Journal of Algebraic Combinatorics 3 (1994), 153–175

ℓ_1-Rigid Graphs

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Received May 11, 1992; Revised December 22, 1992

Abstract. An ℓ_1-graph is a graph whose nodes can be labeled by binary vectors in such a way that the Hamming distance between the binary addresses is, up to scale, the distance in the graph between the corresponding nodes. We show that many interesting graphs are ℓ_1-rigid, i.e., that they admit an essentially unique such binary labeling.

Keywords: ℓ_1-graph, cut cone, rigidity

1. Introduction

In this paper, we consider ℓ_1-graphs, i.e., graphs whose path metric is isometrically embeddable in some ℓ_1-space. In fact, ℓ_1-graphs are exactly those admitting a binary addressing such that, up to scale, the Hamming distance between the binary addresses of two nodes coincides with their distance in the graph. Namely, a graph $G = (V, E)$ is an *ℓ_1-graph* if, for some integer $m \geq 1$, there exists a map $\varphi : V \longrightarrow \{0, 1\}^m$ and an integer scalar λ such that

$$\lambda d_G(u, v) = d_H(\varphi(u), \varphi(v))$$

for all nodes $u, v \in V$, where d_G denotes the path metric of G and d_H the Hamming distance, i.e., the path metric of the m-dimensional hypercube $H(m, 2)$. Such map φ is called an *embedding with scale* λ, or a *λ-embedding* of G into the hypercube $H(m, 2)$. Hence, isometric embeddings into the hypercube correspond to the case $\lambda = 1$. The minimum λ for which there exists a λ-embedding of G into a hypercube is called the *scale* of G. Here, we consider more specifically *ℓ_1-rigid* graphs, i.e., ℓ_1-graphs admitting essentially a unique ℓ_1-embedding.

The ℓ_1-graphs have been characterized in [24] and later in [10]; namely, a graph is an ℓ_1-graph if and only if it is an isometric subgraph of a product of half-cubes and cocktail party graphs. The proof of [10] is shorter, since it relies on the theory of Delaunay polytopes in lattices (see, e.g., [11] for details); moreover, it applies to a larger class of metrics than graphic ones. The proof of [24] provides some insight on the structure of ℓ_1-graphs. Moreover, it permits to establish that ℓ_1-graphs can be recognized in polynomial time. Note that, for general metrics, the problem of testing ℓ_1-embeddability is NP-hard [20].

More precisely, Shpectorov [24] proved the following result.

THEOREM 1.1. *Let G be an ℓ_1-graph on n vertices. Then, there exist a graph Γ and an isometric embedding φ from G into Γ such that*

(i) Γ is a Cartesian product of complete graphs, cocktail party graphs and half-cubes,

(ii) if ψ is a λ-embedding from G into the hypercube $H(m, 2)$, then there exists a λ-embedding $\widetilde{\psi}$ from Γ into the same hypercube $H(m, 2)$ such that $\psi = \widetilde{\psi}\varphi$.

Moreover, the scale λ of G is either equal to 1, or is even, with $\lambda \leq n - 2$ if $n \geq 4$.

As a consequence of Theorem 1.1, G is ℓ_1-rigid if and only if the graph Γ is ℓ_1-rigid. Since Γ is a Cartesian product, it is ℓ_1-rigid if and only if all its factors are ℓ_1-rigid (see Fact (2.2) below). This is equivalent to the condition that no factor is isomorphic to a complete graph or a cocktail party graph on at least 4 vertices. Therefore,

COROLLARY 1.1 [24]. *Every ℓ_1-rigid graph G is an isometric subgraph of a half-cube, i.e., G has scale 1 or 2.*

Moreover, if G is not ℓ_1-rigid, then the variety of its ℓ_1-embeddings arises from that of the complete graph (since the variety of the ℓ_1-embeddings of the cocktail party graph comes from that of the complete graph, see Facts (2.6) and (2.9)).

It is proved in [23] that, if G is bipartite, then G is an ℓ_1-graph if and only if G is an isometric subgraph of a hypercube, i.e., G has scale 1.

On the other hand, every isometric subgraph of a hypercube is ℓ_1-rigid; see Proposition 3.1. Actually, this result is also implied by the following result from [22] (see Section 4.1 for more details). Let G be a bipartite graph, then its path metric d_G is *metrically rigid*, i.e., there is a unique way of writing d_G as a sum of metrics.

The above discussion can be summarized by the following (strict) implications:

$$\begin{array}{c} G \text{ is an isometric subgraph of a hypercube} \\ \Downarrow \\ G \text{ is } \ell_1\text{-rigid} \\ \Downarrow \\ G \text{ is an isometric subgraph of a half-cube} \end{array}$$

Therefore, ℓ_1-rigidity is an important concept lying in between the well-studied concept of isometric hypercube embeddability (scale 1) and its relaxation to isometric half-cube embeddability (scale 1 or 2).

The questions of isometric embedding into a hypercube have been extensively studied; they have many applications, in particular, for addressing problems in communication networks (see, e.g., [19] and references there). Actually, the question of isometric ℓ_1-embedding (i.e., binary addressing up to scale) was already considered in [3].

The isometric subgraphs of hypercubes have been characterized by Djoković [18]. Given a graph $G = (V, E)$ and an edge $e = (i, j)$ of G, set

$$G(i, j) = \{u \in V : d_G(i, u) < d_G(j, u)\},$$

i.e., $G(i, j)$ consists of the nodes which are closer to i than to j. If G is bipartite, then the sets $G(i, j)$ and $G(j, i)$ form a partition of V, for each edge $e = (i, j)$ of G.

THEOREM 1.2 [18]. *The graph G is an isometric subgraph of a hypercube if and only if G is bipartite and, for every edge $e = (i, j)$ of G, both sets $G(i, j)$ and $G(j, i)$ are closed under taking shortest paths.*

On the other hand, no similar characterization is known for the isometric subgraphs of half-cubes.

In this paper, we present several classes of ℓ_1-rigid graphs. This permits us to obtain some classifications for ℓ_1-rigid graphs within several interesting classes of graphs, especially within distance regular graphs.

For definitions, notation and quoting of results on graphs, we rely entirely on the monography [4].

The paper is organized as follows. In Section 2, we give a catalog of graphs for which we group information on their ℓ_1-embeddings, mainly, on rigidity and scale. We also present some operations which preserve ℓ_1-embeddability and ℓ_1-rigidity. Section 3 deals with bipartite graphs and some weighted graphs. We group in Section 4 all the proofs and tools. Actually, the main tool we use is the fact that ℓ_1-metrics are exactly the points of the cut cone and ℓ_1-rigid metrics are those lying on a simplicial face of this cone. In fact, all the examples of ℓ_1-rigid graphs that we present here share the property that they are lying on a simplicial face of the cut cone, which is entirely defined by triangle inequalities. In Section 5, via some classification theorems from [4], and as application of Sections 2 and 3, we specify ℓ_1-graphs and their rigidity within some important classes of graphs.

2. A catalog of ℓ_1-graphs

Let d be a function defined on the pairs of $V = \{1, \ldots, n\}$; d is a *metric* on V if d satisfies the following triangle inequality

$$d(i, j) - d(i, k) - d(j, k) \leq 0 \quad (1)$$

for all $i, j, k \in V$. The metric cone $\mathcal{M}_n$ is the cone defined by the triangle inequalities (1).

Then, d is an *ℓ_1-metric* if d is isometrically ℓ_1-embeddable, i.e., there exist an integer $m \geq 1$ and n vectors $x_1, \ldots, x_n \in \mathbb{R}^m$ such that $d(i, j) = \|x_i - x_j\|$ for all $i, j \in V$. We say that $x_1, \ldots, x_n$ form an *ℓ_1-embedding* of d in $\mathbb{R}^m$. (For $y = (y_h)_{1 \leq h \leq m} \in \mathbb{R}^m$, $\|y\| = \sum_{1 \leq h \leq m} |y_h|$ denotes the ℓ_1-norm.) When the embedding is binary, i.e., $x_1, \ldots, x_n \in \{0, 1\}^m$, then the ℓ_1-distance $\|x_i - x_j\|$ coincides with the Hamming distance $d_H(x_i, x_j)$.

Let d be an ℓ_1-metric and let $x_1, \ldots, x_n$ be an ℓ_1-embedding of d in $\mathbb{R}^m$. Consider the $n \times m$ matrix M whose rows are the vectors $x_1, \ldots, x_n$; M is called an *ℓ_1-realization matrix* of d. Consider the following three operations on the realization matrix M:

- add to (or delete from) M a column with all identical entries,
- add an arbitrary vector $a \in \mathbb{R}^m$ to all the rows of M (translation),
- permute the columns of M.

Call two ℓ_1-embeddings *equivalent* if their realization matrices can be obtained from one another via the above three operations. Then, a metric is *ℓ_1-rigid* if it admits, up to equivalence, a unique ℓ_1-embedding.

Fact (2.1) [2]. Let d be rational valued. Then, d is an ℓ_1-metric if and only if λd admits a binary ℓ_1-embedding for some integer $\lambda \geq 1$. We call the smallest such integer λ the *scale* of d.

For ℓ_1-metrics with scale 1, i.e., metrics admitting a binary ℓ_1-embedding, the following weaker notion of rigidity has been considered. Call d h-rigid if d admits a unique (up to equivalence) binary ℓ_1-embedding. Hence, if d is ℓ_1-rigid, then d is h-rigid, but not vice versa. For example, given an integer $t \geq 1$, Deza [9] proved that the equidistant metric $2td(K_n)$ (taking the value $2t$ on all pairs of $\{1, \ldots, n\}$) is h-rigid for $n \geq t^2 + t + 3$. However, $2td(K_n)$ is not ℓ_1-rigid for $n \geq 4$ and $t \geq 1$, while $2td(K_4)$ is already not h-rigid for $t \geq 1$. In fact, $2d(K_4)$ has exactly two (up to equivalence) binary ℓ_1-embeddings.

We shall consider in this paper only metrics arising from graphs, or weighted graphs. Let $G(V, E)$ be a graph and let $w = (w_e)_{e \in E}$ be nonnegative weights assigned to the edges of G. The *path metric* $d_{(G,w)}$, or $d(G, w)$, of the weighted graph (G, w) is defined as follows: $d_{(G,w)}(i, j)$ is the shortest w-length $\sum_{e \in P} w_e$ of a path P joining i and j in G. In the unweighted case: $w_e = 1$ for all edges $e \in E$, we simply denote the path metric by d_G, or $d(G)$. When the path metric $d_{(G,w)}$ is an ℓ_1-metric, we say that (G, w) is an *ℓ_1-graph* and, when $d_{(G,w)}$ is ℓ_1-rigid, we say that the graph (G, w) is ℓ_1-rigid.

The ℓ_1-graphs with scale 1, i.e., the graphs admitting a binary ℓ_1-embedding, are precisely the isometric subgraphs of a cube. The ℓ_1-graphs with scale 2 are the isometric subgraphs of a half-cube; they are called code graphs in [21].

A useful operation on metrics is the direct product operation. Let d_a be a metric on V_a, for $a = 1, 2$. Consider the function d defined on $V_1 \times V_2$ by:

$$d((i_1, i_2), (j_1, j_2)) = d_1(i_1, j_1) + d_2(i_2, j_2)$$

for all $(i_1, i_2), (j_1, j_2) \in V_1 \times V_2$. Then, d is a metric on $V_1 \times V_2$, called the *direct product* of d_1, d_2. Note that, for path metrics, this corresponds to the Cartesian product of graphs. It is well known and easy to see that ℓ_1-embeddability is preserved by direct product. Moreover,

Fact (2.2). Let d be the direct product of d_1 and d_2. Then, d is ℓ_1-rigid if and only if d_1 and d_2 are ℓ_1-rigid.

The following 1-sum operation also preserves ℓ_1-metrics and ℓ_1-rigidity. Let V_1 and V_2 be two sets which intersect in exactly one element, $V_1 \cap V_2 = \{k\}$. Let d_a be a metric defined on the set V_a, for $a = 1, 2$. Their *1-sum* is the metric d defined on the set $V_1 \cup V_2$ by

$$\begin{cases} d(i, j) = d_a(i, j) & \text{if } i, j \in V_a \text{ for some } a = 1, 2 \\ d(i, j) = d_1(i, k) + d_2(j, k) & \text{if } i \in V_1, j \in V_2. \end{cases}$$

Fact (2.3). Let d be the 1-sum of d_1 and d_2. Then, d is an ℓ_1-metric if and only if d_1 and d_2 are ℓ_1-metrics. Moreover, d is ℓ_1-rigid if and only if d_1 and d_2 are ℓ_1-rigid.

Another useful operation on metrics is the antipodal extension operation (for details, see [13]). Given a metric d on $V = \{1, \dots, n\}$ and $\alpha \in \mathbb{R}$, we define its *antipodal extension* $ant_\alpha(d)$ on $V \cup \{n + 1\}$ as follows:

$$\begin{cases} ant_\alpha(d)_{1,n+1} & = \alpha \\ ant_\alpha(d)_{i,n+1} & = \alpha - d_{1i} \quad \textit{for } 2 \le i \le n \\ ant_\alpha(d)_{ij} & = d_{ij} \quad \textit{for } 1 \le i < j \le n. \end{cases}$$

So, $ant_{\alpha(d)}$ is an extension of d obtained by adding a new node $n+1$ at distance α from node 1 and $\alpha - d_{1i}$ from the other nodes. This operation can be repeated, up to doubling of all the nodes; then, we obtain a metric on $2n$ nodes, denoted as $Ant_\alpha(d)$.

The nice feature of the antipodal extension operation is that, under some condition on α, it preserves ℓ_1-embeddability and ℓ_1-rigidity. Namely,

Fact (2.4) [2], [13]. $ant_\alpha(d)$ is an ℓ_1-metric if and only if d is an ℓ_1-metric and admits an ℓ_1-embedding of size less than or equal to α; moreover, $ant_\alpha(d)$ is ℓ_1-rigid if and only if d is ℓ_1-rigid. (See Fact (4.4), and Section 4.1 for the notion of size.)

In this paper, we study ℓ_1-rigidity for the following classes of graphs:

- K_n, the complete graph (1-skeleton of the simplex α_{n-1})
- $K_{n\times 2}$, the cocktail party graph (1-skeleton of the cross polytope β_n)
- $H(n, d) \simeq (K_d)^n$, the Hamming graph and, in particular,
  $H(n, 2)$, the cube (1-skeleton of the hypercube γ_n)
  $H(2, d)$, the $d \times d$-grid
- $\frac{1}{2}H(n, 2)$, the half-cube
- $J(n, d)$, the Johnson graph
- DO_{2n+1}, the double odd graph
- $\mathcal{A}(n, t)$ (considered in [21])
- O_3, the Petersen graph
- the Shrikhande graph
- the (1-skeleton of the) dodecahedron
- the (1-skeleton of the) icosahedron
- C_n, the cycle of length n.

We will also consider in the next section ℓ_1-rigidity for bipartite graphs and weighted cycles.

All the graphs listed above are ℓ_1-graphs and all of them, except $K_{n\times 2}$ for $n \geq 5$, have scale 1 or 2. In fact, the ℓ_1-graphs with scale 1 are the bipartite ℓ_1-graphs, including even cycles, cubes, double odd graphs. We specify below which graphs are ℓ_1-rigid or not.

Fact (2.5). K_n is not ℓ_1-rigid, except for $n = 2, 3$. See [15] for the study of the variety of ℓ_1-embeddings of K_n and connections with design theory.

Fact (2.6). $K_{n\times 2}$ is not ℓ_1-rigid, except for $n = 2, 3$. In fact, its variety of ℓ_1-embeddings entirely comes from the variety of ℓ_1-embeddings of K_n (via the antipodal extension operation since $d(K_{n\times 2}) = Ant_2(d(K_n))$, see Section 4.3).

Fact (2.7). Any Doob graph (i.e., product of copies of K_4 and of the Shrikhande graph) is not ℓ_1-rigid whenever it involves some K_4; its variety of ℓ_1-embeddings comes from that of K_4 (by the direct product operation, since the Shrikhande graph is ℓ_1-rigid).

Fact (2.8). The Hamming graph $H(n, d)$ is not ℓ_1-rigid, except for $d \leq 3$ (again its variety of embeddings comes from that of K_d, by direct product).

Fact (2.9). The half-cube $\frac{1}{2}H(n, 2)$ is always ℓ_1-rigid, except if $n = 3, 4$; indeed, $\frac{1}{2}H(3, 2) \simeq K_4$ and $\frac{1}{2}H(4, 2) \simeq K_{4\times 2}$.

Fact (2.10). The Johnson graph $J(n, d)$ is always ℓ_1-rigid, except in the case $d = 1$ and $n \geq 4$; indeed, $J(n, 1) \simeq K_n$.

Fact (2.11). The double odd graph DO_{2n+1} is always ℓ_1-rigid (by Proposition 3.1, since it is bipartite).

Fact (2.12). The graph $\mathcal{A}(n, t)$ is ℓ_1-rigid for $t \geq 3$.

Fact (2.13). The Petersen, Shrikhande graphs, the skeletons of the dodecahedron and of the icosahedron are all ℓ_1-rigid, with scale 2.

The proofs of the statements presented in this section will be given in Section 4. We refer also to Section 4 for a precise description of all graphs and of their ℓ_1-embeddings.

3. Bipartite and weighted ℓ_1-graphs

Given a graph $G(V, E)$, consider the relation θ defined on the edge set E of G as follows: Given two edges $e = (i, j)$, $e' = (i', j')$ of G,

$$e\theta e' \text{ if } d_G(i, i') + d_G(j, j') \neq d_G(i, j') + d_G(j, i').$$

The relation θ is reflexive, symmetric, but not transitive in general. An equivalent formulation of Theorem 1.2 is that G is an isometric subgraph of a hypercube if and only if G is bipartite and the relation θ is transitive. It is observed in [26] that every isometric subgraph of a hypercube admits (up to equivalence) a unique binary ℓ_1-embedding (i.e., is h-rigid). In fact, we have the following stronger result.

PROPOSITION 3.1. *Every bipartite ℓ_1-graph is ℓ_1-rigid.*

This result can be extended to weighted bipartite graphs as follows. Let $w = (w_e)_{e \in E}$ be nonnegative edge weights assigned to the edges of G; the weighting w is said to be *compatible with the relation* θ if $w_e = w_{e'}$ holds whenever $e\theta e'$.

PROPOSITION 3.2. *Let G be a bipartite ℓ_1-graph and let w be a nonnegative weighting of the edges of G which is compatible with the relation θ. Then, the weighted graph (G, w) is ℓ_1-rigid.*

As we shall explain in Section 4, a metric d on n points is ℓ_1-rigid if it lies on a simplicial face of the cut cone $\mathcal{C}_n$. Hence, if G is a bipartite ℓ_1-graph on n nodes, then its path metric d_G lies on a simplicial face $\mathcal{F}_G$ of $\mathcal{C}_n$. The dimension of the face $\mathcal{F}_G$ is equal to the number of equivalence classes of the relation θ. Moreover, every other metric d lying on $\mathcal{F}_G$ is the path metric of (G, w) for some (compatible) nonnegative weighting w of G.

In particular, if G is a tree, then (G, w) is ℓ_1-rigid for every nonnegative weighting w (since θ is the identity relation).

In fact, a similar result holds for cycles, as we now see. Note, however, that θ is not the identity relation for even cycles (any two opposite edges on the cycle are in relation by θ).

PROPOSITION 3.3. *Let $C = (V, E)$ be a cycle and let $w = (w_e)_{e \in E}$ be nonnegative integer edge weights. Then, (C, w) is ℓ_1-rigid; moreover, (C, w) admits a binary ℓ_1-embedding if and only if $\sum_{e \in E} w_e \equiv 0$ (mod 2).*

As example of application of the above results, we obtain that every connected graph with largest eigenvalue less than or equal to 2 is ℓ_1-rigid. Indeed, such graphs are classified (see Theorem 3.2.5 in [4]) as subgraphs of extended Dynkin diagrams which are trees or cycles.

4. Tools and proofs

4.1. ℓ_1-metrics and the cut cone

All the proofs use extensively the following correspondence between ℓ_1-metrics and the points of the cut cone.

Given a subset S of $V = \{1, \dots, n\}$, let $\delta(S)$ denote the *cut metric* on V defined by: $\delta(S)(i, j) = 1$ if $|S \cap \{i, j\}| = 1$, and $\delta(S)(i, j) = 0$ otherwise, for $1 \leq i < j \leq n$. Hence, $\delta(S) = \delta(V - S)$ and $\delta(\emptyset) = \delta(V)$ is identically zero. The *cut cone* C_n is the cone in $\mathbb{R}^{\binom{n}{2}}$ generated by the cut metrics $\delta(S)$ for $S \subseteq V$. (For general information on the cut cone, see e.g., [14] and the survey [12].) A face $\mathcal{F}$ of C_n is said to be *simplicial* if the nonzero cut metrics lying on $\mathcal{F}$ are linearly independent. For any metric d on V, the following assertions hold:

Fact (4.1). d is an ℓ_1-metric if and only if d belongs to the cut cone C_n, i.e., $d = \sum_S \lambda_S \delta(S)$, with $\lambda_S \geq 0$ for all $S \subseteq V$.

Fact (4.2). d is an ℓ_1-metric with scale 1, i.e., d admits a binary ℓ_1-embedding, if and only if $d = \sum_S \lambda_S \delta(S)$, with $\lambda_S \in \mathbb{Z}_+$ for all $S \subset V$.

Fact (4.3). d is ℓ_1-rigid if and only if d lies on a simplicial face of the cut cone C_n, i.e., d admits a unique decomposition as a nonnegative sum of cut metrics.

For Facts (4.1) and (4.2), see [2] and for (4.3) see [16]. Note that Fact (2.1) follows immediately from (4.1) and (4.2). Any decomposition of d as a nonnegative sum of cut metrics is called a $\mathbb{R}_+$*-realization* of d and any decomposition of d as a nonnegative integer sum of cut metrics is called a $\mathbb{Z}_+$*-realization*. Moreover, if $d = \sum_S \lambda_S \delta(S)$ with $\lambda_S \geq 0$, then $\sum_S \lambda_S$ is called the *size* of the realization.

For instance, (4.2) can be checked as follows. Let d be an ℓ_1-metric with scale 1 and let $x_1, \ldots, x_n \in \{0, 1\}^m$ be a binary embedding of d. Consider the associated realization matrix M whose rows are $x_1, \ldots, x_n$. Each column of M can be seen as the incidence vector of a subset S_j of V. Then, $d = \sum_{1 \le j \le m} \delta(S_j)$ holds. Therefore, the rows of M provide the binary labeling of the nodes while the columns of M provide the cut metrics whose sum gives d.

All the maximal simplicial faces of the cut cone $\mathcal{C}_5$ are described in [17]; each of them contains a graphic metric. This yields a characterization on the ℓ_1-rigid metrics on at most five points.

We can also precise the statement (2.4) on the antipodal extension operation. Namely, if d is a metric on V and $\alpha \in \mathbb{R}_+$, then:

Fact (4.4). $ant_\alpha(d)$ is an ℓ_1-metric, i.e., $ant_\alpha(d) \in \mathcal{C}_{n+1}$, if and only if d is an ℓ_1-metric, i.e., $d \in \mathcal{C}_n$, and d admits a $\mathbb{R}_+$-realization of size less than or equal to α; moreover, any $\mathbb{R}_+$-realization of $ant_\alpha(d)$ is of the form:

$$ant_\alpha(d) = \sum_{S:1 \in S} \lambda_S \delta(S) + \sum_{S:1 \notin S} \lambda_S \delta(S \cup \{n+1\}) + (\alpha - \sum_S \lambda_S)\delta(\{n+1\})$$

where $d = \sum_S \lambda_S \delta(S)$ is a $\mathbb{R}_+$-realization of d with $\sum_S \lambda_S \le \alpha$.

Let us now indicate the pattern that our proofs of ℓ_1-rigidity will follow.

Given a metric d on V, how to prove that d is ℓ_1-rigid? By (4.3), this amounts to proving that d lies on a simplicial face of the cut cone $\mathcal{C}_n$. We proceed as follows.

Fact (4.5). First, we check that d is an ℓ_1-metric.

For this, we exhibit a binary ℓ_1-embedding for λd, for some scalar λ; in all the cases treated here (except for $K_{n \times 2}$), λ is equal to 1 or 2. In other words, we exhibit a decomposition of λd as integer sum of cuts, say $\lambda d = \sum_{S \in \mathcal{S}} \lambda_S \delta(S)$ with $\lambda_S \in \mathbb{Z}$, $\lambda_S > 0$, for $S \in \mathcal{S}$.

Then, let $\mathcal{F}_d$ denote the face of smallest dimension of the cut cone $\mathcal{C}_n$ containing d. Hence, the cut metrics $\delta(S)$, $S \in \mathcal{S}$, belong to $\mathcal{F}_d$. We show that the face $\mathcal{F}_d$ is simplicial by checking that the following Facts (4.6) and (4.7) hold.

Fact (4.6). The cut metrics $\delta(S)$, $S \in \mathcal{S}$, are linearly independent.

Fact (4.7). The only cut metrics lying on $\mathcal{F}_d$ are the cut metrics $\delta(S)$ for $S \in \mathcal{S}$.

Actually, in all the cases treated here, we shall show the following property, which is stronger than (4.7): The only cut metrics which satisfy the same triangle equalities as d are the cut metrics $\delta(S)$, for $S \in \mathcal{S}$. This means that the face $\mathcal{F}_d$ is completely determined by triangle inequalities.

More precisely, define also the smallest face $\mathcal{F}_d^M$ of the metric cone $\mathcal{M}_n$ containing d. Then, $\mathcal{F}_d \subseteq \mathcal{F}_d^M \cap \mathcal{M}_n$ holds. The equality $\mathcal{F}_d = \mathcal{F}_d^M \cap \mathcal{M}_n$ holds if the face $\mathcal{F}_d$ is completely defined as an intersection of triangle faces. Clearly, if the face $\mathcal{F}_d^M$ is simplicial, then the face $\mathcal{F}_d$ too is simplicial. Lomonossov and Sebö [22] have shown that the path metric of a bipartite graph is metrically rigid, i.e., lies on a simplicial face of the metric cone, which, therefore, implies the result from Proposition 3.1. Note that $K_6 - P_3$, $K_5 - P_3$ are examples of ℓ_1-rigid graphs which are not metrically rigid.

4.2. Proof of (2.2) and (2.3)

We first show the statement (2.2). Let d_a be a metric on V_a, for $a = 1, 2$, and let d denote the direct product of d_1, d_2, defined on $V_1 \times V_2$. It is easy to see that:

(i) If d_1, d_2 are ℓ_1-metrics with $\mathbb{R}_+$-realizations, $d_1 = \sum_{S \in \mathcal{S}} \alpha_S \delta(S)$, $d_2 = \sum_{T \in \mathcal{T}} \beta_T \delta(T)$, respectively, then d is an ℓ_1-metric with $\mathbb{R}_+$-realization

$$d = \sum_{S \in \mathcal{S}} \alpha_S \delta(S \times V_2) + \sum_{T \in \mathcal{T}} \beta_T \delta(V_1 \times T).$$

(ii) Conversely, if d is an ℓ_1-metric with $\mathbb{R}_+$-realization $d = \sum_A \lambda_A \delta(A)$, then

$$\begin{cases} d_1 = \sum_A \lambda_A \delta(A_1) \\ d_2 = \sum_A \lambda_A \delta(A_2) \end{cases}$$

where $A_1 = \{i \in V_1 : (i, i_2) \in A\}$, $A_2 = \{i \in V_2 : (i_1, i) \in A\}$, for some fixed $i_1 \in V_1$, $i_2 \in V_2$.

Therefore, from (i), if d is ℓ_1-rigid, then both d_1, d_2 are ℓ_1-rigid. Conversely, assume that d_1, d_2 are ℓ_1-rigid; we show that d is ℓ_1-rigid. The proof follows the steps (4.6) and (4.7). We suppose that the $\mathbb{R}_+$-realizations of d_1, d_2 are as in (i).

First, it is immediate to see that the cut metrics $\delta(S \times V_2)$, $S \in \mathcal{S}$, and $\delta(V_1 \times T)$, $T \in \mathcal{T}$, are linearly independent.

We check (4.7). Take $A \subseteq V_1 \times V_2$ such that $\delta(A)$ satisfies the same triangle equalities as d and, thus, the following equalities:

$$d((i_1, i_2), (j_1, j_2)) = d((i_1, i_2), (j_1, i_2)) + d((j_1, i_2), (j_1, j_2))$$

for all $i_1, j_1 \in V_1$, $i_2, j_2 \in V_2$. We show that A is one of $S \times V_2$ for $S \in \mathcal{S}$ or $V_1 \times T$ for $T \in \mathcal{T}$. Suppose, e.g., that, for some $i_2 \in V_2$, $A \cap (V_1 \times \{i_2\})$ is a proper nonempty subset of $V_1 \times \{i_2\}$. Then, by the above triangle equalities, we deduce that $\{i\} \times V_2 \subseteq A$ for any $i \in V_1$ such that $(i, i_2) \in A$, and $A \cap (\{i\} \times V_2) = \emptyset$ for any $i \in V_1$ such that $(i, i_2) \notin A$. Therefore, A is of the form $S \times V_2$ and, since d_1 is ℓ_1-rigid, $S \in \mathcal{S}$. This concludes the proof.

We now show that the statement (2.3) holds. Let d_a be a metric on V_a, for $a = 1, 2$, where V_1 and V_2 have exactly one common element, $V_1 \cap V_2 = \{k\}$. Let d denote the 1-sum of d_1 and d_2. It is easy to check that:

(i) If d_1, d_2 are ℓ_1-metrics with $\mathbb{R}_+$-realizations, $d_1 = \sum_{S \in \mathcal{S}} \alpha_S \delta(S)$, $d_2 = \sum_{T \in \mathcal{T}} \beta_T \delta(T)$, respectively, where we can suppose that $k \notin S, T$ for all $S \in \mathcal{S}$ and $T \in \mathcal{T}$, then d is an ℓ_1-metric with $\mathbb{R}_+$-realization

$$d = \sum_{S \in \mathcal{S}} \alpha_S \delta(S) + \sum_{T \in \mathcal{T}} \beta_T \delta(T).$$

(ii) Conversely, if d is an ℓ_1-metric with $\mathbb{R}_+$-realization $d = \sum_{A \in \mathcal{A}} \lambda_A \delta(A)$, where we can suppose that $k \notin A$ for all $A \in \mathcal{A}$, then, we have $A \subseteq V_1$, or $A \subseteq V_2$ for each $A \in \mathcal{A}$ and

$$\begin{cases} d_1 = \sum_{A \in \mathcal{A}, A \subseteq V_1} \lambda_A \delta(A) \\ d_2 = \sum_{A \in \mathcal{A}, A \subseteq V_2} \lambda_A \delta(A) \end{cases}$$

and, therefore, d_1, d_2 are ℓ_1-metrics.

An immediate consequense of (i), (ii) above is that d is ℓ_1-rigid if and only if d_1 and d_2 are ℓ_1-rigid.

4.3. Proof of (2.5)–(2.8)

Let us first observe that the complete graph K_n is an ℓ_1-graph. Indeed, a binary ℓ_1-embedding of $2d(K_n)$ is obtained by labeling the nodes of K_n by the unit vectors of $\mathbb{R}^n$ or, equivalently, $2d(K_n) = \sum_{1 \le i \le n} \delta(\{i\})$. For $n = 2, 3$, the cone C_n is simplicial and, thus, K_n is ℓ_1-rigid. However, for $n \geq 4$, K_n admits many other ℓ_1-embeddings. For example, for $1 \le q \le \frac{n}{2}$, $\sum_{S \subseteq \{1,\dots,n\}, |S|=q} \delta(S) = 2\binom{n-2}{q-1} d(K_n)$; for $q = \lfloor \frac{n}{2} \rfloor$, this gives an ℓ_1-embedding of minimum possible size $\frac{n(n-1)}{2\lceil \frac{n}{2} \rceil \lfloor \frac{n}{2} \rfloor} < 2$.

Consider now the cocktail party graph $K_{n \times 2}$ defined on the $2n$ nodes of $\{1, \dots, 2n\}$. Its path metric has all its values equal to 1 except values 2 on the n pairs $(i, n+i)$ for $1 \le i \le n$. Therefore, $d(K_{n\times 2})$ can be expressed in terms of $d(K_n)$ using the antipodal extension operation; namely, $d(K_{n\times 2}) = Ant_2(d(K_n))$. Therefore, by (2.4), $K_{n\times 2}$ is an ℓ_1-graph and is ℓ_1-rigid if and only if K_n is ℓ_1-rigid, i.e., for $n = 2, 3$.

The statements (2.7) and (2.8) follow by application of (2.2).

4.4. Proof of (2.9) and (2.10)

We first show that the half-cube $\frac{1}{2}H(n, 2)$ is ℓ_1-rigid for any $n \geq 5$.

Let us fix some notation. Let E_n denote the family of the subsets of $V = \{1, \dots, n\}$ of even cardinality. Then, $\frac{1}{2}H(n, 2)$ is the graph with node set E_n and with edges the pairs (A, B) such that $A, B \in E_n$ and $|A \Delta B| = 2$.

For any $i \in V$, define the set S_i consisting of all the sets $A \in E_n$ with $i \in A$. It is easy to see that $2d(\frac{1}{2}H(n, 2)) = \sum_{1 \leq i \leq n} \delta(S_i)$ holds; this corresponds to the binary ℓ_1-embedding of $2d(\frac{1}{2}H(n, 2))$ obtained by assigning to each node $A \in E_n$ its incidence vector.

Observe that the subgraph of $\frac{1}{2}H(n, 2)$ induced by any subset S_i is isomorphic to $\frac{1}{2}H(n-1, 2)$. This simple observation will permit us to use induction on n.

For showing that the half-cube $\frac{1}{2}H(n, 2)$ is ℓ_1-rigid for $n \geq 5$, we follow the steps (4.6) and (4.7). It is easy to check that the cuts $\delta(S_i)$, for $1 \leq i \leq n$, are linearly independent.

Then, by induction on $n \geq 5$, we show that the only nonzero cut metrics on E_n satisfying the same triangle equalities as $d(\frac{1}{2}H(n, 2))$ are the n cut metrics $\delta(S_i)$ for $1 \leq i \leq n$. The case $n = 5$ can be checked directly. We describe only the induction step. We suppose that the property holds for $n - 1$ and we show that it holds for n.

Take $S \subseteq E_n$ such that the cut metric $\delta(S)$ satisfies the same triangle equalities as $d(\frac{1}{2}H(n, 2))$. For any $i \in V$, by looking at the local structure on S_i and applying the induction assumption, we deduce that $S \cap S_i = \emptyset$, S_i, $S_i \cap S_j$ or $S_i - S_j$ for some $j \in V$.

We shall use the fact that $d := d(\frac{1}{2}H(n, 2))$ satisfies the following triangle equalities:

$$d(A, C) = d(A, B) + d(B, C)$$

for any $A, B, C \in E_n$ such that $|A \Delta C| = 4$, $|A \Delta B| = |B \Delta C| = 2$.

Let us first suppose, for example, that $S_1 \subseteq S$. We show that, if $S \neq S_1$, then $S = E_n$. We use the following assertions (i)–(ii).

(i) If $A \notin S$ with $2 \leq |A| \leq n - 2$, then $A - \{a, b\} \notin S$ for all distinct $a, b \in A$. Indeed, take $c \in V - A$, $c \neq 1$, and use the triangle equality: $d(A - \{a, b\}, A \cup \{1, c\}) = d(A - \{a, b\}, A) + d(A, A \cup \{1, c\})$.

(ii) If $A \notin S$ with $|A| \leq n - 4$, then $A \cup \{a, b\} \notin S$ for all distinct $a, b \in V - A$, $a, b \neq 1$. Indeed, take $c \in V - A$, $c \neq a, b, 1$, and use the triangle equality: $d(A \cup \{a, b\}, A \cup \{1, c\}) = d(A, A \cup \{a, b\}) + d(A, A \cup \{1, c\})$.

Let $A \in E_n$ of minimum cardinality such that $A \notin S$. If $A = \emptyset$, then, by (ii), $S = S_1$ holds. If $|A| \leq n - 2$, then we have a contradiction with (i) since $\emptyset \in S$. Hence, $|A| = n - 1$, $A = [2, n] \notin S$ while $A - \{2, 3\}$, $A - \{n - 1, n\} \in S$, contradicting the triangle equality: $d(A - \{2, 3\}, A - \{n - 1, n\}) = d(A, A - \{2, 3\}) + d(A, A - \{n - 1, n\})$.

Therefore, S_i is not contained in S for all i. We can suppose, for instance, that $S \cap S_1 = S_1 \cap S_2$. We deduce that $S \cap S_2 = S_1 \cap S_2$ and $S \cap S_3 = S_3 \cap S_i$ for some $i \neq 3$. By taking the intersection of both sides of $S \cap S_3 = S_3 \cap S_i$ with S_1 and S_2, we obtain that i should coincide with 2 and 1, respectively, hence a contradiction.

This concludes the proof.

We now show that the Johnson graph $J(n, d)$ is ℓ_1-rigid for $d \geq 2$.

Let $\mathcal{P}_n^d$ denote the family of the subsets of $V = \{1, \dots, n\}$ of cardinality d. The Johnson graph $J(n, d)$ is the graph with node set $\mathcal{P}_n^d$ and with edges the pairs (A, B) for $A, B \in \mathcal{P}_n^d$ and $|A \Delta B| = 2$.

A binary ℓ_1-embedding of $2d(J(n, d))$ is obtained, simply, by labeling each node A by its incidence vector. This corresponds to the following $\mathbb{Z}_+$-realization of $2d(J(n, d)) : 2d(J(n, d)) = \sum_{1 \leq i \leq n} \delta(S_i)$, where S_i is the subset of $\mathcal{P}_n^d$ consisting of all $A \in \mathcal{P}_n^d$ with $i \in A$, for $1 \leq i \leq n$.

The subgraph of $J(n, d)$ induced by the node set S_i is isomorphic to $J(n - 1, d - 1)$. This observation permits us, as in the case of the half-cube, to apply induction.

The proof of ℓ_1-rigidity of the Johnson graph $J(n, d)$ (for $d \geq 2$) is similar to the proof of ℓ_1-rigidity for the half-cube $\frac{1}{2}H(n, 2)$ (for $n \geq 5$), so we omit the details.

4.5. Proof of (2.12)

Let us recall the definition of the graph $\mathcal{A}(n, t)$, taken from [21].

Let $V = \{1, \dots, n\}$ and $n \geq 2t$, $t \geq 3$. We define the families:

- X_{00}, consisting of the sets A, for $A \subseteq V, |A| = t$
- X_{11}, consisting of the sets $A \cup \{n + 1, n + 2\}$, for $A \subseteq V$, $|A| = t$
- X_{10}, consisting of the sets $A \cup \{n + 1\}$, for $A \subseteq V$, $|A| = t - 1$
- X_{01}, consisting of the sets $A \cup \{n + 2\}$, for $A \subseteq V, |A| = t - 1$

$\mathcal{A}(n, t)$ is the graph with node set $X_{00} \cup X_{11} \cup X_{10} \cup X_{01}$ and with edges the pairs (C, D) with $|C \Delta D| = 2$.

The subgraph of $\mathcal{A}(n, t)$ induced by any of the sets X_{00}, X_{11} (resp. X_{01}, X_{10}) is isomorphic to the Johnson graph $J(n, t)$ (resp. $J(n, t - 1)$). Hence, it is ℓ_1-rigid since we assume that $t \geq 3$. For $1 \leq i \leq n$, denote by S_i (resp. T_i, U_i, W_i) the subset of X_{00} (resp. X_{11}, X_{10}, X_{01}) consisting of the members containing the element i.

A binary ℓ_1-embedding of $2d(\mathcal{A}(n, t))$ is obtained, simply, by labeling each node by its incidence set. Correspondingly, we have the following $\mathbb{Z}_+$-realization of $2d(\mathcal{A}(n, t))$:

$$2d(\mathcal{A}(n, t)) = \delta(X_{00} \cup X_{01}) + \delta(X_{00} \cup X_{10}) + \sum_{1 \leq i \leq n} \delta(S_i \cup T_i \cup U_i \cup W_i).$$

We show that $\mathcal{A}(n, t)$ is ℓ_1-rigid.

First, it is easy to check that the cut metrics $\delta(X_{00} \cup X_{01})$, $\delta(X_{00} \cup X_{10})$ and $\delta(S_i \cup T_i \cup U_i \cup W_i)$, for $1 \leq i \leq n$, are linearly independent (using, for instance, the corresponding independence result for the Johnson graph).

Let $\delta(S)$ be a nonzero cut metric satisfying the same triangle equalities as $d(\mathcal{A}(n, t))$. We show that $\delta(S)$ is one of the cut metrics $\delta(X_{00} \cup X_{01})$, $\delta(X_{00} \cup X_{10})$ or $\delta(S_i \cup T_i \cup U_i \cup W_i)$, for $1 \leq i \leq n$.

We shall use, in particular, the following triangle equalities:

(i) $d(A, B \cup \{n+1, n+2\}) = d(A, A \cup \{n+1, n+2\}) + d(A \cup \{n+1, n+2\}, B \cup \{n+1, n+2\})$
(ii) $d(A, B \cup \{n+1, n+2\}) = d(A, B) + d(B, B \cup \{n+1, n+2\})$ for $A, B \subseteq V$, $|A| = |B| = t$, and $|A \Delta B| = 2$,
(iii) $d(A, (A-a) \cup \{b, n+1, n+2\}) = d(A, (A-a) \cup \{n+1\}) + d((A-a) \cup \{n+1\}, (A-a) \cup \{b, n+1, n+2\})$
(iv) $d(A, (A-a) \cup \{b, n+1, n+2\}) = d(A, (A-a) \cup \{n+2\}) + d((A-a) \cup \{n+2\}, (A-a) \cup \{b, n+1, n+2\})$ for $A \subseteq V$, $|A| = t$, $a \in A$, $b \in V - A$,
(v) $d((A-a) \cup \{n+1\}, (A-b) \cup \{n+2\}) = d((A-a) \cup \{n+1\}, A) + d(A, (A-b) \cup \{n+2\})$.
(vi) $d((A-a) \cup \{n+1\}, (A-b) \cup \{n+2\}) = d((A-a) \cup \{n+1\}, A \cup \{n+1, n+2\}) + d(A \cup \{n+1, n+2\}, (A-b) \cup \{n+2\})$

The subgraph induced by $\mathcal{A}(n, t)$ on X_{00} is the ℓ_1-rigid graph $J(n, t)$; hence, from the fact that the ℓ_1-embedding of $J(n, t)$ uses precisely the cut metrics $\delta(S_i)$ (see Section 4.4), we deduce that $S \cap X_{00} = \emptyset$, X_{00}, S_i, or $X_{00} - S_i$ for some $1 \leq i \leq n$. Similarly, $S \cap X_{11} = \emptyset$, X_{11}, T_i, or $X_{11} - T_i$ for some i; $S \cap X_{10} = \emptyset$, X_{10}, U_i, or $X_{10} - U_i$ for some i; $S \cap X_{01} = \emptyset$, X_{01}, W_i, or $X_{01} - W_i$ for some i.

We can suppose, e.g., that $S \cap X_{00} = X_{00}$ or S_i. We assume first that $S \cap X_{00} = X_{00}$. If $X_{11} \cap S \neq \emptyset$, then $S = X_{00} \cup X_{11} \cup X_{10} \cup X_{01}$, using the triangle equalities (i)–(iv) above. So we can suppose that $S \cap X_{11} = \emptyset$. From the triangle equalities (v), (vi), S contains exactly one of $A - a \cup \{n+1\}, A - b \cup \{n+2\}$, for all $a, b \in A$, $|A| = t$. One deduces easily that $S = X_{00} \cup X_{01}$, or $X_{00} \cup X_{10}$.

We now assume that $S \cap X_{00} = S_i$ for some i. Using again some triangle equalities, one can check that $S = S_i \cup T_i \cup U_i \cup W_i$.

This concludes the proof.

4.6. Proof of (2.13)

Here, we show that the Petersen graph, the Shrikhande graph, the dodecahedron and the icosahedron are ℓ_1-rigid.

Figures 1 and 2 show, respectively, the Petersen graph O_3 and the Shrikhande graph G_{Sh} together with an isometric embedding in the half-cube $\frac{1}{2}H(6, 2)$ (up to reflection in the first coordinate), i.e., a binary ℓ_1-embedding of $2d(G)$ for $G = O_3$ and G_{Sh}. Actually, the binary ℓ_1-embeddings we describe in Figures 1 and 2 are equivalent to the ones given in Section 3.11 from [4]. Observe that, in

1	0 0 0 1 1 1
2	0 1 0 1 1 0
3	0 1 1 0 1 0
4	0 1 1 0 0 1
5	1 1 0 0 0 1
6	1 0 0 0 1 1
7	0 0 1 1 0 1
8	1 1 0 1 0 0
9	1 0 1 0 1 0
10	1 0 1 1 0 0

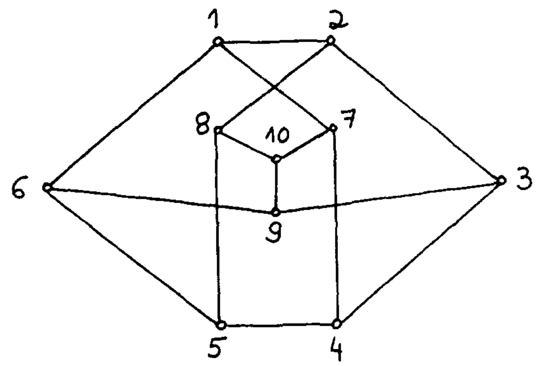


Figure 1. The Petersen graph.

1	0 0 0 0 0 0
2	0 0 1 0 0 1
3	1 1 1 0 0 1
4	1 1 0 0 0 0
5	0 1 0 1 0 0
6	0 0 0 1 0 1
7	1 0 1 1 0 1
8	1 1 1 1 0 0
9	0 1 1 1 1 0
10	0 1 0 1 1 1
11	1 0 0 1 1 1
12	1 0 1 1 1 0
13	0 0 1 0 1 0
14	0 1 1 0 1 1
15	1 1 0 0 1 1
16	1 0 0 0 1 0

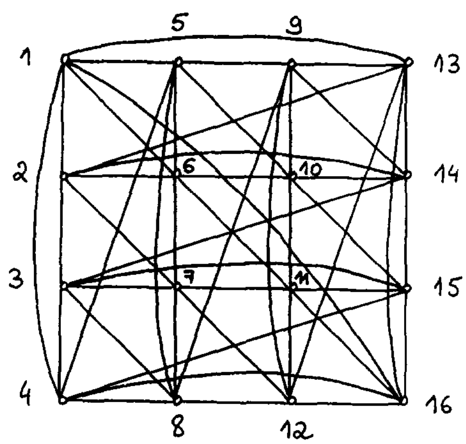


Figure 2. The Shrikhande graph.

1	0 0 0 0 0
2	1 1 0 0 0
3	0 1 1 0 0
4	0 0 1 1 0
5	0 0 0 1 1
6	1 0 0 0 1

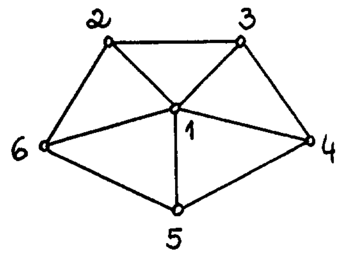


Figure 3. ∇C_5.

both cases, the binary labels are the blocks of a design, namely, of $S_2(2, 3, 6)$ for the Petersen graph, and $S_4(2, \{2, 4\}, 6)$ (where the blocks of size 2 form a cycle) for the Shrikhande graph. The corresponding $\mathbb{Z}_+$-realizations are given by:

$$2d(O_3) = \delta(\{5, 6, 8, 9, 10\}) + \delta(\{2, 3, 4, 5, 8\}) + \delta(\{3, 4, 7, 9, 10\}) + \delta(\{1, 2, 7, 8, 10\}) + \delta(\{1, 2, 3, 6, 9\}) + \delta(\{(1, 4, 5, 6, 7\})$$

and

$$2d(G_{Sh}) = \delta(\{3, 4, 7, 8, 11, 12, 15, 16\}) + \delta(\{3, 4, 5, 8, 9, 10, 14, 15\}) + \delta(\{2, 3, 7, 8, 9, 12, 13, 14\}) + \delta(\{5, 6, 7, 8, 9, 10, 11, 12\}) + \delta(\{9, 10, 11, 12, 13, 14, 15, 16\}) + \delta(\{2, 3, 6, 7, 10, 11, 14, 15\}).$$

Again, one shows that O_3, G_{Sh} are ℓ_1-rigid by checking that the conditions (4.6) and (4.7) hold. We omit the details.

We consider now the icosahedron and the dodecahedron, whose path metrics are denoted, respectively, by d_{ico} and d_{dod}; these two graphs can be found, e.g., in chapter 1 from [4]. A useful observation is that both graphs can be expressed in terms of smaller graphs (one-half number of nodes) using the antipodal extension operation.

Namely, consider the 5-wheel ∇C_5, i.e., the graph obtained by adding a node adjacent to all nodes of the 5-cycle; ∇C_5 and a binary ℓ_1-embedding of $2d(\nabla C_5)$ are shown in Figure 3. Then, $d_{ico} = Ant_3(d(\nabla C_5))$, or $2d_{ico} = Ant_6(2d(\nabla C_5))$. It is easy to check that ∇C_5 is ℓ_1-rigid. Therefore, by (2.4), the icosahedron is ℓ_1-rigid. A binary ℓ_1-embedding for $2d_{ico}$ follows from that of $2d(\nabla C_5)$ using (4.4).

Note that there are other ways to express the icosahedron as antipodal extension of some smaller ℓ_1-rigid graph. For instance, $d_{ico} = Ant_3(d(G_i))$ where G_1 and G_2 are the graphs shown in Figures 4 and 5, respectively (G_1 is weighted, with weights 1 or 2, the edges with weight 2 are indicated by a double line).

Consider also the weighted graph G_3 shown in Figure 6; its edges have weight 1 or 2 (edges with weight 2 being indicated by a double line). Then,

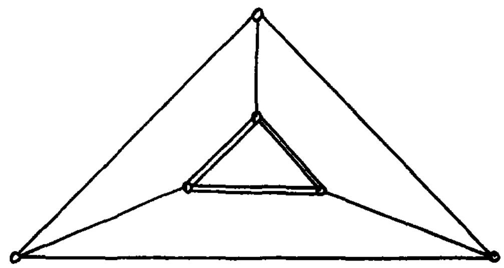


Figure 4. G_1.

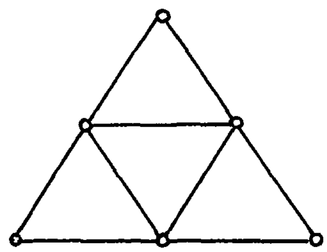


Figure 5. G_2.

$d_{dod} = Ant_5(d(G_3))$ holds. Using the binary ℓ_1-embedding of $2d(G_3)$ shown in Figure 6, it is easy to check that the graph G_3 is ℓ_1-rigid. Therefore, the dodecahedron is ℓ_1-rigid.

4.7. *Proof of Propositions 3.1 and 3.2*

We prove that every bipartite ℓ_1-embeddable graph is ℓ_1-rigid.

We first recall (from [18]) how the binary ℓ_1-embedding is constructed.

Let $G = (V, E)$ be a bipartite ℓ_1-graph. Let $\overline{E}$ denote the set of equivalence classes of the relation θ introduced in Section 3. Recall that, given two edges $e = (i, j)$ and $e' = (i', j')$ of G, $e\theta e'$ holds if and only if the two partitions of V into $G(i, j) \cup G(j, i)$ and $G(i', j') \cup G(j', i')$ are identical.

A binary ℓ_1-embedding of d_G is constructed as follows. Choose a node u_0 in V. For each node $u \in V$, let $F(u)$ denote the subset of $\overline{E}$ consisting of the classes $\overline{e}$ for which u and u_0 belong to distinct sets in the partition $V = G(i, j) \cup G(j, i)$, if $e = (i, j)$. Then, labeling the node u_0 by the zero vector and each other node u by the incidence vector of the set $F(u)$, we obtain a binary ℓ_1-embedding of d_G of dimension $|\overline{E}|$. This corresponds to the following $\mathbb{Z}_+$-realization of d_G:

1	1 1 0 0 0 0 0 0 0 0
2	0 1 1 0 0 0 0 0 0 0
3	0 0 1 1 0 0 0 0 0 0
4	0 0 0 1 1 0 0 0 0 0
5	1 0 0 0 1 0 0 0 0 0
6	1 1 0 0 0 1 1 0 0 0
7	0 1 1 0 0 0 1 1 0 0
8	0 0 1 1 0 0 0 1 1 0
9	0 0 0 1 1 0 0 0 1 1
10	1 0 0 0 1 1 0 0 0 1

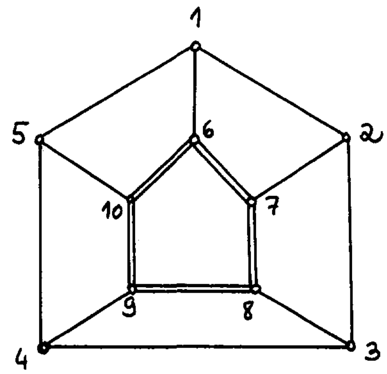


Figure 6. G_3.

$$d_G = \sum_{\bar{e}\in\overline{E}} \delta(S_{\bar{e}})$$

where $S_{\bar{e}} = G(i, j)$ if $e = (i, j)$ is any representant of the class $\bar{e}$.

We now show that G is ℓ_1-rigid. First, the family of the cut metrics $\delta(S_{\bar{e}})$, $\bar{e} \in \overline{E}$, is trivially linearly independent. Then, consider a nonzero cut metric $\delta(S)$ which satisfies the same triangle equalities as d_G; we show that $S = S_{\bar{e}}$ for some $\bar{e} \in \overline{E}$. Consider a shortest path P between two nodes i and j in G, $P = (i_1 = i, i_2, \ldots, i_k = j)$, and suppose that $i \notin S$ and $j \in S$. Let $2 \leq h \leq k-1$ be the largest index such that $i_h \notin S$, so $i_{h+1} \in S$. Since $\delta(S)$ satisfies the triangle equality: $d(i_h, i_p) = d(i_h, i_{h+1}) + d(i_{h+1}, i_p)$ for $h + 2 \leq p \leq k$, we deduce that $i_p \in S$ for all $h + 1 \leq p \leq k$. Hence, the nodes of P belonging to S, as well as the nodes of P not belonging to S, form an interval on P. From this, it is easy to obtain that S is necessarily of the form $S_{\bar{e}}$ for some $e \in E$.

So we have shown that G is ℓ_1-rigid. In fact, we have also shown that the smallest face $\mathcal{F}_G$ of the cut cone $\mathcal{C}_n$ containing d_G is generated by the cut metrics $\delta(S_{\bar{e}})$, $\bar{e} \in \overline{E}$. Every metric $d \in \mathcal{F}_G$ is of the form: $d = \sum_{\bar{e}\in\overline{E}} w_{\bar{e}}\delta(S_{\bar{e}})$ for some scalars $w_{\bar{e}} \geq 0$. Let us extend w to E by setting $w_f = w_{\bar{e}}$ for $f \in E$, $f\theta e$. By construction, w is compatible with θ and d coincides with the path metric of the weighted graph (G, w). This shows Proposition 3.2.

4.8. Proof of Proposition 3.3

Let $C = (1, 2, \ldots, n)$ be a cycle on $V = \{1, \ldots, n\}$ and w_i be a nonnegative integer weight assigned to the edge $(i, i + 1)$, for $1 \leq i \leq n$, the indices being taken modulo n.

We first show that (C, w) is an ℓ_1-graph. Let C^* denote the cycle obtained by replacing each edge e of C by a path of length w_e. Hence, the projection of

$d(C^*)$ on V coincides with $d(C, w)$. If $\sum_e w_e$ is even, then C^* is an even cycle and C^* is ℓ_1-embeddable with scale 1, otherwise C^* is ℓ_1-embeddable with scale 2. In both cases, an ℓ_1-embedding for $d(C, w)$ is obtained from an ℓ_1-embedding for $d(C^*)$ by projection. We now show that $d(C, w)$ is ℓ_1-rigid, i.e., lies on a simplicial face of $\mathcal{C}_n$.

If $w_n > w_1 + \cdots + w_{n-1}$, then no shortest path in C uses the edge $(1, n)$. Hence, $d(C, w)$ coincides with the path metric of the path $P = (1, \ldots, n)$ with weights $w_1, \ldots, w_{n-1}$ on its edges; therefore, by Proposition 3.2, (C, w) is ℓ_1-rigid.

We can now assume that $\max(w_i : 1 \le i \le n) \le \frac{1}{2}\sum_{1\le i\le n} w_i$. Hence, $d(i, i + 1) = w_i$ for all $1 \le i \le n$. Define i^* as the largest index i, $2 \le i \le n$, such that $w_1 + \cdots + w_{i-1} \le w_i + \cdots + w_n$. Hence, $w_1 + \cdots + w_{i^*-1} \le w_{i^*} + \cdots + w_n$ and $w_1 + \cdots + w_{i^*} > w_{i^*+1} + \cdots + w_n$. Let (P, w) (resp. (Q, w)) denote the path $(1, \ldots, i^*)$ (resp. $(i^* + 1, \ldots, n, 1)$) with the weights w_e being assigned to its edges. By construction of i^*, the projection of $d(C, w)$ on P (resp. Q) coincides with $d(P, w)$ (resp. $d(Q, w)$). From Section 4.7, $d(P, w) = \Sigma_{2\le h\le i^*} w_h\delta(\{h, h + 1, \ldots, i^* - 1, i^*\})$ and $d(Q w) = \Sigma_{i^*+1\le k\le n} w_k\delta(\{i^* + 1, i^* + 2, \ldots, k - 1, k\})$ and they are ℓ_1-rigid.

Let $\delta(S)$ be a cut metric satisfying the same triangle equalities as $d(C, w)$. Since the paths (P, w) and (Q, w) are ℓ_1-rigid, we obtain that $S \cap \{1, \ldots, i^*\} = \emptyset, \{1, \ldots, i^*\}, \{1, 2, \ldots, h - 2, h - 1\}$, or $\{h, h + 1, \ldots, i^*\}$ for some $2 \le h \le i^*$, and $S \cap \{i^*+1, \ldots, n-1, n, 1\} = \emptyset, \{i^*+1, \ldots, n-1, n, 1\}, \{i^*+1, \ldots, k-1, k\}$, or $\{k+1, \ldots, n-1, n, 1\}$ for some $i^* + 1 \le k \le n$. We can suppose, for instance, that $1 \notin S$. Then, S is one of the following sets: $\emptyset$, $A_k := \{i^* + 1, \ldots, k - 1, k\}$ for $i^* + 1 \le k \le n$, $B_h := \{h, h + 1, \ldots, i^*\}$ for $2 \le h \le i^*$, and $S_{h,k} := \{h, h + 1, \ldots, i^*\} \cup \{i^* + 1, \ldots, k - 1, k\}$ for $2 \le h \le i^*$ and $i^* + 1 \le k \le n$.

Let $\mathcal{F}$ denote the smallest face of $\mathcal{C}_n$ containing $d(C, w)$. Every nonzero cut metric lying on $\mathcal{F}$ must be one of the cut metrics $\delta(A_k)$, $\delta(B_h)$, $\delta(S_{h,k})$, for $2 \le h \le i^*$ and $i^* + 1 \le k \le n$. If we show that the cut metrics $\delta(A_k)$, $\delta(B_h)$ and $\delta(S_{h,k})$ are linearly independent, then this will imply that the face $\mathcal{F}$ is simplicial and, thus, that (C, w) is ℓ_1-rigid.

Indeed, suppose that

$$0 = \sum_k a_k\delta(A_k) + \sum_h b_h\delta(B_h) + \sum_{h,k} s_{h,k}\delta(S_{h,k}).$$

Computing the value on the edges $(h_0 - 1, h_0)$, $(k_0, k_0 + 1)$ and (h_0, k_0), for $2 \le h_0 \le i^*$, $i^* + 1 \le k_0 \le n$, yields, respectively, the relations:

(i) $b_{h_0} + \sum_k s_{h_0,k} = 0$,
(ii) $a_{k_0} + \sum_h s_{h,k_0} = 0$,
(iii) $\sum_{k_0\le k\le n} a_k + \sum_{2\le h\le h_0} b_h + \sum_{2\le h\le h_0,\, i^*+1\le k\le k_0-1} s_{h,k}$
$+ \sum_{h_0+1\le h\le i^*,\, k_0\le k\le n} s_{h,k} = 0$.

Using (i) and (ii), (iii) can be rewritten as

$$\sum_{2 \le h \le h_0,\, k_0 \le k \le n} s_{h,k} = 0,$$

from which one obtains that $s_{h,k} = 0$ for all h, k and, thus, $a_k = 0$, $b_h = 0$ for all h, k.

This concludes the proof.

5. Applications

In application of the results from Sections 2 and 3, we group here information on ℓ_1-rigidity for some important classes of graphs, whose classification is provided, in particular, in [4].

Fact (5.1). The only strongly regular ℓ_1-graphs are $K_{n\times 2}$, $H(2, d)$, $J(n, 2)$, C_5, the Petersen graph O_3 and the Shrikhande graph [21, Corollary 3.9]. All, except $K_{n\times 2}$ and the grid $H(2, n)$ for $n \ge 4$, are ℓ_1-rigid.

Fact (5.2). The 1-skeletons of all regular polytopes, except the 24-cell, the 600-cell, and the undecided case of the 120-cell, are ℓ_1-graphs [1]. Among them, all are ℓ_1-rigid except for the simplex α_n for $n \ge 3$ and the cross-polytope β_n for $n \ge 4$.

Remark that the Johnson graph and the half-cube are also the 1-skeletons of some polytopes, in fact, of some L-polytopes (or Delaunay polytopes). (Recall that an L-polytope is the convex hull of the lattice points lying on the boundary of a hole in a lattice L, see, e.g., [5].) For $J(n, 1)$ and $\frac{1}{2}H(n, 2)$ with $n = 2, 3$, the L-polytope is regular (it is a simplex); else, it is not regular. For $J(5, 2) \simeq T(5)$, the L-polytope is 0_{21}, and for $\frac{1}{2}H(5, 2)$ (the Clebsch graph), it is 1_{21}, which are both semiregular (see [6]).

Using Section 3.15 from [4], we obtain (5.3) and (5.4) below.

Fact (5.3). The only distance regular ℓ_1-graphs with $\mu \ge 2$ and $d \ge 3$ are $\frac{1}{2}H(n, 2)$ for $n \ge 6$, $J(n, d)$, $H(n, d)$ for $n \ge 3$, the icosahedron and the Doob graphs.

Fact (5.4). The only amply regular ℓ_1-graphs with $\mu \ge 2$ are $K_{n\times 2}$, $\frac{1}{2}H(n, 2)$, $J(n, d)$, $H(n, d)$ and direct products of icosahedra and Doob graphs.

Fact (5.5). Using Table 10.6 from [4], among distance regular finite Coxeter graphs, the only ℓ_1-graphs are $K_{n\times 2}$, $\frac{1}{2}H(n, 2)$, $J(n, d)$, $H(n, 2)$, C_n for $n \ge 5$, the icosahedron and the dodecahedron.

Fact (5.6). Using Theorem 7.5.1 from [4], all distance regular cubic ℓ_1-graphs are K_4, the Petersen graph O_3, $H(3, 2)$, the double odd graph DO_5 and the dodecahedron.

Fact (5.7). We consider now ℓ_1-graphs among distance regular antipodal graphs. Koolen ([21], Lemma 5.7) observed that any distance regular antipodal ℓ_1-graph is a double cover and, moreover, classified them among the double covers of complete graphs, i.e., among Taylor graphs (Lemma 3.13); namely, they are $J(6, 3)$, $\frac{1}{2}H(6, 2)$, the icosahedron and the two bipartite graphs: $C_6 \simeq DO_3$, $H(3, 2)$, and, thus, they are all ℓ_1-rigid. Other examples of distance regular antipodal ℓ_1-graphs include the dodecahedron (double cover of the Petersen graph O_3) and the three bipartite graphs: C_{2n} (double cover of C_n), $H(n, 2)$ (double cover of the folded cube) and DO_{2n+1} (double cover of the odd graph O_{2n+1}). Actually, the only distance regular graphs that are ℓ_1-graphs with scale 1 are C_{2n}, DO_{2n+1} and $H(n, 2)$ ([21], [25]).

Fact (5.8). Using Theorem 2.8 from [21], the only root graphs with $\mu \geq 3$ and root representation in $\mathbb{Z}^n$ which are ℓ_1-graph are: complete multipartite graphs with classes of size 1 or 2, $\frac{1}{2}H(n, 2)$, the suspension of $J(n, 2) \simeq T(n)$, $J(n, d)$, $\mathcal{A}(n, t)$, the suspension of the $p \times q$-grid and $L_{s,t}$. Moreover, except the undecided case of $L_{s,t}$, we know which ones are ℓ_1-rigid. Namely, $\frac{1}{2}H(n, 2)$, $\mathcal{A}(n, t)$, the suspension of $T(n)$ or of a $p \times q$-grid are ℓ_1-rigid (this is easy to check in the last two cases). A complete multipartite graph with p classes of size 1 or 2 is ℓ_1-rigid if and only if $p \leq 3$ (this is easy to see using (2.4)).

Fact (5.9). The complement of $J(n, 2)$ is not an ℓ_1-graph if $n \geq 6$ (since it contains $K_{2,3}$ as an isometric subgraph) and it is ℓ_1-rigid if $n = 5$ (since it is isomorphic to the Petersen graph).

Fact (5.10). Let Q be a generalized quadrangle and let G_Q denote its collinearity graph.

(i) If Q is degenerate, then G_Q is an ℓ_1-graph with scale 1 if all lines have size 2 and with scale 2 otherwise. Moreover, G_Q is ℓ_1-rigid if and only if all the lines of Q have size at most 3.

(ii) If Q is a generalized quadrangle with all lines of size 3, then G_Q is an ℓ_1-graph if and only if Q is degenerate or Q is the 3×3-grid $H(2, 3)$. In both cases, G_Q is ℓ_1-rigid.

Proof. (ii) follows from the classification of generalized quadrangles with line size 3 (see Section 1.15 in [4]) and from (*i*). (*i*) follows by applying (2.3). Indeed, let Q be a degenerate generalized quadrangle with point set X with lines $L_1, \ldots, L_k$ and let x_0 denote their common point. Observe that $d(G_Q)$ arises as the 1-sum of the metrics d_a, where d_a is the path metric of the complete graph on L_a, for $1 \leq a \leq k$. Therefore, from (2.3), G_Q is an ℓ_1-graph with scale 1 if all lines have size 2, and with scale 2 otherwise. Moreover, G_Q is ℓ_1-rigid if and only if all d_a are ℓ_1-rigid, i.e., all lines have size at most 3. □

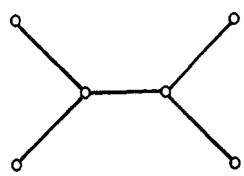
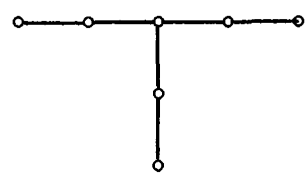
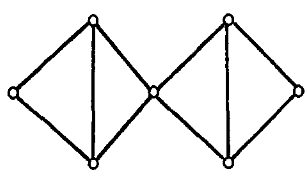
Fact (5.11). Given $p, q \geq 2$, the complement of the $p \times q$-grid $K_p \times K_q$ is not an ℓ_1-graph if $q \geq 5$, or $p \geq 3$ and $q \geq 4$ (since it contains $K_{2,3}$ as isometric subgraph) and it is ℓ_1-rigid for $(p, q) = (2, 3), (2, 4), (3, 3)$ (since it is isomorphic to C_6, $H(3, 2)$, $K_3 \times K_3$, respectively).

Fact (5.12). There are 30 connected graphs on $2 \leq n \leq 5$ nodes (see [8]). Among them, there are

- three graphs which are not ℓ_1-graphs, namely, $\nabla K_{1,3}$, $K_5 - (P_2 \cup P_3)$, $K_{2,3}$
- four nonrigid ℓ_1-graphs with scale 2, namely, K_4, K_5, $K_5 - P_2$, K_4 plus a pendant edge
- nine ℓ_1-rigid graphs with scale 1, namely, C_4, C_4 plus a pendant edge and all seven trees.

The remaining 15 graphs are ℓ_1-rigid with scale 2.

Fact (5.13). Among the integral graphs on $n \leq 7$ nodes (i.e., the eigenvalues of their adjacency matrix are all integral; see [7] for their list), the ℓ_1-graphs are:

- K_4, K_5, K_6
  and the following ones which are ℓ_1-rigid:
- C_2, C_3, C_4, C_6, $K_{3\times 2}$, the 2×3-grid,
- the trees $K_{1,4}$,  and 
- the degenerate generalized quadrangle on 7 nodes with line size 3 (i.e., three triangles sharing a common vertex),
- the graph 

Fact (5.14). We conclude with a remark on (k, g)-cages. All known (k, g)-cages of even girth are bipartite; therefore, by Proposition 3.1, they are ℓ_1-rigid whenever they are ℓ_1-graphs. Actually, all known ℓ_1-graphs among (k, g)-cages are the $(2, g)$-cage C_g, the $(k, 3)$-cage K_{k+1} and the $(3, 5)$-cage O_3 (Petersen graph).

Acknowledgment

We are grateful to the referees for their careful reading and valuable comments.

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