J Elast (2010) 98: 117–140
DOI 10.1007/s10659-009-9220-6

Ellipsoidal Domain with Piecewise Nonuniform Eigenstrain Field in One of Joined Isotropic Half-Spaces

R. Avazmohammadi · R. Hashemi · H.M. Shodja · M.H. Kargarnovin

Received: 20 June 2009 / Published online: 16 October 2009

Abstract Consider an arbitrarily oriented ellipsoidal domain near the interface of an isotropic bimaterial space. It is assumed that a general class of piecewise nonuniform dilatational eigenstrain field is distributed within the ellipsoidal domain. Two theorems relevant to prediction of the nature of the induced displacement field for the interior and exterior points of the ellipsoidal domain are stated and proved. As a resultant the exact analytical expression of the elastic fields are obtained rigorously. In this work a new Eshelby-like tensor, **A** is introduced. In particular, the closed-form expressions for **A** associated with the interior points of spherical and cylindrical inclusion are derived. The stress field is presented for a single ellipsoidal inclusion which undergoes a Gaussian distribution of eigenstrain field and one of the principal axes of the domain is perpendicular to the interface. For the limiting case of spherical inclusion the closed-form solution is obtained and the associated strain energy is discussed. For further demonstration, two examples of two concentric spheres and three concentric cylinders with eigenstrain field distributions which are descriptive of the general class of functions defined in this paper. The effect of some parameters such as distance between the inclusion and the interface, and the ratio of the shear moduli of the two media on the induced elastic fields are examined.

Keywords Micromechanics · Bimaterial · Nonuniform piecewise eigenstrain field · Ellipsoidal domain

Mathematics Subject Classification (2000) 74M25

R. Avazmohammadi · R. Hashemi · M.H. Kargarnovin
Department of Mechanical Engineering, Sharif University of Technology, Tehran, Iran

H.M. Shodja (✉)
Department of Civil Engineering, Sharif University of Technology, Tehran, Iran
e-mail: shodja@sharif.edu

H.M. Shodja
Institute for Nanoscience and Nanotechnology, Sharif University of Technology, Tehran, Iran

1 Introduction

Ever since the pioneering contribution of Eshelby [1–3], the subject of ellipsoidal inclusion and related problems, due to its numerous valuable engineering applications, has become one of the most attractive topics of solid mechanics. In addition to the practical examples of inclusion given by Mura [4], one may mention new implications in isolation trenches in large scale integrated circuits and strained semiconductor laser devices where residual stresses induced by thermal or lattice mismatch strains affect electronic performance of devices. In the past, the micromechanics of inclusion with uniform eigenstrain and its related problems have received considerable attentions, as evidenced by the reviews given by Mura [4], Mura et al. [5], and Nemat-Nasser and Hori [6]. On the other hand, Eshelby [2], Sendeckij [7] and Rahman [8] performed analytical works on the ellipsoidal inclusion in an infinite domain with eigenstrain field characterized by a polynomial of arbitrary order. In particular, Eshelby [3] has shown that if the eigenstrain field in the inclusion is a homogenous polynomial of degree $n = 1$, the disturbance strain for the interior points will be also a homogenous polynomial of degree 1. If the distribution of eigenstrain field is a homogeneous polynomial of degree $n \geq 2$, the corresponding disturbance strain field inside the inclusion is a inhomogeneous polynomial whose terms are of degree n, $(n-2)$, $(n-4)$, ... This brilliant conclusion was extensively utilized by Moschovidis and Mura [9], Shodja and Sarvestani [10], Shodja et al. [11], and Shodja and Ojaghnejhad [12] in dealing with various fundamental problems of single and multi -inhomogeneity systems.

Nevertheless several problems in physics and mechanics of materials naturally give rise to eigenstrain field which is best described by piecewise continuous nonuniform functions other than polynomials. For example, a nonuniform temperature distribution in the vicinity of a point heat within electronic chips leads to thermal eigenstrains which have Gaussian or exponential form. Single or multilayered films containing strained islands encountered in nano-electro-mechanical systems (NEMS) is another area of technological interest where one may deal with piecewise continuous nonuniform eigenstrain distribution. For a rather general treatment, it is proposed that the ellipsoidal inclusion is subjected to a class of piecewise continuous nonuniform eigenstrain field which can encompass a wide range of distribution. The particular class of eigenstrain field utilized is due to Shodja and Shokrolahi-Zadeh [13] who considered an ellipsoidal inclusion embedded in an infinite domain. Some special cases of such eigenstrain distributions include Fourier series, spherical harmonics, and series expansions in terms of special functions like Bessel's function, Legendre polynomial, etc. The nonuniform distribution under consideration has been instrumented for treatment of nested infinitely extended elastic cylindrical media with general cylindrical anisotropy embedded in an unbounded elastic isotropic medium under general remote loading, Shokrolahi-Zadeh and Shodja [14]. Subsequently, they introduced the notions of Eshelby-Fourier tensor and spectral consistency conditions for the first time.

Formation of an ellipsoidal inclusion near the interface of a bimaterial either deliberately or undesirably is of great concern. For instance, in many practical cases such as those encountered in electronic packaging, it is interesting to consider the thermo-mechanical problems associated with an inclusion buried within one of the two joined elastic half-spaces. But most available studies are related to an infinite medium rather than joined half-spaces. There are a few contributions in the literature which address the inclusion problem associated with joined half-spaces but deal with uniform eigenstrain field. For example, Guell and Dunders [15] utilized the Papkovitch-Neuber displacement potentials for determination of the elastic fields of a spherical region in one of the joined elastic half-paces with constant dilatational eigenstrain field. For the first time, a general stress vector function was applied by

Yu and Sanday [16, 17] to obtain an analytical solution of the ellipsoidal inclusion problem for isotropic joined half-spaces. Subsequently, Yu et al. [18] extended this approach to the case where the half-spaces are transversely isotropic. A closed-form solution for the case of a spherical inclusion with uniform thermal dilatational eigenstrain field was obtained by Yu et al. [19]. Referring to the method of images, the Green's function for the two joined half-spaces was presented by Walpole [20]. Subsequently, Walpole [21] revisited the problem of an inclusion with uniform eigenstrain in one of two joined isotropic elastic half-spaces and evaluated the elastic strain energy which is of great importance in physical applications.

Overall, it appears that the studies on Eshelby's problem in joined half-spaces are limited to inclusions with simple distribution of eigenstrains, and the analysis of bimaterial spaces in which one of the media contains a subdomain with piecewise nonuniform eigenstrain field has not been reported in the literature. The present work aims to provide an analytical approach for determining the elastic fields for an arbitrarily oriented ellipsoidal inclusion near the interface of two joined elastic half-spaces. In this problem, a general class of piecewise nonuniform eigenstrain field is prescribed within the confocal ellipsoidal inclusions. In the next section some important Theorems on prediction of the nature of disturbance displacement field are stated and proved. The displacement field due to the dilatational distribution of eigenstrains inside a single ellipsoidal inclusion is predictable via Theorem 1. As a consequence the displacements due to an inclusion with piecewise nonuniform distribution of eigenstrain field may be predicted by employing Theorem 2. Section 3 is devoted to derivation of the closed-form expressions for a new tensor **A** which is introduced in Theorem 1. In Sects. 4 and 5 the robustness of present solution is established by proposing some illustrative examples. First, the example of a single ellipsoidal inclusion with one of its principle axes perpendicular to the interface of the joined half-spaces is considered. Then a spherical inclusion with Gaussian distribution of eigenstrain, because of its numerous applications is examined. Closing section contains two examples of two concentric spheres and three concentric cylinders. Prediction of the elastic fields due to the proposed distributions of eigenstrain field in these examples is made possible via the presented theories.

2 The General Formulation

Consider a bimaterial body with perfect interface as shown in Fig. 1. Let μ^{I} and ν^{I} denote the shear modulus and Poisson's ratio for the medium I and μ^{II} and ν^{II} indicate the corresponding material properties for medium II; the media are assumed to be linear elastic isotropic materials. The Cartesian coordinates x_i, $i = 1, 2, 3$ are oriented in such a way that the plane $x_3 = 0$ coincides with the interface plane, and so $x_3 > 0$ and $x_3 < 0$ are the half-spaces occupied by media I and II, respectively.

Define an ellipsoidal sub-domain $\Omega(\xi)$, $\xi \geq 0$

$$\Omega(\xi) = \left\{ \bar{\mathbf{x}} | \bar{\mathbf{x}} \in \mathbb{R}^3, \sum_{p=1}^{3} \frac{\bar{x}_p^2}{a_p^2} \leq \xi^2 \right\}, \tag{2.1}$$

inside medium I. The dimensionless parameter ξ indicates the size of $\Omega(\xi)$ and a_p, $p = 1, 2, 3$ are the principal half axes of $\Omega(\xi)$. Determination of the elastic fields of the ascribed inclusion problem with piecewise nonuniform eigenstrain field via method of image is of particular interest. Suppose that $\Omega^m(\xi)$ is the mirror image of $\Omega(\xi)$. The local Cartesian coordinates of the ellipsoidal subdomain, $\Omega(\xi)$ and $\Omega^m(\xi)$ are respectively, denoted by $\bar{x}_p$ and $\tilde{x}_p$, $p = 1, 2, 3$ with their origins at $\bar{\mathrm{o}}$ and $\tilde{\mathrm{o}}$, Fig. 1. The coordinate axes $\bar{x}_p$

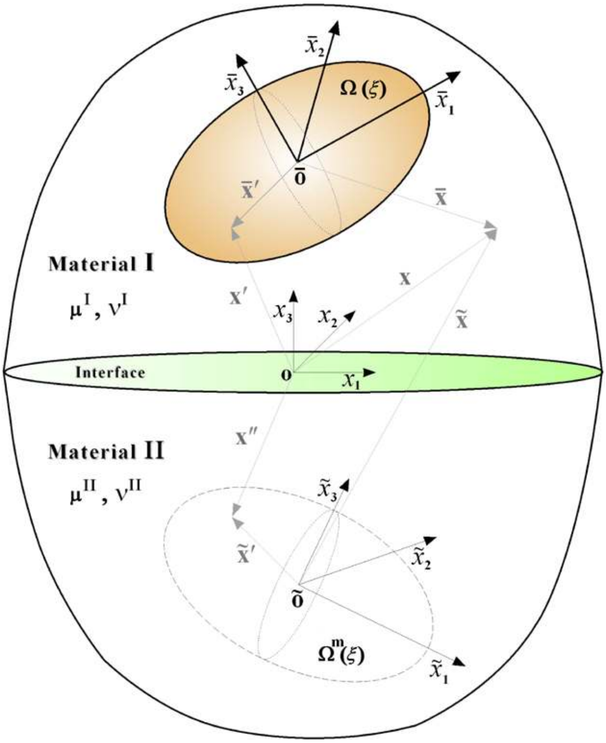

Fig. 1 Ellipsoidal inclusion $\Omega(\xi)$ and its mirror image $\Omega^m(\xi)$ in two joined semi-infinite solids

and $\tilde{x}_p$ coincide with the corresponding principle axes. Assume that $\bar{o}$ is located at ($x_1 = 0$, $x_2 = 0$, $x_3 = h$). The orientation of the coordinate system $\bar{x}_p$ relative to the global coordinate system x_p are described by the corresponding Eulerian angles α_p. The relationship between the coordinate systems x_p and $\bar{x}_p$ can be written in the form

$$\begin{aligned} x_1 &= \bar{x}_1 C_{12} + \bar{x}_2(C_1 S_{23} - S_1 C_3) + \bar{x}_3(C_{13} S_2 + S_{13}), \\ x_2 &= \bar{x}_1 S_1 C_2 + \bar{x}_2(S_{123} + C_{13}) + \bar{x}_3(S_{12} C_3 - C_1 S_3), \\ x_3 &= -\bar{x}_1 S_2 + \bar{x}_2 C_2 S_3 + \bar{x}_3 C_{23} + h, \end{aligned} \tag{2.2}$$

where $C_{i\ldots k} = \cos\alpha_i \ldots \cos\alpha_k$ and $S_{i\ldots k} = \sin\alpha_i \ldots \sin\alpha_k$. In a similar manner x_p are related to $\tilde{x}_p$, except that the associated Eulerian angles are replaced by α_1, $-\alpha_2$ and $-\alpha_3$ respectively and h is replaced by $-h$.

Now, suppose that the ellipsoidal inclusion consists of a set of nested similar ellipsoidal domains $\{\Omega_1, \Omega_2, \ldots, \Omega_N\}$,

$$\Omega_t = \Omega(\xi_t), \quad t = 1, 2, \ldots, N, \tag{2.3}$$

where

$$\xi_0 = 0 < \xi_1 < \xi_2 < \cdots < \xi_N. \tag{2.4}$$

In this manner, N regions denoted by Γ_t are defined

$$\Gamma_1 = \Omega_1 \quad \text{and} \quad \Gamma_t = \Omega_t - \Omega_{t-1} \quad \text{for } t = 2, 3, \ldots, N. \tag{2.5}$$

A general piecewise nonuniform dilatational eigenstrain field

$$\varepsilon_{ij}^{*}(\bar{\mathbf{x}})=\begin{cases} f_{kl\ldots m}^{(t)}\left(\sum_{p=1}^{3}\frac{\bar{x}_p^2}{a_p^2}\right)\bar{x}_k\bar{x}_l\ldots\bar{x}_m\delta_{ij}, & \bar{\mathbf{x}}\in\Gamma_t,\\ 0, & \bar{\mathbf{x}}\in(\mathbb{R}^3-\Omega_N)\end{cases} \tag{2.6}$$

is proposed, where $\bar{x}_k\bar{x}_l\ldots\bar{x}_m$ is of order n, and $f_{kl\ldots m}^{(t)}$ represents $N(n+1)(n+2)/2$ different piecewise continuous functions whose arguments are $\sum_{p=1}^{3}\bar{x}_p^2/a_p^2$.

Throughout this paper, "$ij\ldots k$" which appear as indices of a quantity indicate the partial derivatives of that quantity with respect to the coordinate $x_i, x_j, \ldots, x_k$.

Theorem 1 *If the distribution of eigenstrains in the local coordinate system* $\{\bar{x}_1,\bar{x}_2,\bar{x}_3\}$ *has the form given by* (2.6), *then the displacement components for points inside materials* I *and* II *are*

$$u_i^{\mathrm{I}}(\mathbf{x})=\sum_{t=1}^{N}A_{ikl\ldots m}(\mathbf{x};\xi_t)f_{kl\ldots m}^{(t)}\left(\xi_t^2\right)-\sum_{t=1}^{N}A_{ikl\ldots m}(\mathbf{x};\xi_{t-1})f_{kl\ldots m}^{(t)}\left(\xi_{t-1}^2\right)$$
$$-\sum_{t=1}^{N}\int_{\xi_{t-1}}^{\xi_t}A_{ikl\ldots m}(\mathbf{x};\xi)\,\mathrm{d}f_{kl\ldots m}^{(t)}\left(\xi^2\right), \tag{2.7}$$

and

$$u_i^{\mathrm{II}}(\mathbf{x})=-\frac{(1+\nu^{\mathrm{I}})\mu^{\mathrm{I}}}{\pi(\mu^{\mathrm{I}}+\kappa\mu^{\mathrm{II}})}\times\left\{\sum_{t=1}^{N}f_{kl\ldots m}^{(t)}\left(\xi_t^2\right)\phi_{kl\ldots m,i}(\bar{\mathbf{x}};\xi_t)-\sum_{t=1}^{N}f_{kl\ldots m}^{(t)}\left(\xi_{t-1}^2\right)\phi_{kl\ldots m,i}(\bar{\mathbf{x}};\xi_{t-1})\right.$$
$$\left.-\sum_{t=1}^{N}\frac{\partial}{\partial x_i}\int_{\xi_{t-1}}^{\xi_t}\phi_{kl\ldots m}(\bar{\mathbf{x}};\xi)\,\mathrm{d}f_{kl\ldots m}^{(t)}\left(\xi^2\right)\right\}, \tag{2.8}$$

respectively, where $\phi_{kl\ldots m}(\bar{\mathbf{x}},\xi)$ *is the Newtonian potential function*

$$A_{ikl\ldots m}(\mathbf{x};\xi)=-\frac{(1+\nu^{\mathrm{I}})}{4\pi(1-\nu^{\mathrm{I}})}\left\{\phi_{kl\ldots m,i}(\bar{\mathbf{x}};\xi)\right.$$
$$\left.+\frac{\mu^{\mathrm{I}}-\mu^{\mathrm{II}}}{\mu^{\mathrm{I}}+\kappa\mu^{\mathrm{II}}}(-1)^K[\kappa(1-2\delta_{3i})\phi_{kl\ldots m,i}(\tilde{\mathbf{x}};\xi)+2x_3\phi_{kl\ldots m,3i}(\tilde{\mathbf{x}};\xi)]\right\}, \tag{2.9}$$

$\kappa=3-4\nu^{\mathrm{I}}$ *and* K *is the number of* 3*'s which appear in the indices of* ϕ.

Proof The displacement components due to an arbitrary distribution of eigenstrain field $\varepsilon_{ij}^{*}(\mathbf{x})$ over the region $\Omega_N=\Omega(\xi_N)$ are obtained from

$$u_i^{D}(\mathbf{x})=\int_{\Omega_N}C_{jkmn}\varepsilon_{mn}^{*}(\mathbf{x}')\frac{\partial}{\partial x_k'}G_{ij}^{D}(\mathbf{x},\mathbf{x}')\,\mathrm{d}\mathbf{x}', \tag{2.10}$$

where the superscript $D\equiv$ I or II refers to the region, and $G_{ij}^{D}(\mathbf{x},\mathbf{x}')$ is the Green's function for joined half-spaces with perfect bonding. The Green's function for this problem was first given by Rongved [22]. Using the image method, Walpole [20] has presented a more attractive form of the Green's function. For material I, $G_{ij}^{\mathrm{I}}(\mathbf{x},\mathbf{x}')$ is interpreted as the x_i-direction displacement at point $\mathbf{x}$ due to the unit body force applied at $\mathbf{x}'$ in the x_j-direction. Define $R_1 = |\mathbf{x} - \mathbf{x}'|$ and $R_2 = |\mathbf{x} - \mathbf{x}''|$ with $\mathbf{x}''$ the image point of $\mathbf{x}'$, then in view of (2.6) and (2.10)

$$u_i^{\mathrm{I}}(\mathbf{x}) = -\frac{(1+\nu^{\mathrm{I}})}{12\pi(1-\nu^{\mathrm{I}})}\int_{\Omega_N}\varepsilon_{kk}^*(\mathbf{x}')\biggl\{\left(\frac{1}{R_1}\right)_{,i} + \frac{(\mu^{\mathrm{I}}-\mu^{\mathrm{II}})}{(\mu^{\mathrm{I}}+\kappa\mu^{\mathrm{II}})} \times\left[\kappa(1-2\delta_{3i})\left(\frac{1}{R_2}\right)_{,i} + 2x_3\left(\frac{1}{R_2}\right)_{,3i}\right]\biggr\}\,\mathrm{d}\mathbf{x}', \tag{2.11}$$

and

$$u_i^{\mathrm{II}}(\mathbf{x}) = -\frac{(1+\nu^{\mathrm{I}})\mu^{\mathrm{I}}}{3\pi(\mu^{\mathrm{I}}+\kappa\mu^{\mathrm{II}})}\int_{\Omega_N}\varepsilon_{kk}^*(\mathbf{x}')\left(\frac{1}{R_1}\right)_{,i}\mathrm{d}\mathbf{x}'. \tag{2.12}$$

Switching to the local coordinate systems $\bar{x}_p$ and $\tilde{x}_p$ and inserting the expression for the eigenstrains given by (2.6), lead to

$$\begin{aligned} u_i^{\mathrm{I}}(\mathbf{x}) = &-\frac{(1+\nu^{\mathrm{I}})}{4\pi(1-\nu^{\mathrm{I}})} \\ &\times\Biggl\{\sum_{t=1}^{N}\frac{\partial}{\partial x_i}\int_{\Gamma_t}\frac{1}{|\bar{\mathbf{x}}-\bar{\mathbf{x}}'|}f_{kl\ldots m}^{(t)}\left(\sum_{p=1}^{3}\frac{\bar{x}_p'^2}{a_p^2}\right)\bar{x}_k'\bar{x}_l'\ldots\bar{x}_m'\,\mathrm{d}\bar{\mathbf{x}}' \\ &+\frac{\mu^{\mathrm{I}}-\mu^{\mathrm{II}}}{\mu^{\mathrm{I}}+\kappa\mu^{\mathrm{II}}}\sum_{t=1}^{N}\Biggl\{(-1)^{K_t}\left[\kappa(1-2\delta_{3i})\frac{\partial}{\partial x_i}+2x_3\frac{\partial}{\partial x_3\partial x_i}\right] \\ &\times\int_{\Gamma_t^m}\frac{1}{|\tilde{\mathbf{x}}-\tilde{\mathbf{x}}'|}f_{kl\ldots m}^{(t)}\left(\sum_{p=1}^{3}\frac{\tilde{x}_p'^2}{a_p^2}\right)\tilde{x}_k'\tilde{x}_l'\ldots\tilde{x}_m'\,\mathrm{d}\tilde{\mathbf{x}}'\Biggr\}\Biggr\}, \end{aligned} \tag{2.13}$$

and

$$u_i^{\mathrm{II}}(\mathbf{x}) = -\frac{(1+\nu^{\mathrm{I}})\mu^{\mathrm{I}}}{\pi(\mu^{\mathrm{I}}+\kappa\mu^{\mathrm{II}})}\sum_{t=1}^{N}\frac{\partial}{\partial x_i}\int_{\Gamma_t}\frac{1}{|\bar{\mathbf{x}}-\bar{\mathbf{x}}'|}f_{kl\ldots m}^{(t)}\left(\sum_{p=1}^{3}\frac{\bar{x}_p'^2}{a_p^2}\right)\bar{x}_k'\bar{x}_l'\ldots\bar{x}_m'\,\mathrm{d}\bar{\mathbf{x}}', \tag{2.14}$$

where K_t is order of $\bar{x}_3$ in expression of the eigenstrain field over Γ_t, and Γ_t^m is the image of Γ_t. Now, by converting the volume integrals to line integrals, (2.13) becomes

$$\begin{aligned} u_i^{\mathrm{I}}(\mathbf{x}) = &-\frac{(1+\nu^{\mathrm{I}})}{4\pi(1-\nu^{\mathrm{I}})} \\ &\times\Biggl\{\sum_{t=1}^{N}f_{kl\ldots m}^{(t)}(\xi_t^2)\phi_{kl\ldots m,i}(\bar{\mathbf{x}};\xi_t) - \sum_{t=1}^{N}f_{kl\ldots m}^{(t)}(\xi_{t-1}^2)\phi_{kl\ldots m,i}(\bar{\mathbf{x}};\xi_{t-1}) \\ &-\sum_{t=1}^{N}\frac{\partial}{\partial x_i}\int_{\xi_{t-1}}^{\xi_t}\phi_{kl\ldots m}(\bar{\mathbf{x}};\xi)\,\mathrm{d}f_{kl\ldots m}^{(t)}(\xi^2) + \frac{\mu^{\mathrm{I}}-\mu^{\mathrm{II}}}{\mu^{\mathrm{I}}+\kappa\mu^{\mathrm{II}}} \\ &\times\Biggl[\sum_{t=1}^{N}(-1)^{K_t}f_{kl\ldots m}^{(t)}(\xi_t^2)[\kappa(1-2\delta_{3i})\phi_{kl\ldots m,i}(\tilde{\mathbf{x}};\xi_t)+2x_3\phi_{kl\ldots m,3i}(\tilde{\mathbf{x}};\xi_t)] \end{aligned}$$

$$
\begin{aligned}
&-\sum_{t=1}^{N}(-1)^{K_t} f^{(t)}_{kl\ldots m}(\xi^2_{t-1})[\kappa(1-2\delta_{3i})\phi_{kl\ldots m,i}(\tilde{\mathbf{x}};\xi_{t-1})+2x_3\phi_{kl\ldots m,3i}(\tilde{\mathbf{x}};\xi_{t-1})]\\
&-\sum_{t=1}^{N}(-1)^{K_t}\left[\kappa(1-2\delta_{3i})\frac{\partial}{\partial x_i}+2x_3\frac{\partial^2}{\partial x_3\partial x_i}\right]\int_{\xi_{t-1}}^{\xi_t}\phi_{kl\ldots m}(\tilde{\mathbf{x}};\xi)\,\mathrm{d}f^{(t)}_{kl\ldots m}(\xi^2)\Bigg]\Bigg\}.
\end{aligned}
\tag{2.15}
$$

Noting that the first derivative of Newtonian potential function $\phi_{kl\ldots m}(\mathbf{x};\xi)$ is continuous everywhere in $\mathbb{R}^3$ and its second derivative is continuous everywhere except on the boundary of $\Omega(\xi)$ with a finite jump, the order of differentiation and integration may be changed. Subsequently, after some manipulations and use of the definition of $A_{ikl\ldots m}(\mathbf{x};\xi)$ given by (2.9), (2.15) reduces to (2.7). Similarly, changing the volume integral in (2.14) to line integral, leads to (2.8). □

Theorem 2 *Suppose that the piecewise nonuniform dilatational eigenstrain field is defined by*

$$
\varepsilon^*_{ij}(\bar{\mathbf{x}})=\begin{cases} f^{(t)}\big(\sum_{p=1}^{3}\frac{\bar{x}_p^2}{a_p^2}\big)\delta_{ij}+f^{(t)}_k\big(\sum_{p=1}^{3}\frac{\bar{x}_p^2}{a_p^2}\big)\bar{x}_k\delta_{ij}+\cdots, & \bar{\mathbf{x}}\in\Gamma_t,\\ 0, & \bar{\mathbf{x}}\in(\mathbb{R}^3-\Omega_N)\end{cases}
\tag{2.16}
$$

with respect to the local coordinates, then the corresponding displacement field in the joined isotropic half-spaces become:

$$
\begin{aligned}
u^{\mathrm{I}}_i(\mathbf{x})=&-\frac{(1+\nu^{\mathrm{I}})}{4\pi(1-\nu^{\mathrm{I}})}\\
&\times\Bigg\{\sum_{t=1}^{N}\big[A_i(\mathbf{x};\xi_t)f^{(t)}(\xi_t^2)+A_{ik}(\mathbf{x};\xi_t)f^{(t)}_k(\xi_t^2)+\cdots\big]\\
&-\sum_{t=1}^{N}\big[A_i(\mathbf{x};\xi_{t-1})f^{(t)}(\xi_{t-1}^2)+A_{ik}(\mathbf{x};\xi_{t-1})f^{(t)}_k(\xi_{t-1}^2)+\cdots\big]\\
&-\sum_{t=1}^{N}\int_{\xi_{t-1}}^{\xi_t}\big[A_i(\mathbf{x};\xi)\,\mathrm{d}f^{(t)}(\xi^2)+A_{ik}(\mathbf{x};\xi)\,\mathrm{d}f^{(t)}_k(\xi^2)+\cdots\big]\Bigg\}
\end{aligned}
\tag{2.17}
$$

and

$$
\begin{aligned}
u^{\mathrm{II}}_i(\mathbf{x})=&-\frac{(1+\nu^{\mathrm{I}})\mu^{\mathrm{I}}}{\pi(\mu^{\mathrm{I}}+\kappa\mu^{\mathrm{II}})}\\
&\times\Bigg\{\sum_{t=1}^{N}\big[f^{(t)}(\xi_t^2)\phi_{,i}(\bar{\mathbf{x}};\xi_t)+f^{(t)}_k(\xi_t^2)\phi_{k,i}(\bar{\mathbf{x}};\xi_t)+\cdots\big]\\
&-\sum_{t=1}^{N}\big[f^{(t)}(\xi_{t-1}^2)\phi_{,i}(\bar{\mathbf{x}};\xi_{t-1})+f^{(t)}_k(\xi_{t-1}^2)\phi_{k,i}(\bar{\mathbf{x}};\xi_{t-1})+\cdots\big]\\
&-\sum_{t=1}^{N}\frac{\partial}{\partial x_i}\int_{\xi_{t-1}}^{\xi_t}\big[\phi(\bar{\mathbf{x}};\xi)\,\mathrm{d}f^{(t)}(\xi^2)+\phi_k(\bar{\mathbf{x}};\xi)\,\mathrm{d}f^{(t)}_k(\xi^2)+\cdots\big]\Bigg\}.
\end{aligned}
\tag{2.18}
$$

Proof The proof of this theorem readily follows Theorem 1 and principle of superposition. □

3 Closed-Form Expressions for the Tensors $A_{ikl\ldots m}$ Associated with the Interior Points of Spherical and Cylindrical Inclusions

In the next two sections the exact closed-form solutions of some inclusion problems, for which the distribution of eigenstrains inside the inclusion is described by a piecewise continuous function are addressed. To this end, the closed-form expressions for the corresponding tensors $A_{ikl\ldots m}(\mathbf{x};\xi)$ are required. In order to evaluate its components for the interior and exterior points of $\Omega(\xi)$, following Ferres [23] and Dyson [24], the potential functions $\phi_{kl\ldots m}(\mathbf{x};\xi)$ may be linked to elliptic integrals. Subsequently, for the interior points of spherical and cylindrical inclusions the closed form expressions for these tensors are obtained. Whereas for the exterior points is not possible to derive such brief expressions. In this regard, for the interior points of the spherical inclusion with radius a only the following components of the first and second order tensors are needed:

$$
\begin{aligned}
A_i(\mathbf{x};\xi) &= \frac{1+\nu^{\mathrm{I}}}{3\tilde{r}^5(1-\nu^{\mathrm{I}})}\left[\frac{(\mu^{\mathrm{I}}-\mu^{\mathrm{II}})}{(\mu^{\mathrm{I}}+\kappa\mu^{\mathrm{II}})}a^3\xi^3\tilde{x}_i(\kappa\tilde{r}^2-6x_3\tilde{x}_3)+\bar{x}_i\tilde{r}^5\right], \quad i=1,2,\\
A_3(\mathbf{x};\xi) &= \frac{1+\nu^{\mathrm{I}}}{3\tilde{r}^5(1-\nu^{\mathrm{I}})}\left\{\frac{(\mu^{\mathrm{I}}-\mu^{\mathrm{II}})}{(\mu^{\mathrm{I}}+\kappa\mu^{\mathrm{II}})}a^3\xi^3\left[(2x_3-\kappa\tilde{x}_3)\tilde{r}^2-6x_3\tilde{x}_3^2\right]+\bar{x}_3\tilde{r}^5\right\},\\
A_{11}(\mathbf{x};\xi) &= -\frac{1+\nu^{\mathrm{I}}}{30(1-\nu^{\mathrm{I}})}\left\{(5a^2\xi^2-6\bar{x}_1^2-3\bar{r}^2)\right.\\
&\quad \left.-2\frac{(\mu^{\mathrm{I}}-\mu^{\mathrm{II}})}{(\mu^{\mathrm{I}}+\kappa\mu^{\mathrm{II}})}\frac{a^5\xi^5}{\tilde{r}^7}\left[6x_3\tilde{x}_3(\tilde{r}^2-5\tilde{x}_1^2)-\kappa\tilde{r}^2(\tilde{r}^2-3\tilde{x}_1^2)\right]\right\},\\
A_{21}(\mathbf{x};\xi) &= \frac{1+\nu^{\mathrm{I}}}{5(1-\nu^{\mathrm{I}})}\left[\bar{x}_1\bar{x}_2-\frac{(\mu^{\mathrm{I}}-\mu^{\mathrm{II}})}{(\mu^{\mathrm{I}}+\kappa\mu^{\mathrm{II}})}\frac{a^5\xi^5}{\tilde{r}^7}\tilde{x}_1\tilde{x}_2(10x_3\tilde{x}_3-\kappa\tilde{r}^2)\right],\\
A_{31}(\mathbf{x};\xi) &= \frac{1+\nu^{\mathrm{I}}}{5(1-\nu^{\mathrm{I}})}\left\{\bar{x}_1\bar{x}_3-\frac{(\mu^{\mathrm{I}}-\mu^{\mathrm{II}})}{(\mu^{\mathrm{I}}+\kappa\mu^{\mathrm{II}})}\frac{a^5\xi^5}{\tilde{r}^7}\tilde{x}_1\left[\tilde{r}^2(\kappa\tilde{x}_3-2x_3)+10x_3\tilde{x}_3^2\right]\right\},
\end{aligned}
\tag{3.1}
$$

where $\bar{r}=|\bar{\mathbf{x}}|$, $\tilde{r}=|\tilde{\mathbf{x}}|$. For the interior points of the circular cylindrical inclusion with radius a, the following components of the first and third order tensors would be used:

$$
\begin{aligned}
A_1(\mathbf{x};\xi) &= \frac{1+\nu^{\mathrm{I}}}{2\tilde{r}^4(1-\nu^{\mathrm{I}})}\left[\frac{(\mu^{\mathrm{I}}-\mu^{\mathrm{II}})}{(\mu^{\mathrm{I}}+\kappa\mu^{\mathrm{II}})}a^2\xi^2\tilde{x}_1(4x_3\tilde{x}_3-\kappa\tilde{r}^2)-\bar{x}_1\tilde{r}^4\right],\\
A_3(\mathbf{x};\xi) &= \frac{1+\nu^{\mathrm{I}}}{2\tilde{r}^4(1-\nu^{\mathrm{I}})}\left\{\frac{(\mu^{\mathrm{I}}-\mu^{\mathrm{II}})}{(\mu^{\mathrm{I}}+\kappa\mu^{\mathrm{II}})}a^2\xi^2\left[2x_3(\tilde{x}_3^2-\tilde{x}_1^2)+\kappa\tilde{x}_3\tilde{r}^2\right]-\bar{x}_3\tilde{r}^4\right\},\\
A_{111}(\mathbf{x};\xi) &= -\frac{1+\nu^{\mathrm{I}}}{24(1-\nu^{\mathrm{I}})}\left\{\bar{x}_1(3a^2\xi^2-7\bar{x}_1^2-3\bar{x}_3^2)\right.\\
&\quad +\frac{(\mu^{\mathrm{I}}-\mu^{\mathrm{II}})}{(\mu^{\mathrm{I}}+\kappa\mu^{\mathrm{II}})}\frac{a^4\xi^4}{\tilde{r}^8}\tilde{x}_1\left\{3\kappa\tilde{r}^6-4(3x_3\tilde{x}_3+\kappa a^2\xi^2)\tilde{r}^4\right.\\
&\quad \left.\left.+12x_3\tilde{x}_3a^2\xi^2\tilde{r}^2+a^2\xi^2\left[\kappa(5\tilde{x}_1^4+6\tilde{x}_1^2\tilde{x}_3^2+\tilde{x}_3^4)+12x_3\tilde{x}_3(\tilde{x}_3^2-3\tilde{x}_1^2)\right]\right\}\right\},
\end{aligned}
$$

$$
\begin{aligned}
A_{311}(\mathbf{x};\xi) = -\frac{1+\nu^{\mathrm{I}}}{24(1-\nu^{\mathrm{I}})}\bigg\{ & \bar{x}_3(3a^2\xi^2 + 3\bar{x}_1^2 - \bar{x}_3^2) + \frac{(\mu^{\mathrm{I}}-\mu^{\mathrm{II}})}{(\mu^{\mathrm{I}}+\kappa\mu^{\mathrm{II}})}\frac{a^4\xi^4}{\tilde{r}^8}\big\{3(\kappa\tilde{x}_3 - 3x_3)\tilde{r}^6 \\
& + \big[(3\tilde{x}_1^2 + 15\tilde{x}_3^2 + 4a^2\xi^2)x_3 - 2\kappa\tilde{x}_3a^2\xi^2\big]\tilde{r}^4 - 12x_3\tilde{x}_3^2a^2\xi^2\tilde{r}^2 \\
& + a^2\xi^2\big[\kappa\tilde{x}_3(5\tilde{x}_1^4 + 6\tilde{x}_1^2\tilde{x}_3^2 + \tilde{x}_3^4) + 2x_3(\tilde{x}_3^4 + 20\tilde{x}_1^2\tilde{x}_3^2 - 5\tilde{x}_1^4)\big]\big\}\bigg\}, \\
A_{113}(\mathbf{x};\xi) = -\frac{1+\nu^{\mathrm{I}}}{24(1-\nu^{\mathrm{I}})}\bigg\{ & \bar{x}_3\big[(3a^2\xi^2 - 2\bar{r}^2) - 4\bar{x}_1^2\big] \\
& - \frac{(\mu^{\mathrm{I}}-\mu^{\mathrm{II}})}{(\mu^{\mathrm{I}}+\kappa\mu^{\mathrm{II}})}\frac{a^6\xi^6}{\tilde{r}^8}\big[(6x_3 - \kappa\tilde{x}_3)\tilde{r}^4 + 4\kappa\tilde{x}_3\tilde{x}_1^2\tilde{r}^2 - 48x_3\tilde{x}_1^2\tilde{x}_3^2\big]\bigg\}, \\
A_{313}(\mathbf{x};\xi) = -\frac{1+\nu^{\mathrm{I}}}{24(1-\nu^{\mathrm{I}})}\bigg\{ & \bar{x}_1\big[(3a^2\xi^2 - 2\bar{r}^2) - 4\bar{x}_3^2\big] \\
& + \frac{(\mu^{\mathrm{I}}-\mu^{\mathrm{II}})}{(\mu^{\mathrm{I}}+\kappa\mu^{\mathrm{II}})}\frac{a^6\xi^6}{\tilde{r}^8}\tilde{x}_1\big[\kappa\tilde{r}^4 + 4\tilde{x}_3(6x_3 - \kappa\tilde{x}_3)\tilde{r}^2 - 48x_3\tilde{x}_3^3\big]\bigg\}.
\end{aligned}
\tag{3.2}
$$

4 A Dilatational Eigenstrain Field Over a Single Ellipsoidal Inclusion in One of the Joined Half-Spaces

An ellipsoidal region in a full space undergoing nonuniform dilatational eigenstrain field which is described by Gaussian or exponential distribution has been considered by Sharma and Sharma [25]. Their proposed Gaussian distribution falls within the family of functions described by the first term of the expression (2.16). Sharma and Sharma [25] gave a concise formulation of the stress field for the interior points of a spherical inclusion, but their analysis for the exterior points contains an unfortunate mistake. Shodja and Shokrolahi-Zadeh [13] re-examined the problem as a special case of one of their theorems on prediction of the nature of the disturbance strain due to a broad distribution of piecewise nonuniform eigenstrain field over an ellipsoidal domain, which is embedded in an unbounded medium.

This section extends the formulation to a perfectly bonded bimaterial, where an ellipsoidal region of one of the joined half-spaces contains the piecewise nonuniform eigenstrain field described by the first term of the expression (2.16). The interaction of the ellipsoidal inclusion with the interface gives rise to physically interesting behavior of the interior and exterior elastic fields. For this problem, as in the case of a full space made of the same material, a closed-form solution is not available; however when the inclusion is spherical shape with Gaussian distribution of eigenstrain field, the closed-form solution is possible.

In the context of the present work, suppose that the ellipsoidal region $\Omega(\xi)$ is situated near the interface of a bimaterial in such a way that the Euler's angles $\alpha_1 = \alpha_2 = \alpha_3 = 0$. Moreover, consider

$$
\varepsilon_{ij}^*(\bar{\mathbf{x}}) = \begin{cases} f\big(\sum_{p=1}^{3}\frac{\bar{x}_p^2}{a_p^2}\big)\delta_{ij}, & \bar{\mathbf{x}} \in \Omega, \\ 0, & \bar{\mathbf{x}} \in (\mathbb{R}^3 - \Omega). \end{cases}
\tag{4.1}
$$

Using Theorem 2 along with the eigenstrain field defined by (4.1), the displacement fields (2.17) and (2.18) are expressed as

$$u_i^{\mathrm{I}}(\mathbf{x}) = \left[A_i(\mathbf{x}; 1) f(1) - \int_0^1 A_i(\mathbf{x}; \xi)\, \mathrm{d} f(\xi^2) \right], \tag{4.2}$$

and

$$u_i^{\mathrm{II}}(\mathbf{x}) = -\frac{(1+\nu^{\mathrm{I}})\mu^{\mathrm{I}}}{\pi(\mu^{\mathrm{I}} + \kappa\mu^{\mathrm{II}})} \left[f(1)\phi_{,i}(\bar{\mathbf{x}}; 1) - \frac{\partial}{\partial x_i} \int_0^1 \phi(\bar{\mathbf{x}}; \xi)\, \mathrm{d} f(\xi^2) \right], \tag{4.3}$$

where

$$\phi(\bar{\mathbf{x}}; \xi) = \pi a_1 a_2 a_3 \int_{\bar{\Lambda}}^{\infty} \left(\xi^2 - \sum_{p=1}^{3} \frac{\bar{x}_p^2}{a_p^2 + s} \right) \frac{\mathrm{d}s}{\Delta(s)}, \tag{4.4}$$

in which $\Delta(s) = \sqrt{(a_1^2+s)(a_2^2+s)(a_3^2+s)}$ and $\bar{\Lambda}$ is equal to zero if $\mathbf{x} \in \Omega(\xi)$, otherwise $\bar{\Lambda}$ is the largest positive root of the equation

$$\sum_{p=1}^{3} \frac{\bar{x}_p^2}{a_p^2 + \bar{\Lambda}} = \xi^2. \tag{4.5}$$

After some manipulations, making the change of variable $\xi^2 = t$ and change of the order of integration, the displacement field in media I and II reduce to

$$u_i^{\mathrm{I}}(\mathbf{x}) = -\frac{(1+\nu^{\mathrm{I}})}{4\pi(1-\nu^{\mathrm{I}})} \left[F_{,i}(\bar{\mathbf{x}}) + F_{,\Box i}(\tilde{\mathbf{x}}) \right], \tag{4.6}$$

$$u_i^{\mathrm{II}}(\mathbf{x}) = -\frac{(1+\nu^{\mathrm{I}})\mu^{\mathrm{I}}}{\pi(\mu^{\mathrm{I}} + \kappa\mu^{\mathrm{II}})} F_{,i}(\bar{\mathbf{x}}), \tag{4.7}$$

where

$$\frac{\partial}{\partial \Box} = \frac{(\mu^{\mathrm{I}} - \mu^{\mathrm{II}})}{(\mu^{\mathrm{I}} + \kappa\mu^{\mathrm{II}})} \left[\kappa(1 - 2\delta_{3i}) \frac{\partial}{\partial x_i} + 2x_3 \frac{\partial^2}{\partial x_3 \partial x_i} \right], \tag{4.8}$$

$$F(\bar{\mathbf{x}}) = \pi a_1 a_2 a_3 \int_{s=\bar{\lambda}}^{\infty} \frac{\mathrm{d}s}{\Delta(s)} \int_{t=\bar{h}(s)}^{1} f(t)\, \mathrm{d}t, \tag{4.9}$$

in which

$$\bar{h}(s) = \sum_{p=1}^{3} \frac{\bar{x}_p^2}{a_p^2 + s}. \tag{4.10}$$

The expression for $F(\tilde{\mathbf{x}})$ is similar to that of $F(\bar{\mathbf{x}})$, except that, in (4.9), $\bar{\lambda}$ and $\bar{h}(s)$ must be replaced by $\tilde{\lambda}$ and $\tilde{h}(s)$, respectively. Note that, $\bar{\lambda}$ is equal to zero for the interior points, $\mathbf{x} \in \Omega$ and is equal to the largest positive root of equation $\bar{h}(\bar{\lambda}) = 1$ for the exterior points, $\mathbf{x} \notin \Omega$. Whereas $\tilde{\lambda}$ is the largest positive root of $\tilde{h}(\tilde{\lambda}) = 1$ in the entire domain, $\mathbf{x} \in \mathbb{R}^3$.

In presence of eigenstrain field, Hooke's law reads $\sigma_{ij} = C_{ijkl}(\varepsilon_{kl} - \varepsilon_{kl}^*)$. Accordingly, the stress field in the entire domain due to the proposed dilatational eigenstrain field which is distributed over an ellipsoidal subdomain of medium I is obtained

$$\sigma_{ij}^{\mathrm{I}}(\mathbf{x}) = 2\mu^{\mathrm{I}}\left\{\frac{\nu^{\mathrm{I}}}{1-2\nu^{\mathrm{I}}}\left[\frac{1+\nu^{\mathrm{I}}}{2(1-\nu^{\mathrm{I}})}a_1a_2a_3H^{\mathrm{In}}(\mathbf{x}) - 3f\left(\sum_{p=1}^{3}\frac{\bar{x}_p^2}{a_p^2}\right)\right]\delta_{ij}\right.$$

$$\left. + \frac{1+\nu^{\mathrm{I}}}{2(1-\nu^{\mathrm{I}})}a_1a_2a_3H_{ij}^{\mathrm{In}}(\mathbf{x}) - f\left(\sum_{p=1}^{3}\frac{\bar{x}_p^2}{a_p^2}\right)\delta_{ij}\right\}, \quad \mathbf{x}\in\Omega, \tag{4.11}$$

$$\sigma_{ij}^{\mathrm{I}}(\mathbf{x}) = \frac{\mu^{\mathrm{I}}(1+\nu^{\mathrm{I}})a_1a_2a_3}{(1-\nu^{\mathrm{I}})}\left[\frac{\nu^{\mathrm{I}}}{1-2\nu^{\mathrm{I}}}H^{\mathrm{Ex}}(\mathbf{x})\delta_{ij} + H_{ij}^{\mathrm{Ex}}(\mathbf{x})\right], \quad \mathbf{x}\notin\Omega, \tag{4.12}$$

$$\sigma_{ij}^{\mathrm{II}}(\mathbf{x}) = -\frac{2\mu^{\mathrm{I}}\mu^{\mathrm{II}}(1+\nu^{\mathrm{I}})a_1a_2a_3}{\pi(\mu^{\mathrm{I}}+\kappa\mu^{\mathrm{II}})}\left[\frac{\nu^{\mathrm{II}}}{1-2\nu^{\mathrm{II}}}L^{\mathrm{Ex}}(\bar{\mathbf{x}})\delta_{ij} + L_{ij}^{\mathrm{Ex}}(\bar{\mathbf{x}})\right], \tag{4.13}$$

where σ_{ij}^{I} and $\sigma_{ij}^{\mathrm{II}}$ denote the stress components in media I and II, respectively. In the above expressions, the corresponding functions H^{In}, H^{Ex}, H_{ij}^{In}, H_{ij}^{Ex}, L^{Ex} and L_{ij}^{Ex} are defined in Appendix A.

4.1 Some Limiting Cases

Using the above formulations, the closed-form solution pertinent to a spherical inclusion in medium I of two jointed half-spaces where the eigenstrain distribution is Gaussian can be obtained. The Gaussian type eigenstrain distribution is defined as

$$\varepsilon_{ij}^{*}(\bar{\mathbf{x}}) = \begin{cases}\exp\left(-\frac{k^2}{a^2}\sum_{p=1}^{3}\bar{x}_p^2\right)\delta_{ij}, & \bar{\mathbf{x}}\in\Omega,\\ 0, & \bar{\mathbf{x}}\in(\mathbb{R}^3-\Omega),\end{cases} \tag{4.14}$$

where k is a constant and a is the radius of the spherical inclusion. Under the above assumptions the functions H^{In}, H_{ij}^{In}, H^{Ex}, H_{ij}^{Ex}, L^{Ex} and L_{ij}^{Ex} have closed-form expressions. This is due to the fact that the integrals involved in the former problem (Appendix A) can be integrated exactly for the latter special case (Appendix B).

Define normalized stress components $\hat{\sigma}_{ij} = [3(1-\nu^{\mathrm{I}})/2(1+\nu^{\mathrm{I}})\mu^{\mathrm{I}}]\sigma_{ij}$ and let $\nu^{\mathrm{I}} = \nu^{\mathrm{II}} = 0.3$, $h/a = 1.5$ and $k = 1$. Figures 2a and 2b show the variations of, respectively, $\hat{\sigma}_{11}$ and $\hat{\sigma}_{33}$ along the x_3-axis for several ratios of shear moduli, $\mu^{\mathrm{I}}/\mu^{\mathrm{II}} = 0.1$, 1, 10 and ∞. Note that the limiting cases $\mu^{\mathrm{I}}/\mu^{\mathrm{II}} = 1$ and ∞ correspond to the problem of spherical inclusion in infinite and semi-infinite media, respectively. Interestingly, the results pertinent to the case of full-space are in exact agreement with the results presented by Shodja and Shokrolahi-Zadeh [13] for the dilatational Gaussian eigenstrain. As it is expected, the stress component $\hat{\sigma}_{33}$ is continuous across the inclusion-matrix interface as well as the interface between media I and II while the stress component $\hat{\sigma}_{11}$ has jumps at those interfaces. Note that $\hat{\sigma}_{33}$ is compressive everywhere, while $\hat{\sigma}_{11}$ is compressive for the interior points and abruptly changes to tension at the inclusion-matrix interface. Away from the inclusion the stress components decay to satisfy the zero traction boundary conditions at infinity. For the case of half-space $\mu^{\mathrm{I}}/\mu^{\mathrm{II}} = \infty$ the stress $\hat{\sigma}_{33}$ satisfies the traction free boundary condition on $x_3 = 0$ plane; whereas $\hat{\sigma}_{11}$ takes on its maximum value on this plane.

To this end it is noteworthy that some other special cases like the problem of spherical inclusion in half-space with uniform eigenstrains ($k = 0$) considered by Mindlin and Cheng [26] and Seo and Mura [27] has been verified.

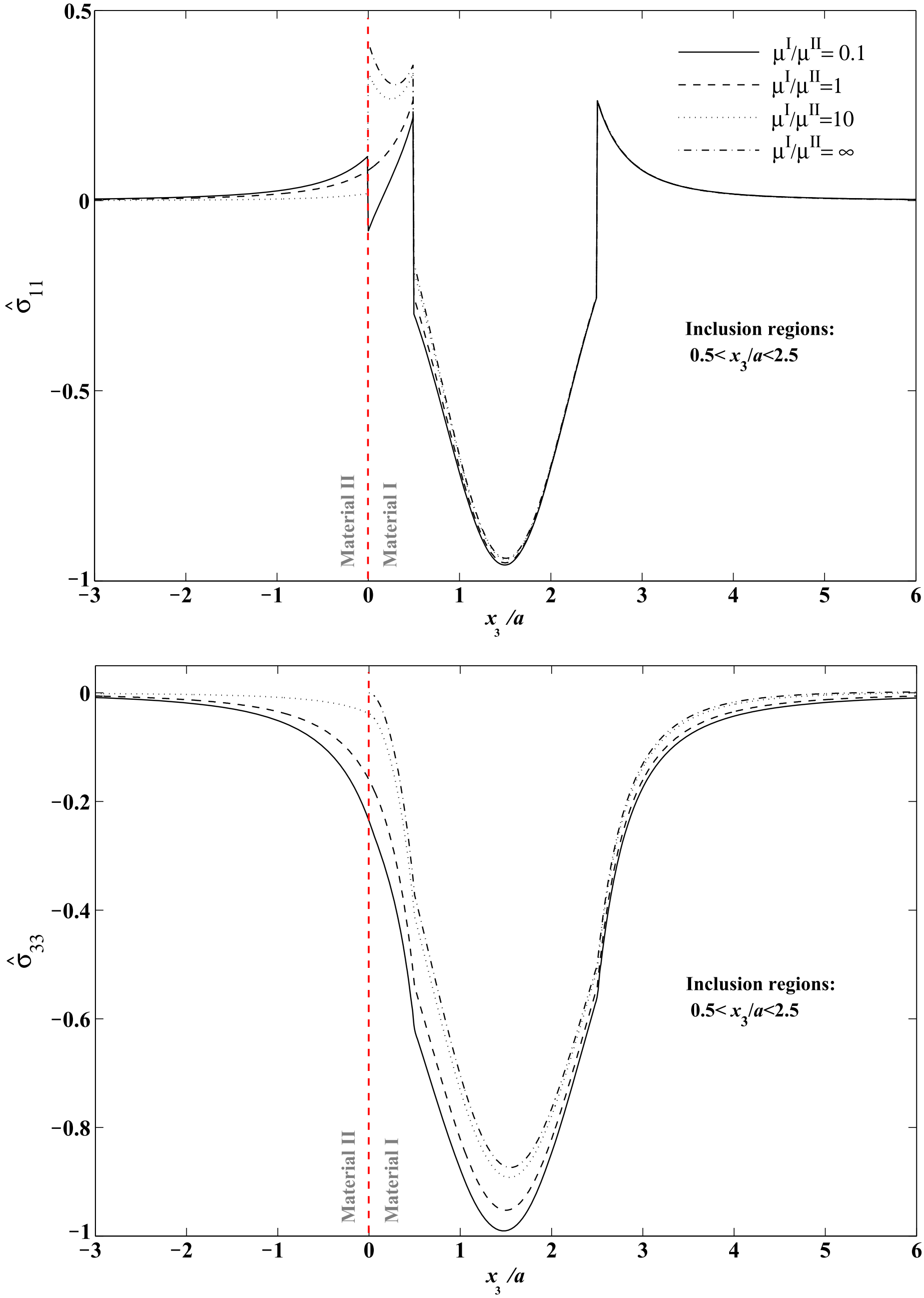

Fig. 2 Stress distribution along the x_3-axis for dilatational Gaussian distribution of eigenstrain. (**a**) The normalized stress $\hat{\sigma}_{11}$; (**b**) The normalized stress $\hat{\sigma}_{33}$

4.2 Energy Consideration

Let us now examine the elastic strain energy stored in bimaterial body D which contains a spherical inclusion Ω (in medium I) with Gaussian eigenstrain field (4.14). Using the divergence theorem as well as the traction free condition at the remote boundary and equilibrium condition, the total (elastic) strain energy of the bimaterial becomes:

$$W = -\frac{1}{2}\int_{\Omega} \sigma^{\mathrm{I}}_{ii}\varepsilon^{*}_{ii}(\bar{\mathbf{x}})\,\mathrm{d}\bar{\mathbf{x}} = W^{\mathrm{Inf}} + W^{\mathrm{Bimat}}, \tag{4.15}$$

where the term W^{Inf} is the elastic strain energy for the case of an infinite space occupied by medium I only. The second term W^{Bimat} is the correction needed to account for the replacement of medium II with medium I in the bimaterial problem of interest. After some manipulations, it may be shown that W^{Inf} and W^{Bimat} can be expressed by

$$\begin{aligned} W^{\mathrm{Inf}} = \frac{4\pi}{k^3}\Bigg\{ &\frac{9(1+\nu^{\mathrm{I}})a^3\mu^{\mathrm{I}}}{16(1-2\nu^{\mathrm{I}})}\left\{\sqrt{2}\left[\sqrt{\pi}-\gamma(0.5;2k^2)\right]-4ke^{-2k^2}\right\} \\ &+\int_0^a \frac{(1+\nu^{\mathrm{I}})\Xi}{\bar{r}}\left[3\gamma(1.5;k^2\bar{r}^2/a^2)-2\gamma(2.5;k^2\bar{r}^2/a^2)\right]\mathrm{d}\bar{r}\Bigg\}, \end{aligned} \tag{4.16}$$

$$\begin{aligned} W^{\mathrm{Bimat}} = 2\pi \int_0^a \int_{-\sqrt{a^2-\rho^2}}^{\sqrt{a^2-\rho^2}} &\frac{(\mu^{\mathrm{I}}-\mu^{\mathrm{II}})a\,\Xi}{(a^2\tilde{r}^2+\tilde{r}^2-a^2)(\mu^{\mathrm{I}}+\kappa\mu^{\mathrm{II}})\tilde{r}^5} \\ &\times\big\{6\nu^{\mathrm{I}}a^2\tilde{r}^2(a^2-a^2\tilde{r}^2-\tilde{r}^2)\gamma(2.5;k^2)/k + a^2\tilde{r}^5(a^2\tilde{r}^2+\tilde{r}^2-a^2) \\ &\times\big\{4x_3\tilde{x}_3(1+\nu^{\mathrm{I}})[2\gamma(3.5;k^2)-5\gamma(2.5;k^2)]/k^3\tilde{r}^5 + (1+7\nu^{\mathrm{I}}) \\ &\times\gamma(1.5;k^2)/k^2\tilde{r}^3 - 2(1-2\nu^{\mathrm{I}})(\rho^2-\tilde{x}^2)\gamma(2.5;k^2)/k^2\tilde{r}^5\big\} \\ &-2e^{-k^2}\big\{a^2\kappa(1-2\nu^{\mathrm{I}})(\rho^2-\tilde{x}^2)\tilde{r}^4 + \big[10(1+\nu^{\mathrm{I}})(1+a^2)x_3\tilde{x}_3 \\ &-3\nu^{\mathrm{I}}\kappa(a^2-2)(a^2\tilde{r}^2+\tilde{r}^2-a^2)\big]\tilde{r}^2 - 2(1+\nu^{\mathrm{I}})e^{-k^2}a^2x_3\tilde{x}_3 \\ &\times[5a^2+(5+4k^2)(a^2\tilde{r}^2+\tilde{r}^2-a^2)]\big\}\big\}\rho\,\mathrm{d}\bar{x}_3\,\mathrm{d}\rho \end{aligned} \tag{4.17}$$

where $\rho=\sqrt{x_1^2+x_2^2}$, $\bar{r}$ and $\tilde{r}$ have been defined in Sect. 3, $\Xi=-\frac{3\mu^{\mathrm{I}}(1+\nu^{\mathrm{I}})}{2(1-\nu^{\mathrm{I}})(1-2\nu^{\mathrm{I}})}e^{-k^2\bar{r}^2/a^2}$, and the function $\gamma(x;n)$ is the incomplete gamma function of x of order n.

The expression for W^{Inf} and W^{Bimat} corresponding to the limiting case where the eigenstrain field is uniform ($k=0$) are in exact agreement with those given by Walpole [21]. Sharma and Sharma [25] considered the strain energy expression for a dilatational Gaussian distribution of eigenstrain within a single spherical domain which is embedded in an infinite body. In their work, the expression for W^{Inf} is not fully correct and has some evident mistypes; the correct form of W^{Inf} can be directly obtained from (4.16).

For $\nu^{\mathrm{I}}=\nu^{\mathrm{II}}=0.3$, $k=1$ and selected stiffness ratios $\mu^{\mathrm{I}}/\mu^{\mathrm{II}}=0$, 0.5, 1, 2 and ∞, the effect of relative depth of the inclusion center from the interface on the normalized total strain energy ($\hat{W}=W/W^{\mathrm{Inf}}$) can be observed in Fig. 3. Two extreme limits of the stiffness ratios, i.e., $\mu^{\mathrm{I}}/\mu^{\mathrm{II}}=0$ and ∞ correspond to the rigid-elastic bimaterial and semi-infinite elastic solid, respectively. As expected for the case of a full space, the normalized strain energy $\hat{W}=1$ for all values of h/a. For the cases where medium II is stiffer and softer than medium I, $\hat{W}$ takes on, respectively, its maximum and minimum values when the spherical

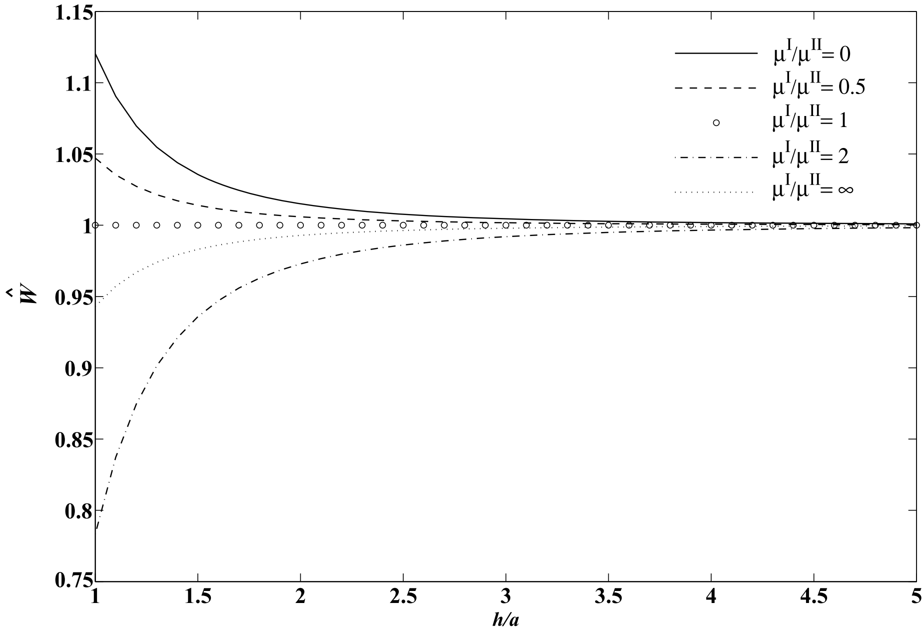

Fig. 3 The normalized strain energy $\hat{W}$ as function of the depth parameter h/a

inclusion becomes tangent to the interface. Moreover, in all cases, as h/a becomes larger $\hat{W}$ approaches the value it has in the case of the full space.

5 Examples Pertinent to Multiple Spherical/Cylindrical Inclusions

In this section, further illustrations of applicability of Theorem 2 are provided through consideration of two examples involving piecewise nonuniform eigenstrain distributions. The closed-form solutions will be obtained with the aid of the tensors $A_{ikl...m}$ discussed in Sect. 3.

The examples of Sects. 5.1 and 5.2 are, respectively, pertinent to two concentric spheres with radii a and qa and three concentric cylinders with radii a, q_1a, and q_2a. For the sake of brevity, only certain components of the stress field are given in the following subsections. Also, in the plots given in this section the Poisson's ratio of the materials I and II are taken as $\nu^{\mathrm{I}} = \nu^{\mathrm{II}} = 0.3$. In the expressions of the eigenstrain field distribution two constant parameters, ε_1 and ε_2 are introduced for the sake of tracking the contribution of the corresponding terms of eigenstrain field to the stress field. However, in the plots $\varepsilon_1 = \varepsilon_2 = 1$ is used for simplicity.

5.1 Two Concentric Spheres

Consider two concentric spherical regions Ω_1 and Ω_2 in medium I of the joined half-spaces. Suppose the eigenstrain field over those regions is prescribed as

$$\varepsilon^*_{ij}(\bar{\mathbf{x}}) = \begin{cases} \varepsilon_1 P_1(\bar{r}/a)\delta_{ij}, & \bar{\mathbf{x}} \in \Gamma_1, \\ \varepsilon_2 P_2(\bar{r}/a)\bar{x}_1/a\delta_{ij}, & \bar{\mathbf{x}} \in \Gamma_2, \\ 0, & \text{otherwise}, \end{cases} \tag{5.1}$$

where $P_n(\bar{r})$ is the Legendre polynomial of order n. For this problem, among all $A_{ikl\dots m}$ for the interior points, only closed-form expressions of the tensors A_i and A_{ij} are needed. By employing Theorem 2, the displacement field and subsequently the stress field are determined. For brevity, only the normal component of the stress field perpendicular to the interface is given here:

$$\begin{aligned}
\sigma_{33}^{\mathrm{I}} = {} & \frac{2\mu^{\mathrm{I}}}{(1-2\nu^{\mathrm{I}})\tilde{r}^7}\bigg\{\varepsilon_1\bigg\{-\frac{(\mu^{\mathrm{I}}-\mu^{\mathrm{II}})(1+\nu^{\mathrm{I}})a^3}{(\mu^{\mathrm{I}}+\kappa\mu^{\mathrm{II}})(1-\nu^{\mathrm{I}})} \\
& \times \big\{\kappa\tilde{r}^4 + \big\{[4(1-\nu^{\mathrm{I}})-3\kappa]\tilde{x}_3^2 - 2(1-\nu^{\mathrm{I}})\tilde{\rho}^2 + 4(2+\nu^{\mathrm{I}})x_3\tilde{x}_3\big\}\tilde{r}^2 \\
& - 10\big[2(1-\nu^{\mathrm{I}})\tilde{x}_3^2 - (1-4\nu^{\mathrm{I}})\tilde{\rho}^2\big]x_3\tilde{x}_3\big\} - (1+\nu^{\mathrm{I}})\xi\tilde{r}^7\bigg\} \\
& - \frac{\varepsilon_2(\mu^{\mathrm{I}}-\mu^{\mathrm{II}})(1+\nu^{\mathrm{I}})a^4\tilde{x}_1}{70(\mu^{\mathrm{I}}+\kappa\mu^{\mathrm{II}})(1-\nu^{\mathrm{I}})\tilde{r}^2} \\
& \times \big\{7\kappa Q_2\tilde{r}^4 + \big\{\big\{24[\kappa-2(1-\nu^{\mathrm{I}})]Q_1 + 7[8(1-\nu^{\mathrm{I}})-5\kappa]Q_2\big\}\tilde{x}_3^2 \\
& + \big\{6[2(1-\nu^{\mathrm{I}})-\kappa]Q_1 - 14Q_2(1-\nu^{\mathrm{I}})\big\}\tilde{\rho}^2 + 14Q_2(8+5\nu^{\mathrm{I}})x_3\tilde{x}_3\big\}\tilde{r}^2 \\
& - 2\big\{[90(1-2\nu^{\mathrm{I}})Q_1 + 49(5\nu^{\mathrm{I}}-1)Q_2]\tilde{\rho}^2 \\
& + [49(4-5\nu^{\mathrm{I}})Q_2 - 120(1-2\nu^{\mathrm{I}})Q_1]\tilde{x}_3^2\big\}x_3\tilde{x}_3\big\}\bigg\}, \\
& \mathbf{x}\in\Gamma_1,
\end{aligned} \tag{5.2}$$

$$\begin{aligned}
\sigma_{33}^{\mathrm{I}} = {} & \frac{\mu^{\mathrm{I}}}{(1-2\nu^{\mathrm{I}})a^3\bar{r}^5\tilde{r}^7}\bigg\{-\frac{a^3(1+\nu^{\mathrm{I}})\varepsilon_1}{2(1-\nu^{\mathrm{I}})}\bigg\{\frac{(\mu^{\mathrm{I}}-\mu^{\mathrm{II}})}{(\mu^{\mathrm{I}}+\kappa\mu^{\mathrm{II}})} \\
& \times \big\{\kappa\tilde{r}^4 + \big\{\big[4(1-\nu^{\mathrm{I}})-3\kappa\big]\tilde{x}_3^2 - 2(1-\nu^{\mathrm{I}})\tilde{\rho}^2 + 4x_3\tilde{x}_3(2+\nu^{\mathrm{I}})\big\}\tilde{r}^2 \\
& - 10x_3\tilde{x}_3\big[(4\nu^{\mathrm{I}}-1)\tilde{\rho}^2 + 2(1-\nu^{\mathrm{I}})\tilde{x}_3^2\big]\big\}\bar{r}^5 - \big\{\big[\bar{r}^2(1+\nu^{\mathrm{I}}) - 3\nu^{\mathrm{I}}\bar{\rho}^2 - 3(1-\nu^{\mathrm{I}})\bar{x}_3^2\big]\tilde{r}^2 \\
& + 4\big[\nu^{\mathrm{I}}(\bar{x}_1\tilde{x}_1 + \bar{x}_2\tilde{x}_2 - \bar{x}_3\tilde{x}_3) + \bar{x}_3\tilde{x}_3\big]\bar{r}^2\big\}\tilde{r}^5\bigg\} + \frac{\varepsilon_2}{105a^3\bar{r}^2\tilde{r}^2}\bigg\{\frac{(1+\nu^{\mathrm{I}})(\mu^{\mathrm{I}}-\mu^{\mathrm{II}})a^7\tilde{x}_1}{(1-\nu^{\mathrm{I}})(\mu^{\mathrm{I}}+\kappa\mu^{\mathrm{II}})} \\
& \times \big\{\kappa\big[98\nu^{\mathrm{I}}q^5(1-3q^2) + 7(14\nu^{\mathrm{I}}-3)Q_2 + 4\nu^{\mathrm{I}}(9Q_1+49)\big]\tilde{r}^4 \\
& + \big\{\big\{42Q_2(1-\nu^{\mathrm{I}}) + 18Q_1\big[\kappa(1-2\nu^{\mathrm{I}}) - 2(1-\nu^{\mathrm{I}})\big]\big\}\tilde{\rho}^2 \\
& + 3\big\{12Q_1\big[4(1-\nu^{\mathrm{I}}) - (2+\nu^{\mathrm{I}})\kappa\big] + 14Q_2\big[4(\nu^{\mathrm{I}}-1)-\kappa\big] \\
& + 49\big[q^5(3q^2-1)-2\big]\kappa\big\}\tilde{x}_3^2 - 42Q_2(8+5\nu^{\mathrm{I}})x_3\tilde{x}_3\big\}\tilde{r}^2 \\
& + 6\big\{\big\{90Q_1(1-2\nu^{\mathrm{I}}) - 49(5\nu^{\mathrm{I}}-1)\big[2-q^5(3q^2-1)\big]\big\}\tilde{\rho}^2 \\
& - \big\{120(1-2\nu^{\mathrm{I}})Q_1 + 49(4-5\nu^{\mathrm{I}})\big[2-q^5(3q^2-1)\big]\big\}\tilde{x}_3^2\big\}x_3\tilde{x}_3\big\}\bar{r}^7 \\
& + \bar{x}_1\bigg\{\frac{(1+\nu^{\mathrm{I}})}{(1-\nu^{\mathrm{I}})}\big\{3\big[15(1+5\nu^{\mathrm{I}})\bar{\rho}^2 + 15(3+\nu^{\mathrm{I}})\bar{x}_3^2 - 7a^2(1+3\nu^{\mathrm{I}})\big]\bar{r}^7 \\
& + 14a^7(3+4\nu^{\mathrm{I}})\bar{r}^2 + 2a^7\big[(9+52\nu^{\mathrm{I}})\bar{\rho}^2 + (69-68\nu^{\mathrm{I}})\bar{x}_3^2\big]\big\} \\
& - 105a^2(1+\nu^{\mathrm{I}})(3\xi^2-1)\bar{r}^7\bigg\}\tilde{r}^9\bigg\}\bigg\}, \\
& \mathbf{x}\in\Gamma_2.
\end{aligned} \tag{5.3}$$

In medium I but outside of spherical region Ω_2

$$
\begin{aligned}
\sigma_{33}^{\mathrm{I}} = & -\frac{\mu^{\mathrm{I}}(1+\nu^{\mathrm{I}})a^3}{2(1-2\nu^{\mathrm{I}})(1-\nu^{\mathrm{I}})\bar{r}^5\tilde{r}^7} \\
& \times \left\{\varepsilon_1\left\{\frac{(\mu^{\mathrm{I}}-\mu^{\mathrm{II}})}{(\mu^{\mathrm{I}}+\kappa\mu^{\mathrm{II}})}\left\{\kappa\tilde{r}^4-\left\{2(1-\nu^{\mathrm{I}})\tilde{\rho}^2+\left[3\kappa-4(1-\nu^{\mathrm{I}})\right]\tilde{x}_3^2-4(2+\nu^{\mathrm{I}})x_3\tilde{x}_3\right\}\tilde{r}^2\right.\right.\right. \\
& \left.\left.+10\left[(1-4\nu^{\mathrm{I}})\tilde{\rho}^2-2(1-\nu^{\mathrm{I}})\tilde{x}_3^2\right]x_3\tilde{x}_3\right\}\bar{r}^5-\left[(1+\nu^{\mathrm{I}})\bar{r}^2-3\nu^{\mathrm{I}}\bar{\rho}^2-3(1-\nu^{\mathrm{I}})\bar{x}_3^2\right]\tilde{r}^7\right\} \\
& -\frac{2a\varepsilon_2}{105\bar{r}^2\tilde{r}^2}\left\{\frac{(\mu^{\mathrm{I}}-\mu^{\mathrm{II}})\tilde{x}_1}{(\mu^{\mathrm{I}}+\kappa\mu^{\mathrm{II}})}\left\{3\kappa(6Q_1-7Q_2)\tilde{r}^4\right.\right. \\
& -3\left\{2(1-\nu^{\mathrm{I}})(6Q_1-7Q_2)\tilde{\rho}^2+(6Q_1-7Q_2)\left[5\kappa-8(1-\nu^{\mathrm{I}})\right]\tilde{x}_3^2\right. \\
& \left.+14Q_2(5+8\nu^{\mathrm{I}})x_3\tilde{x}_3\right\}\tilde{r}^2 \\
& +6\left\{\left[90(1-2\nu^{\mathrm{I}})Q_1+49(5\nu^{\mathrm{I}}-1)Q_2\right]\tilde{\rho}^2\right. \\
& \left.\left.-\left[120(1-2\nu^{\mathrm{I}})Q_1-49(4-5\nu^{\mathrm{I}})Q_2\right]\tilde{x}_3^2\right\}x_3\tilde{x}_3\right\}\bar{r}^7 \\
& +\bar{x}_1\left\{7(3+4\nu^{\mathrm{I}})Q_2\bar{r}^2-2\left[35Q_2\nu^{\mathrm{I}}+9Q_1(1-2\nu^{\mathrm{I}})\right]\bar{\rho}^2\right. \\
& \left.\left.\left.+\left[72Q_1(1-2\nu^{\mathrm{I}})-35(3-4\nu^{\mathrm{I}})Q_2\right]\bar{x}_3^2\right\}\tilde{r}^9\right\}\right\},
\end{aligned}
\tag{5.4}
$$

and for medium II

$$
\begin{aligned}
\sigma_{33}^{\mathrm{II}} = & \frac{2(1+\nu^{\mathrm{I}})\mu^{\mathrm{I}}\mu^{\mathrm{II}}a^3}{(1-2\nu^{\mathrm{II}})(\mu^{\mathrm{I}}+\kappa\mu^{\mathrm{II}})\bar{r}^5}\left\{\varepsilon_1\left[(1+\nu^{\mathrm{II}})\bar{r}^2-3\nu^{\mathrm{II}}\bar{\rho}^2-3(1-\nu^{\mathrm{II}})\bar{x}_3^2\right]+\frac{2a\varepsilon_2\bar{x}_1}{35\bar{r}^2}\right. \\
& \times\left\{7(1+3\nu^{\mathrm{II}})Q_2\bar{r}^2-\left\{\left[6(1-2\nu^{\mathrm{II}})Q_1+35\nu^{\mathrm{II}}Q_2\right]\bar{\rho}^2\right.\right. \\
& \left.\left.\left.-\left[24(1-2\nu^{\mathrm{II}})Q_1-35(1-\nu^{\mathrm{II}})Q_2\right]\bar{x}_3^2\right\}\right\}\right\},
\end{aligned}
\tag{5.5}
$$

where $\bar{\rho}^2=\bar{x}_1^2+\bar{x}_2^2$, $\tilde{\rho}^2=\tilde{x}_1^2+\tilde{x}_2^2$ and

$$
Q_1=(q-1)\sum_{k=1}^{6}q^k,
$$

$$
Q_2=(q-1)\left[3(q^6+q^5)+2\sum_{k=1}^{6}q^k\right].
$$

The following results are obtained for $q=2$. Figure 4 shows the variations of the normalized stress components $\hat{\sigma}_{11}$ and $\hat{\sigma}_{33}$ along the x_3-axis at selected stiffness ratios $\mu^{\mathrm{I}}/\mu^{\mathrm{II}}=0.1$ and 10 when the inclusion makes contact with the interface ($h=2a$). From this figure, it can be seen that the stresses σ_{11}^{I} and σ_{33}^{I} attain their maximum magnitudes at the points $x_3/a=1,3$ on the boundary of the inner sphere. Note that, all the normal components of the eigenstrain field along the segment of x_3-axis which is in Γ_1 equal zero. As a result the $\hat{\sigma}_{11}$ component of stress field remains continuous across the boundary of $x_3/a=4$. Needless to mention, $\hat{\sigma}_{33}$ is continuous along the entire x_3-axis as it ought to be. Also, the results show

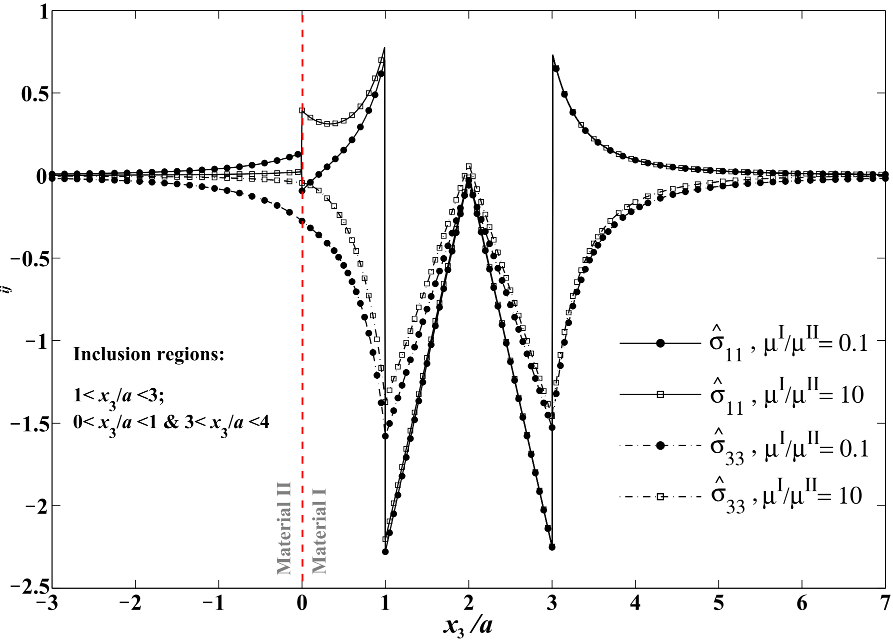

Fig. 4 Stress distribution along the x_3-axis for eigenstrain given by (5.1)

that a spherical region with nonuniform thermal strain in the lower semi-infinite space will not cause any noticeable stress in the upper medium, especially when the lower space is stiffer than the upper one.

The distribution of the normalized stress components $\hat{\sigma}_{11}$, $\hat{\sigma}_{33}$ and $\hat{\sigma}_{13}$ along the x_1-axis under the conditions $\mu^{\mathrm{I}}/\mu^{\mathrm{II}} = 5$ and $h = 3a$ are illustrated in Fig. 5. It is observed that for the eigenstrain distribution described by (5.1) $\hat{\sigma}_{33}$ has the largest magnitude at the $x_1 = \pm 2a$ on the boundary of the outer spherical region as compared with $\hat{\sigma}_{11}$ and $\hat{\sigma}_{13}$. It is interesting to note that the prescribed eigenstrain field is continuous at $x_1 = a$, while it is discontinuous at $x_1 = -a$. Consequently, the normalized stress component $\hat{\sigma}_{33}$ along x_1-axis follows a similar behavior at $x_1 = \pm a$.

The variation of the dimensionless displacement component, u_3^{I}/a over the interface for $h/a = 2.5$ and $\mu^{\mathrm{I}}/\mu^{\mathrm{II}} = 2$ is shown in Fig. 6. This variation resembles a doublet formation on the interface directly above the inclusion. If the sphere is projected on to the interface, the maximum and minimum values of u_3^{I}/a correspond to the boundary between Γ_1 and Γ_2, and $u_3^{\mathrm{I}}/a = 0$ corresponds to the center of the spherical region. Also, by distancing from this place to $|x_1/a| > 4$ or $|x_2/a| > 4$, the displacement magnitude is rapidly vanishing.

5.2 Three Concentric Cylinders

In the second example, the results are specialized for three concentric cylinders. Take $\bar{x}_2$-axis as the axis of cylinder and parallel to the x_2-axis. The following non-unifrom distribution of eigenstrain field is prescribed

$$\varepsilon_{ij}^{*}(\bar{\mathbf{x}}) = \begin{cases} \varepsilon_1 P_1(\bar{\rho}/a)\bar{x}_1^2/a^2\delta_{ij}, & \bar{\mathbf{x}} \in \Gamma_2, \\ \varepsilon_2 \exp(\bar{\rho}/a)\bar{x}_1\bar{x}_3/a^2\delta_{ij}, & \bar{\mathbf{x}} \in \Gamma_3, \\ 0, & \text{otherwise}, \end{cases} \tag{5.6}$$

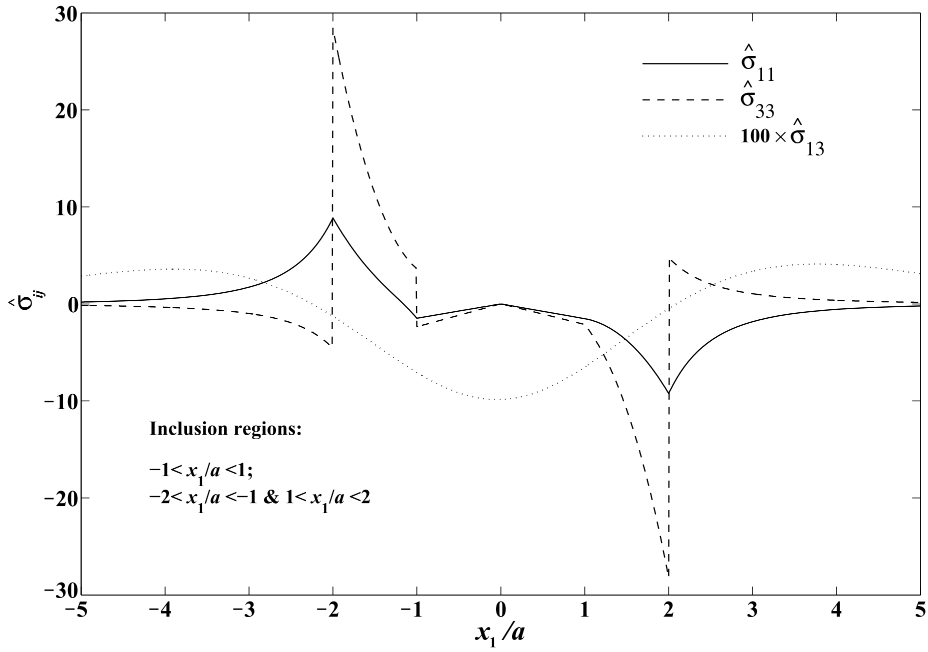

Fig. 5 Stress distribution along the x_1-axis at $x_3 = h$ for eigenstrain given by (5.1)

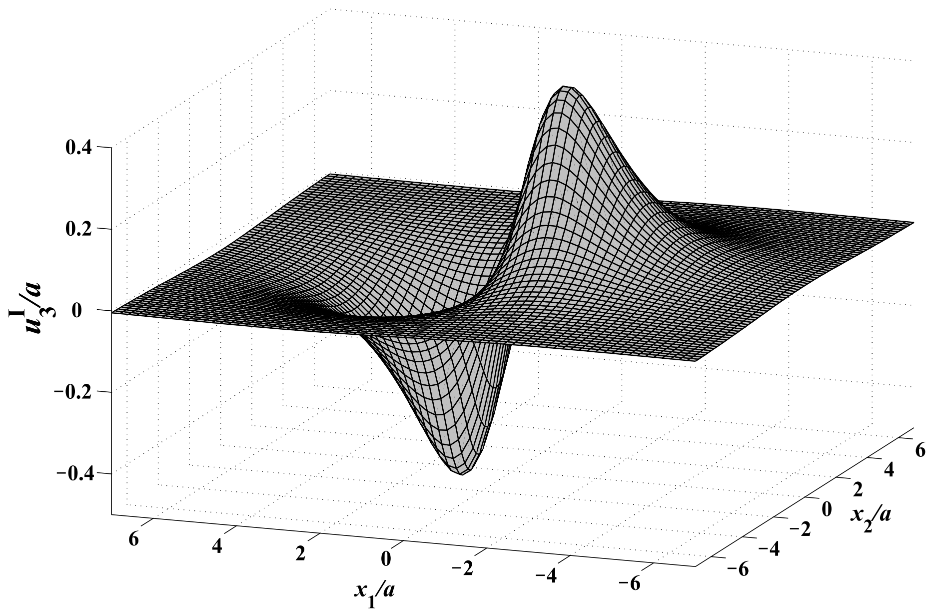

Fig. 6 The normalized displacement component, u_3^{I}/a on the interface as a function of the coordinates x_1/a and x_2/a

where $\exp(\cdot)$ is the exponential function, and $\bar{\rho}^2 = \bar{x}_1^2 + \bar{x}_3^2$. The stress field of this multi-inclusion interacting with the interface can be readily obtained by utilizing Theorem 2 and the closed-form expression of the third order tensors A_{ikl} derived in Sect. 3. For example, the stress component σ_{13}^{I} for the points $\mathbf{x} \in \Gamma_1$ is

$$
\begin{aligned}
\sigma_{13}^{\mathrm{I}} = {} & \frac{\mu^{\mathrm{I}}(1+\nu^{\mathrm{I}})}{12(1-\nu^{\mathrm{I}})a^2\tilde{\rho}^{10}} \left\{ \frac{2(\mu^{\mathrm{I}}-\mu^{\mathrm{II}})\varepsilon_1}{35(\mu^{\mathrm{I}}+\kappa\mu^{\mathrm{II}})} \{ 84a^4 J_4 \tilde{x}_1(\tilde{x}_3+5x_3)\tilde{\rho}^6 \right. \\
& - 21a^4\tilde{x}_1\big[5a^2 J_6(\tilde{x}_3+2x_3) + J_4 x_3(12\tilde{x}_1^2+44\tilde{x}_3^2)\big]\tilde{\rho}^4 \\
& - 5a^6 J_6\tilde{x}_1\big[3\tilde{x}_3(5\tilde{x}_3^2-19\tilde{x}_1^2)+14x_3(3\tilde{x}_3^2+11\tilde{x}_1^2)\big]\tilde{\rho}^2 \\
& + 20a^6 J_6\tilde{x}_1 x_3(67\tilde{x}_1^4+111\tilde{x}_3^4-110\tilde{x}_1^2\tilde{x}_3^2)\} - \varepsilon_2 \left\{ \frac{(\mu^{\mathrm{I}}-\mu^{\mathrm{II}})a^6}{(\mu^{\mathrm{I}}+\kappa\mu^{\mathrm{II}})} \{4(\tilde{x}_3^2-\tilde{x}_1^2) \right. \\
& \times \big\{ S_6\kappa - \big[2(5\tilde{x}_3^4+110\tilde{x}_1^2\tilde{x}_3^2+9\tilde{x}_1^4)S_6 + 36x_3\tilde{x}_3(5\tilde{x}_1^4-10\tilde{x}_1^2\tilde{x}_3^2+\tilde{x}_3^4)S\big]\tilde{\rho}^4 \big\} \\
& + \big[8S_6 x_3\tilde{x}_3(9\tilde{x}_1^2+5\tilde{x}_3^2)+18S(\tilde{x}_1^4-6\tilde{x}_1^2\tilde{x}_3^2+\tilde{x}_3^4)\big]\tilde{\rho}^2 + 768S_6 x_3\tilde{x}_1^2\tilde{x}_3^3 \} \\
& \left.\left. + 6a^2(S_1-S_0)\tilde{\rho}^{10} \right\}\right\},
\end{aligned} \tag{5.7}
$$

such that $\tilde{\rho}^2 = \tilde{x}_1^2 + \tilde{x}_3^2$ and

$$
J_p = (q_1 - 1)\sum_{k=1}^{p} q_1^k,
$$

$$
S_k = (q_1^k \exp(q_1) - q_2^k \exp(q_2)), \quad k = 0, 1, \dots, 6
$$

and

$$
S = S_5 - 5S_4 + 20S_3 - 60S_2 + 120S_1 - 120S_0.
$$

According to Theorem 4 of Shodja and Shokrolahi-Zadeh [13] when the eigenstrain of the form given by (5.6) is considered in an infinite medium, the resulting stresses inside the core Γ_1 must be uniform. This is confirmed by $\mu^{\mathrm{II}} = \mu^{\mathrm{I}}$ in the pertinent formulae obtained from the present theory:

$$
\sigma_{11}^{\mathrm{I}} = \frac{(1+\nu^{\mathrm{I}})(1-q_1^3)\mu^{\mathrm{I}}\varepsilon_1}{6(1-\nu^{\mathrm{I}})}, \quad \mathbf{x} \in \Gamma_1, \tag{5.8}
$$

$$
\sigma_{33}^{\mathrm{I}} = -\frac{(1+\nu^{\mathrm{I}})(1-q_1^3)\mu^{\mathrm{I}}\varepsilon_1}{6(1-\nu^{\mathrm{I}})}, \quad \mathbf{x} \in \Gamma_1, \tag{5.9}
$$

$$
\sigma_{13}^{\mathrm{I}} = -\frac{(1+\nu^{\mathrm{I}})[(q_2-1)\exp(q_2)-(q_1-1)\exp(q_1)]\mu^{\mathrm{I}}\varepsilon_2}{2(1-\nu^{\mathrm{I}})}, \quad \mathbf{x} \in \Gamma_1. \tag{5.10}
$$

Thus, the stress field is uniform within the core region.

In the remainder of this section, let $q_1 = 1.5$ and $q_2 = 2$. Figure 7 illustrates the variation of the calculated stresses $\hat{\sigma}_{11}$ and $\hat{\sigma}_{13}$ along the x_1-axis for the extreme limits of the stiffness ratios $\mu^{\mathrm{I}}/\mu^{\mathrm{II}} = 0$ and ∞. It is observed that, the interaction of multi-inclusion with the interface results in nonuniform distribution of $\hat{\sigma}_{11}$ and $\hat{\sigma}_{33}$ within the core, in which the eigenstrains are zero. Also, it is seen that, except for the core region, the stress distribution

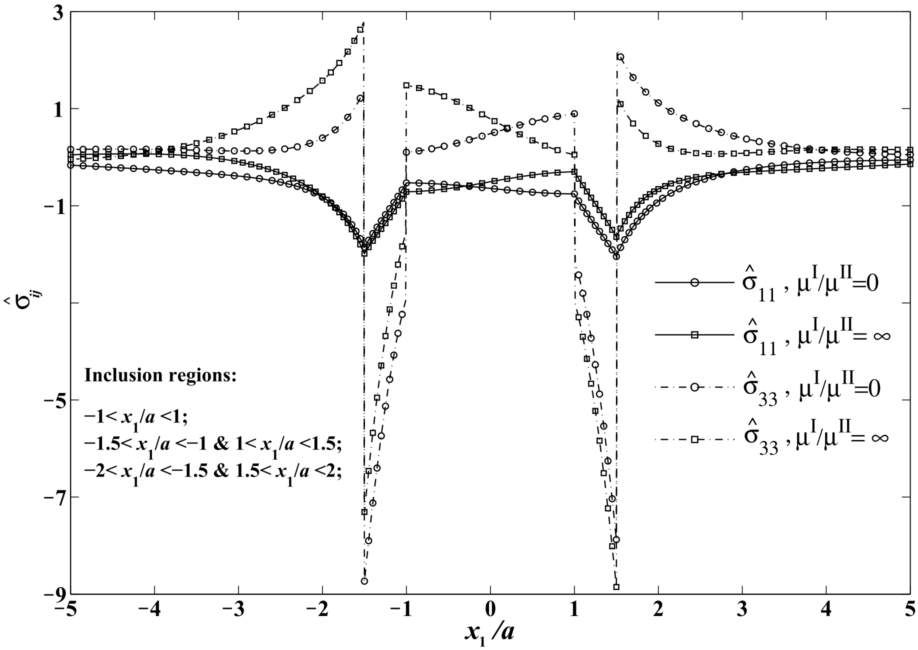

Fig. 7 Stress distribution along the x_1-axis at $x_3 = h$ for eigenstrain given by (5.6)

for the fixed and free surface conditions, $\mu^{\mathrm{I}}/\mu^{\mathrm{II}} = 0$ and ∞,respectively follow the same trend.

For the case of $\mu^{\mathrm{I}}/\mu^{\mathrm{II}} = 1/3$, the influence of the relative depth h/a on the distribution of the stress $\hat{\sigma}_{33}$ within the interface along the x_1-axis is displayed in Fig. 8. The eigenstrain distribution (5.6) yields maximum tensile stress $\hat{\sigma}_{33}$ for $x_1 > 0$ and maximum compressive stress for $x_1 < 0$. This specific distribution of the stress $\hat{\sigma}_{33}$ produces a considerable bending at the interface, especially when the inclusion touches the interface.

6 Summary

The problem of an ellipsoidal domain with a general class of piecewise nonuniform dilatational eigenstrain distribution associated with two joined isotropic half-spaces having a perfect planar interface was proposed. The stated theorems provide a rigorous and exact solution of the pertinent problems. The multi-inclusion problems associated with the half-spaces having fixed or free surface boundary conditions, and full space can be treated as the limiting case. When one of the principal axes of the ellipsoidal inclusion is parallel to the interface an exact closed-form solution was obtained. A special interface-inclusion interaction problem involving a spherical inclusion with Gaussian distribution of eigenstrain field was addressed. A concise note on the calculations of the relevant elastic strain energy was given; followed by some numerical remarks. For further demonstrations of the theorems, in the example sections two multi-inclusion problems involving two concentric spheres and three concentric cylinders have been proposed and solved. In these examples, the closed-form expressions for the stress field were obtained.

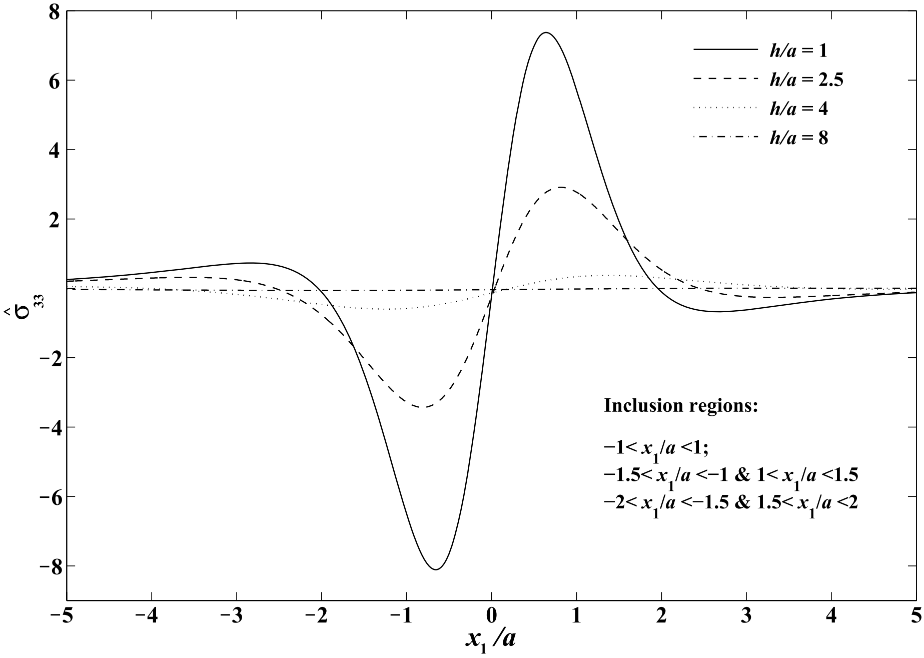

Fig. 8 Stress distribution along the x_1-axis at the interface for eigenstrain given by (5.6)

Finally, it should be emphasized that by finding elastic fields due to dilatational multi-inclusions embedded in bimaterials and prescribed by a more general eigenstrains than polynomial ones, the current paper can offer a wide range of applications in the solid state physics and material science to deal with problems such as transient thermal strains in electronic chips, diffusion–induced type eigenstrains in materials, and so on in which a localized source of misfit strains gives rise to a strongly nonlinear distribution of eigenstrains.

Appendix A

$$H^{\text{In,Ex}}(\mathbf{x}) = L^{\text{In,Ex}}(\bar{\mathbf{x}}) + \frac{(\mu^{\text{I}} - \mu^{\text{II}})}{(\mu^{\text{I}} + \kappa\mu^{\text{II}})}\left[\kappa\left(2\sum_{k=1}^{3}\tilde{x}_k^2 F_{KK}(\tilde{\mathbf{x}}) + \sum_{k=1}^{3} F_k(\tilde{\mathbf{x}})\right)\right.$$
$$\left. + 2x_3 G(\tilde{\mathbf{x}}) + \frac{2\kappa f(1)}{\Delta(\tilde{\lambda})}\left(1 - \frac{2\Psi_2(\tilde{\mathbf{x}};\tilde{\lambda})}{\Psi_3(\tilde{\mathbf{x}};\tilde{\lambda})}\right)\right],$$

$$H_{ij}^{\text{In,Ex}}(\mathbf{x}) = L_{ij}^{\text{In,Ex}}(\bar{\mathbf{x}}) + \frac{(\mu^{\text{I}} - \mu^{\text{II}})}{(\mu^{\text{I}} + \kappa\mu^{\text{II}})}\left[\kappa(2\tilde{x}_i\tilde{x}_j F_{IJ}(\tilde{\mathbf{x}}) + \delta_{ij} F_I(\tilde{\mathbf{x}}))(1 - 2\delta_{3i})\right.$$
$$\left. + 2x_3 G_{ij}(\tilde{\mathbf{x}}) - \frac{2\kappa f(1)(1 - 2\delta_{3i})\tilde{x}_i\tilde{x}_j}{\Delta(\tilde{\lambda})(a_I^2 + \tilde{\lambda})(a_J^2 + \tilde{\lambda})\Psi_3(\tilde{\mathbf{x}};\tilde{\lambda})}\right],$$

$$L^{\text{In}}(\bar{\mathbf{x}}) = 2\sum_{k=1}^{3}\bar{x}_k^2 F_{KK}(\bar{\mathbf{x}}) + \sum_{k=1}^{3} F_k(\bar{\mathbf{x}}),$$

$$L^{\mathrm{Ex}}(\bar{\mathbf{x}}) = 2\sum_{k=1}^{3} \bar{x}_k^2 F_{KK}(\bar{\mathbf{x}}) + \sum_{k=1}^{3} F_k(\bar{\mathbf{x}}) - \frac{2f(1)}{\Delta(\bar{\lambda})},$$

$$L_{ij}^{\mathrm{In}}(\bar{\mathbf{x}}) = 2\bar{x}_i\bar{x}_j F_{IJ}(\bar{\mathbf{x}}) + \delta_{ij} F_I(\bar{\mathbf{x}}),$$

$$L_{ij}^{\mathrm{Ex}}(\bar{\mathbf{x}}) = 2\bar{x}_i\bar{x}_j F_{IJ}(\bar{\mathbf{x}}) + \delta_{ij} F_I(\bar{\mathbf{x}}) - \frac{2\bar{x}_i\bar{x}_j f(1)}{\Delta(\lambda)(a_I^2+\bar{\lambda})(a_J^2+\bar{\lambda})\Psi_3(\bar{\mathbf{x}};\bar{\lambda})},$$

$$\begin{aligned} G_{ij}(\tilde{\mathbf{x}}) = {} & 2[2\tilde{x}_i\tilde{x}_j\tilde{x}_3 F_{ij3}(\tilde{\mathbf{x}}) + F_{i3}(\tilde{\mathbf{x}})(\delta_{3i}\tilde{x}_j + \delta_{3j}\tilde{x}_i + \delta_{ij}\tilde{x}_3)] \\ & - \frac{1}{(a_I^2+\tilde{\lambda})\Delta(\tilde{\lambda})}\left[\frac{2}{(a_J^2+\tilde{\lambda})} f'(1)\tilde{x}_i\tilde{x}_j\tilde{\lambda}_{,3} + f(1)(\delta_{3i}\tilde{\lambda}_{,j} + \delta_{ij}\tilde{\lambda}_{,3})\right] \\ & - \frac{\tilde{x}_i f(1)}{(a_I^2+\tilde{\lambda})\Delta(\tilde{\lambda})}(\tilde{\lambda}_{,3j} - \Lambda_{ij}), \end{aligned}$$

$$\begin{aligned} G(\tilde{\mathbf{x}}) = {} & 2\tilde{x}_3\left\{2\left[\tilde{x}_1^2 F_{113}(\tilde{\mathbf{x}}) + \tilde{x}_2^2 F_{223}(\tilde{\mathbf{x}}) + \tilde{x}_3^2 F_{333}(\tilde{\mathbf{x}})\right] + F_{13}(\tilde{\mathbf{x}}) + F_{23}(\tilde{\mathbf{x}}) + 3F_{33}(\tilde{\mathbf{x}})\right\} \\ & - \frac{\tilde{\lambda}_{,3}}{\Delta(\tilde{\lambda})}\left\{2\Psi_3(\tilde{\mathbf{x}};\tilde{\lambda}) f'(1) + \left[\frac{1}{(a_1^2+\tilde{\lambda})} + \frac{1}{(a_2^2+\tilde{\lambda})} + \frac{2}{(a_3^2+\tilde{\lambda})}\right] f(1)\right\} \\ & - \frac{f(1)}{\Delta(\tilde{\lambda})}\left[\frac{\tilde{x}_1}{(a_1^2+\tilde{\lambda})}(\tilde{\lambda}_{,13} - \Lambda_{11}) + \frac{\tilde{x}_2}{(a_2^2+\tilde{\lambda})}(\tilde{\lambda}_{,23} - \Lambda_{22}) + \frac{\tilde{x}_3}{(a_3^2+\tilde{\lambda})}(\tilde{\lambda}_{,33} - \Lambda_{33})\right], \end{aligned}$$

in which

$$\Psi_k(\bar{\mathbf{x}};\bar{\lambda}) = \sum_{p=1}^{k} \bar{x}_p^2 \Big/ (a_p^2+\bar{\lambda})^2,$$

$$\Lambda_{ij} = \left[\frac{1}{a_i^2+\tilde{\lambda}} + \frac{1}{2}\sum_{p=1}^{3}\frac{1}{a_p^2+\tilde{\lambda}}\right]\tilde{\lambda}_{,3}\tilde{\lambda}_{,j}$$

and the integrals are

$$F_i(\bar{\mathbf{x}}) = \int_{\bar{\lambda}}^{\infty} \frac{1}{(a_i^2+s)\Delta(s)} f\left(\sum_{p=1}^{3}\frac{\bar{x}_p^2}{a_p^2+s}\right)\mathrm{d}s,$$

$$F_{ij}(\bar{\mathbf{x}}) = \int_{\bar{\lambda}}^{\infty} \frac{1}{(a_i^2+s)(a_j^2+s)\Delta(s)} f'\left(\sum_{p=1}^{3}\frac{\bar{x}_p^2}{a_p^2+s}\right)\mathrm{d}s,$$

$$F_{ijk}(\bar{\mathbf{x}}) = \int_{\bar{\lambda}}^{\infty} \frac{1}{(a_i^2+s)(a_j^2+s)(a_k^2+s)\Delta(s)} f''\left(\sum_{p=1}^{3}\frac{\bar{x}_p^2}{a_p^2+s}\right)\mathrm{d}s$$

where

$$f'(\chi) = \frac{\mathrm{d}f(\chi)}{\mathrm{d}\chi}, \qquad f''(\chi) = \frac{\mathrm{d}^2 f(\chi)}{\mathrm{d}\chi^2}.$$

Note that $\Psi_k(\tilde{\mathbf{x}};\tilde{\lambda})$, $F_i(\tilde{\mathbf{x}})$, $F_{ij}(\tilde{\mathbf{x}})$ and $F_{ijk}(\tilde{\mathbf{x}})$ are similar to that of $\Psi_k(\bar{\mathbf{x}};\bar{\lambda})$, $F_i(\bar{\mathbf{x}})$, $F_{ij}(\bar{\mathbf{x}})$ and $F_{ijk}(\bar{\mathbf{x}})$, except that, $\bar{\lambda}$ and $\bar{\mathbf{x}}$ must be replaced by $\tilde{\lambda}$ and $\tilde{\mathbf{x}}$, respectively. Also, the Einstein summation convention is valid only for the repeated lower case indices. The upper case indices have the same values as the lower cases, but are not summed.

Appendix B

The results for stress field are specialized for a spherical inclusion with radius a, for which the closed-form expressions of the corresponding integrals are found to be

$$F_i(\bar{\mathbf{x}}) = \frac{\gamma(1.5; k^2\bar{r}^2/a^2)}{k^3\bar{r}^3},$$

$$F_{ij}(\bar{\mathbf{x}}) = -\frac{\gamma(2.5; k^2\bar{r}^2/a^2)}{k^3\bar{r}^5},$$

$$F_{ijk}(\bar{\mathbf{x}}) = \frac{\gamma(3.5; k^2\bar{r}^2/a^2)}{k^3\bar{r}^7}, \quad \mathbf{x} \in \Omega,$$

and

$$F_i(\bar{\mathbf{x}}) = \frac{\gamma(1.5; k^2)}{k^3\bar{r}^3},$$

$$F_{ij}(\bar{\mathbf{x}}) = -\frac{\gamma(2.5; k^2)}{k^3\bar{r}^5},$$

$$F_{ijk}(\bar{\mathbf{x}}) = \frac{\gamma(3.5; k^2)}{k^3\bar{r}^7}, \quad \mathbf{x} \notin \Omega,$$

where the function, $\gamma(x, n)$ is the incomplete gamma function of x of order n. Also, the functions $F_i(\tilde{\mathbf{x}})$, $F_{ij}(\tilde{\mathbf{x}})$ and $F_{ijk}(\tilde{\mathbf{x}})$ can be effortlessly obtained from corresponding functions by replacing $\bar{r}$ with $\tilde{r}$ in the entire domain, $\mathbf{x} \in \mathbb{R}^3$.

References

1. Eshelby, J.D.: The determination of the elastic field of an ellipsoidal inclusion and related problems. Proc. R. Soc. A **241**, 376–396 (1957)
2. Eshelby, J.D.: The elastic field outside an ellipsoidal inclusion. Proc. R. Soc. A **252**, 561–569 (1959)
3. Eshelby, J.D.: Elastic inclusions and inhomogeneities. In: Sneddon, I.N., Hill, R. (eds.) Progress in Solid Mechanics 2, pp. 89–140. North-Holland, Amsterdam (1961)
4. Mura, T.: Micromechanics of Defects in Solids. Martinus Nijhoff, Hague (1987)
5. Mura, T., Shodja, H.M., Hirose, Y.: Inclusion problems. Appl. Mech. Rev. **49**, 118–127 (1996)
6. Nemat-Nasser, S., Hori, M.: Micromechanics: Overall Properties of Heterogeneous Solids. Elsevier, New York (1999)
7. Sendeckyj, G.P.: Ellipsoidal Inhomogeneity Problem. Ph.D. dissertation, Northwestern University, Evanston, IL (1967)
8. Rahman, M.: The isotropic ellipsoidal inclusion with a polynomial distribution of eigenstrain. J. Appl. Mech. **69**, 593–601 (2002)
9. Moschovidis, Z.A., Mura, T.: Two-ellipsoidal inhomogeneities by the equivalent inclusion method. J. Appl. Mech. **42**, 847–852 (1975)
10. Shodja, H.M., Sarvestani, A.S.: Elastic fields in double inhomogeneity by the equivalent inclusion method. J. Appl. Mech. **68**, 3–10 (2001)
11. Shodja, H.M., Rad, I.Z., Soheilifard, R.: Interacting cracks and ellipsoidal inhomogeneities by the equivalent inclusion method. J. Mech. Phys. Solids **51**, 945–960 (2003)
12. Shodja, H.M., Ojaghnezhad, F.A.: General unified treatment of lamellar inhomogeneities. Eng. Frac. Mech. **74**, 1499–1510 (2007)

13. Shodja, H.M., Shokrolahi-Zadeh, B.: Ellipsoidal domains: piecewise nonuniform and impotent eigenstrain fields. J. Elast. **86**, 1–18 (2007)
14. Shokrolahi-Zadeh, B., Shodja, H.M.: Spectral equivalent inclusion method: anisotropic cylindrical multiinhomogeneities. J. Mech. Phys. Solids **56**, 3565–3573 (2008)
15. Guell, D.L., Dunders, J.: Further results on centres of dilatation and residual stresses in joined elastic half spaces. In: Shaw, W.A. (ed.) Developments in Theoretical and Applied Mechanics, vol. 3, pp. 105–115. Pergamon, Oxford (1966)
16. Yu, H.Y., Sanday, S.C.: Elastic fields in joined half-space due to nuclei of strain. Proc. R. Soc. A **434**, 503–519 (1991)
17. Yu, H.Y., Sanday, S.C.: Elastic fields in joined semi-infinite solids with an inclusion. Proc. R. Soc. A **434**, 520–529 (1991)
18. Yu, H.Y., Sanday, S.C., Rath, B.B., Chang, C.I.: Elastic fields due to defects in transversely isotropic bimaterials. Proc. R. Soc. A **449**, 1–30 (1994)
19. Yu, H.Y., Sanday, S.C., Rath, B.B.: Thermoelastic stresses in bimaterials. Philos. Mag. **65**, 1049–1064 (1992)
20. Wapole, L.J.: An elastic singularity in joined half-spaces. Int. J. Eng. Sci. **34**, 629–638 (1996)
21. Walpole, L.J.: An inclusion in one of two joined isotropic half-spaces. IMA J. Appl. Math. **59**, 193–209 (1997)
22. Rongved, L.: Force interior to one of two jointed semi-infinite solids. In: Proc. 2nd Midwestern Conf. Solid Mech., pp. 1–13 (1955)
23. Ferres, N.M.: On the potentials of ellipsoids, ellipsoidal shells, elliptic laminae and elliptic rings of variable densities. Q. J. Pure Appl. Math. **14**, 1–22 (1877)
24. Dyson, F.W.: The potentials of ellipsoids with variable density. Q. J. Pure Appl. Math. **XXV**, 259–288 (1891)
25. Sharma, P., Sharma, R.: On the Eshelby's inclusion problem for ellipsoids with nonuniform dilatational Gaussian and exponential eigenstrains. J. Appl. Mech. **70**, 418–425 (2003)
26. Mindlin, R.D.: Force at a point in the interior of a semi-infinite solid. In: Proceedings of First Midwestern Conference on Solid Mechanics, pp. 55–59 (1953)
27. Seo, K., Mura, T.: The elastic fields in a half-space due to an ellipsoidal inclusion with uniform dilatational eigenstrains. J. Appl. Mech. **46**, 568–572 (1979)