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Ellipsoidal Unfalsified Control: Stability

Jeroen van Helvoort, Bram de Jager and Maarten Steinbuch

Abstract—Unfalsified Control is a direct data-driven, plant-model-free controller design method, which recursively falsifies controllers that fail to meet the required performance specification, making them ineligible to actually control the plant. In this paper it is shown that sufficient conditions for stability can be derived for Unfalsified Control with an ellipsoidal Unfalsified set, Ellipsoidal Unfalsified Control (EUC), under the mild assumption that there exists at least some region in the original candidate controller pool, which contains controllers that meet the performance specifications. One of these conditions is a finite number of controller switches, which is guaranteed by imposing a maximum volume ratio between two consecutive ellipsoidal Unfalsified sets.

I. INTRODUCTION

The concept of Unfalsified Control is introduced in [1], as a data-driven plant-model-free control approach. It recursively falsifies control parameter sets that fail to satisfy a performance specification, given measured data and specified control law. Although in early works the parameter space was gridded (resulting in a finite, but often large, set of candidate controllers), this restriction was lifted by applying a quadratic performance specification to a control law where the control parameters appear affine [2], [3]. As a result, a continuous region of unfalsified control parameter sets can be regarded, hence, with infinitely many controllers. In Ellipsoidal Unfalsified Control [3], the continuous region of controllers is described by an ellipsoid, resulting in simple algebraic equations to describe the entire set (An introduction to Ellipsoidal Unfalsified Control will be provided in Section II).

A crucial element for any control design method is the notion of stability. Especially for a plant-model-free control design method, no a priori statements can be made whether a *specific* controller is stabilizing the closed loop system. In [4] it was shown that for general *adaptive controller schemes*, sufficient conditions for stability can be derived, under the assumption that there exists at least one robustly-stabilizing and -performing controller in the candidate controller pool. The conditions imply that the cost functional is cost-detectable (i.e., the cost goes to infinity if the controller is unstable) and that the number of controller switches is finite. The latter condition is satisfied by imposing a monotone non-decreasing cost functional which is bounded from above and some minimum improvement in the cost functional between two consecutive controller

switches. Furthermore, if a continuous set of controllers is regarded, some neighborhood around a controller with similar performance is required (hence, which is falsified simultaneously). See also [5], [6] for applications of these results to unfalsified multiple-controller adaptive control schemes.

The main result of this paper is the introduction of a finite number of controller switches to Ellipsoidal Unfalsified Control theory. This is done, not by imposing restrictions on the (implicit) cost functional, but by exploiting the (explicit) decrease in volume of the Unfalsified set. It is shown that, under the mild assumption that there exists a region E in the candidate controller pool of some volume $e > 0$ with stabilizing and performing controllers, stability of Ellipsoidal Unfalsified Control is guaranteed with only a minor adaptation to the algorithm as introduced in [3]. Hence, even with infinitely many controllers, a finite number of controller switches is guaranteed.

An introduction to Ellipsoidal Unfalsified Control is given in Section II. In Section III, sufficient conditions for Ellipsoidal Unfalsified Control to be stable are provided. The conditions for stability are elaborated in Section IV (cost detectability) and Section V (finite number of controller switches). A simulation example is provided in Section VI, and the conclusions are presented in Section VII.

II. ELLIPSOIDAL UNFALSIFIED CONTROL

In this research, the Ellipsoidal Unfalsified Control (EUC) approach is considered, as developed in [3]. This data-driven, plant-model-free controller design method recursively falsifies controller parameter sets that fail to satisfy a performance specification, given measured data and a specified control law. In this section, an overview of Ellipsoidal Unfalsified Control is given.

A. Data Acquisition

The only “plant information” required by EUC is measurement data. The EUC algorithm is applied each time new measurement data becomes available.

B. Candidate Controllers

A “cloud” of candidate controllers is selected, the candidate controller set. When no measurement data is available yet, no controllers have been falsified, and the candidate controller set is, trivially, equal to the initial candidate controller set. When measurement data is

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available, though, candidate controllers might get falsified, and the approximation of the set of currently unfalsified controllers, the Unfalsified set, is used as candidate controller set.

Definition 1: The *True Unfalsified set* is the set of controllers, which are currently unfalsified by all available measurement data.

Definition 2: The *Unfalsified set* is the approximation of the True Unfalsified set.

The need for gridding of the candidate controllers is overcome by describing the Unfalsified set with a continuous region. In Ellipsoidal Unfalsified Control, the Unfalsified set is defined by an ellipsoid, see [2], which allows for the evaluation of the entire set with simple algebraic equations. The Unfalsified set at time t_{k-1} is described by the ellipsoid

$$\mathcal{E}(t_{k-1}) = \{\theta | (\theta - \theta_c(t_{k-1}))^T \Sigma^{-1}(t_{k-1}) (\theta - \theta_c(t_{k-1})) \leq 1\} \quad (1)$$

with $\theta \in \mathbb{R}^p$ the controller parameters, $\theta_c(t_k) \in \mathbb{R}^p$ the center of the ellipsoid and $\Sigma(t_k) \in \mathbb{R}^{p \times p}$ the matrix, which describes the shape of the ellipsoid.

C. Fictitious Reference

For every controller in the candidate controller pool, a “fictitious reference” r_{fict} is constructed. The “fictitious reference” is an abstract notion, but it can be thought of as a *controller parameter dependent* reference, that would have resulted in exactly the measured input and output, if that controller would have been in the loop during the measurements.

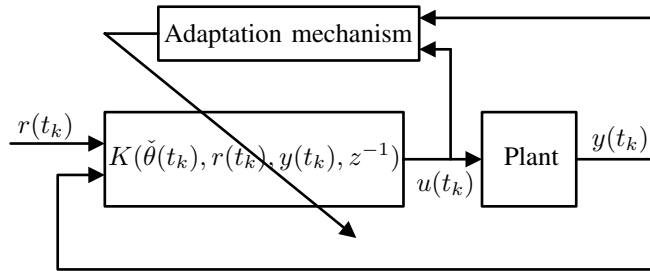


Fig. 1. General setup of closed loop feedback system with adaptation of controller parameters

As an example, consider the general closed loop adaptive feedback system as given in Fig. 1. Here, $r(t_k)$ is the (actual) reference, $u(t_k)$ is the plant input and $y(t_k)$ is the plant output. The currently implemented controller parameters are denoted with $\tilde{\theta}(t_k)$ and z^{-1} is the discrete time shift operator. Assume that plant input $u(t_k)$ can be written as

$$\begin{aligned} u(t_k) &= K(\tilde{\theta}(t_k), r(t_k), y(t_k), z^{-1}) \\ &= K_r(\tilde{\theta}(t_k), z^{-1}) * r(t_k) + K_y(\tilde{\theta}, y(t_k), z^{-1}) \end{aligned} \quad (2)$$

with $*$ a discrete-time convolution. Then, for a given $u(t_k)$ and $y(t_k)$, the *controller parameter dependent* fictitious reference $r_{\text{fict}}(t_k)$ is given by

$$r_{\text{fict}}(\theta, t_k) = K_r(\theta, z^{-1})^{-1} * (u(t_k) - K_y(\theta, y(t_k), z^{-1})) \quad (3)$$

As can be seen from (2) and (3), for $\theta = \tilde{\theta}(t_k)$, $r_{\text{fict}}(\theta, t_k)$ exactly results in the actual reference $r(t_k)$, provided that $K_r(\theta)$ is causally-left-invertible. Of course, the restriction that $K_r(\theta, z^{-1})$ is invertible limits the selection of candidate controllers. However, still a large class of controllers is available.

Let the controller structure be chosen such, that $r_{\text{fict}}(\theta, t_k)$ is affine in the controller parameters θ . Then (3) can be rewritten as

$$r_{\text{fict}}(\theta, t_k) = w(u(t_k), y(t_k), z^{-1})^T \theta \quad (4)$$

Note that the concept of a fictitious reference enables the evaluation of controllers, even if they were not in the loop at the time of the measurement.

D. Unfalsification

Given a desired performance specification, and exploiting the fictitious reference, a region can be constructed of controller parameters which are unfalsified by current measurement data.

Let the performance specification be defined as a time-dependent maximum allowed tracking error $\Delta(t_k)$, as in [3]. Then the region of controller parameters, which is unfalsified by current measurement data at time t_k , is given by

$$\begin{aligned} \mathcal{U}(t_k) &= \{\theta \mid -\Delta(t_k) \leq \\ &\quad G_m(z^{-1}) * r_{\text{fict}}(\theta, t_k) - y(t_k) \leq \Delta(t_k)\} \\ &= \{\theta \mid -1 \leq \\ &\quad \frac{G_m(z^{-1}) * w(u(t_k), y(t_k), z^{-1})^T \theta - \frac{y(t_k)}{\Delta(t_k)}}{\Delta(t_k)} \leq 1\} \end{aligned} \quad (5)$$

with $G_m(z^{-1})$ the desired closed loop dynamics. From (6) it is clear to see, that $\mathcal{U}(t_k)$ defines two parallel half-spaces in the controller parameter space.

E. Update Unfalsified set

The region of controllers, that is unfalsified by all available measurement data (hence, including all past and present measurement data), is given by the intersection of the candidate controllers $\mathcal{E}(t_{k-1})$ from section II-B (the controllers that are unfalsified by past measurement data) and the controllers $\mathcal{U}(t_k)$ from section II-D (the controllers that are unfalsified by the present measurement data).

To maintain an ellipsoidal Unfalsified set, the intersection $\mathcal{E}(t_{k-1}) \cap \mathcal{U}(t_k)$ is approximated by a minimum-volume outer-bounding ellipsoid $\mathcal{E}(t_k)$. Since $\mathcal{U}(t_k)$ defines two parallel half-spaces, this approximation can be computed

analytically, as was shown in [7]. To compute $\mathcal{E}(t_k)$, define the variables

$$y_k = \frac{y(t_k)}{\Delta(t_k)} \quad (7)$$

$$\phi_k = \frac{G_m(z^{-1}) * w(u(t_k), y(t_k), z^{-1})}{\Delta(t_k)} \quad (8)$$

$$g = \phi_k^T \Sigma(t_{k-1}) \phi_k \quad (9)$$

$$a_+ = \max\left(\frac{y_k - \phi_k^T \theta_c(t_{k-1}) - 1}{\sqrt{g}}, -1\right) \quad (10)$$

$$a_- = \max\left(\frac{-y_k + \phi_k^T \theta_c(t_{k-1}) - 1}{\sqrt{g}}, -1\right) \quad (11)$$

If $a_+ a_- \geq 1/p$ (Recall from (1) that p is the number of controller parameters), $\mathcal{E}(t_{k-1})$ is the minimum-volume outer-bounding ellipsoid of the intersection, hence, $\mathcal{E}(t_k) = \mathcal{E}(t_{k-1})$. Consequently, $\Sigma(t_k) = \Sigma(t_{k-1})$ and $\theta_c(t_k) = \theta_c(t_{k-1})$, with $\Sigma(t_k)$ and $\theta_c(t_k)$ as in (1).

For $a_+ a_- < 1/p$ and $a_+ \neq a_-$, $\mathcal{E}(t_k)$ is defined by (see [7])

$$\Sigma(t_k) = \delta \left(\Sigma(t_{k-1}) - \frac{\sigma}{g} \Sigma(t_{k-1}) \phi_k \phi_k^T \Sigma(t_{k-1}) \right) \quad (12)$$

$$\theta_c(t_k) = \theta_c(t_{k-1}) + \frac{\sigma(a_+ - a_-)}{2\sqrt{g}} \Sigma(t_{k-1}) \phi_k \quad (13)$$

with

$$\delta = \frac{p^2}{p^2 - 1} \left(1 - \frac{a_+^2 + a_-^2 - \rho/p}{2} \right) \quad (14)$$

$$\sigma = \frac{1}{p+1} \cdot \left[p + \frac{2}{(a_+ - a_-)^2} \left(1 - a_+ a_- - \frac{\rho}{2} \right) \right] \quad (15)$$

$$\rho = \sqrt{4(1 - a_+^2)(1 - a_-^2) + p^2(a_+^2 - a_-^2)^2} \quad (16)$$

If $a_+ = a_-$, (15) becomes unbounded. Therefore, for $a_+ a_- < 1/p$ and $a_+ = a_- = a$, $\mathcal{E}(t_k)$ is defined by

$$\Sigma(t_k) = \frac{p(1 - a^2)}{p - 1} \left(\Sigma(t_{k-1}) - \frac{1 - pa^2}{(1 - a^2)g} \Sigma(t_{k-1}) \phi_k \phi_k^T \Sigma(t_{k-1}) \right) \quad (17)$$

$$\theta_c(t_k) = \theta_c(t_{k-1}) \quad (18)$$

F. Controller Selection

A controller, that is unfalsified by the available measurement data, is to be selected to be inserted in the loop. Or in other words, one controller inside the new Unfalsified set $\mathcal{E}(t_k)$, as derived in the previous section, is to be implemented.

Consider

$$\gamma = \phi_k^T \check{\theta}(t_{k-1}) - y_k \quad (19)$$

with $\check{\theta}(t_{k-1})$ the currently implemented controller parameters (as in (2)). Note from (6) through (8) that for $|\gamma| > 1$, $\check{\theta}(t_{k-1})$ is falsified by current measurement

data.

Lemma 1: $\theta_c(t_k)$ is always unfalsified by $\mathcal{U}(t_k)$, i.e., $\theta_c(t_k) \in \mathcal{U}(t_k)$.

Proof: As explained in Section II-E, $\mathcal{E}(t_k) \supset (\mathcal{E}(t_{k-1}) \cap \mathcal{U}(t_k))$. Assume that there is some ellipsoid \mathcal{E}^* with center θ_c^* , such that $\mathcal{E}^* \supset (\mathcal{E}(t_{k-1}) \cap \mathcal{U}(t_k))$ and $\theta_c^* \notin \mathcal{U}(t_k)$. Because $\theta_c^* \notin \mathcal{U}(t_k)$, it holds that $a_+ > 0$ or $a_- > 0$ (from (6) through (8): $|\phi_k^T \theta_c^* - y_k| > 1$). Then $a_+ a_- \leq 0 < 1/p$, hence, $\exists \mathcal{E}^{**} \supset (\mathcal{E}^* \cap \mathcal{U}(t_k)) \supset ((\mathcal{E}(t_{k-1}) \cap \mathcal{U}(t_k)) \cap \mathcal{U}(t_k)) = (\mathcal{E}(t_{k-1}) \cap \mathcal{U}(t_k))$ of smaller volume than \mathcal{E}^* . Hence, for $\mathcal{E}(t_k)$ to be of minimal volume, $\theta_c(t_k)$ has to be in $\mathcal{U}(t_k)$. ■

From Lemma 1, it can be concluded that $\theta_c(t_k)$ is both in $\mathcal{E}(t_k)$ (trivially) and in $\mathcal{U}(t_k)$. Therefore, $\theta_c(t_k)$ is a legitimate choice for the controller parameter set $\theta(t_k)$, if the old controller parameter set $\check{\theta}(t_{k-1})$ is falsified. This is implemented in the controller parameter update algorithm

$$\check{\theta}(t_k) = \begin{cases} \check{\theta}(t_{k-1}) & \text{if } |\gamma| \leq 1 \\ \theta_c(t_k) & \text{if } |\gamma| > 1 \end{cases} \quad (20)$$

III. STABILITY OF ADAPTIVE SYSTEMS

Ellipsoidal Unfalsified Control only considers the external, or input-output, behavior of a plant, in contrast to the internal, or state-space, behavior. This naturally leads to the stability concept of bounded-input bounded-output stability.

Definition 3 (BIBO stability): A system is called bounded-input bounded-output (BIBO) stable if the system has bounded gain [8, p.218].

From definition 3 it can be concluded, that the output of a BIBO stable system will remain bounded for all time, for any finite input (and initial condition). Since only finite time data is considered, the stability of a system (with a fixed controller) can at best be unfalsified. That is, at best it can *not* be concluded from available data, that the system is *not* BIBO stable.

A. Stability of adaptive systems

In [4], properties are imposed on a feedback adaptive control system, such that the adaptive system is stable. Crucial assumption is the ‘‘feasibility assumption,’’ i.e., that the adaptive control problem is feasible.

Assumption 1 (Feasibility): It is assumed that there exists at least one controller in the candidate controller pool, which is robustly-stabilizing and -performing, hence, which satisfies given performance and stability constraints at all times.

Definition 4 (Cost-detectability): Let $J(K, z_{\text{data}}, t_k)$ be a cost functional, with controller K , measurement data z_{data} , and time t_k . A system is said to be cost detectable if, whenever stability of the system with controller K in the loop

is falsified by data z_{data} , then $\lim_{\tau \rightarrow \infty} J(K, z_{\text{data}}, \tau) = \infty$ (See [4], Definition 11).

Lemma 2 (Stability): Consider an adaptive scheme with an associated cost functional $J(K, z_{\text{data}}, t_k)$. Suppose that the adaptive control problem is feasible. Then stability of the adaptive scheme is unfalsified, if the cost-detectability property is satisfied and if the maximum number of controller switches is finite (See [4], Lemma 2).

Remark: In [4], restrictions are imposed on the cost functional $J(K, z_{\text{data}}, t_k)$ to guarantee a finite number of controller switches. Namely, $J(K, z_{\text{data}}, t_k)$ has to be bounded from above, be monotone increasing and a minimum improvement in the cost functional has to be achieved before the controller is updated.

B. Stability of Ellipsoidal Unfalsified Control

Consider the feasibility assumption 1. To better suit the Ellipsoidal Unfalsified Control framework, the assumption is somewhat altered.

Assumption 2 (Feasibility EUC): It is assumed that there exists at least a region E in the candidate controller pool of some volume $e > 0$ with stabilizing and performing controllers, hence, which satisfy the performance specification at all times.

Remark 1: The assumption that there exists a region of some volume is more natural to Ellipsoidal Unfalsified Control than the assumption of one controller, which corresponds to an ellipsoid with zero volume.

Remark 2: By the definition of the performance specification, E is a convex set.

Theorem 1 (Stability EUC): Consider Ellipsoidal Unfalsified Control, as introduced in Section II, with performance specification as in (5) and controller switching algorithm as in (20). Suppose that the EUC problem is feasible. Furthermore, impose a maximum volume ratio $\delta_V \leq \nu < 1$ on two consecutive ellipsoidal Unfalsified sets, if the Unfalsified set changes. Then the EUC system is BIBO stable.

Proof: Stability of the adaptive scheme is unfalsified, if the cost-detectability property is satisfied and if the maximum number of controller switches is finite (Lemma 2). As will be shown in Section IV, with the performance specification (5), the cost detectability property is satisfied. Furthermore, as will be shown in Section V, with the controller selection as in (20) and a maximum volume ratio δ_V between two consecutive ellipsoidal Unfalsified sets, the maximum number of controller switches is finite. ■

IV. COST-DETECTABILITY

As stated in definition 4, a system is said to be cost detectable if, whenever stability of the system with controller $K(\theta)$ in the loop is falsified by data z_{data} , then

$\lim_{\tau \rightarrow \infty} J(K(\theta), z_{\text{data}}, \tau) = \infty$. However, the cost functional for Ellipsoidal Unfalsified Control is only specified implicitly. Therefore, consider the cost functional $J(K(\theta), z_{\text{data}}, t_k)$, defined by

$$\begin{cases} J(K(\theta), z_{\text{data}}, t_k) = (\theta - \theta_c(t_k))^T \Sigma^{-1}(t_k) (\theta - \theta_c(t_k)) \\ \quad \text{for } K(\theta) \text{ unfalsified} \\ J(K(\theta), z_{\text{data}}, t_k) = \infty \\ \quad \text{for } K(\theta) \text{ falsified} \end{cases} \quad (21)$$

with $\theta_c(t_k)$ and $\Sigma(t_k)$ defined by the data z_{data} at time t_k . Should a destabilizing controller $K(\theta)$ be unfalsified at time t , and future data would indeed falsify BIBO stability at some time t^+ , then the performance specification (5) will not be satisfied for controller $K(\theta)$ at time t^+ . Hence, the unstable controller $K(\theta)$ will be falsified at time t^+ . As a consequence, the cost functional (21) will be ∞ . Hence, if BIBO stability is falsified for a given controller $K(\theta)$, the corresponding cost will be ∞ and the adaptive system is cost detectable.

V. FINITE NUMBER OF CONTROLLER SWITCHES

Ellipsoidal Unfalsified Control uses an ellipsoidal description of the Unfalsified set, which is continuous in the controller parameter space. Hence, an infinite number of candidate controllers is considered. Note, however, that the volume of the ellipsoidal Unfalsified set is non-increasing. Furthermore, the volume is lower bounded by e , the volume of the region E , containing the stabilizing and performing controllers. Hence, in stead of regarding a cost functional to limit the maximum number of controller switches, the volume of the Unfalsified set is regarded.

A. Decrease of volume

Lemma 3: The volume ratio $\delta_V(t_k)$ between two consecutive ellipsoids for $a_+ \neq a_-$ is given by

$$\delta_V(t_k) = \sqrt{\delta^p(1 - \sigma)} \quad (22)$$

Proof: The volume $V(t_{k-1})$ of the Unfalsified set $\mathcal{E}(t_{k-1})$ is given by

$$V(t_{k-1}) = \text{vol}(\mathcal{E}(t_{k-1})) = V_p \sqrt{\det(\Sigma(t_{k-1}))} \quad (23)$$

with V_p the volume of the unit ball in \mathbb{R}^p and $\Sigma(t_{k-1})$ from (1). The volume ratio $\delta_V(t_k)$ between two consecutive ellipsoids is given by

$$\delta_V(t_k) = \frac{\text{vol}(\mathcal{E}(t_k))}{\text{vol}(\mathcal{E}(t_{k-1}))} = \sqrt{\frac{\det(\Sigma(t_k))}{\det(\Sigma(t_{k-1}))}} \quad (24)$$

Consider $a_+ \neq a_-$. Using (9) and (12), $\det(\Sigma(t_k))$ can be expressed in terms of $\det(\Sigma(t_{k-1}))$:

$$\begin{aligned} \det(\Sigma(t_k)) &= \det\left(\delta\left(\Sigma(t_{k-1}) - \frac{\sigma}{g}\Sigma(t_{k-1})\phi_k\phi_k^T\Sigma(t_{k-1})\right)\right) \\ &= \delta^p \det(\Sigma(t_{k-1})) \det\left(\mathbb{I} - \frac{\sigma}{g}\phi_k\phi_k^T\Sigma(t_{k-1})\right) \\ &= \delta^p \det(\Sigma(t_{k-1})) \left(1 - \text{Tr}\left(\frac{\sigma}{g}\phi_k\phi_k^T\Sigma(t_{k-1})\right)\right) \\ &= \delta^p(1 - \sigma) \det(\Sigma(t_{k-1})) \end{aligned} \quad (25)$$

$$\Rightarrow \delta_V(t_k) = \sqrt{\frac{\det(\Sigma(t_k))}{\det(\Sigma(t_{k-1}))}} = \sqrt{\delta^p(1 - \sigma)} \quad \blacksquare$$

Corollary 1: The volume ratio $\delta_V(t_k)$ between two consecutive ellipsoids is given by

$$\delta_V(t_k) = -a \sqrt{p^p \left(\frac{1 - a^2}{p - 1}\right)^{p-1}} \quad (26)$$

for $a_+ = a_- = a \leq 0$.

Proof: The result for $a_+ = a_- = a$ is obtained by using (17) to express $\det(\Sigma(t_k))$ in terms of $\det(\Sigma(t_{k-1}))$. The remainder of the derivation is similar to the proof of Lemma 3 \blacksquare

From Lemma 3 it can be concluded that the volume of the ellipsoids decreases when $\delta^p(1 - \sigma) < 1$.

B. Conditions for a finite number of controller switches

In the previous subsection, the volume ratio between two consecutive ellipsoids is given in (22) and (26). This subsection specifies a condition, such that a maximum on the volume ratio between two consecutive ellipsoids is guaranteed.

Consider a maximum volume ratio $\delta_V \leq \nu$ for some positive $\nu < 1$. From (22), it can be seen that a sufficient condition to ensure $\delta_V \leq \nu$ between two consecutive ellipsoids is to require that $a_+a_- \leq \epsilon(\nu)/p$ for some $\epsilon(\nu) < 1$. The value of $\epsilon(\nu)$ can be derived from (22). It is observed, that for $\epsilon(\nu)$ close to 1,

$$\arg \max_{a_+a_- = \epsilon(\nu)/p} \delta_V(t_k) = \begin{cases} \{a_+ = -1, a_- = -\epsilon(\nu)/p\} \\ \{a_- = -1, a_+ = -\epsilon(\nu)/p\} \end{cases} \quad (27)$$

for a fixed p .

If $\epsilon(\nu)/p < a_+a_- \leq 1/p$, the additionally falsified region is neglected and the Unfalsified set is not changed.

Remark: If ν is chosen close to 0, a small volume ratio between two consecutive ellipsoidal Unfalsified sets is enforced. It also induces, though, that intersections which would lead to a volume ratio larger than ν are dismissed. Therefore, to not throw away valuable falsification data, ν should not be chosen too small.

To express the maximum number of ellipsoidal Unfalsified sets, consider the volume $V(t_0)$ of the initial Unfalsified set, which is the largest possible volume of the Unfalsified set. The volume of the n^{th} ellipsoid is upperbounded by $V(t_0)\nu^n$. Next, consider the smallest possible volume e of the Unfalsified set, which is given by the volume of region E with stabilizing and performing controllers. Then the maximum number of ellipsoids n_{max} is limited by

$$n_{\text{max}} \leq \frac{\log(e/V(t_0))}{\log(\nu)} \quad (28)$$

As is shown, by imposing the constraint $a_+a_- \leq \epsilon(\nu)/p$ with $\epsilon(\nu) < 1$ before updating the ellipsoidal Unfalsified set, the number of controller switches is limited. A maximum number of ellipsoids, in turn, implies a maximum number of controller switches with controller update algorithm (20). This can be seen from the fact that if the final ellipsoid is reached (in a finite number of steps), the currently implemented controller parameters can at most be falsified one more time. Then, the center of the final ellipsoid is chosen, which can not be falsified, for then the ellipsoid has to be updated. This is in contradiction with the assumption that the final ellipsoid is reached.

VI. SIMULATION

In simulations, the effect of Ellipsoidal Unfalsified Control has been evaluated on a fourth order system, which is given by

$$\dot{x} = Ax + Bu \quad (29)$$

$$y = Cx \quad (30)$$

with

$$x = [x_1 \quad \dot{x}_1 \quad x_2 \quad \dot{x}_2]^T \quad (31)$$

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -c/J_1 & -d/J_1 & c/J_1 & d/J_1 \\ 0 & 0 & 0 & 1 \\ c/J_2 & d/J_2 & -c/J_2 & -d/J_2 \end{bmatrix} \quad (32)$$

$$B = [0 \quad 1/J_1 \quad 0 \quad 0]^T \quad (33)$$

$$C = [0 \quad 0 \quad 1 \quad 0] \quad (34)$$

The parameter-values are chosen as $J_1 = 1.56 \cdot 10^{-4}$, $J_2 = 1.95 \cdot 10^{-4}$, $d = 0.9 \cdot 10^{-3}$ and $c = 8.64$. The plant is sampled at 1 kHz with a zero order hold and a bounded output disturbance with noise power 10^{-8} is present. The controller structure $w(u(t_k), y(t_k), z^{-1})$ is chosen as

$$w(u(t_k), y(t_k), z^{-1}) = \begin{bmatrix} u(t_k) \\ 10^{-3} \frac{1+z^{-1}}{1-z^{-1}} u(t_k) \\ y(t_k) \\ 10^{-3} \frac{1+z^{-1}}{1-z^{-1}} y(t_k) \\ y^2(t_k) \end{bmatrix} \quad (35)$$

The last element of $w(u(t_k), y(t_k), z^{-1})$ is chosen, to underline that EUC is not limited to linear controllers. The maximum volume ratio is constrained by setting $\epsilon(\nu) =$

$0.99 < 1$. This corresponds to $\nu = 0.999989$, which is close to, but still smaller than, 1.

The reference trajectory is $r(t_k) = \text{sign}(\sin(0.5\pi(t_k)))$ and the reference model is $G_m(z^{-1}) = \frac{2 \cdot 10^{-4}(z^{-1} + z^{-2})}{1 - 1.96z^{-1} + 0.9604z^{-2}}$. The performance bound $\Delta(t_k) = 0.02 + e^{-t_k}$. Here, the lower bound on $\Delta(t_k)$ is included, to guarantee feasibility in the presence of output disturbance.

The algorithm is initialized with

$$\Sigma(0) = 10^4 \mathbb{I}_{5 \times 5} \quad (36)$$

$$\check{\theta}(0) = \theta_c(0) = [100 \ 0 \ 1 \ 0 \ 0]^T \quad (37)$$

The initial value $\check{\theta}(0)$ corresponds to a P-controller with gain 0.01 ($1/\theta_1$) (which, in fact, is destabilizing the system due to the phase lag caused by the zero order hold).

In Fig. 2, the tracking error $G_m(z^{-1}) * r(t_k) - y(t_k)$ of the EUC adaptive system is shown. After 10 seconds the EUC algorithm has found a controller parameter set which is unfalsified for $\Delta = 0.02$. In Fig. 3, controller parameters $\check{\theta}(t_k)$ are shown as a function of time, together with the center of the ellipsoidal Unfalsified set $\theta_c(t_k)$. If the tracking error of Fig. 2 is within the performance bounds, the controller parameters are unchanged. The center $\theta_c(t_k)$ on the other hand changes almost continuously.

In Fig. 4, $\det(\Sigma(t_k))$ is shown as a function of time, which is proportional to the volume of the Unfalsified set (see (23)). The volume is monotone decreasing and tends to a stationary value.

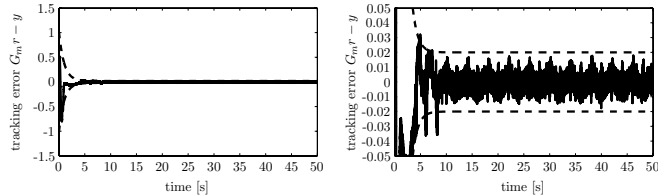


Fig. 2. Tracking error of fourth order plant with EUC, with control structure (35).

VII. CONCLUSIONS

Stability for Ellipsoidal Unfalsified Control is established, even though Ellipsoidal Unfalsified Control uses a continuous region of unfalsified control parameter sets, hence, with infinitely many controllers. It is assumed that the adaptive control problem is feasible, i.e., that there is a region E in the candidate controller pool with stabilizing and performing controllers.

By explicitly defining a cost functional, it is shown that Ellipsoidal Unfalsified Control is cost detectable. A finite number of controller switches is guaranteed by imposing a constraint on the update of the Unfalsified set. The constraint guarantees a maximum volume ratio between two consecutive (ellipsoidal) Unfalsified sets. A sufficient condition to fulfill the constraint is implemented by a simple check on the update variables. Since the volume is lower bounded and monotone decreasing, the number of Unfalsified sets is finite,

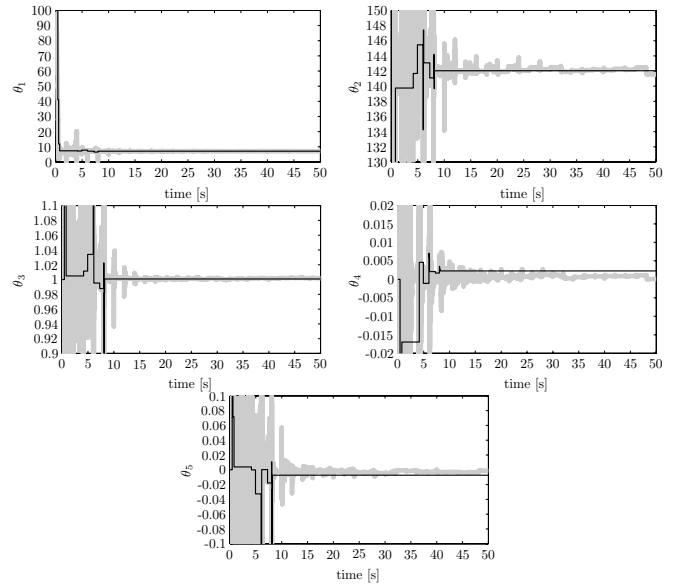


Fig. 3. Plot of the controller parameters $\check{\theta}(t_k)$ as a function of time (black), together with the center of the ellipsoidal Unfalsified set $\theta_c(t_k)$ (grey).

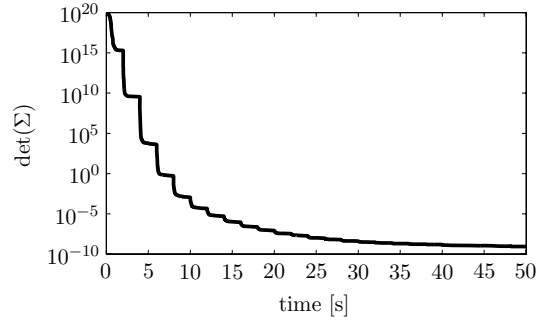


Fig. 4. Plot of $\det(\Sigma(t_k))$ as a function of time, which is proportional to the volume of the Unfalsified set.

and, hence, the number of controllers is finite.

In a simulation example with a fourth order system, the effectiveness of the proposed method is shown.

REFERENCES

- [1] M. Safonov and T.-C. Tsao, "The unfalsified control concept and learning," *IEEE Trans. Automatic Control*, vol. 42, no. 6, pp. 843–847, June 1997.
- [2] F. Cabral and M. Safonov, "Unfalsified model reference adaptive control using the ellipsoid algorithm," in *Proc. Conf. Dec. & Contr.*, Maui, Hawaii, USA, December 2003, pp. 3250–3255.
- [3] J. v. Helvoort, B. d. Jager, and M. Steinbuch, "Unfalsified control using an ellipsoidal unfalsified region applied to a motion system," in *Proc. IFAC World Congress*, Prague, Czech Republic, July 2005.
- [4] M. Stefanovic, A. Paul, and M. Safonov, "Safe adaptive switching through infinite controller set: Stability and convergence," in *Proc. IFAC World Congress*, Prague, Czech Republic, July 2005.
- [5] A. Paul, "Multi-controller adaptive control (mcac): Cost detectability, stability and some applications," Ph.D. dissertation, University of Southern California, Los Angeles, August 2005.
- [6] R. Wang and M. Safonov, "Stability of unfalsified adaptive control using multiple controllers," in *Proc. Amer. Contr. Conf.*, Portland, Oregon, USA, June 2005, pp. 3162–3167.
- [7] L. Ros, A. Sabater, and F. Thomas, "An ellipsoidal calculus based on propagation and fusion," *IEEE Trans. Systems Man Cybernet.*, vol. 32, no. 4, pp. 430–442, August 2002.
- [8] K. Åström and B. Wittenmark, *Adaptive Control*, 2nd ed. Reading, Massachusetts, USA: Addison Wesley, 1995.