_0^t R(s,\varepsilon)(X_{s+\varepsilon} - X_s)^2 ds. \end{aligned}$$

Clearly we have

$$\sup_{t \leq T} |I_2(t,\varepsilon)| \leq \sup_{s \leq T} |R(s,\varepsilon)| \frac{1}{\varepsilon}\int_0^T (X_{s+\varepsilon} - X_s)^2 ds.$$

Since $[X]$ exists then $I_2(\cdot,\varepsilon) \stackrel{\text{ucp}}{\longrightarrow} 0$. The result will follow if we establish

$$\frac{1}{\varepsilon}\int_0^{\cdot} Y_s d\mu_\varepsilon(s) \stackrel{\text{ucp}}{\longrightarrow} \int_0^{\cdot} Y_s d[X,X]_s \tag{4.1}$$

where $\mu_\varepsilon(t) = \int_0^t \frac{ds}{\varepsilon}(X_{s+\varepsilon} - X_s)^2$ and Y is a continuous process. It is not difficult to verify that a.s., $\mu_\varepsilon(dt)$ converges to $d[X,Y]$, when $\varepsilon \to 0$. This finally implies (4.1). ■

4.2 Itô formulae for finite quadratic variation processes

Even if all the Itô formulae that we will consider can be stated in the multidimensional case, see for instance [48], we will only deal here with dimension 1. Let $X = (X_t)_{t \geq 0}$ be a continuous process.

Proposition 4.2 *Suppose that $[X, X]$ exists and let $f \in C^2(\mathbb{R})$. Then*

$$\int_0^{\cdot} f'(X) d^- X \quad \text{and} \quad \int_0^{\cdot} f'(X) d^+ X \quad \text{exist.} \tag{4.2}$$

Moreover

a) $f(X_t) = f(X_0) + \int_0^t f'(X) d^{\mp} X \pm \frac{1}{2} \int_0^t f''(X_s) d[X, X]_s,$

b) $f(X_t) = f(X_0) + \displaystyle\int_0^t f'(X) d^{\mp} X \pm \frac{1}{2} [f'(X), X]_t,$

c) $f(X_t) = f(X_0) + \displaystyle\int_0^t f'(X) d^{\circ} X.$

Proof. c) follows from b) summing up $+$ and $-$.

b) follows from a), since Proposition 4.1 implies that

$$[f'(X), X]_t = \int_0^t f''(X) d[X, X].$$

a) and (4.2) follow by similar methods as Proposition 4.1 proceeding this time with Taylor expansion up to second order. ■

We emphasize that existence of the quadratic variation is closely connected with existence of some related forward and backward integrals.

Lemma 4.3 *Let X be a continuous process. Then $[X, X]$ exists $\Longleftrightarrow \int_0^{\cdot} X d^- X$ exists $\Longleftrightarrow \int_0^{\cdot} X d^+ X$ exists.*

Proof. We start with identity

$$(X_{s+\varepsilon} - X_s)^2 = X_{s+\varepsilon}^2 - X_s^2 - 2X_s (X_{s+\varepsilon} - X_s). \tag{4.3}$$

We observe that, when $\varepsilon \to 0$,

$$\frac{1}{\varepsilon} \int_0^t (X_{s+\varepsilon}^2 - X_s^2) ds \to X_t^2 - X_0^2.$$

Integrating (4.3) from 0 to t and dividing by ε, we easily obtain the equivalence between the two first assertions.

The equivalence between the first and the third one follows replacing ε, with $-\varepsilon$ in (4.3). ■

Lemma 4.3 admits the following generalization.

Corollary 4.4 *Let X be a continuous process. The following properties are equivalent*

a) $[X, X]$ *exists;*

b) $\int_0^{\cdot} g(X) d^- X$ *exists* $\forall g \in C^1$*;*

c) $\int_0^{\cdot} g(X) d^+ X$ *exists* $\forall g \in C^1$*.*

Proof. The Itô formula stated in Proposition 4.2 1) implies a) $\Rightarrow$ b). b) $\Rightarrow$ a) follows setting $g(x) = x$ and using Lemma 4.3.

b) $\Leftrightarrow$ c) because of Proposition 2.5 1) which states that

$$\int_0^{\cdot} g(X) d^+ X = \int_0^{\cdot} g(X) d^- X + [g(X), X],$$

and Proposition 4.1 saying that $[g(X), X]$ exists. ■

Previous Itô formula becomes as follows in the case when X is a semimartingale.

Proposition 4.5 *Let $(S_t)_{t \geq 0}$ be a continuous $(\mathcal{F}_t)-$semimartingale, $f \in C^2(\mathbb{R})$. We have the following.*

1.

$$f(S_t) = f(S_0) + \int_0^t f'(S_u) dS_u + \frac{1}{2} \int_0^t f''(S_u) d[S, S]_u$$

2. Let (S_t^0) be another continuous $(\mathcal{F}_t)$-semimartingale. The following integration by parts holds:

$$S_t S_t^0 = S_0 S_0^0 + \int_0^t S_u dS_u^0 + \int_0^t S_u^0 dS_u + [S, S^0]_t.$$

Proof. We recall that Itô and forward integrals coincide, see Proposition 3.16. Therefore point 1. is a consequence of Proposition 4.2.

2. is a consequence of integration by parts formula Proposition 2.5 4).

■

4.3 Lévy area

4.8

At Corollary 4.4, we have seen that $\int_0^t g(X)d^-X$ exists when X is a one-dimensional finite quadratic variation process and $g \in C^1(\mathbb{R})$.

Let X is a two-dimensional so that $X = (X^1, X^2)$ and has all its mutual covariations and consider $g \in C^1(\mathbb{R}^2; \mathbb{R}^2)$. We naturally define, if it exists,

$$\int_0^t g(X) \cdot d^-X = \lim_{\varepsilon \to 0^+} I^-(\varepsilon, g(X) \cdot dX)(t),$$

where

$$I^-(\varepsilon, g(X) \cdot dX)(t) = \int_0^t g(X)(s) \cdot \frac{X(s+\varepsilon) - X(s)}{\varepsilon} ds; \quad 0 \leq t \leq T, \tag{4.4}$$

and $\cdot$ denotes the scalar product in $\mathbb{R}^2$.

Formulating a 2-dimensional Itô formula of the same type as Proposition 4.2, it is possible to show that $\int_0^t g(X) \cdot d^-X$ exists if $g = \nabla u$ where u is a potential of class C^2. If g is a general $C^1(\mathbb{R}^2)$ function, we cannot expect in general that $\int_0^t g(X) \cdot d^-X$ exists.

T. Lyons rough paths theory approach, see for instance [35, 34, 30, 27, 8] has considered in detail the problem of the existence of integrals of the type $\int_0^t g(X) \cdot dX$. In this theory, the concept of Lévy area plays a significant role. Translating this in our context one would say that the essential assumption is that $X = (X^1, X^2)$ has a Lévy area type process. This section will only make some basic observations on that topic from the perspective of stochastic calculus via regularization.

Given two classical semimartingales S^1, S^2, the classical notion of Lévy area is given by

$$L(S^1, S^2)_t = \int_0^t S^1 dS^2 - \int_0^t S^2 dS^1,$$

where previous integrals are of Itô type.

Definition 4.6 *Given two processes X and Y, we denote*

$$L(X, Y)_t = \lim_{\varepsilon \to 0^+} \int_0^t \frac{X_s Y_{s+\varepsilon} - X_{s+\varepsilon} Y_s}{\varepsilon} ds.$$

where the limit is understood in the ucp sense. $L(X, Y)$ is called the **Lévy area** *of processes X and Y.*

Remark 4.7 *The following properties are easy to establish.*

1. $L(X, X) \equiv 0$.

2. *The Lévy area is an antisymmetric operation, i.e.*

$$L(X,Y) = -L(Y,X).$$

Using the approximation of symmetric integral we can easily prove the following.

Proposition 4.8 *$\int_0^{\cdot} X d^\circ Y$ exists if and only if $L(X,Y)$ exists. Moreover*

$$2\int_0^t X d^\circ Y = X_t Y_t - X_0 Y_0 + L(X,Y)_t$$

Recalling the convention that an equality among three objects implies that at least two among the three are defined, we have the following.

Proposition 4.9 1. $L(X,Y)_t = \int_0^t X d^\circ Y - \int_0^t Y d^\circ X$.

2. $L(X,Y)_t = \int_0^t X d^- Y - \int_0^t Y d^- X$.

Proof.

1. From Proposition 4.8 applied to X, Y and Y, X, and by antisymmetry of Lévy area we have

$$\begin{aligned} 2\int_0^t X d^\circ Y &= X_t Y_t - X_0 Y_0 + L(X,Y)_t, \\ 2\int_0^t Y d^\circ X &= X_t Y_t - X_0 Y_0 - L(X,Y)_t. \end{aligned}$$

   Taking the difference of the two lines, 1. follows.

2. follows from the definition of forward integrals.

■

Remark 4.10 *If $[X,Y]$ exists, point 2. of Proposition 4.9 is a consequence of point 1. and of Proposition 2.5 1., 2.*

For a real valued process $(X_t)_{t\geq 0}$, Lemma 4.3 says that

$$[X,X] \quad \text{exists} \Leftrightarrow \int_0^{\cdot} X d^- X \quad \text{exists.}$$

Given a vector of processes $\underline{X} = (X^1, X^2)$ we may ask wether the following statement is true:

(X^1, X^2) has all its mutual brackets if and only if

$$\int_0^\cdot X^i d^- X^j \quad \text{exists},$$

for $i, j = 1, 2$. In fact the answer is negative if the two-dimensional process X does not have a Lévy area.

Remark 4.11 *Let us suppose that (X^1, X^2) has all its mutual covariations. Let $* = \circ, -, +$. The following are equivalent.*

1. *The Lévy area $L(X^1, X^2)$ exists.*
2. *$\int_0^\cdot X^i d^* X^j$ exists for any $i, j = 1, 2$.*

By Lemma 4.3, we first observe that $\int X^i d^\circ X^i$ exists since X^i is a finite quadratic variation process. In point 2. the equivalence between the cases $* = \circ, -, +$ is obvious using Proposition 2.5 1) 2). Equivalence between the existence of $\int_0^\cdot X^1 d^\circ X^2$ and $L(X^1, X^2)$ has already been established in Proposition 4.8.

5 Weak Dirichlet processes

5.1 Generalities

Weak Dirichlet processes constitute a natural generalization of Dirichlet processes, which naturally extend semimartingales. Dirichlet processes have been considered by many authors, see for instance [20, 2].

Let $(\mathcal{F}_t)_{t \geq 0}$ be a fixed filtration fulfilling the usual conditions. In the present section 5, (W_t) will denote a classical $(\mathcal{F}_t)$-Brownian motion. We will remain for simplicity, in the framework of continuous processes.

Definition 5.1

1. *An $(\mathcal{F}_t)$-**Dirichlet process** is the sum of an $(\mathcal{F}_t)$- local martingale M plus a zero quadratic variation process A.*
2. *An $(\mathcal{F}_t)$-**weak Dirichlet process** is the sum of a $(\mathcal{F}_t)$- local martingale M plus a process A such that $[A, N] = 0$ for any continuous $(\mathcal{F}_t)$- local martingale N.*

In both cases, we will suppose $A_0 = 0$ a.s.

Remark 5.2

1. *Process (A_t) in previous decomposition is an $(\mathcal{F}_t)$-adapted process.*

2. *An* $(\mathcal{F}_t)$*-semimartingale is an* $(\mathcal{F}_t)$*-Dirichlet process.*

The statement of the following proposition is essentially contained in [12].

Proposition 5.3 1. *An* $(\mathcal{F}_t)$*-Dirichlet process is an* $(\mathcal{F}_t)$*-weak Dirichlet process.*

2. *The decomposition* $M + A$ *is unique.*

Proof. Point 1. follows from Proposition 2.5 6).

Concerning point 2., let X be a weak Dirichlet process with decompositions $X = M^1 + A^1 = M^2 + A^2$. Then $0 = M + A$ where $M = M^1 - M^2, A = A^1 - A^2$. We evaluate the covariation of both members against M to obtain

$$0 = [M] + [M, A^1] - [M, A^2] = [M].$$

Since $M_0 = A_0 = 0$ and M is a local martingale, Corollary 3.8 gives $M = 0$.

■

The class of semimartingales with respect to a given filtration is known to be stable with respect to C^2 transformations, as Proposition 4.5 implies. Proposition 4.1 says that finite quadratic variation processes are stable through C^1 transformations.

It is possible to show that the class of weak Dirichlet processes with finite quadratic variation (as well as Dirichlet processes) is stable with respect to the same type of transformations.

We start with a result which is a slight improvement (in the continuous case) of a result obtained by [7].

Proposition 5.4 *Let* X *be a finite quadratic variation process which is* $(\mathcal{F}_t)$*- weak Dirichlet,* $f \in C^1(\mathbb{R})$*. Then* $f(X)$ *is again weak Dirichlet.*

Proof. Let $X = M + A$ be the corresponding decomposition. We express $f(X_t) = M^f + A^f$ where

$$M_t^f = f(X_0) + \int_0^t f'(X) dM, \quad A_t^f = f(X_t) - M_t^f.$$

Let N be a local martingale. We have to show that $[f(X) - M^f, N] = 0$.

By additivity of the covariation, and the definition of weak Dirichlet process, $[X, N] = [M, N]$ so that Proposition 4.1 implies that $[f(X), N]_t = \int_0^t f'(X_s) d[M, N]_s$.

On the other hand, Proposition 3.22 gives

$$[M^f, N]_t = \int_0^t f'(X_s) d[M, N]_s,$$

and the result follows. ■

Remark 5.5 1. *If X is an a $(\mathcal{F}_t)$- Dirichlet process, it can be proved similarly that $f(X)$ is an $(\mathcal{F}_t)$- Dirichlet process, see for details [2] and [50].*

2. *The class of Lyons-Zheng processes introduced in [50] consitutes a natural generalization of reversible semimartingales, see Definition 5.12. The authors proved that this class is also stable through C^1 transformation.*

We also report a Girsanov type theorem established by [7] at least in a discretization framework.

Proposition 5.6 *Let $X = (X_t)_{t\in[0,T]}$ be an $(\mathcal{F}_t)$-weak Dirichlet process. Let Q a probability equivalent to P on $\mathcal{F}_T$. Then $X = (X_t)_{t\in[0,T]}$ is $(\mathcal{F}_t)$-weak Dirichlet process with respect to Q.*

Proof. We set $D_t = \frac{dQ}{dP}|_{\mathcal{F}_t}$. D is a positive local martingale.

Let L be the local martingale such that $D_t = \exp(L_t - \frac{1}{2}[L]_t)$. Let $X = M + A$ be the corresponding decomposition. It is well-known that $\tilde{M} = M - [M, L]$ is a local martingale under Q. So, X is a Q- weak Dirichlet process. ■

As mentioned earlier, Dirichlet processes are stable with respect to C^1- transformations. In applications, in particular to control theory, one would need to know the nature of process $(u(t, D_t))$ where $u \in C^{0,1}(\mathbb{R}_+ \times \mathbb{R})$ and D is a Dirichlet process. The following result was established in [23].

Proposition 5.7 *Let (S_t) be a continuous $(\mathcal{F}_t)$-weak Dirichlet process with finite quadratic variation. Let $u \in C^{0,1}(\mathbb{R}_+ \times \mathbb{R})$. Then $(u(t, S_t))$ is a $(\mathcal{F}_t)$-weak Dirichlet process.*

Remark 5.8 *There is no reason for $(u(t, S_t))$ to be a finite quadratic variation process since the dependence of u from the first argument t may be very rough. A fortiori $(u(t, S_t))$ will not be Dirichlet. Consider for instance u only depending on time, deterministic, with no finite quadratic variation.*

Examples of Dirichlet processes (respectively weak Dirichlet processes) arise directly from classical Brownian motion W.

Example 5.9 *Let f be of class $C^0(\mathbb{R})$, $u \in C^{0,1}(\mathbb{R}_+ \times \mathbb{R})$.*

1. *If f is C^1, then $X = f(W)$ is a $(\mathcal{F}_t)$-Dirichlet process.*
2. *$u(t, W_t)$ is an $(\mathcal{F}_t)$-weak Dirichlet process wich in general is not Dirichlet.*
3. *$f(W)$ is not always a Dirichlet process, not even of finite quadratic variation as shows Proposition 5.17.*

Previous Example and Remark easily show that the class of $(\mathcal{F}_t)$-Dirichlet processes strictly include the class of $(\mathcal{F}_t)$-semimartingales.

More sophisticated examples of weak Dirichlet processes may be found in the class of the so called *Volterra* type processes, se e.g. [11, 12]

Example 5.10 *Let $(N_t)_{t\geq 0}$ be an $(\mathcal{F}_t)$-local martingale, $G : \mathbb{R}_+ \times \mathbb{R}_+ \times \Omega \longrightarrow \mathbb{R}$ continuous random field such that $G(t, \cdot)$ is $(\mathcal{F}_s)$-*adapted *for any t. We set*

$$X_t = \int_0^t G(t, s) dN_s.$$

Then (X_t) is an $(\mathcal{F}_t)$-weak Dirichlet process with decomposition $M + A$, where $M_t = \int_0^t G(s, s) dN_s$.

Suppose that $[G(\cdot, s_1); G(\cdot, s_2)]$ exists for any s_1, s_2. With some additional technical assumption, one can show that A is a finite quadratic variation process with

$$[A]_t = 2 \int_0^t \left(\int_0^{s_2} [G(\cdot, s_1); G(\cdot, s_2)] \circ dM_{s_1} \right) \circ dM_{s_2}.$$

Previous iterated Stratonovich integral can be expressed as the sum $C_1(t) + C_2(t)$ where

$$\begin{aligned} C_1(t) &= \int_0^t [G(\cdot, s); G(\cdot, s)] d[M]_s, \\ C_2(t) &= 2 \int_0^t \left(\int_0^{s_2} [G(\cdot, s_1); G(\cdot, s_2)] dM_{s_1} \right) dM_{s_2}. \end{aligned}$$

Example 5.11 *Suppose that N is a classical Brownian motion W and $G(t, s) = B_{t-s}$ where B is a Brownian motion for positive indices and 0 for negative ones; we suppose B independent of W. Then $[A] = \int_0^t (t - s) ds = \frac{t^2}{2}$.*

One significant motivation for considering Dirichlet (respectively weak Dirichlet) processes comes from the study of generalized diffusion processes, typically solutions of stochastic differential equations with distributional drift.

Such processes were investigated using stochastic calculus via regularization by [17, 18]. We try to express here just a guiding idea. The following particular case of such equations is motivated by random media modelization:

$$dX_t = dW_t + b'(X_t)dt, \quad X_0 = x_0 \tag{5.1}$$

where b is a continuous function.

b could be the realization of a continuous process, independent of W, stopped outside a finite interval.

We do not want to recall the precise sense of the solution of (5.1). In [17, 18] the authors give a precise sense to a solution (in the distribution laws) and they show existence and uniqueness for any initial conditions.

Here we can just convince the reader that the solution is a Dirichlet process. For this we define the real function h of class C^1 defined by

$$h(x) = \int_0^x e^{-b(y)} dy.$$

We set $\sigma_0 = h' \circ h^{-1}$. We consider the unique solution in law of the equation

$$dY_t = \sigma_0(Y_t)dW_t, Y_0 = h(x_0)$$

which exists because of classical Stroock-Varadhan arguments ([52]); so Y is clearly a semimartingale, so a Dirichlet process. The process $X = h^{-1}(Y)$ is a Dirichlet process since and h^{-1} is of class C^1. If b were of class C^1, (5.1) would be an ordinary stochastic differential equation, and it could be shown that X is the unique solution of that equation. In the actual case X will still be the solution of (5.1), considered as generalized stochastic differential equation.

We consider now the case when the drift is time inhomogeneous as follows

$$dX_t = dW_t + \partial_x b(t, X_t)dt, X_0 = x_0 \tag{5.2}$$

where $b : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$ is a continuous function of class C^1 in time. Then it is possible to define $k : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$ of class C^1 where the *solution* (X_t) of (5.2) can be expressed as $(k(t, Y_t))$ where Y is a semimartingale and k is of class $C^{0,1}$. In conclusion X will be a $(\mathcal{F}_t)$ weak Dirichlet process. For this and more general situations, see [43].

5.2 Itô formula under weak smoothness assumptions

In this section, we formulate and prove an Itô formula of C^1 type. As for the C^2 type Itô formula, next Theorem is stated in the one-dimensional framework in spite of its validity in the multidimensional case.

Let $(S_t)_{t\geq 0}$ be a semimartingale and $f \in C^2$. We recall the classical Itô formula, as a particular case of Proposition 4.5: :

$$f(S_t) = f(S_0) + \int_0^t f'(S_s)dS_s + \frac{1}{2}\int_0^t f''(S_s)d[S,S]_s.$$

Using Proposition 3.16 and Stratonovich integral Definition 3.27, we obtain

$$\begin{aligned} f(S_t) &= f(S_0) + \int_0^t f'(S_s)dS_s + \frac{1}{2}[f'(S),S]_t \\ &= f(S_0) + \int_0^t f'(S) \circ dS. \end{aligned} \tag{5.3}$$

We observe that in formulae (5.3), only the first derivative of f appears. Besides, we know that $f(S)$ is a Dirichlet process if $f \in C^1(\mathbb{R})$.
At this point we may ask if formulae (5.3) remains valid when f is only $\in C^1(\mathbb{R})$ only. A partial answer will be given in Theorem 5.13 below.

Definition 5.12 *Let (S_t) be a continuous semimartingale. We set $\hat{S}_t = S_{T-t}, t \in [0,T]$, S is said to be a* **reversible semimartingale** *if $(\hat{S}_t)_{t\in[0,T]}$ is again a semimartingale.*

Theorem 5.13 *([44]) Let S be a reversible semimartingale indexed by $[0,T]$ and $f \in C^1(\mathbb{R})$. Then, we have*

$$\begin{aligned} f(S_t) &= f(S_0) + \int_0^t f'(S)dS + R_t \\ &= f(S_0) + \int_0^t f'(S) \circ dS. \end{aligned}$$

where $R = \frac{1}{2}[f'(S),S]$.

Remark 5.14 *After the pioneering work of [5], which expressed the remainder term (R_t) with the help of generalized integral with respect to local time, two papers appeared: [21] in the case of Brownian motion and [21] and [44] for multidimensional reversible semimartingales. Later, an incredible amount of contributions on* *that topic have been published. The present paper cannot give precisely the content of each paper. A non-exhaustive list of papers is given by [1, 13, 14, 22, 23, 38, 39]. Among the C^1-type Itô formula in the framework of generalized Stratonovich integral with respect to Lyons-Zheng processes, it is also important to quote [32, 33, 50].*

Example 5.15 *i) Classical $(\mathcal{F}_t)$-Brownian motion W is a reversible semimartingale, see for instance [21, 40, 18]. More precisely we have $\hat{W}_t = W_T + \beta_t + \int_0^t \frac{\hat{W}_s}{T-s} ds$, where β is a $(\mathcal{G}_t)$-Brownian motion and $(\mathcal{G}_t)$ is the natural filtration associated with $\hat{W}_t$.*

ii) Let (X_t) be the solution of the stochastic differential equation

$$dX_t = \sigma(t, X_t)dW_t + b(t, X_t)dt,$$

with $\sigma, b : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ Lipschitz with at most linear growth, $\sigma \geq c > 0$. Then (X_t) is a reversible semimartingale, see for instance [18]. Moreover if $f \in W^{1,2}_{\mathrm{loc}}$, in [18] it is proved that $(f(X_t))$ is an $(\mathcal{F}_t)$-Dirichlet process.

Proof (of Theorem 5.13). We use in an essential way the Banach-Steinhaus theorem for F- spaces, see for instance [9] chap. 2.1.

We define the following maps $T_\varepsilon^\pm$ going from the F- space $C^0(\mathbb{R})$ into the F- space of continuous processes indexed by $[0, T]$ which is denoted by $\mathcal{C}([0, T])$:

$$T_\varepsilon^- g = \int_0^\cdot g(S_s) \frac{S_{s+\varepsilon} - S_s}{\varepsilon} ds,$$

$$T_\varepsilon^+ g = \int_0^\cdot g(S_s) \frac{S_s - S_{s-\varepsilon}}{\varepsilon} ds,$$

Those operators are linear and continuous. Moreover, for each $g \in C^0$ we have

$$\lim_{\varepsilon \to 0} T_\varepsilon^- g = \int_0^\cdot g(S) dS,$$

because of Proposition 3.16 which says that $\int_0^t g(S)dS$ coincides with Itô's integral. Since $\hat{S}$ is a semimartingale, for the same reasons as above,

$$\int_{T-t}^T g(\hat{S}) d^- \hat{S} \tag{5.4}$$

also exists and it equals Itô's integral.

Using Proposition 2.5 3), it follows that $\int_0^\cdot g(S) d^+ S$ also exists.

Therefore Banach-Steinhaus theorem implies that

$$g \mapsto \int_0^{\cdot} g(S)d^- S, \quad g \mapsto \int_0^{\cdot} g(S)d^+ S,$$

and by additivity

$$g \mapsto [g(S), S], \quad g \mapsto \int_0^{\cdot} g(S)d^\circ S,$$

are continuous maps from $C^0(\mathbb{R})$ to $\mathcal{C}([0,T])$.

Let $f \in C^1(\mathbb{R})$, $(\rho_\varepsilon)_{\varepsilon>0}$ be a family of mollifiers converging to the Dirac measure at zero. We set $f_\varepsilon = f \star \rho_\varepsilon$ where $\star$ denotes convolution. Since f_ε is of class C^2, by the "smooth" Itô formula stated at Proposition 4.5 and Proposition 2.5 1) and 2), we have

$$\begin{aligned} f_\varepsilon(S_t) &= f_\varepsilon(S_0) + \int_0^t f'_\varepsilon(S)dS + \frac{1}{2}[f'_\varepsilon(S), S], \\ f_\varepsilon(S_t) &= f_\varepsilon(S_0) + \int_0^t f'_\varepsilon(S)d^\circ S \end{aligned}$$

Since f'_ε goes to f' in $C^0(\mathbb{R})$, we can take the limit term by term and

$$\begin{aligned} f(S_t) &= f(S_0) + \int_0^t f'(S)dS + \frac{1}{2}[f'(S), S], \\ f(S_t) &= f(S_0) + \int_0^t f'(S)d^\circ S. \end{aligned} \tag{5.5}$$

Remark 3.28 says that the previous symmetric integral is in fact a Stratonovich integral. ■

Corollary 5.16 *If $(S_t)_{t\in[0,T]}$ be a reversible semimartingale and $g \in C^0(\mathbb{R})$, then $[g(S), S]$ exists and it is a zero quadratic variation process.*

Proof. Let $g \in C^0(\mathbb{R})$ and let $S = M + V$ be the decomposition of S as a sum of a local martingale M and a finite variation process V, such that $V_0 = 0$. Let $f \in C^1(\mathbb{R})$ such that $f' = g$. We know that $f(S)$ is a Dirichlet process with local martingale part

$$M_t^f = f(S_0) + \int_0^t g(S)dM.$$

Let A^f be its zero quadratic variation component. Using Thereom 5.13, we have

$$A_t^f = \int_0^t g(S)dV + \frac{1}{2}[g(S), S].$$

$\int_0^\cdot g(S)dV$ has finite variation, therefore it has zero quadratic variation; since A^f is also a zero finite quadratic variation process, the result follows immediately. ■

Proposition 5.17 *Let $g \in C^0(\mathbb{R})$ such that $g(W)$ is a finite quadratic variation process. Then g has bounded variation.*

Proof. Suppose that $g(W)$ is of finite quadratic variation. We already know that W is a reversible semimartingale. By Corollary 5.16, $[W, g(W)]$ exists and it is a zero quadratic variation process. Since $[W]$ exists, we deduce that $(g(W), W)$ has all its mutual covariations. In particular $[g(W), W]$ has bounded variation because of Remark 2.4. Let f be such $f' = g$; Theorem 5.13 implies that $f(W)$ is a semimartingale. A celebrated result of Çinlar, Jacod, Protter and Sharpe, [6] implies that $f(W)$ is a $(\mathcal{F}_t)$-semimartingale if and only if f is difference of two convex functions; this allow finally to conclude that g is of bounded variation. ■

Remark 5.18 *Given two processes X and Y, the covariations $[X]$ and $[X, Y]$ may exist even if Y is not of finite quadratic variation. In particular (X, Y) may not have all its mutual covariations. Consider for instance $X = W$, $Y = g(W)$ where g is continuous but not of bounded variation and we apply Proposition 5.17.*

Remark 5.19 *([21] In the case of S being a classical Brownian motion, it is possible to see that Theorem 5.13 and Corollary 5.16 are valid respectively for $f \in W^{1,2}_{\mathrm{loc}}(\mathbb{R})$ and $g \in L^2_{\mathrm{loc}}(\mathbb{R})$*

6 Final remarks

We conclude this paper with some considerations about calculus related to processes having no quadratic variation. The reader can consult for this [12, 25, 26]. In [12] one defines a notion of $n-$ covariation $[X^1, \ldots, X^n]$ of n processes $X^1, \ldots, X^n$ and the n-variation of a process X.

We recall some basic significant results related to those papers.

1. Given a process X having a 3- variation, it is possible to express an Itô formula of the type

$$f(X_t) = f(X_0) + \int_0^t f'(X_s)d^\circ X_s - \frac{1}{12}\int_0^t f^{(3)}(X_s)d[X, X, X]_s.$$

   Moreover one-dimensional stochastic differential equations driven by a strong 3-variation were considered in[12].

2. Let $B = B^H$ a fractional Brownian motion with Hurst index $H > \frac{1}{6}$, f of class C^6. In [25, 26], it was shown that

$$f(B_t) = f(B_0) + \int_0^t f'(B) d^\circ B.$$

3. Other types of Itô formulae can be expressed when H is any number in $]0,1[$ using more sophisticated integrals via regularization, see [26].

4. In [24], the authors show that stochastic calculus via regularization is *almost pathwise*. Suppose for instance that X is a semimartingale or a fractional Brownian motion, with Hurst index $H > \frac{1}{2}$; then its quadratic variation $[X]$ is not only a limit of $C(\varepsilon, X, X)$ (see notation (2.5)) in the uniform convergence in probability sense but also *uniformly a.s.*. Similarly if X is semimartingale and Y is a suitable integrand, the Itô integral $\int_0^\cdot Y dX$ is not only limit of $I^-(\varepsilon, Y, dX)$, see (2.2) but also uniformly a.s.

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References

[1] Xavier Bardina and Maria Jolis. An extension of Ito's formula for elliptic diffusion processes. *Stochastic Process. Appl.*, 69(1):83–109, 1997.

[2] Jean Bertoin. Les processus de Dirichlet en tant qu'espace de Banach. *Stochastics*, 18(2):155–168, 1986.

[3] Jean Bertoin. Sur une intégrale pour les processus à α-variation bornée. *Ann. Probab.*, 17(4):1521–1535, 1989.

[4] F. Biagini and B. Øksendal. Minimal variance hedging for insider trading. *Preprint Oslo 2004.*

[5] Nicolas Bouleau and Marc Yor. Sur la variation quadratique des temps locaux de certaines semimartingales. *C. R. Acad. Sci. Paris Sér. I Math.*, 292(9):491–494, 1981.

[6] E. Çinlar, J. Jacod, P. Protter, and M. J. Sharpe. Semimartingales and Markov processes. *Z. Wahrsch. Verw. Gebiete*, 54(2):161–219, 1980.

[7] Jakubowki A. Mémin J. Coquet, F. and Slomiński L. Natural decomposition of processes and weak Dirichlet processes. *To appear: Séminaire de Probabilités.*

[8] Laure Coutin and Zhongmin Qian. Stochastic analysis, rough path analysis and fractional Brownian motions. *Probab. Theory Related Fields*, 122(1):108–140, 2002.

[9] Nelson Dunford and Jacob T. Schwartz. *Linear operators. Part I.* Wiley Classics Library. John Wiley & Sons Inc., New York, 1988. General theory, With the assistance of William G. Bade and Robert G. Bartle, Reprint of the 1958 original, A Wiley-Interscience Publication.

[10] Nathalie Eisenbaum. Integration with respect to local time. *Potential Anal.*, 13(4):303–328, 2000.

[11] Mohammed Errami and Francesco Russo. Covariation de convolution de martingales. *C. R. Acad. Sci. Paris Sér. I Math.*, 326(5):601–606, 1998.

[12] Mohammed Errami and Francesco Russo. n-covariation, generalized Dirichlet processes and calculus with respect to finite cubic variation processes. *Stochastic Process. Appl.*, 104(2):259–299, 2003.

[13] Mohammed Errami, Francesco Russo, and Pierre Vallois. Itô's formula for $C^{1,\lambda}$-functions of a càdlàg process and related calculus. *Probab. Theory Related Fields*, 122(2):191–221, 2002.

[14] Denis Feyel and Arnaud de La Pradelle. On fractional Brownian processes. *Potential Anal.*, 10(3):273–288, 1999.

[15] Franco Flandoli and Francesco Russo. Generalized calculus and SDEs with non regular drift. *Stoch. Stoch. Rep.*, 72(1-2):11–54, 2002.

[16] Franco Flandoli and Francesco Russo. Generalized integration and stochastic ODEs. *Ann. Probab.*, 30(1):270–292, 2002.

[17] Franco Flandoli, Francesco Russo, and Jochen Wolf. Some SDEs with distributional drift. I. General calculus. *Osaka J. Math.*, 40(2):493–542, 2003.

[18] Franco Flandoli, Francesco Russo, and Jochen Wolf. Some SDEs with distributional drift. II. Lyons-Zheng structure, Itô's formula and semimartingale characterization. *Random Oper. Stochastic Equations*, 12(2):145–184, 2004.

[19] H. Föllmer. Calcul d'Itô sans probabilités. In *Seminar on Probability, XV (Univ. Strasbourg, Strasbourg, 1979/1980) (French)*, volume 850 of *Lecture Notes in Math.*, pages 143–150. Springer, Berlin, 1981.

[20] H. Föllmer. Dirichlet processes. In *Stochastic integrals (Proc. Sympos., Univ. Durham, Durham, 1980)*, volume 851 of *Lecture Notes in Math.*, pages 476–478. Springer, Berlin, 1981.

[21] Hans Föllmer, Philip Protter, and Albert N. Shiryaev. Quadratic covariation and an extension of Itô's formula. *Bernoulli*, 1(1-2):149–169, 1995.

[22] R. Ghomrasni and G. Peskir. Local time-space calculus and extensions of Itô's formula. In *High dimensional probability, III (Sandjberg, 2002)*, volume 55 of *Progr. Probab.*, pages 177–192. Birkhäuser, Basel, 2003.

[23] F. Gozzi and F. Russo. Weak Dirichlet processes with a stochastic control perspective, journal = Preprint LAGA-Paris 13, 2005,.

[24] Mihai Gradinaru and Ivan Nourdin. Approximation at first and second order of m-order integrals of the fractional Brownian motion and of certain semimartingales. *Electron. J. Probab.*, 8:no. 18, 26 pp. (electronic), 2003.

[25] Mihai Gradinaru, Francesco Russo, and Pierre Vallois. Generalized covariations, local time and Stratonovich Itô's formula for fractional Brownian motion with Hurst index $H \geq \frac{1}{4}$. *Ann. Pobab.*, 31(4):1772–1820, 2003.

[26] Nourdin I. Russo F. Gradinaru, M. and P. Vallois. m-order integrals and generalized I tô's formula: the case of fractional Brownian motion with any Hurst index. *Preprint LAGA, 2003-07. To appear: Annales de l'Institut Henry Poincaré.*

[27] Massimiliano Gubinelli. Controlling rough paths. *J. Funct. Analysis*, 216, 2004.

[28] Th. Jeulin and M. Yor, editors. *Grossissements de filtrations: exemples et applications*, volume 1118 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin, 1985. Papers from the seminar on stochastic calculus held at the Université de Paris VI, Paris, 1982/1983.

[29] Ioannis Karatzas and Steven E. Shreve. *Brownian motion and stochastic calculus*, volume 113 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, second edition, 1991.

[30] Antoine Lejay. An introduction to rough paths. In *Séminaire de Probabilités XXXVII*, volume 1832 of *Lecture Notes in Math.*, pages 1–59. Springer, Berlin, 2003.

[31] Jorge A. León, Reyla Navarro, and David Nualart. An anticipating calculus approach to the utility maximization of an insider. *Math. Finance*, 13(1):171–185, 2003. Conference on Applications of Malliavin Calculus in Finance (Rocquencourt, 2001).

[32] T. J. Lyons and T. S. Zhang. Decomposition of Dirichlet processes and its application. *Ann. Probab.*, 22(1):494–524, 1994.

[33] Terence J. Lyons and Wei An Zheng. A crossing estimate for the canonical process on a Dirichlet space and a tightness result. *Astérisque*. Colloque Paul Lévy sur les Processus Stochastiques (Palaiseau, 1987).

[34] Terry Lyons and Zhongmin Qian. *System control and rough paths*. Oxford Mathematical Monographs. Oxford University Press, Oxford, 2002. Oxford Science Publications.

[35] Terry J. Lyons. Differential equations driven by rough signals. *Rev. Mat. Iberoamericana*, 14(2):215–310, 1998.

[36] H. P. McKean, Jr. *Stochastic integrals*. Probability and Mathematical Statistics, No. 5. Academic Press, New York, 1969.

[37] E. J. McShane. *Stochastic calculus and stochastic models*. Academic Press, New York, 1974. Probability and Mathematical Statistics, Vol. 25.

[38] S. Moret and D. Nualart. Quadratic covariation and Itô's formula for smooth nondegenerate martingales. *J. Theoret. Probab.*, 13(1):193–224, 2000.

[39] S. Moret and D. Nualart. Generalization of Itô's formula for smooth nondegenerate martingales. *Stochastic Process. Appl.*, 91(1):115–149, 2001.

[40] É. Pardoux. Grossissement d'une filtration et retournement du temps d'une diffusion. In *Séminaire de Probabilités, XX, 1984/85*, volume 1204 of *Lecture Notes in Math.*, pages 48–55. Springer, Berlin, 1986.

[41] Edwin Perkins. Stochastic integrals and progressive measurability—an example. In *Seminar on probability, XVII*, volume 986 of *Lecture Notes in Math.*, pages 67–71. Springer, Berlin, 1983.

[42] Philip Protter. *Stochastic integration and differential equations*, volume 21 of *Applications of Mathematics (New York)*. Springer-Verlag, Berlin, 1990. A new approach.

[43] F. Russo and G. Trutnau. Time inhomogeneous one-dimensional stochastic differential equations with distributional drift. *In preparation*.

[44] F. Russo and P. Vallois. Itô formula for C^1-functions of semimartingales. *Probab. Theory Related Fields*, 104(1):27–41, 1996.

[45] Francesco Russo and Pierre Vallois. Intégrales progressive, rétrograde et symétrique de processus non adaptés. *C. R. Acad. Sci. Paris Sér. I Math.*, 312(8):615–618, 1991.

[46] Francesco Russo and Pierre Vallois. Forward, backward and symmetric stochastic integration. *Probab. Theory Related Fields*, 97(3):403–421, 1993.

[47] Francesco Russo and Pierre Vallois. Noncausal stochastic integration for làd làg processes. In *Stochastic analysis and related topics (Oslo, 1992)*, volume 8 of *Stochastics Monogr.*, pages 227–263. Gordon and Breach, Montreux, 1993.

[48] Francesco Russo and Pierre Vallois. The generalized covariation process and Itô formula. *Stochastic Process. Appl.*, 59(1):81–104, 1995.

[49] Francesco Russo and Pierre Vallois. Stochastic calculus with respect to continuous finite quadratic variation processes. *Stochastics Stochastics Rep.*, 70(1-2):1–40, 2000.

[50] Francesco Russo, Pierre Vallois, and Jochen Wolf. A generalized class of Lyons-Zheng processes. *Bernoulli*, 7(2):363–379, 2001.

[51] Elias M. Stein. *Singular integrals and differentiability properties of functions.* Princeton Mathematical Series, No. 30. Princeton University Press, Princeton, N.J., 1970.

[52] Daniel W. Stroock and S. R. Srinivasa Varadhan. *Multidimensional diffusion processes*, volume 233 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences].* Springer-Verlag, Berlin, 1979.

[53] Jochen Wolf. An Itô formula for local Dirichlet processes. *Stochastics Stochastics Rep.*, 62(1-2):103–115, 1997.

[54] Jochen Wolf. Transformations of semimartingales and local Dirichlet processes. *Stochastics Stochastics Rep.*, 62(1-2):65–101, 1997.

[55] Jochen Wolf. A representation theorem for continuous additive functionals of zero quadratic variation. *Probab. Math. Statist.*, 18(2, Acta Univ. Wratislav. No. 2111):385–397, 1998.

[56] L. C. Young. An inequality of hölder type, connected with stieltjes integration. *Acta Math.*, 67:251–282, 1936.

[57] M. Zähle. Integration with respect to fractal functions and stochastic calculus. II. *Math. Nachr.*, 225:145–183, 2001.