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ELLIPTIC BOUNDARY VALUE PROBLEMS ON MANIFOLDS WITH A PIECEWISE SMOOTH BOUNDARY

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Abstract. In this paper general boundary value problems for second-order elliptic differential equations are considered on manifolds with edges. It is assumed that in the neighborhood of an edge point the manifold is diffeomorphic to the interior of a convex dihedral angle. Effective conditions for normal solvability of these boundary value problems are obtained and the parametrix is constructed. The methods make use of the theory of analytic functions of several variables and automorphic functions.

Bibliography: 17 items.

Introduction

In this paper boundary value problems are considered on a compact n-dimensional manifold $\mathfrak{M}$ with a piecewise smooth boundary. These problems are of the form

$$\begin{aligned} Au_0(x) + Ku_2(x) &= f_0(x), \quad x \in \mathcal{M}_0, \\ Bu_0(x) + Lu_2(x) &= f_1(x), \quad x \in \mathcal{M}_1, \\ Tu_0(x) + Ju_2(x) &= f_2(x), \quad x \in \mathcal{M}_2. \end{aligned} \tag{0.1}$$

Here $\mathfrak{M}_0$ is the interior of $\mathfrak{M}$, $\mathfrak{M}_1$ is the smooth part of the boundary and $\mathfrak{M}_2$ is the edge of codimension 2. It is assumed that $\mathfrak{M}_2$ is a smooth $(n-2)$-dimensional manifold and that the tangent spaces to $\mathfrak{M}_1$ at points of $\mathfrak{M}_2$ are intersected transversally. An example of an admissible manifold $\mathfrak{M}$ is represented in Figure 1.

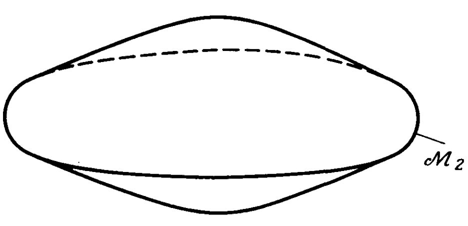

Figure 1

AMS (MOS) subject classifications (1970). Primary 35J25, 58G99; Secondary 30A88, 32N05.

The letter A in (0.1) denotes a second-order differential opeartor on $\mathfrak{M}$ with smooth coefficients; B denotes a differential operator of arbitrary order whose coefficients are smooth on $\mathfrak{M}_1$ but may have discontinuities of the first kind on $\mathfrak{M}_2$. In turn, K and L are operators of potential type, T is a boundary operator of codimension two (Definition 7.3) and J is a pseudodifferential operator on the edge. All functions occurring in (0.1) belong to spaces of Sobolev type.

We assume that A is an elliptic operator and that the pair (A, B) satisfies the classical Šapiro-Lopatinskiĭ condition on the smooth parts of the boundary up to the edge when approaching it from each side.

Problem (0.1) has not been considered in full generality up to now. The class of problems of the form

$$\begin{aligned} Au_0(x) &= f_0(x), \quad x \in \mathcal{M}_0, \\ Bu_0(x) &= f_1(x), \quad x \in \mathcal{M}_1, \end{aligned} \tag{0.2}$$

has been partially studied.

In the case $n = 2$ problem (0.2) has been investigated in a whole series of papers: S. L. Sobolev [1], V. V. Fufaev [2], G. E. Šilov [3], N. I. Mushelišvili [4], M. Š. Birman and G. E. Skvorcov [5], E. A. Volkov [6]. The most complete results for $n = 2$ were obtained by G. I. Èskin [7] and V. A. Kondrat'ev [8]. For $n \geq 3$ problem (0.2) was partially studied by M. S. Hanna and K. T. Smith [10], Kondrat'ev [13] and V. S. Maz'ja and B. A. Plamenevskiĭ [16]. In [10] there are isolated qualitative results on the Dirichlet problem for the Laplace operator in convex polytopes. In [13] results are obtained on the smoothness of the solution of the Dirichlet problem for a second-order operator with real coefficients. Finally, in [16] effective conditions were first indicated in order that problem (0.2) be Noetherian when A is the Beltrami-Laplace operator and B is a differential operator of first order at most with real coefficients (see also [17]).

Such contrast between the cases $n = 2$ and $n = 3$ stems from the fact that to establish the Noetherian property for problem (0.2) when $n \geq 3$ it is necessary to construct the inverse operator for a problem of the same type with a parameter and constant coefficients in a quadrant of the plane. Consequently for problem (0.2) with constant coefficients in a plane quadrant it is necessary to be able to find the kernel and cokernel exactly. But from results of [1]–[8] only the Noetherian property of such problems follows. Exact computation of their kernel and cokernel involves considerable difficulties. In the special case when A is a second-order operator with real coefficients and B is an operator of first order at most with real coefficients, this problem was solved in [16].

In this paper, for a second-order operator A we find, in the general case itself, the kernel and cokernel of the problem with a parameter and constant coefficients in a quadrant. To do this we apply a new method based on variables dual to x in the complex plane and the use of automorphic functions. With the aid of this method we find the kernel and cokernel of the problem with a parameter, and explicit formulas for its solution. This permits us to investigate problems (0.1) and (0.2) for a second-order operator A without assuming that coefficients are real and without restriction on the order of the boundary operator B. We assume that $n \geq 3$, since problem (0.2) has been well studied in the case $n = 2$.

We note that the method of automorphic functions, which we apply to partial differential equations, was first applied to difference equations in a quadrant by V. A. Malyšev [12].

For lack of space we omit proofs of some assertions in the first and seventh sections. Detailed proofs of these assertions will be published elsewhere by the author.

We shall employ the following notation: ∂ is the differentiation vector $(\partial_1, \cdots, \partial_n)$ in $\mathbf{R}^n$ and in $\mathbf{C}^n$, and $D = i\partial$. $\mathbf{N}$ denotes the set $\{0, 1, 2, \cdots\}$, $\mathbf{R}^+$ the real half-line $x > 0$ and $\mathbf{C}^\pm$ the complex half-plane $\operatorname{Im} z \gtrless 0$. For a real number s, $[s]$ denotes its integral part. If X and Y are smooth spaces, $C^\infty_{\text{in}}(X, Y)$ is the space of smooth imbeddings of X in Y. A set $B \subset C^\infty_{\text{in}}(X, Y)$ is called bounded if $\overline{B} \subset C^\infty_{\text{in}}(X, Y)$ and the derivatives of arbitrary order of functions effecting imbeddings of B in local coordinates are uniformly bounded on every compact set. As usual, $\mathcal{E}(X) = C^\infty(X)$ and $\mathcal{D}(X) = C^\infty_0(X)$; $\mathcal{D}'(X)$ is the dual of $\mathcal{D}(X)$ of generalized functions (distributions). The Fourier transform of the generalized function u in $\mathbf{R}^n$ is denoted by $\tilde{u}$ or Fu. If K is a set in $\mathbf{R}^n$, $\mathcal{S}'(\mathbf{R}^n, K)$ is the space of tempered distributions in $\mathbf{R}^n$ concentrated in K.

For $\Omega \subset \mathbf{C}^n$ we denote by $\mathcal{O}_\Omega$ the ring of functions holomorphic on Ω. For $A \in \mathcal{O}_{\mathbf{C}^n}$, $V(A)$ denotes the set of zeros of the function A. If P is a polynomial in one variable, $\deg P$ denotes its degree.

We define the norm for elements of a finite-dimensional linear space to be any of the equivalent norms. For example, for polynomials of bounded degree the sum of the moduli of the coefficients can be taken as norm.

If X and Y are two sets and $h: X_0 \to Y$ is a mapping of $X_0 \subset X$ into Y, $X \cup_h Y$ denotes the union $X \cup Y$ in which the points $x_0 \in X_0$ and $hx_0 \in Y$ are identified. If $X = Y$, then $X \cup_h Y$ is denoted by X/h.

By a *covering* we mean a fibering in Serre's sense with discrete fiber.

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§1. Boundary value problems in a quadrant of the plane

In this section we reduce the general boundary value problem itself in a quadrant to an equivalent system of equations on a Riemann surface. Let $\mathbf{K}$ be the first quadrant of the plane: $\mathbf{K} = \mathbf{R}^+ \times \mathbf{R}^+$. In $\mathbf{K}$ we consider a general boundary value problem of the form

$$\begin{aligned} &A(D)u(x) = f(x), \quad x_1 > 0,\ x_2 > 0, \\ &B_1(D)u(x_1, 0) = f_1(x_1), \quad x_1 > 0, \\ &B_2(D)u(0, x_2) = f_2(x_2), \quad x_2 > 0. \end{aligned} \tag{1.1}$$

Here $A(D)$ is a scalar differential operator of order m and $B_l(D)$, for $l = 1, 2$, is a differential operator with constant vector coefficients:

$$B_l(D) = (B_l^1(D), \ldots, B_l^{n_l}(D)), \tag{1.2}$$

where the n_l are natural numbers and the $B_l^j(D)$ are scalar operators of order m_l^j. We assume that $f \in H_{s-m}(\overline{\mathbf{K}})$, where $s - m > -\frac{1}{2}$. Suppose also that $f_l \in \bigoplus_j H_{s-m_l^j-\frac{1}{2}}(\overline{\mathbf{R}}^+)$, with $s - m_l - \frac{1}{2} > 0$, where $m_l = \max_j m_l^j$. We shall seek a solution u of problem (1.1) in $H_s(\overline{\mathbf{K}})$.

Let us extend the function u to $\mathbf{R}^2$ by setting it equal to zero on $\mathbf{R}^2 \backslash \mathbf{K}$. Then

$$A(D)u(x) = f'(x), \quad x \in \mathbf{R}^2, \tag{1.3}$$

where $f' \in \mathcal{S}'(\mathbf{R}^2, \mathbf{K})$. Fourier transformation carries (1.3) into

$$A(z)\widetilde{u}(z) = \widetilde{f}'(z), \quad z \in \mathbf{R}^2. \tag{1.4}$$

We shall assume that $A(D)$ is a strongly elliptic operator, i.e. that for $z \in \mathbf{R}^2$

$$|A(z)| \geqslant C(1 + |z|)^m. \tag{1.5}$$

Under this condition $\widetilde{u}$ is uniquely determined by (1.4). Consequently, to find the general solution of (1.1) it sufficies in this case to describe the image of the operator $A(D)$: $H_s(\overline{\mathbf{K}}) \to \mathcal{S}'(\mathbf{R}^2, \mathbf{K})$. We give such a description at once by using the Fourier transform of f' and the complex characteristics of $A(D)$.

We introduce the notation

$$a^\alpha = \frac{1}{\alpha!} D^\alpha A(0), \quad a_1^\alpha(z_1) = \frac{1}{\alpha!} D_2^\alpha A(z_1, 0) \quad \text{and} \quad a_2^\alpha(z_2) = \frac{1}{\alpha!} D_1^\alpha A(0, z_2).$$

Then

$$A(z) = \sum_\alpha a^\alpha(-iz)^\alpha = \sum_\alpha a_1^\alpha(z_1)(-iz_2)^\alpha = \sum_\alpha a_2^\alpha(z_2)(-iz_1)^\alpha, \tag{1.6}$$

or

$$A(D) = \sum_\alpha a^\alpha \partial^\alpha = \sum_\alpha a_1^\alpha(D_1)\partial_2^\alpha = \sum_\alpha a_2^\alpha(D_2)\partial_1^\alpha. \tag{1.7}$$

Let $\mathbf{CK}^*$ be the tubular domain dual to $\mathbf{K}$:

$$\mathbf{CK}^* = \{z \in \mathbf{C}^2 \mid \operatorname{Im} z_1 > 0, \operatorname{Im} z_2 > 0\}. \tag{1.8}$$

We denote by $\mathcal{J}^*(A)$ the principal ideal $A(z)\mathcal{O}_{\mathbf{CK}^*}$ of the ring of functions holomorphic in $\mathbf{CK}^*$. Now we are able completely to describe the space $A(D)H_s(\overline{\mathbf{K}})$. We note that $\widetilde{f}'(z)$ is holomorphic in $\mathbf{CK}^*$.

Theorem 1.1. (i). *If $s - m > \frac{1}{2}$ and $f' \in A(D)H_s(\overline{\mathbf{K}})$, then $\widetilde{f}' \in \mathcal{J}^*(A)$ and for $z \in \mathbf{CK}^*$*

$$f'(z)=\sum_{0\leqslant\beta<\alpha\leqslant m} a_2^\alpha(z_2)\tilde{v}_2^\beta(z_2)(-iz_1)^{\alpha-\beta-1}$$
$$-\sum_{0\leqslant|k|\leqslant m-2} f^k(-iz)^k+\sum_{0\leqslant\beta<\alpha\leqslant m} a_1^\alpha(z_1)\tilde{v}_1^\beta(z_1)(-iz_2)^{\alpha-\beta-1}+\tilde{f}(z), \tag{1.9}$$

where $f\in H_{s-m}(\overline{\mathbf{K}})$, $v_l^\beta\in H_{s-\beta-1/2}(\overline{\mathbf{R}}^+)$, *the* f^k *are complex numbers and the following compatibility conditions hold: there exist constants* $v^\beta\in\mathbf{C}$, $0\leq|\beta|<s-1$, *satisfying the system of equations*

$$f^k=\sum_{\alpha-\beta-I=k} a^\alpha v^\beta,\quad 0\leqslant|k|\leqslant m-2,$$
$$\partial_1^{\beta_1}v_1^{\beta_2}(0)=v^\beta,\quad s-|\beta|>1,\ 0\leqslant\beta_2\leqslant m-1,$$
$$\partial_2^{\beta_2}v_2^{\beta_1}(0)=v^\beta,\quad s-|\beta|>1,\ 0\leqslant\beta_1\leqslant m-1, \tag{1.10}$$
$$\partial^k f(0)=\sum_{\alpha-\beta=k} a^\alpha v^\beta,\quad s-m-|k|>1,$$

where $I=(1,1)$.

(ii). *Conversely, suppose* $\tilde{f}'\in\mathcal{T}^*(A)$ *and conditions* (1.9) *and* (1.10) *hold. If* $A(D)$ *is a strongly elliptic operator of order* m *and* s *is not an integer, then* $f'=A(D)u$, *where* $u\in H_s(\overline{\mathbf{K}})$. *Here the functions* v_l^β *are Cauchy data for* u:

$$v_1^\beta(x_1)=\partial_2^\beta u(x_1,0),\quad x_1>0,\ 0\leqslant\beta\leqslant m-1,$$
$$v_2^\beta(x_2)=\partial_1^\beta u(0,x_2),\quad x_2>0,\ 0\leqslant\beta\leqslant m-1, \tag{1.11}$$

and the constants v^β *are equal to the derivatives of* u *at the vertex of the quadrant:*

$$v^\beta=\partial^\beta u(0). \tag{1.12}$$

The function u *is determined by* (1.4); *moreover*

$$\|u\|_s\leqslant C\Big(\|f\|_{s-m}+\sum_{l,\beta}\|v_l^\beta\|_{s-\beta-1/2}+\sum_k|f^k|\Big). \tag{1.13}$$

Lemma 1.1. *Let* $A(D)$ *be a strongly elliptic operator of order* m. *Then the system* (1.10) *of linear equations is solvable if and only if* $m\cdot[s]$ *linearly independent orthogonality conditions hold on the constants* f^k *and on the derivatives at zero of the functions* f *and* v_l^β, $0\leq\beta\leq m-1$, $l=1,2$.

In consequence of Theorem 1.1 and Lemma 1.1, for a strongly elliptic operator $A(D)$ of order m the first equation of (1.1) is equivalent to the system of conditions

$$\tilde{f}'\in\mathcal{T}^*(A),$$
$$L_j(\mathbf{f},\partial v_1(0),\partial v_2(0),\partial f(0))=0,\quad 1\leqslant j\leqslant m[s], \tag{1.14}$$

where f' is of the form (1.9) and the L_j are independent linear functionals. In (1.4) we have used the notation

$$
\begin{aligned}
\mathfrak{f} &= \{f^k \mid 0 \leqslant |k| \leqslant m-2\}, \\
\partial v_1(0) &= \{\partial_1^{\beta_1} v_1^{\beta_2}(0) \mid s - |\beta| > 1,\ 0 \leqslant \beta_2 \leqslant m-1\}, \\
\partial v_2(0) &= \{\partial_2^{\beta_2} v_2^{\beta_1}(0) \mid s - |\beta| > 1,\ 0 \leqslant \beta_1 \leqslant m-1\}, \\
\partial f(0) &= \{\partial^k f(0) \mid s - m - |k| > 1\}.
\end{aligned} \tag{1.15}
$$

A one-to-one correspondence between solutions of the system (1.14) and the first equation of (1.1) is effected by (1.4).

From (1.11) it follows that besides the known function f and the constants f^k only the Cauchy data for the solution u figure in the first condition of the system (1.14). But the operator $A(D)$ is elliptic and consequently does not have real characteristics. Therefore all normal derivatives of u on the boundary of the quadrant can be expressed in terms of the Cauchy data and the derivatives of f. We set $u_1^\beta(x_1) = \partial_2^\beta u(x_1, 0)$ for $x_1 > 0$ and $u_2^\beta(x_2) = \partial_1^\beta u(0, x_2)$ for $x_2 > 0$. Then, as in the Cauchy-Kowalewski theorem, from the first equation of (1.1) for $\beta \geq m$ we obtain the recurrence relation

$$
u_1^\beta(x_1) = \frac{1}{a_1^m(0)} \left(\partial_2^{\beta-m} f(x_1, 0) - \sum_{\alpha=0}^{m-1} a_1^\alpha(D_1) u_1^{\beta-m+\alpha}(x_1) \right), \quad x_1 > 0. \tag{1.16}
$$

A similar relation also holds for the functions $u_2^\beta(x_2)$.

From what has been said it follows that the boundary conditions for (1.1) can be expressed in terms of the Cauchy data for the solution in the following way: for $l = 1, 2$

$$
\sum_{0 \leqslant \beta \leqslant m-1} P_{l\beta}(D_l) v_l^\beta(x_l) = \varphi_l(x_l), \quad x_l > 0, \tag{1.17}
$$

where $P_{l\beta}(D_l) = (P_{l\beta}^1(D_l), \cdots, P_{l\beta}^{n_l}(D_l))$ are differential operators with constant vector coefficients belonging to $\mathbf{C}^{n_l}$ and ϕ_l is the sum of f_l and the derivatives of f on the boundary. For example, if $m_1 \leq m - 1$, then

$$
P_{1\beta}(z_1) = \frac{1}{\beta!} (D_2^\beta B_1)(z_1, 0), \tag{1.18}
$$

and $\phi_1 = f_1$. In the general case the symbols $P_{l\beta}(z_l)$ and functions $\phi_l(x_l)$ are computed using the recurrence relations (1.16).

Remark 1.1. (i) From (1.16) it follows that

$$
\sum_{0 \leqslant \beta \leqslant m-1} P_{1\beta}(z_1)(-iz_2)^\beta = B_1(z) \bmod A(z) \tag{1.19}
$$

and similarly

$$
\sum_{0 \leqslant \beta \leqslant m-1} P_{2\beta}(z_2)(-iz_1)^\beta = B_2(z) \bmod A(z). \tag{1.20}
$$

(ii) If $f = 0$, then $\phi_l = f_l$ for $l = 1, 2$.

(iii) The degree of the polynomial $P^j_{l\beta}$ does not exceed $m^j_l - \beta$.

Applying the Fourier transformation to (1.17), we obtain

$$\sum_{0\leqslant\beta\leqslant m-1} P_{l\beta}(z_l)\widetilde{v}_l^{\beta}(z_l) = \varphi_l(z_l) + \sum_{0\leqslant k\leqslant m_l-1} \varphi_{lk} z_l^k, \quad z_l \in \mathbf{C}^+, \tag{1.21}$$

for $l = 1, 2$, where $\phi_{lk} \in \mathbf{C}^{n_l}$.

Finally, we note that the first condition in the system (1.14) is equivalent to the equality $\widetilde{f}'(z) = 0$ on the complex characteristics of $A(D)$ situated in $\mathbf{CK}^*$, provided the polynomial $A(z)$ is irreducible. Then we arrive at the following equivalent statement of the boundary value problem (1.1). We denote by $V^*(A)$ the part of the characteristic $A(z) = 0$ lying in $\mathbf{CK}^*$.

Theorem 1.2. *Let $A(D)$ be a strongly elliptic operator of order m and suppose its symbol is irreducible. Then*

(i). *The boundary value problem* (1.1) *is equivalent to the following problem. Find functions $v_l^{\beta} \in H_{s-\beta-1/2}(\overline{\mathbf{R}}^+)$, $l = 1, 2$, z and $0 \leq \beta \leq m-1$, satisfying the system of equations*

$$\begin{aligned} \widetilde{f}'(z) = \sum_{0\leqslant\beta<\alpha\leqslant m} a_2^{\alpha}(z_2)\widetilde{v}_2^{\beta}(z_2)(-iz_1)^{\alpha-\beta-1} - \sum_{0\leqslant|k|\leqslant m-2} f^k(-iz)^k \\ + \sum_{0\leqslant\beta<\alpha\leqslant m} a_1^{\alpha}(z_1)\widetilde{v}_1^{\beta}(z_1)(-iz_2)^{\alpha-\beta-1} + \widetilde{f}(z) = 0, \ z \in V^*(A), \\ \sum_{0\leqslant\beta\leqslant m-1} P_{l\beta}(z_l)\widetilde{v}_l^{\beta}(z_l) = \widetilde{\varphi}_l(z_l) + P_l(z_l), \quad z_l \in \mathbf{C}^+, \ l = 1, 2, \\ L_j(\mathbf{f}, \partial v_1(0), \partial v_2(0), \partial f(0)) = 0, \quad 1 \leqslant j \leqslant m[s], \end{aligned} \tag{1.22}$$

where the f^k are arbitrary constants and the P_l are arbitrary polynomials of degree not exceeding $m_l - 1$, with coefficients in $\mathbf{C}^{n_l}$.

(ii). *A one-to-one correspondence between solutions of the problems* (1.22) *and* (1.1) *is effected by the formula*

$$u = F^{-1}\frac{\widetilde{f}'(z)}{A(z)}; \tag{1.23}$$

moreover the estimate (1.13) *is valid.*

Proof. If u is a solution of (1.1), then by virtue of (1.11) and (1.12) the functions $v_l^{\beta} = u_l^{\beta}$ and constants $f^k = \Sigma_{\alpha-\beta-l=k} a^{\alpha}\partial^{\beta}u(0)$ satisfy (1.22). Conversely, from the first equation of (1.22) and the compatibility conditions $L_j(\ldots) = 0$ it follows by Theorem 1.1 and Lemma 1.1 that u, defined by formula (1.23), belongs to $H_s(\mathbf{K})$ and satisfies the first equation of (1.1). But by (1.11) we have $v_l^{\beta} = u_l^{\beta}$. Therefore, from the second and third equations of (1.22) it follows that the boundary conditions of problem (1.1) are also satisfied. Theorem 1.2 is proved.

§2. Reduction of the boundary value problem in a quadrant for a second-order elliptic equation to an algebraic equation with a shift

In §§2–6 we shall fully investigate the boundary value problem (1.1) for a second-order strongly elliptic operator $A(D)$ and scalar operators $B_l(D)$ of arbitrary order m_l. Thus we now consider the problem

$$\begin{aligned} &A(D)u(x) = f(x), \quad x_1 > 0,\ x_2 > 0, \\ &B_1(D)u(x_1, 0) = f_1(x_1), \quad x_1 > 0, \\ &B_2(D)u(0, x_2) = f_2(x_2), \quad x_2 > 0, \end{aligned} \tag{2.1}$$

in which $f \in H_{s-2}(\overline{\mathbf{K}})$ and $f_l \in H_{s-m_l-\frac{1}{2}}(\overline{\mathbf{R}}^+)$, with $s > 3/2$, $s - m_l - \frac{1}{2} > 0$ and s nonintegral. There is a solution u in the space $H_s(\overline{\mathbf{K}})$. We set

$$\mathcal{H}_{(s)}(\overline{\mathbf{K}}) = H_{s-2}(\overline{\mathbf{K}}) \oplus H_{s-m_1-1/2}(\overline{\mathbf{R}}^+) \oplus H_{s-m_2-1/2}(\overline{\mathbf{R}}^+). \tag{2.2}$$

Then (2.1) can be written briefly in the form

$$\mathcal{A}u = \mathcal{F}, \tag{2.3}$$

where $\mathcal{F} \in \mathcal{H}_{(s)}(\overline{\mathbf{K}})$.

Since $m = 2$, strong ellipticity of the operator $A(D)$ means that for $z \in \mathbf{R}^2$

$$|A(z)| \geqslant C(1 + |z|)^2. \tag{2.4}$$

From (2.4) follows the irreducibility of $A(z)$. In fact, if $A(z) = A_1(z)A_2(z)$, where the $A_l(z)$ are polynomials, then the A_l are strongly elliptic and therefore have even degree, equal to 0 or 2.

For brevity we denote $V^*(A)$ by V^* and $V(A)$ by V. Since $A(z)$ is irreducible, from Theorem 1.2 it follows that problem (2.1) is equivalent to the following system of algebraic equations:

$$\begin{aligned} &\tilde{f}(z) + \tilde{v}_1^0(z_1)(a_1^1(z_1) - iz_2 a_1^2) + \tilde{v}_1^1(z_1)a_1^2 \\ &+ \tilde{v}_2^0(z_2)(a_2^1(z_2) - iz_1 a_2^2) + \tilde{v}_2^1(z_2)a_2^2 - f^0 = 0, \quad z \in V^*, \\ &P_{11}(z_1)\tilde{v}_1^1(z_1) + P_{10}(z_1)\tilde{v}_1^0(z_1) = \tilde{\varphi}_1(z_1) + P_1(z_1), \quad z_1 \in \mathbf{C}^+, \\ &P_{21}(z_2)\tilde{v}_2^1(z_2) + P_{20}(z_2)\tilde{v}_2^0(z_2) = \tilde{\varphi}_2(z_2) + P_2(z_2), \quad z_2 \in \mathbf{C}^+, \\ &L_j(f^0, \partial v_1(0), \partial v_2(0), \partial f(0)) = 0, \quad 1 \leqslant j \leqslant 2[s]. \end{aligned} \tag{2.5}$$

Here the $\tilde{\varphi}_l \in \tilde{H}_{s-m_l-\frac{1}{2}}(\overline{\mathbf{R}}^+)$ and in correspondence with (1.15)

$$\begin{aligned} &\partial v_1(0) = \{\partial_1^{\beta_1} v_1^{\beta_2}(0) \mid s - |\beta| > 1,\ \beta_2 = 0, 1\}, \\ &\partial v_2(0) = \{\partial_2^{\beta_2} v_2^{\beta_1}(0) \mid s - |\beta| > 1,\ \beta_1 = 0, 1\}, \\ &\partial f(0) = \{\partial^k f(0) \mid s - 2 - |k| > 1\}. \end{aligned} \tag{2.6}$$

It is assumed that the $v_l^\beta \in H_{s-\beta-\frac{1}{2}}(\overline{\mathbf{R}}^+)$, f^0 is an arbitrary constant and the P_l are arbitrary polynomials of degree not exceeding $m_l - 1$, with complex coefficients.

In view of statement (i) of Remark 1.1, for $z \in V$

$$P_{11}(z_1)(-iz_2) + P_{10}(z_1) = B_1(z). \tag{2.7}$$

By virtue of (iii) of the same remark, $\deg P_{11} \leq m_1 - 1$ and $\deg P_{10} \leq m_1$. We now note that for the a priori estimate

$$\|u\|_{s+\varepsilon} \leqslant C\left(\|f\|_{s+\varepsilon-2} + \sum_{l=1,2} \|f\ \|_{s+\varepsilon-m_l-1/2} + \|u\|_s\right) \tag{2.8}$$

to hold for at least one $\epsilon > 0$ it is necessary that problem (2.1) satisfy the Šapiro-Lopatinskiĭ condition on the smooth parts of the boundary of the quadrant. For (2.1) this condition on $x_2 = 0$ is equivalent to

$$B_{10}(z) \neq 0 \tag{2.9}$$

for $A_0(z) = 0$, $z_1 \in \mathbf{R}\setminus\{0\}$ and $z_2 \in \mathbf{C}^-$, while on $x_1 = 0$ it is equivalent to

$$B_{20}(z) \neq 0 \tag{2.10}$$

for $A_0(z) = 0$, $z_2 \in \mathbf{R}\setminus\{0\}$ and $z_1 \in \mathbf{C}^-$. Here B_{l0} denotes the leading homogeneous part of degree m_l of the polynomial B_l, and A_0 is the leading homogeneous part of A. From (2.7) and (2.9) it follows that equality holds in at least one of the relations $\deg P_{11} \leq m_1 - 1$ and $\deg P_{10} \leq m_1$. Suppose, for example,

$$\deg P_{10} = m_1. \tag{2.11}$$

Let us also assume that

$$\deg P_{20} = m_2. \tag{2.12}$$

The cases when P_{10} and P_{21} or P_{11} and P_{21} have maximum degree are examined similarly.

Thus, suppose (2.11) and (2.12) are satisfied. We eliminate the functions $\tilde{v}_l^0$ from the first three equations of (2.5). Then we obtain one equation in two unknown functions:

$$S_1(z)\tilde{v}_1^1(z_1) + S_2(z)\tilde{v}_2^1(z_2) = g(z), \quad z \in V^*. \tag{2.13}$$

Here we have used the notation

$$S_1 = -P_{20}a_1^2\left(P_{11}\left(-\frac{a_1^1}{a_1^2} + iz_2\right) + P_{10}\right),$$

$$S_2 = -P_{10}a_2^2\left(P_{21}\left(-\frac{a_2^1}{a_2^2} + iz_1\right) + P_{20}\right), \tag{2.14}$$

$$g = P_{10}P_{20}(\tilde{f} - f^0) + P_{20}(a_1^1 - iz_2a_1^2)\tilde{\varphi}_1' + P_{10}(a_2^1 - iz_1a_2^2)\tilde{\varphi}_2',$$

where $\tilde{\phi}_l' = \tilde{\phi}_l + P_l$. Note that $a_l^2 \neq 0$ for $l = 1, 2$ in view of (2.4).

As an algebraic equation in the functions $\tilde{v}_l'$, relation (2.13) is underdetermined. But it can be raised to a well-posed problem if we take into account the fact that $\tilde{v}_l'$ is holomorphic in $V_l^+ = \{z \in V | z_l \in \mathbf{C}^+\}$ and depends only on z_l. The latter property is expressed analytically as the invariance of $\tilde{v}'_l$ with respect to the monodromy group of the covering p_l: $V_l^+ \to \mathbf{C}^+$ effected by the coordinate function:

$$p_l z = z_l. \tag{2.15}$$

The monodromy group of the covering p_l is isomorphic to Z_2. In fact, its generator h_l transposes the roots of the equation $A(z) = 0$ with the same coordinates z_l. Consequently by Viète's formulas

$$h_1 z = \left(z_1, -i\frac{a_1^1(z_1)}{a_1^2} - z_2\right) \tag{2.16}$$

and similarly

$$h_2 z = \left(-i\frac{a_2^1(z_2)}{a_2^2} - z_1, z_2\right). \tag{2.17}$$

The monodromy group of p_l acts transitively in its fibers; consequently (2.13) is equivalent to the system

$$\begin{aligned} S_1(z)v_1(z) + S_2(z)v_2(z) &= \varphi(z), \quad z \in V^*, \\ v_1^{h_1}(z) &= v_1(z), \quad z \in V_1^+, \\ v_2^{h_2}(z) &= v_2(z), \quad z \in V_2^+, \end{aligned} \tag{2.18}$$

under the assumption that $v_l(z_l) \in \tilde{H}_{s-3/2}(\overline{\mathbf{R}}^+)$. In (2.18) we have used the exponential notation for the action of the automorphisms h_l in the space of functions on V_l^+:

$$v_l^{h_l}(z) = v_l(h_l z). \tag{2.19}$$

To simplify the investigation of (2.18) we reduce the consideration of (2.1) to the case $f = 0$. For this we find a particular solution of the equation

$$A(D)u_+(x) = f(x), \quad x_1 > 0, \ x_2 > 0, \tag{2.20}$$

belonging to $H_s(\overline{\mathbf{K}})$. Since $u_+ \in H_s(\overline{\mathbf{K}})$, the boundary value problem (2.1) is equivalent to

$$\begin{aligned} A(D)u^0(x) &= 0, \quad x_1 > 0, \ x_2 > 0, \\ B_1(D)u^0(x_1, 0) &= f_1^0(x_1), \quad x_1 > 0, \\ B_2(D)u^0(0, x_2) &= f_2^0(x_2), \quad x_2 > 0, \end{aligned} \tag{2.21}$$

where $u^0 = u - u_+$, $f_1^0(x_1) = f_1(x_1) - B_1(D)u_+(x_1, 0)$ and $f_2^0(x_2) = f_2(x_2) - B_2(D)u_+(0, x_2)$.

Thus we may assume that $f = 0$. In addition, in view of statement (ii) of Remark 1.1

$$g = -P_{10}P_{20}f^0 + P_{20}(a_1^1 - iz_2a_1^2)\tilde{f}'_1 + P_{10}(a_2^1 - iz_1a_2^2)\tilde{f}'_2. \tag{2.22}$$

Let us find the general solution of (2.18) for such a function g. First we find functions v_{1l} meromorphic in V_l^+ and satisfying (2.18) in the case $\tilde{f}'_z = 0$. Then we compute the general solution v_{2l} of the same system in the case $f^0 = 0$ and $\tilde{f}'_1 = 0$ under the assumption that $v_{1l}(z_l) + v_{2l}(z_l) \in \tilde{H}_{s-3/2}(\overline{\mathbf{R}}^+)$ for $l = 1, 2$. Then the sum $v_{1l} + v_{2l} = v_l$ will be the general solution of (2.18). As we shall show in the remaining part of this section, v_{12} and v_{21} are solutions of linear algebraic equations with a shift on Riemann surfaces.

First of all consider the case $\tilde{f}_2' = 0$. Then

$$g(z) = g_1(z) = -P_{10}P_{20}f^0 + P_{20}(a_1^1 - iz_2a_1^2)\tilde{f}_1'(z_1). \tag{2.23}$$

Obviously g_1 is analytic in V_1^+. The function v_{11} is meromorphic in V_1^+ by assumption. Therefore, from the first equation of the system (2.18) it follows that S_2v_{12} is also meromorphic in V_1^+. But from (2.14), the irreducibility of $A(z)$ and the conditions (2.11) and (2.12), by the Hilbert Nullstellensatz it follows that

$$S_l \not\equiv 0 \tag{2.24}$$

on V for $l = 1, 2$. Consequently v_{12} can be continued to a meromorphic function in $W^+ = V_2^+ \cup V_1^+$, and as before the equality

$$S_1(z)v_{11}(z) + S_2(z)v_{12}(z) = g_1(z), \quad z \in W^+, \tag{2.25}$$

holds.

Let us apply the automorphism h_1 to both sides of (2.25). Then in view of the second equation of (2.18) we obtain

$$S_1^{h_1}v_{11}(z) + S_2^{h_1}(z)v_{12}^{h_1}(z) = g_1^{h_1}(z), \quad z \in V_1^+. \tag{2.26}$$

Taking account of (2.24) we find that under (2.23) the system (2.18) is equivalent to

$$\begin{gathered} S_1(z)v_{11}(z) + S_2(z)v_{12}(z) = g_1(z), \quad z \in V_1^+, \\ S_1^{h_1}(z)v_{11}(z) + S_2^{h_1}(z)v_{12}^{h_1}(z) = g_1^{h_1}(z), \quad z \in V_1^+, \\ v_{12}^{h_2}(z) = v_{12}(z), \quad z \in V_2^+. \end{gathered} \tag{2.27}$$

Eliminating v_{11} from the first two equations of (2.27), we arrive at the equivalent (via (2.24)) system

$$\begin{gathered} S_1(z)v_{11}(z) + S_2(z)v_{12}(z) = g_1(z), \quad z \in V_1^+, \\ S_1^{h_1}(z)S_2(z)v_{12}(z) - S_1(z)S_2^{h_1}(z)v_{12}^{h_1}(z) = S_1^{h_1}(z)g_1(z) - S_1(z)g_1^{h_1}(z), \quad z \in V_1^+, \\ v_{12}^{h_2}(z) = v_{12}(z), \quad z \in V_2^+. \end{gathered} \tag{2.28}$$

From the first equation of (2.18) we have deduced that v_{12} is meromorphic in $W^+ = V_1^+ \cup V_2^+$. The second equation of this system has also been used to derive (2.26). We now consider the third equation of (2.18), i.e. the invariance of v_{12} with respect to h_2 in V_2^+. We shall show that from this invariance it follows that v_{12} is meromorphic in the multisheeted region $\hat{W}_2 = W^+ \cup_{h_2'} h_2W^+$, where h_2' is the restriction of h_2 to V_2^+.

The space $\hat{W}_2$ is a covering of $W_2 = W^+ \cup h_2W^+$ with projection p. We set $\overline{\hat{W}}_2 = \overline{W^+} \cup_{h_2''} \overline{h_2W^+}$, where h_2'' is the restriction of h_2 to $\overline{V_2^+}$. If i_1: $\overline{W^+} \to \overline{\hat{W}}_2$ and i_2: $\overline{h_2W^+} \to \overline{\hat{W}}$ are the natural imbeddings, then

$$pw = \begin{cases} i_1^{-1}w, & w \in i_1\overline{W^+}, \\ i_2^{-1}w, & w \in i_2h_2\overline{W^+}. \end{cases} \tag{2.29}$$

Let $\hat{h}_2$ be a lifting of h_2 to $\overline{\hat{W}}_2$, i.e. an automorphism that makes the diagram

$$\begin{array}{ccc} \overline{\hat{W}_2} & \xrightarrow{\hat{h}_2} & \overline{\hat{W}_2} \\ \downarrow p & & \downarrow p \\ \overline{W_2} & \xrightarrow{h_2} & \overline{W_2} \end{array} \tag{2.30}$$

commutative. Obviously $\hat{h}_2$ is, like h_2, an involution:

$$\hat{h}_2^2 = 1. \tag{2.31}$$

We denote by $\hat{V}_l^+$ the sheet of the covering $\hat{W}_2$ lying in $i_1 W^+$ over V_l^+. The automorphism h_1 carries $\overline{V_1^+}$ into itself and therefore it can be lifted to $\hat{h}_1$ on $\hat{V}_1^+$. In this connection the diagram

$$\begin{array}{ccc} \overline{\hat{V}_1^+} & \xrightarrow{\hat{h}_1} & \overline{\hat{V}_1^+} \\ \downarrow p & & \downarrow p \\ \overline{V_1^+} & \xrightarrow{h_1} & \overline{V_1^+} \end{array} \tag{2.32}$$

is commutative and $\hat{h}_1$ is an involution:

$$\hat{h}_1^2 = 1. \tag{2.33}$$

Let $\hat{v}_{12}$ be a lifting of v_{12} to $i_1 W^+$:

$$\hat{v}_{12}(w) = v_{12}(pw), \quad w \in i_1 W^+. \tag{2.34}$$

From the commutativity of (2.30) it follows that $\hat{v}_2$ on $\hat{V}_2^+$ is invariant with respect to $\hat{h}_2$. Therefore $\hat{v}_{12}$ can be continued to a meromorphic function on W_2, invariant with respect to $\hat{h}_2$:

$$\hat{v}_{12}^{\hat{h}_2}(w) = \hat{v}_{12}(w), \quad w \in \hat{W}_2. \tag{2.35}$$

Now we proceed to examine (2.28). Let us write its second equation in the form

$$Q_1(z) v_{12}(z) - Q_1^{h_1} v_{12}^{h_1}(z) = G_1(z), \quad z \in V_1^+. \tag{2.36}$$

Obviously

$$G_1^{h_1}(z) = -G_1(z), \quad z \in V_1^+. \tag{2.37}$$

We now use (2.35) to reduce (2.36) to an algebraic equation on $\hat{W}_2$ with shift $\hat{h} = \hat{h}_2 \hat{h}_1$. In §3 we shall show that for the fundamental domain $\hat{\Pi}_1$ of the group generated by $\hat{h}$ this equation is the Haseman problem [11] and by passage to the quotient space $\overline{\hat{\Pi}_1}/\hat{h}$ it can be reduced to the Riemann problem. In §4 we shall solve the latter problem in quadratures.

Remark 2.1. Relation (2.36) itself is an equation with shift h_1. But after identifying the boundary of V_1^+ by means of h_1 a nonanalytic space is obtained, since h_1 carries V_1^+ into itself. Therefore (2.36) cannot be reduced to the Riemann problem by the method of conformal identification. Equation (2.36) is called a one-sided Carleman problem and it can be reduced to a Fredholm equation of the first kind [11].

Let us lift equation (2.36) to $\hat{W}_2$. Let $\hat{Q}_1$ be a lifting of Q_1 to $\hat{W}_2$ and $\hat{G}_1$ a lifting of G_1 to $\hat{V}_1^+$. Taking account of (2.35), we transform (2.36) into

$$\hat{Q}_1(w)\,\hat{v}_{12}(w) - \hat{Q}_1^{\hat{h}_1}(w)\,\hat{v}_{12}^{\hat{h}}(w) = \hat{G}_1(w), \quad w \in \hat{V}_1^+, \tag{2.38}$$

where $\hat{h} = \hat{h}_2\hat{h}_1$. In (2.38) we have used the commutativity of (2.32).

Obviously the system of equations (2.38) and (2.35) is equivalent to the last two equations in (2.28). Solving (2.38) and (2.35) simultaneously, we then determine v_{11} from the first equation of (2.28), since $S_1 \neq 0$ on V.

We shall show that in fact (2.35) "follows" from the structure of (2.38).

Theorem 2.1. *If the function* $\hat{v}_2$ *satisfies* (2.38), *then* $\hat{v}_2^{\hat{h}_2}$ *is also a solution of this equation, and the function* $\frac{1}{2}(\hat{v}_2 + \hat{v}_2^{\hat{h}_2})$ *satisfies* (2.38) *as well as* (2.35).

Proof. In view of (2.31), (2.33) and (2.37), for $w \in \hat{V}_1^+$

$$\hat{Q}_1(w)\,\hat{v}_2^{\hat{h}_2}(w) - \hat{Q}_1^{\hat{h}_1}(w)\,\hat{v}_2^{\hat{h}_2}(\hat{h}w) = \hat{Q}_1(w)\,\hat{v}_2(\hat{h}_2 w) - \hat{Q}_1(\hat{h}_1 w)\,\hat{v}_2(\hat{h}_1 w) = -\hat{G}_1(\hat{h}_1 w) = \hat{G}_1(w). \tag{2.39}$$

Thus it suffices to find a solution of (2.38) meromorphic in $\hat{W}$. Hence the consideration of (2.18) in the case $f_z' = 0$ under the assumption that the v_{1l} are meromorphic in V_l^+ has brought us to the linear algebraic equation (2.38) with shift $\hat{h}$ on the Riemann surface $\hat{W}_2$. Since in the case $f_1' = 0$ we shall seek a solution v_{2l} of (2.18) under the assumption that $\tilde{v}_{1l} + \tilde{v}_{2l} \in H_{s-3/2}(\overline{\mathbf{R}}^+)$, the functions v_{2l} are also meromorphic in V_l^+. Consequently v_{21} satisfies a linear equation with a shift similar to (2.38) on the corresponding Riemann surface $\hat{W}_1$.

§3. Reduction of a linear algebraic equation with a shift to the Riemann conjugation problem

To solve equation (2.38) it is necessary to inquire into the topology of the space $\hat{W}_2$ and the action of the automorphism $\hat{h}$ on $\hat{W}_2$.

We first consider the algebraic curve V. The covering $V_l^+ \overset{p_l}{\to} \mathbf{C}^+$ can be extended to a covering $V \overset{p_l}{\to} \mathbf{C}$ in accordance with formula (2.15). Let $\overline{V}$ be the projectivization of the affine algebraic curve V. Then the covering can in turn be extended to a covering $\overline{V} \overset{p_l}{\to} \mathbf{CP}^1$. Obviously the last covering is two-sheeted and has two double branch points z_l' and z_l''. These points are distinct, since $A(z)$ is irreducible. They are not real (because $A(z)$ is strongly elliptic) and consequently finite. The monodromy groups of the coverings $V_l^+ \overset{p_l}{\to} \mathbf{C}$ and $\overline{V} \overset{p_l}{\to} \mathbf{CP}^1$ are isomorphic (and isomorphic to Z_2). Their "common" generator h_l acts according to formulas (2.16) and (2.17).

Lemma 3.1. (i) *The projective algebraic curve* $\overline{V}$ *is birationally equivalent to* $\mathbf{CP}^1$. *The equivalence is given by a mapping* $\chi\colon \mathbf{CP}^1 \to \overline{V}$ *in accordance with the formulas*

$$z_l = p_l\chi(\lambda) = \omega_l\left(\theta_l\lambda + \frac{1}{\theta_l\lambda}\right) + \sigma_l, \quad l = 1, 2. \tag{3.1}$$

The inverse mapping has the form

$$\lambda = \psi(z) = a_1 z_1 + a_2 z_2 + a_0. \tag{3.2}$$

(ii) *The automorphisms* h_l *go into* $h_l^* = \psi h_l \chi$ *under this equivalence*:

$$h_l^*\lambda = \frac{1}{\theta_l^2 \lambda}. \tag{3.3}$$

(iii) *The mapping* ψ *establishes a holomorphic equivalence of the affine algebraic curve* V *with the Riemann surface* $\mathbf{C}\setminus\{0\}$.

Proof. By means of the affine transformation

$$z = c\zeta + d \tag{3.4}$$

the polynomial $A(z)$ is brought into the form

$$A_1(\zeta) = \zeta_1^2 + \zeta_2^2 + b_0^2, \tag{3.5}$$

where $b_0 \neq 0$. In fact, the quadratic form occurring in $A(z)$ can be reduced to a sum of squares by a nonsingular linear transformation. The form is nondegenerate by virtue of the strong ellipticity of A, since the polynomial

$$\zeta_1^2 + \varepsilon z_2^2 + a z_2 + b \tag{3.6}$$

can have real roots in z_2 for any ζ_1 and arbitrarily small complex ϵ. In consequence of the nondegeneracy of the quadratic form, the linear terms in the symbol disappear under a suitable shift. Furthermore, from the irreducibility of $A(z)$ follows the irreducibility of $A_1(\zeta)$; therefore $b_0 \neq 0$.

Thus the surface $\overline{V}$ is the graph of the root $\zeta_1 = \sqrt{-\zeta_2^2 - b_0^2}$ of $A_1(\zeta)$. Therefore it is a double covering of $\mathbf{CP}^1$ with projection q_2: $\zeta \to \zeta_2$ and double branch points $\pm ib_0$. Now it is easy to establish a homeomorphism ψ of the covering q_2 with the double covering j: $\mathbf{CP}^1 \to \mathbf{CP}^1$ effected by Žukov's function

$$\zeta_2 = j(\lambda) = \frac{ib_0}{2}\left(\lambda + \frac{1}{\lambda}\right). \tag{3.7}$$

In fact j (as well as q_2) branch at $\pm ib_0$; consequently there exists a homeomorphism ψ: $\overline{V} \to \overline{\mathbf{C}}\mathbf{P}^1$ for which the diagram

$$\begin{array}{ccc} \overline{V} & \xrightarrow{\psi} & \mathbf{CP}^1 \\ & {\scriptstyle q_2}\searrow \quad \swarrow{\scriptstyle j} & \\ & \mathbf{CP}^1 & \end{array} \tag{3.8}$$

is commutative. Except for $q_2^{-1}(\pm ib_0)$ and $j^{-1}(\pm ib_0)$ the local coordinates of both covering spaces are obviously equivalent to ζ_2, and at the exceptional points they are equivalent to $\sqrt{\zeta_2 \mp ib_0}$. Consequently ψ is a holomorphic equivalence.

Let us find formulas for the mapping $\chi = \psi^{-1}$. Since $\zeta_2 = j(\lambda) = ib_0(\lambda + 1/\lambda)/2$, it follows that

$$\zeta_1 = \sqrt{\frac{b_0^2}{4}\left(\lambda - \frac{1}{\lambda}\right)^2} = \pm\frac{b_0}{2}\left(\lambda - \frac{1}{\lambda}\right). \tag{3.9}$$

To define χ uniquely, let $\chi(i) = (ib_0, 0)$. Then

$$\zeta_1 = \frac{b_0}{2}\left(\lambda - \frac{1}{\lambda}\right). \tag{3.10}$$

From (3.7) and (3.10) it follows that

$$\lambda = \frac{\zeta_1 - i\zeta_2}{b_0}. \tag{3.11}$$

Thus the birational equivalence of $\overline{V}$ and $\mathbf{CP}^1$ is established. In addition, from (3.11) and the invertibility of the substitution (3.4) follows (3.2).

From (3.4), by virtue of (3.7) and (3.10), we obtain

$$z = c\zeta + d = c_1\lambda + \frac{c_2}{\lambda} + d. \tag{3.12}$$

The covering $p_l\chi$ is two-sheeted; therefore the vectors c_1 and c_2 have nonzero components and consequently (3.12) can be transformed into the form (3.1).

To prove formula (3.3) we note that the automorphism h_l^* permutes the roots of the equation

$$z_l = \omega_l\left(\theta_l\lambda + \frac{1}{\theta_l\lambda}\right) + \sigma_l. \tag{3.13}$$

Consequently by Viète's formulas

$$\lambda \cdot h_l^*\lambda = \frac{1}{\theta_l^2}. \tag{3.14}$$

Lemma 3.1 is proved.

Now let us consider the space W^+. For convenience we identify V with $\mathbf{CP}^1$ by means of the birational equivalence ψ. Thus h_l is identified with h_l^*. We introduce the notation

$$\Gamma_1^\pm = p_1^{-1}\mathbf{R} \cap V_2^\pm, \quad \Gamma_2^\pm = p_2^{-1}\mathbf{R} \cap V_1^\pm. \tag{3.15}$$

Evidently

$$h_l\Gamma_l^\pm = \Gamma_l^\mp \tag{3.16}$$

for $l = 1, 2$. From the strong ellipticity of $A(z)$ it follows that $\Gamma_l^\pm$ are smooth connected curves. Let $\Gamma_l = \Gamma_l^+ \cup \Gamma_l^-$.

Lemma 3.2. (i) *For $l = 1, 2$ the curves $\Gamma_l^\pm$ join the points 0 and ∞ in* $\mathbf{CP}^1$ *and do not intersect.*

(ii) *For $l = 1, 2$ the curves $\overline{\Gamma}_l$ are smooth and intersect transversally at the points 0 and ∞.*

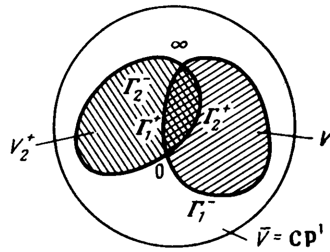

Figure 2

(iii) *The branch points of the covering* p_l *lie on different sides of the real axis.*

Proof. From (3.15) it follows that

$$\overline{\Gamma}_l = p_l^{-1}\mathbf{RP}^1. \tag{3.17}$$

Since the branch points of p_l are finite and not real, $\overline{\Gamma}_l$ are smooth curves. Each of the curves $\overline{\Gamma}_l$ passes through 0 and ∞ and has at most two connected components. But from the strong ellipticity of $A(z)$ it follows that the curves $\overline{\Gamma}_1$ and $\overline{\Gamma}_2$ cannot intersect at any points different from 0 and ∞ and cannot be tangent at these intersection points. This is possible only if they are connected. In this case the points 0 and ∞ split $\overline{\Gamma}_l$ into two nonintersecting curves $\Gamma_l^{\pm}$, which proves (i). Assertion (ii) has already been proved, and (iii) follows from the connectedness of $\overline{\Gamma}_l$ and formula (3.1). The lemma is proved.

Now we can reduce equation (2.38) to Riemann's conjugation problem. From (3.16) it follows that for $h = h_2 h_1$

$$h\Gamma_1^{\pm} \subset V_2^{\mp}. \tag{3.18}$$

Therefore

$$h\Gamma_1^{\pm} \cap \Gamma_1^{\pm} = \varnothing. \tag{3.19}$$

Next, by virtue of (3.3)

$$h\lambda = \theta^2\lambda, \tag{3.20}$$

where $\theta = \theta_1/\theta_2$. From (ii) of Lemma 3.2 and from (3.18) it follows that

$$\arg\theta \neq 0. \tag{3.21}$$

In addition, it is possible to assume that

$$|\arg\theta| < \pi. \tag{3.22}$$

Equality (3.20) means that $h\Gamma_1^-$ as well as Γ_1^- join 0 and ∞. But, in view of (i) of Lemma 3.2 the space $\widehat{W}_2$ is contractible. We denote by $\widehat{\Pi}_1$ the region in $\widehat{W}_2$ included between $\widehat{\Gamma}_1^-$ and $\hat{h}\widehat{\Gamma}_1^-$, where $\widehat{\Gamma}_1^-$ is the counterimage of Γ_1^- in $i_1\overline{W}^+$. Then equation (2.38) is equivalent to the following

$$\widehat{Q}_1(w)\,\hat{v}_{12}(w) - \widehat{Q}_1^{\hat{h}_1}(w)\,\hat{v}_{12}^{\hat{h}}(w) = \widehat{\mathscr{F}}_1(w), \quad w \in \widehat{\Gamma}_1^-, \tag{3.23}$$

under the assumption that $\hat{v}_{12}$ is meromorphic in $\hat{\Pi}_1$. Here

$$\hat{\mathcal{G}}_1(w) = \hat{G}_1(w) + \sum_{0 \leqslant p < \hat{p}_k} \hat{G}_{kp}\delta^{(p)}(\hat{h}w - w_k), \tag{3.24}$$

where the w_k are the poles of $\hat{v}_{12}$ on $\hat{h}\hat{\Gamma}_1^-$ of orders $\hat{p}_k$. The boundary values of the function $\hat{v}_{12}$ on $\hat{\Gamma}_1^-$ and $\hat{h}\hat{\Gamma}_1^-$ are regarded as the limiting values approaching from within $\hat{\Pi}_1$ in the space $\mathfrak{D}'(\partial\hat{\Pi}_1)$.

Let us state the equivalence of problems (2.38) and (3.23) more precisely.

Theorem 3.1. (i) *Every solution of equation* (3.23) *can be extended to a meromorphic function in* $\hat{W}_2$ *and satisfies equation* (2.38).

(ii) *If* $\hat{v}_{12}$ *has a finite set of poles in* $\hat{\Pi}_1$ *and the sum in* (3.24) *is finite, then* $\hat{v}_{12}$ *has a finite set of poles in* $\hat{W}_2$.

Proof. Since W^+ and $\hat{W}_2$ are simply connected by virtue of Lemma 3.2, we can identify $\hat{W}_2$ with a region on the universal covering surface $\hat{V}$ of the curve V. From (iii) of Lemma 3.1 it follows that the universal covering for V is the complex plane $\mathbf{C}$, and

$$w = \ln \lambda \tag{3.25}$$

can be taken as uniformizing parameter on V. The automorphisms $\hat{h}_2$ and $\hat{h}_1$ can be extended from $\hat{W}$ and $\hat{V}_1^+$, respectively, to automorphisms of $\hat{V} \simeq \mathbf{C}$, and by virtue of (3.3) they act according to the formulas

$$\hat{h}_l w = -w + 2w_l, \tag{3.26}$$

where $w_l = \ln \theta_l$ is a fixed point of $\hat{h}_l$. Obviously $w_l \in \hat{V}_l^+$ and

$$p_l p w_l = z_l', \tag{3.27}$$

where z_l' is the branch point of the covering p_l lying in $\mathbf{C}^+$. This follows from the fact that w_l is a fixed point of $\hat{h}_l$—it lies over the fixed point of h_l.

Let $\hat{\Gamma}_l^\pm$ be the counterimages of $\Gamma_l^\pm$ in $i_1\overline{W}^+$. From (i) and (ii) of Lemma 3.2 it follows that $\hat{\Gamma}_l^\pm$ are smooth curves in the complex plane having asymptotes parallel to the real axis. Since $\hat{h}_l\hat{V}_l^+ = \hat{V}_l^+$ and $\hat{h}_l\hat{\Gamma}_l^\pm = \hat{\Gamma}_l^\pm$, by (3.26) $\hat{V}_l^+$ is centrally symmetric with respect to w_l and $\hat{\Gamma}_l^+$ and $\hat{\Gamma}_l^-$ are centrally symmetric to each other with respect to w_l. Similarly $\hat{W}_2$ is centrally symmetric with respect to w_2:

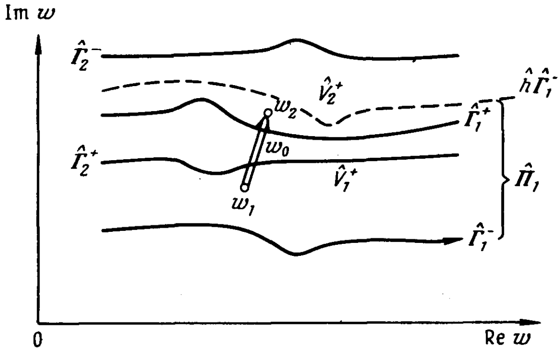

Figure 3

From (3.26) it follows that

$$\hat{h}w = w + 2w_0, \tag{3.28}$$

where $w_0 = w_2 - w_1 = \ln\theta$. From (3.21) and (3.22) it follows that

$$0 < |\operatorname{Im} w_0| < \pi. \tag{3.29}$$

Consequently $\overline{\hat{\Pi}}_1$ is the fundamental domain of the group of translations of the complex plane generated by $\hat{h}$. Since $\hat{\Gamma}_1^+$ and $\hat{h}\hat{\Gamma}_1^-$ lie in $\hat{V}_2^+$, the regions $\hat{V}_1^+$ and $\hat{\Pi}_1$ lie on one side of $\hat{\Gamma}_1^-$. Consequently

$$\hat{V}_1^+ = \bigcup_{0\leqslant k} (\hat{V}_1^+ \cap \hat{h}^k\overline{\hat{\Pi}}_1). \tag{3.30}$$

Now we can prove Theorem 3.1. Obviously it suffices to demonstrate that if $\hat{v}_{12}$ has a finite set of poles in $\hat{\Pi}_1$ and the sum in (3.24) is finite, then $\hat{v}_{12}$ can be extended to a meromorphic function in $\hat{V}_1^+$ and has a finite set of poles there. In fact $\hat{G}_1(w)$ is holomorphic in $\hat{V}_1^+$ by (2.23) and (2.28). Therefore if $\hat{v}_{12}$ is meromorphic in $\hat{V}_1^+$, then from (3.23), using the principle of analytic continuation by continuity, we obtain that $\hat{v}_{12}$ can be continued across $\hat{\Gamma}_1^+$ to a meromorphic function in $\overline{\hat{\Pi}}_1 \cup \hat{h}\hat{V}_1^+ = \hat{W}_2$. Finiteness of the set of poles of $\hat{v}_{12}$ in $\hat{W}_2$ follows from the birational equivalence of $\overline{V}$ and $\mathbf{CP}^1$ and from (2.24), and (2.38) follows from the uniqueness of analytic continuation.

Finally, we show by induction that $\hat{v}_{12}$ can be extended to a meromorphic function in each domain

$$\hat{V}_{1,n}^+ = \bigcup_{1\leqslant k\leqslant n} (V_1^+ \cap \hat{h}^k\overline{\hat{\Pi}}_1) \tag{3.31}$$

with $n \geq 0$ and has a finite set of poles there. For $n = 0$ the assertion follows directly from the hypothesis. Let us assume that it is true for some $n \geq 0$. We shall show that

$$\hat{h}^{-1}(\hat{V}_1^+ \cap \hat{h}^{n+1}\hat{\Pi}_1) \subset \hat{V}_{1,n}^+. \tag{3.32}$$

In fact

$$\hat{h}^{-1}(\hat{V}_1^+ \cap \hat{h}^{n+1}\hat{\Pi}_1) \subset \hat{V}_1^+, \tag{3.33}$$

since otherwise we would have $w' \in \hat{V}_1^+$ for some point $w' \in \hat{V}_1^+ \cap \hat{h}^{n+1}\hat{\Pi}_1$, while $\hat{h}^{-1}w' \notin \hat{V}_1^+$.

But $\hat{h}^{-1}w' \in \hat{h}^n\hat{\Pi}_1$ for $n \geq 0$. Consequently w' and $\hat{h}^{-1}w'$ lie on one side of $\hat{\Gamma}_1^-$ and on different sides of $\hat{\Gamma}_1^+$. But $\hat{h}^{-1}\hat{V}_1^+$ lies on one side of $\hat{\Gamma}_1^+$, since

$$\hat{h}^{-1}\hat{\Gamma}_1^+ \cap \hat{\Gamma}_1^+ = \varnothing \tag{3.34}$$

by (3.19), and $\hat{h}^{-1}\hat{\Gamma}_1^+$ lies on the same side of $\hat{\Gamma}_1^+$ as $\hat{\Gamma}_1^-$. From (3.33), evidently, follows (3.32).

Now we can define $\hat{v}_{12}$ in $\hat{V}_1^+ \cap \hat{h}^{n+1}\hat{\Pi}_1$ by the relation

$$\hat{Q}_1(w)\,\hat{v}_{12}(w) - \hat{Q}_1^{\hat{h}_1}(w)\,\hat{v}_{12}^{\hat{h}}(w) = \hat{G}_1(w), \quad w \in \hat{h}^{-1}(\hat{V}_1^+ \cap \hat{h}^{n+1}\hat{\Pi}_1). \tag{3.35}$$

In $\hat{V}_1^+ \cap \hat{h}^{n+1}\hat{\Pi}_1$ the function $\hat{v}_{12}$ is meromorphic by (2.24) and has a finite number of poles there. We show that $\hat{v}_{12}$ is meromorphic and has a finite set of poles in $\hat{V}_{1,n+1}^+$. In fact the intersection of the boundaries of the regions $\hat{h}^{n+1}\hat{\Pi}_1$ and $\hat{V}_{1,n}^+$ is equal to $\hat{h}^n\hat{\Gamma}_1^- \cap \hat{V}_1^+$. Assume that $n > 0$. Then from (3.23) and the uniqueness of analytic continuation we obtain that

$$\hat{Q}_1(w)\,\hat{v}_{12}(w) - \hat{Q}_1^{\hat{h}_1}(w)\,\hat{v}_{12}^{\hat{h}}(w) = \hat{G}_1(w) + \sum_{0 \leqslant p \leqslant \hat{p}^{n-1}} \hat{G}_{kp}^{n-1}\delta^{(p)}(w - w_k^{n-1}), \; hw \in \hat{h}^n\hat{\Gamma}_1^- \cap \hat{V}_1^+, \tag{3.36}$$

where the w_k^{n-1} are the poles of $\hat{v}_{12}$ on $\hat{h}^{n-1}\hat{\Gamma}_1^-$ of orders $\hat{p}_k^{n-1}$. The boundary values of $\hat{v}_{12}$ are regarded as the limiting values from within $\hat{h}^n\hat{\Pi}_1$ in the space $\mathfrak{D}'(\partial(\hat{h}^n\hat{\Pi}_1))$. Therefore from (3.35) it follows that the difference of the traces of $\hat{v}_{12}$ as $\hat{h}^n\hat{\Gamma}_1^- \cap \hat{V}_1^+$ is approached from different sides is equal to a sum of δ-functions and their derivatives. This sum is finite by the inductive assumption. In consequence of the principle of analytic continuation by continuity, $\hat{v}_{12}$ is meromorphic on $\hat{h}^n\hat{\Gamma}_1^- \cap \hat{V}_1^+$ and has a finite set of poles there. In the case $n = 0$ the proof is completed analogously. Theorem 3.1 is proved.

Problem (3.23) is a Haseman conjugation problem for meromorphic functions on $\hat{\Pi}_1$ with translation $\hat{h}$ (see [11]). It is equivalent to a Riemann conjugation problem for meromorphic functions on the surface $\check{\Pi}_1 = \overline{\hat{\Pi}}_1/\hat{h}$ along the contour $\check{\Gamma}_1^- = \hat{\Gamma}_1^- \cup_{\hat{h}} \hat{h}\hat{\Gamma}_1^-$. We denote the quotient mapping $\overline{\hat{\Pi}}_1 \to \check{\Pi}_1$ by Φ_1. Let $\check{v}_{12}(t)$ be the image of the function $\hat{v}_{12}(w)$ on $\check{\Pi}_1$:

$$\check{v}_{12}(t) = \hat{v}_{12}(\Phi_1^{-1}t), \quad t \in \check{\Pi}_1 \setminus \check{\Gamma}_1^-. \tag{3.37}$$

We orient $\hat{\Gamma}_1^-$ consistently with $\overline{\hat{\Pi}}_1$, i.e. $\hat{\Pi}_1$ is on the left when moving along $\hat{\Gamma}_1^-$. Then the contour $\check{\Gamma}_1 = \Phi_1\hat{\Gamma}_1^-$ is also oriented. For the function $\check{v}$ defined on $\check{\Pi}_1\setminus\check{\Gamma}_1^-$ we shall denote by $\check{v}^\pm$ the limiting values of $\check{v}$ as $\check{\Gamma}_1^-$ is approached from the left and from the right, respectively, in the space $\mathfrak{D}'(\check{\Gamma}_1^-)$. Then it is possible to write problem (3.23) in the form

$$Q_1^+(t)\,\check{v}_{12}^+(t) - Q_1^-(t)\,\check{v}_{12}^-(t) = \check{\mathscr{G}}_1(t), \quad t \in \check{\Gamma}_1^-. \tag{3.38}$$

Here $Q_1^\pm(t)$ are the values of the functions $\hat{Q}_1$ and $\hat{Q}_1^{\hat{h}_1}$, respectively, at the point

$w = \Phi_1^{-1} t \cap \hat{\Gamma}_1^-$, and $\check{\mathcal{G}}_1(t)$ is the "value" of $\check{\mathcal{G}}_1$ at the same point. In other words, for $t \in \check{\Gamma}_1^-$

$$
\begin{aligned}
Q^+(t) &= \hat{Q}_1(\Phi_1^{-1} t), \\
Q^-(t) &= \hat{Q}_1^{\hat{h}_1}(\Phi_1^{-1} t), \\
\check{\mathcal{G}}_1(t) &= \hat{\mathcal{G}}_1(\Phi_1^{-1} t),
\end{aligned}
\tag{3.39}
$$

where the branch with values in $\hat{\Gamma}_1^-$ is taken for Φ_1^{-1}.

§4. Solution of the Riemann conjugation problem for meromorphic functions on a Riemann surface

As in the classical scheme for solving the Riemann problem (see [11]) we first find a nonzero solution $T_2(t)$ of the homogeneous problem

$$Q_1^+(t) T_2^+(t) - Q_1^-(t) T_2^-(t) = 0, \quad t \in \check{\Gamma}_1^-, \tag{4.1}$$

corresponding to (3.38), meromorphic in $\check{\Pi}_1$. Then we obtain $\check{v}_{12}$ as a solution of the saltus problem

$$\left(\frac{\check{v}_{12}}{T_2}\right)^+(t) - \left(\frac{\check{v}_{12}}{T_2}\right)^-(t) = \frac{\check{\mathcal{G}}_1(t)}{(Q_1^+ T_2^+)(t)}, \quad t \in \check{\Gamma}_1^- . \tag{4.2}$$

The problem of factoring (4.1) also reduces by a standard method (taking the logarithm) to the saltus problem.

First of all we consider the problem of factoring (4.1):

$$\frac{T_2^+(t)}{T_2^-(t)} = \frac{Q_1^-(t)}{Q_1^+(t)} = \check{R}_1(t), \quad t \in \check{\Gamma}_1^- . \tag{4.3}$$

From (2.28) it follows that $\check{R}_1(t)$ is a lifting to $\check{\Gamma}_1^-$ of the function

$$R_1(z) = \left(\frac{S_2}{S_1}\right)^{h_1}(z) \left(\frac{S_1}{S_2}\right)(z), \tag{4.4}$$

which is rational on $\bar{V}$. In view of (2.24) this function is not identically zero on $\bar{V}$.

For further analysis of problem (2.1) it is necessary to use its nondegeneracy, i.e. the Šapiro-Lopatinskiĭ condition. Obviously (2.9) and (2.10) are equivalent to the estimates

$$|\mathbf{B}_l(z)| \geqslant C |z|^{m_l}, \quad z \in \Gamma_l^-, \ |z| > C, \tag{4.5}$$

where $\mathbf{C}$ is a constant. By Lemma 3.1 the mapping ψ is a birational equivalence of $\bar{V}$ and $\mathbf{CP}^1$, and by virtue of Lemma 3.2 the curves Γ_l^- join the points 0 and ∞. Therefore from (4.5) and (3.1) it follows that the rational function $B_l(\chi(\lambda))$ has poles of the same order m_l at 0 and ∞. But from (2.14), (2.16) and (2.7) it follows that

$$S_1 = -P_{20} a_1^2 B_1^{h_1}. \tag{4.6}$$

Similarly

$$S_2 = -P_{10}a_2^2B_2^{h_2}. \tag{4.7}$$

Taking (3.3) into account, we find that the functions $S_l(\chi(\lambda))$ and $S^{h_1}(\chi(\lambda))$ have poles of the same order at 0 and ∞, equal to $m_1 + m_2$. Therefore $R_1(\chi(\lambda))$ is regular and different from zero at 0 and ∞, and

$$R_1(\chi(0)) = \frac{1}{R_1(\chi(\infty))} \tag{4.8}$$

by (3.3) and (4.4).

Next, the surface $\check{\Pi}_1$ is holomorphically equivalent to the Riemann sphere less two points. In fact the mapping

$$w \xrightarrow{\mathcal{T}} \tau = e^{\pi i \frac{w}{w_0}}, \tag{4.9}$$

carrying $\bar{\hat{\Pi}}^1$ into $\mathbf{C}\setminus\{0\}$ is by definition equivalent to the quotient mapping $\Phi_1\colon \bar{\hat{\Pi}}_1 \to \check{\Pi}_1$. This means that there exists a holomorphic equivalence $\mathcal{H}\colon \check{\Pi}_1 \to \mathbf{C}\setminus\{0\}$ under which the diagram

$$\begin{array}{ccc} & \bar{\hat{\Pi}}_1 & \\ {}^{\Phi_1}\swarrow & & \searrow{}^{\mathcal{T}} \\ \check{\Pi}_1 & \xrightarrow{\mathcal{H}} & \mathbf{C}\setminus 0 \end{array} \tag{4.10}$$

is commutative. The linear-fractional function $\tau \overset{\mathcal{Q}}{\to} t = (\tau - 1)/(\tau + 1)$ transforms $\mathbf{C}\setminus\{0\}$ into $\mathbf{CP}^1\setminus\{-1, 1\}$. Therefore the mapping

$$w \xrightarrow{\mathcal{Q}\circ\mathcal{T}} t = \frac{e^{\pi i \frac{w}{w_0}} - 1}{e^{\pi i \frac{w}{w_0}} + 1} = i \tan \pi \frac{w}{2w_0} \tag{4.11}$$

is also holomorphically equivalent to Φ_1. We identify $\check{\Pi}_1$ with $\mathbf{CP}^1\setminus\{-1, 1\}$, using the holomorphic equivalence $\mathcal{Q} \circ \mathcal{H}$. Then Φ_1 is identified with $\mathcal{Q} \circ \mathcal{T}$, and $\check{\Gamma}_1^-$ becomes a contour in $\mathbf{CP}^1$ joining the points -1 and 1:

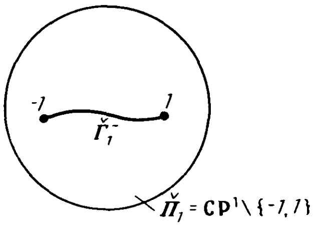

Figure 4

Equality (4.8) passes into the relation

$$\check{R}_1(-1) = \frac{1}{\check{R}_1(+1)}. \tag{4.12}$$

Let us now prove the following factorization theorem.

Theorem 4.1. (i) *There exists a nonzero meromorphic function* $T_2(t)$ *in* $\check{\Pi}_1 \backslash \check{\Gamma}_1^-$, *analytic up to* $\check{\Gamma}_1^-$ *as* $\check{\Gamma}_1^-$ *is approached from both sides, whose limiting values satisfy the relation*

$$\frac{T_2^+(t)}{T_2^-(t)} = \check{R}_1(t), \quad t \in \check{\Gamma}_1^-. \tag{4.13}$$

(ii) *The function* $\hat{T}_2(w) - T_2(t)$ *lifted to* $\bar{\hat{\Pi}}_1$ *– can be extented to a meromorphic function on* $\hat{V}$ *and has a finite number of zeros and poles in* $\hat{\bar{W}}_2$.

(iii) *As* $t \in \check{\Pi}_1 \backslash \check{\Gamma}_1^-$ *approaches* $\check{\Gamma}_1^-$ *from either side the function* $T_2(t)$ *exhibits the asymptotic behavior*

$$T_2(t) \sim (t \mp 1)^{\mu_2} \sum_{\substack{\delta \in \Delta \\ 0 \leqslant p \leqslant \operatorname{Re}\delta}} T_{\delta p}^{\pm} (t \mp 1)^{\delta} \ln^p (t \mp 1), \quad t \to \pm 1, \tag{4.14}$$

where $0 \leq \operatorname{Re} \mu_2 < 1$ *and* $\Delta = \{\nu^+ k + j/2 | \; k, j \in N\}$ *with* $\operatorname{Re} \nu^+ > 0$ *and in addition*

$$T_{00}^{\pm} \neq 0. \tag{4.15}$$

(iv) *A factorization with properties* (i)–(iii) *is unique up to a nonzero constant factor.*

(v) *The function* $\hat{T}(w)$ *is invariant with respect to* $\hat{h}_2$ *up to sign:*

$$\hat{T}^{\hat{h}_2}(w) = \pm \hat{T}(w), \quad w \in \hat{V}. \tag{4.16}$$

Proof. We introduce the function $\mathfrak{z}(t)$—an arbitrary branch, meromorphic in $\mathbf{CP}^1 \backslash \check{\Gamma}_1^-$, of the function inverse to Žukov's function:

$$t = \frac{1}{2}\left(\mathfrak{z}(t) + \frac{1}{\mathfrak{z}(t)}\right). \tag{4.17}$$

Evidently $\mathfrak{z}(t)$ is a holomorphism of $\mathbf{CP}^1 \backslash \bar{\check{\Gamma}}_1^-$ and of the contractible region in $\mathbf{CP}^1$ with boundary $\Gamma_1' = \mathfrak{z}^+(\bar{\check{\Gamma}}_1^-) - \mathfrak{z}^-(\bar{\check{\Gamma}}_1^-)$ not passing through 0 or ∞. By Viète's theorem

$$\mathfrak{z}^+(t)\, \mathfrak{z}^-(t) = 1, \quad t \in \bar{\check{\Gamma}}_1^-. \tag{4.18}$$

Consequently

$$\operatorname*{Var\,arg}_{\Gamma_1'} \mathfrak{z} = \pm 2\pi. \tag{4.19}$$

Let t_- be the initial and t_+ the terminal point of the contour $\check{\Gamma}_1^-$. Clearly $\{t_-, t_+\} = \{1, -1\}$. For definiteness we shall assume that $\mathfrak{z}(\infty) = \infty$. Then by (4.18) and (4.19)

$$\operatorname*{Var\,arg}_{\check{\Gamma}_1^-} \mathfrak{z}^{\pm}(t) = \mp \pi \tag{4.20}$$

in the case when $t_{\pm} = \pm 1$. We define a function $\mathfrak{S}(t)$ in $\mathbf{CP}^1 \backslash \bar{\check{\Gamma}}_1^-$ by

$$\mathscr{E}(t) = \exp\left(\frac{l_-}{2\pi i}\ln\frac{t^2-1}{\mathfrak{z}^2(t)}\right). \tag{4.21}$$

We let $l_- = \ln \check{R}_1(t_-)$ and require that $0 \leq \operatorname{Im} l_- < 2\pi$. From (4.20) and the condition $\mathfrak{z}(\infty) = \infty$ it follows that $\mathscr{E}(t)$ is analytic and the function

$$R_{11}(t) = \frac{\mathscr{E}^-(t)}{\mathscr{E}^+(t)}\check{R}_1(t) \tag{4.22}$$

satisfies the relation

$$R_{11}(-1) = R_{11}(+1) = 1 \tag{4.23}$$

by virtue of (4.12).

Now we "unwind" the symbol $R_{11}(t)$. We denote by $\tilde{\Gamma}_1^-$ a contour in $\mathbf{CP}^1$ that joins -1 with 1 and lies in a sufficiently small neighborhood of $\check{\Gamma}_1^-$. In addition, $\tilde{\Gamma}_1^-$ goes around the zeros of $\check{R}_1$ lying on Γ_1^- to the left and around the poles to the right. Let

$$\varkappa_2 = \frac{1}{2\pi}\operatorname*{Var\,arg}_{\tilde{\Gamma}_1^-} R_{11}(t). \tag{4.24}$$

Then the function

$$R_{10}(t) = \left(\frac{\mathfrak{z}^-(t)}{\mathfrak{z}^+(t)}\right)^{-\varkappa_2} R_{11}(t) \tag{4.25}$$

has zero change of argument along $\tilde{\Gamma}_1^-$ by (4.20), and consequently has a logarithm on $\tilde{\Gamma}_1^-$ that is analytic and equal to zero at the ends. Therefore we obtain a factorization of $R_{10}(t)$ from a solution of the saltus problem

$$\ln T_0^+(t) - \ln T_0^-(t) = \ln R_{10}(t), \quad t \in \check{\Gamma}_1^-. \tag{4.26}$$

By the Sohockii-Plemelj formulas it is possible to set

$$\ln T_0(t) = \frac{1}{2\pi i}\int_{\check{\Gamma}_1^-} \frac{\ln R_{10}(t')}{t'-t}\,dt' \tag{4.27}$$

for $t \in \check{\Pi}_1 \setminus \check{\Gamma}_1^-$. The function $T_0(t)$ is holomorphic on $\check{\Pi}_1 \setminus \bar{\check{\Gamma}}_1^-$ up to $\check{\Gamma}_1^-$ by virtue of the special choice of branch for $\ln R_{10}(t)$.

Let

$$T_2(t) = T_0(t)\frac{\mathscr{E}(t)}{\mathfrak{z}^{\varkappa_2}(t)}. \tag{4.28}$$

Evidently T_2 satisfies (i) of Theorem 4.1. Furthermore, assertion (ii) is the multiplicative analog of Theorem 3.1.

Let us determine the asymptotic behavior of $T_2(t)$ as $t \to \pm 1$. As already noted, the rational function $R_1(\chi(\lambda))$ is regular at zero and at infinity. Consequently

$$R_1(X(\lambda)) \sim \sum_{0\leqslant k} R_k^{\pm}\lambda^{\pm k}, \quad \lambda^{\pm 1}\to 0. \tag{4.29}$$

Let $\nu = w_0/\pi i$. Then

$$\lambda = \tau^{\nu} \sim (t\mp 1)^{\mp\nu}\sum_{0\leqslant j}\lambda_j^{\pm}(t\mp 1)^j, \quad t\to \pm 1, \tag{4.30}$$

for $t \in \check{\Gamma}_1^-$ by (4.9) and (4.11). Hence

$$\check{R}_1(t) \sim \sum_{0\leqslant k,j} R_{kj}^{\pm}(t\mp 1)^{\nu^+ k+j}, \quad t\to + 1. \tag{4.31}$$

In (4.31) we have used the notation

$$\nu^+ = \begin{cases} \nu, & \operatorname{Re}\nu > 0, \\ -\nu, & \operatorname{Re}\nu < 0. \end{cases} \tag{4.32}$$

From (4.22), (4.25), (4.17) and (4.21) it is now easy to obtain the asymptotic formula

$$R_{10}(t) \sim \sum_{0\leqslant k,j} r_{kj}^{\pm}(t\mp 1)^{\nu^+ k+j/2}, \quad t\to \pm 1. \tag{4.33}$$

Correspondingly for $\ln R_{10}(t)$ we have

$$\ln R_{10}(t) \sim \sum_{\substack{0\leqslant k,j\\ 0< k+j}} L_{kj}^{\pm}(t\mp 1)^{\nu^+ k+j/2}, \quad t\to \pm 1. \tag{4.34}$$

Substituting (4.34) in (4.27) and integrating term by term we obtain that for $t \in \check{\Pi}_1\backslash\check{\Gamma}_1^-$

$$\ln T_0(t) \sim \sum_{1\leqslant k} l_k^{\pm}(t\mp 1)^k \ln(t\mp 1) + \sum_{0\leqslant k,j} m_{kj}^{\pm}(t\mp 1)^{\nu+k+j/2}, \quad t\to \pm 1. \tag{4.35}$$

Exponentiating (4.35) and taking account of (4.28), (4.17) and (4.21), we obtain the required asymptotic formula (4.14). Here

$$\mu_2 = \frac{l_-}{2\pi i}, \qquad T_{00}^{\pm} \sim \exp m_{00}^{\pm}. \tag{4.36}$$

Finally, (iv) follows from the principle of analytic continuation by continuity and the theorem on a removable singularity. Assertion (v) is the multiplicative analog of Theorem 2.1 and can be proved using (iv) and the fact that $\hat{h}_2$ is an involution. Theorem 4.1 is proved.

Let us return to equation (3.38). From (4.3) it follows that it is equivalent to

$$\left(\frac{\check{v}_{12}}{T_2}\right)^+(t) - \left(\frac{\check{v}_{12}}{T_2}\right)^-(t) = p.f.\ \frac{\check{\mathscr{G}}_1}{Q_1^+T_2^+}(t), \quad t\in\check{\Gamma}_1^-, \tag{4.37}$$

where an arbitrary solution of

$$Q_1^+T_2^+X(t) = \check{\mathscr{G}}_1(t), \quad t\in\check{\Gamma}_1^-, \tag{4.38}$$

in the space $\mathfrak{D}'(\check{\Gamma}_1^-)$ stands on the right side.

Remark 4.1. Let $\hat{v}_2'$ and $\hat{v}_2''$ be any two solutions of (2.38). Then

$$\hat{v}_2' - \hat{v}_2'' = \hat{T}_2 \hat{\mathfrak{M}}, \tag{4.39}$$

where $\hat{\mathfrak{M}}$ is meromorphic on $\hat{W}_2$. The function $\hat{\mathfrak{M}}$ on $\hat{\Pi}_1$ is a lifting of a function $\mathfrak{M}$ meromorphic on $\check{\Pi}_1$. In fact let $\check{v}_2'$ and $\check{v}_2''$ be the images of $\hat{v}_2'$ and $\hat{v}_2''$ on $\check{\Pi}_1$. Then from (4.37) it follows that $(\check{v}_2' - \check{v}_2'')/T_2$ is meromorphic on $\check{\Pi}_1$. Now it remains to use (ii) of Theorem 4.1.

Since we are free in choosing the constants G_{kp} in (3.24), we set them equal to zero. Then $\mathfrak{G}_1 = G_1$ and by the Sohockiĭ-Plemelj formulas the solution of (4.37) is an integral of Cauchy type

$$\frac{\check{v}_{12}'}{T_2}(t) = \frac{1}{2\pi i} \int_{\Gamma_1^-} \frac{p.f. \dfrac{\check{G}_1}{Q_1^+ T_2^+}(t')}{t' - t} dt'. \tag{4.40}$$

At points within the contour Γ_1^- this integral is regularized in the sense of generalized functions, and in the neighborhood of the points ± 1 it converges absolutely in the Lebesgue sense. To prove absolute convergence of the integral in (4.40) in the neighborhood of the points ± 1 it is essential that $s > 3/2$ and $\operatorname{Re} \mu_2 < 1$.

§5. Investigation of the general solution of the system (2.18)

From Theorem 3.1 it follows that the function $\hat{v}_{21}'(w) = \hat{v}_{12}'(\Phi w)$ can be extended from $\hat{\Pi}_1$ to a function $\hat{v}_{12}'(w)$ meromorphic in $\hat{W}_2$. But by Theorem 2.1 the function

$$\hat{v}_{12}(w) = \frac{1}{2}(\hat{v}_{12}'(w) + \hat{v}_{12}'(\hat{h}_2 w)) \tag{5.1}$$

is a solution of the system of equations (2.38) and (2.35). Consequently the function

$$v_{12}(z) = \hat{v}_{12}(p^{-1}z), \quad z \in W^+, \tag{5.2}$$

satisfies the last two equations of (2.28) if the branch with values in $i_1 W^+$ is taken for p^{-1}. Thus we satisfy (2.28) if we let

$$v_{11}(z) = \frac{g_1(z) - S_2(z) v_{12}(z)}{S_1(z)}, \quad z \in V_1^+. \tag{5.3}$$

Hence we have found a particular solution (v_{11}, v_{12}) of (2.18) for the case $f = 0$, $f_2' = 0$ under the assumption that v_{1l} is meromorphic in V_l^+.

Completely analogously we determine the general solution (v_{21}, v_{22}) of this system for the case $f = 0$, $f^0 = 0$ and $f_1' = 0$ under the assumption that $v_{1l} + v_{2l} \in \tilde{H}_{s-3/2}(\overline{\mathbf{R}}^+)$, $l = 1, 2$. In fact from (4.40), assertion (ii) of Theorem 3.1, assertion (ii) of Theorem 4.1 and the special choice of the constants in (3.24) it follows that v_{12} has a finite set of poles in W^+. Therefore from (5.3) and (2.24) it follows that v_{11} has a finite number of poles in V_1^+. Now from the assumption that $v_{1l} + v_{2l} \in \tilde{H}_{s-3/2}(\overline{\mathbf{R}}^+)$ it evidently follows that v_{2l} is meromorphic and has a finite number of poles in V_l^+. Consequently to determine the functions v_{2l} we can apply the methods used above to find the functions v_{1l}.

In the first place we see that v_{21}, like v_{12}, is meromorphic in W^+ and has a finite number of poles there. Hence we immediately obtain an important corollary. Namely, from (5.3) it follows that the first component v_1 of the general solution of (2.18) in the case $f=0$ has the form

$$v_1 = v_{11} + v_{21} = \frac{g_1(z)}{S_1(z)} - \frac{S_2(z)}{S_1(z)} v_{12}(z) + v_{21}(z). \tag{5.4}$$

Using (2.23) and (4.6) we obtain that, modulo meromorphic functions on $\overline{V}_1^+\backslash\Gamma_1^-$,

$$v_1 = -\frac{a_1^1 - iz_2a_1^2}{a_1^2 B_1^{h_1}} \widetilde{f}_1. \tag{5.5}$$

But by virtue of the strong ellipticity of $A(z)$

$$a_1^1 - iz_2a_1^2 \neq 0 \tag{5.6}$$

for $z \in \Gamma_1^+$, since otherwise

$$A(z_1, 0) = A(z) + iz_2(a_1^1 - iz_2a_1^2) = 0 \tag{5.7}$$

for real z_1. Consequently, in order that problem (2.1) be Noetherian it is necessary that

$$B_1^{h_1}(z) \neq 0 \tag{5.8}$$

for $z \in \Gamma_1^+$. In fact, by (5.5) only in this case will the function $v_1(z_1)$ be locally summable on the real axis when a finite number of orthogonality conditions hold for f_1 and f_2. From (3.16) it follows that (5.8) is equivalent to the condition

$$B_1(z) \neq 0, \quad z \in \Gamma_1^-. \tag{5.9}$$

Now we can formulate the definition of strong ellipticity for problem (2.1).

Definition 5.1. The boundary value probelm (2.1) in a quadrant is called *strongly elliptic* if

$$|B_l(z)| \geqslant C(1+|z|)^{m_l}, \quad z \in \Gamma_l^-, \tag{5.10}$$

for $l = 1, 2$.

By virtue of (4.5), from the arguments presented above it follows that strong ellipticity of problem (2.1) is necessary in order that it be Noetherian in conjunction with the a priori estimate (2.8). In this and the following sections we shall show that, conversely, under condition (5.10) the problem (2.1) is Noetherian and its solution satisfies (2.8) for all s (except a locally finite set) with $\epsilon > 0$ depending on s. For this we find the general form of the solution (v_{21}, v_{22}) of the system (2.18) in the case $f^0 = 0$ and $f_1' = 0$. In this connection

$$g = g_2 = P_{10}(a_2^1 - iz_1a_2^2)\widetilde{f}_2'. \tag{5.11}$$

We trace briefly and with obvious changes in notation the reasoning of §§2–4 adapted to this case. Let $\hat{v}_{21}$ be a lifting of v_{21} from V_1^+ to $\hat{V}_1^+$. Then $\hat{v}_{21}$ is meromorphic on $\hat{W}_1$ and satisfies an algebraic equation with a shift of the type in (2.38). The function $\hat{v}_{21}$ is a lifting of the solution of the conjugation problem of type (3.38) for the meromorphic function $\check{v}_{21}$. If $\mathcal{G}_2 = G_2$, then a particular solution $\check{v}'_{21}$ of this conjugation problem is given by an integral of the type (4.40):

$$\frac{\check{v}'_{21}}{T_1}(t) = \frac{1}{2\pi i} \int\limits_{\check{\Gamma}_2^-} \frac{p.f.\ \frac{\check{G}_2}{Q_2^+ T_1^+}(t)}{t' - t}\, dt'. \tag{5.12}$$

Here $\check{\Gamma}_2^-$ is a contour in $\check{\Pi}_2 \approx \mathbf{CP}^1 \backslash \{-1, 1\}$ joining 1 with -1. Let $\hat{v}'_{21}$ denote a lifting of $\check{v}'_{12}$ to $\hat{\Pi}_2$. By Theorem 3.1 the function $\hat{v}'_{21}$ is meromorphic on $\hat{W}_1$, has a finite number of poles there and satisfies the same algebraic equation with a shift as $\hat{v}_{21}$. As shown in Theorem 2.1, the function

$$\hat{v}^*_{21}(w) = \frac{1}{2}\left(\hat{v}'_{21}(w) + \hat{v}'_{21}(\hat{h}_1 w)\right) \tag{5.13}$$

also satisfies this equation. In consequence of Remark 4.1

$$\hat{v}_{21} = \hat{v}^*_{21} + \hat{T}_1 \hat{\mathfrak{M}}, \tag{5.14}$$

where $\hat{\mathfrak{M}}$ is a lifting to $\hat{\Pi}_2$ of the function $\mathfrak{M}$ meromorphic in $\check{\Pi}_2$.

The function $\hat{v}^*_{21}$ is invariant with respect to $\hat{h}_1$. Therefore the invariance of $\hat{v}_{21}$ with respect to $\hat{h}_1$ is equivalent to

$$\hat{\mathfrak{M}}^{\hat{h}_1}(w) = \pm \hat{\mathfrak{M}}(w), \quad w \in \hat{W}_1, \tag{5.15}$$

by virtue of (4.16). The sign in (5.15) coincides with the sign in (4.16).

In addition, the singularities of $\hat{v}_{21}$ and $\hat{v}^*_{21}$ on $\hat{W}_1$ are known. Therefore the singularities of $\hat{T}_1 \hat{\mathfrak{M}}$ are also known: for each point $w \in \hat{W}_1$ an integer $\hat{p}_w$ is assigned such that

$$\hat{T}_1 \hat{\mathfrak{M}}(w') = \sum_{-\hat{p}_w \leqslant k} c_k (w' - w)^k \tag{5.16}$$

in a neighborhood of w. Let $\hat{d}_w$ denote the multiplicity of the zero of $\hat{T}_1$ at w. Then from (5.16) it follows that $\hat{\mathfrak{M}}$ has a pole at w of order no higher than $\hat{p}_w + \hat{d}_w$. Let $q_t = \hat{p}_w + \hat{d}_w$ for $t = \Phi_2 w$, with $w \in \hat{\Pi}_2 \backslash \hat{\Gamma}_2^-$. Then $\mathfrak{M}$ has a pole at $t \in \check{\Pi}_2$ of order no higher than q_t. Since $\hat{v}_{21}$ and $\hat{v}^*_{21}$ have a finite set of poles on $\hat{W}_1$, the integer $\hat{p}_w$ is positive only for a finite set of points $w \in \hat{\Pi}_2 \backslash \hat{\Gamma}_2^-$. Moreover, analyzing the proof of Theorem 3.1 it is easy to see that $\hat{p}_w$ depends only on the symbols A, B_1 and B_2. In addition

$$\sum_{w \in \hat{W}_1} \hat{p}_w < \infty. \tag{5.17}$$

Next, from (ii) of Theorem 4.1 it follows that $\hat{d}_w > 0$ only for a finite set of points $w \in \hat{W}_1$. Therefore the set of $t \in \check{\Pi}_2$ for which $q_t > 0$ is finite. Below we shall show that from the assumption $v_{11} + v_{21} \in H_{s-3/2}(\overline{\mathbf{R}}^+)$ follows the regularity of $\mathfrak{M}$ at the points ± 1. Thus we arrive at the following assertion.

Lemma 5.1. (i) *The integer q_t depends only on the symbols A, B_1 and B_2, and*

$$\sum_{t\in\check{\Pi}_2} q_t < \infty. \tag{5.18}$$

(ii) *The function $\mathfrak{M}$ belongs to the finite-dimensional space of rational functions on $\mathbf{CP}^1$ with poles of order q_t at $t \in \mathbf{CP}^1$. The dimension of this space is*

$$d = \sum_{t\in\check{\Pi}_2} q_t + 1. \tag{5.19}$$

We have already proved (i), and $\mathfrak{M}$ is holomorphic at ± 1 because $\mathfrak{M}$ is square summable in the neighborhood of these points. To prove this we study the functions v_{1l} and v_{21}^*. We first prove two important lemmas. Let $\mu_2^0 = \operatorname{Re}\mu_2$ and $\nu^0 = \operatorname{Re}\nu^+$.

Lemma 5.2. (i) *The function $\check{v}_{21}'(t)$ is meromorphic in $\mathbf{CP}^1 \backslash \overline{\check{\Gamma}_1^-}$ and has a finite set of poles there; in the neighborhood of the contour $\check{\Gamma}_1^-$ in $\mathbf{CP}^1$ it admits the estimate*

$$|v_{12}'(t)| \leqslant C\rho^{-\alpha}(t, \check{\Gamma}_1^-) \tag{5.20}$$

with constants C and α, where ρ is distance in $\mathbf{CP}^1$.

(ii) *Suppose s is not an integer and is different from $1 + (\mu_2^0 + k)/\nu^0$ for $k = 1, 2, \cdots$. Then in the neighborhood of the points ± 1 there exists a finite expansion*

$$\check{v}_{12}'(t) = \check{v}_0'(t) + \sum_{\substack{\delta\in\Delta_{\mu_2}, \operatorname{Re}\delta\leqslant\nu^0(s-1)\\ 0\leqslant p\leqslant \operatorname{Re}\delta+1}} \check{v}_{p\delta}^{\pm}(t \mp 1)^{\delta}\ln^p(t \mp 1), \tag{5.21}$$

where $\check{v}_0'$ is square summable in the neighborhood of ± 1 with weight $|t^2-1|^{-2\nu^0(s-1)-1}dt$ on any smooth arc γ in $\mathbf{CP}^1\backslash\check{\Gamma}_1^-$ passing through 1 or -1, and

$$\Delta_{\mu_2} = \Delta \cup (\Delta + \mu_2). \tag{5.22}$$

(iii) *If the curve γ passes through 1 or -1, is sufficiently short and belongs to an arbitrary bounded set in $C_{\text{in}}^\infty([0, 1], \mathbf{CP}^1)$, then*

$$\int_\gamma |\check{v}_0'(t)|^2 |t^2 - 1|^{-2\nu^0(s-1)-1} |dt| \leqslant C(\|f_1\|_{s-m_1-1/2} + \|P_1\| + |f^0|) \tag{5.23}$$

and the constants $\check{v}_{p\delta}^{\pm}$ also admit this estimate.

Proof. We apply the Euler substitution $|t \mp 1| = e^\tau$ in the neighborhood of the points ± 1 on the contours γ and $\check{\Gamma}_1^-$. Then the operator (4.40) is transformed into a pseudodifferential operator on the line. Its symbol, computed in the coordinate τ, can be studied by the methods of [15]. Namely, this symbol admits the expansion described in Lemma 4.1 of [15]. Therefore the arguments of §5 of [15] and Theorem A.1 of the

Appendix of this paper yield the required expansion (5.21). The parameter s occurring in the statement of Lemma 5.1 of [15] is equal to zero in our case, and the parameter h equals $\nu^0(s-1)-\mu_2^0$. Consequently the restriction $h \neq 0, 1, 2, \cdots$ of [15] is equivalent to the inequalities $s \neq 1+(\mu_2^0+k)/\nu^0$, $k=0, 1, \cdots$.

Lemma 5.3. (i) *The function* G_1 *is holomorphic in* V_1^+ *and admits the estimate*

$$|G_1(\lambda)| \leqslant C\rho^{-\alpha}(\lambda, \partial V_1^+) \tag{5.24}$$

with constants C *and* α, *where* ρ *is distance in* $\mathbf{CP}^1 \approx \overline{V}$.

(ii) *If* s *is not an integer, then* G_1 *admits a finite expansion*

$$G_1(\lambda) = G_{10}(\lambda) + \sum_{2(m_1+m_2)-s+1 \leqslant k \leqslant 2(m_1+m_2)} G_1^k \lambda^{\mp k} \tag{5.25}$$

in the neighborhood of the points $0^{\pm 1}$ *in* V_1^+, *where* G_{10} *is square summable in the neighborhood of* $0^{\pm 1}$ *with weight* $\lambda^{\pm(s-3/2-2(m_1+m_2))}d\lambda^{\mp 1}$ *on any smooth arc* γ *in* V_1^+ *passing through* 0 *or* ∞, *and* k *is an integer.*

(iii) *If the curve* γ *passes through* 0 *or* ∞, *is sufficiently short and belongs to an arbitrary bounded set in* $C_{\mathrm{in}}^{\infty}([0, 1], \overline{V}_1^+)$, *then*

$$\int_{\gamma} |G_{10}(\lambda)|^2 |\lambda^{\mp 2(s-3/2-2(m_1+m_2))} d\lambda^{\mp 1}| \leqslant C(\|f_1\|_{s-m_1-1/2} + \|P_1\| + |f^0|) \tag{5.26}$$

and the constants G_1^k *also admit this estimate.*

Proof. Recall that $G_1(z) = S_1^{h}{}^1(z)g_1(z) - S_1(z)g_1^{h}{}^1(z)$. The expansion (5.25) follows from Theorem A.1 and from (2.23). The estimate (5.24) is trivial and (5.26) follows from properties of an intergral of Cauchy type, as in Lemma 5.2.

Now we can completely describe the properties of the functions v_{12}.

Theorem 5.1. (i) *The function* v_{12} *is meromorphic in* W^+, *has a finite set of poles there and admits the estimate*

$$\Delta_{\mu_2} = \Delta \cup (\Delta + \mu_2) \tag{5.27}$$

in the neighborhood of the points $\lambda = 0^{\pm 1}$, *with constants* C *and* α.

(ii) *If* s *is not an integer and is different from* $1+(\mu_2^0+k)/\nu^0$, $k=0, 1, 2, \cdots$, *then there exists a finite expansion*

$$v_{12}(\lambda) = v_0(\lambda) + \sum_{\substack{\delta \in \Delta^2, \mathrm{Re}\,\delta \leqslant s-1 \\ 0 \leqslant p \leqslant p_\delta^{12}}} v_{p\delta}^{\pm} \lambda^{\pm\delta} \ln^p \lambda \tag{5.28}$$

in the neighborhood of $\lambda = 0^{\pm 1}$, *where* v_0 *is square summable in the neighborhood of* $\lambda = 0^{\pm 1}$ *with weight* $\lambda^{\mp 2(s-3/2)}d\lambda^{\mp 1}$ *on any smooth arc* γ *in* $\overline{W}^+$ *passing through* 0 *or* ∞, p_δ^{12} *and* p *are integers, and*

$$\Delta^2 = \left(k + \frac{k}{2\nu^+}\right) \cup \left(k + \frac{k}{2\nu^+} + \left(\frac{\mu_2}{\nu}\right)^+\right), \quad k = 0, 1, 2, \cdots, \tag{5.29}$$

where $(\mu_2/\nu)^+$ *is defined analogously to* (4.32).

(iii) *If the curve* γ *is sufficiently short, belongs to an arbitrary bounded set in* $C^\infty_{in}([0, 1], \overline{W}^+)$ *and passes through* 0 *or* ∞, *then*

$$\int_\gamma |v_0(\lambda)|^2 |\lambda^{\mp 2(s-3/2)} d\lambda^{\mp 1}| \leqslant C(\|f_1\|_{s-m_1-1/2} + \|P_1\| + |f^0|) \tag{5.30}$$

and the constants $v^\pm_{p\delta}$ *also satisfy this estimate.*

Proof. For the function v'_{12} in $\hat{\Pi}_1$ outside a neighborhood of $\hat{h}\hat{\Gamma}^-_1$ the estimate (5.27), expansion (5.28) and estimate (5.30) follow from Lemmas 5.2 and 5.3 and from (5.10). In a neighborhood of $\hat{h}\hat{\Gamma}^-_1$ and in $\hat{W}_2\backslash\hat{\Pi}_1$ the assertions (5.27)–(5.30) are easy to prove by analyzing the proof of Theorem 3.1. All of these assertions are carried over to v_{12} using (5.1).

Let us now complete the proof of Lemma 5.1. Let γ be a short arc in $\check{\Pi}_2\backslash\check{\Gamma}^-_2$ issuing from the point ± 1. We assert that $\check{v}_{11}$ and $\check{v}^*_{21}$ are square summable on γ with weight $|t^2-1|^{-\mu^0_1}dt$. This is easy to prove for $\check{v}_{11}$ using (5.3) and the description of v_{12} obtained in Theorem 5.1. The function $\check{v}^*_{21}$ is studied in exactly the same way as $\check{v}_{12}$. Furthermore, by assumption the sum $v_1 = v_{11} + v_{12}$ belongs to $\tilde{H}_{s-3/2}$. Hence it is easy to deduce that $\check{v}_1$ and consequently $T_1\mathfrak{M} = \check{v}_1 - \check{v}_{11} - \check{v}^*_{21}$ are square summable on γ with weight $|t^2-1|^{-\mu^0_1}dt$. Therefore $\mathfrak{M}$ is square summable on γ with weight $|t^2-1|^{\mu^0_1}dt$. Moreover, integrals of $|\mathfrak{M}|^2|t^2-1|^{\mu^0_1}|dt|$ along arcs γ are bounded if γ is sufficiently short, belongs to an arbitrary bounded set in $C^\infty_{in}([0, 1], \mathbf{CP}^1)$ and passes through 1 or -1. Thus it follows that $\mathfrak{M}$ is square summable in the neighborhood of ± 1 with weight $|t \mp 1|^{\mu^0_1-1}dtd\bar{t}$. Since $\mu^0_1 < 1$, $\mathfrak{M}$ is analytic at ± 1.

Hence $\mathfrak{M}$ is regular at ± 1 and consequently the general solution of the system (2.18) depends on d arbitrary constants. These constants are subject to condition (5.15) involving the nonbranching of v_{21}. In addition, $\mathfrak{M}$ is subject to the condition $v_l \in \tilde{H}_{s-3/2}(\overline{\mathbf{R}}^+)$, $l = 1, 2$. Let

$$\mathbf{S} = \left\{1 + \frac{\mu^0_l + k}{\nu^0} \,\Big|\, l = 1, 2;\ k \in \mathbf{N}\right\}. \tag{5.31}$$

From Theorem 5.1 we obtain, by virtue of (5.4), the following basic assertion.

Theorem 5.2. *Suppose problem* (2.1) *is strongly elliptic. Then* (i) *the function* $v_1(z_1) = v_{11} + v^*_{21} + T_1\mathfrak{M}$ *is meromorphic in* $\mathbf{C}^*$ *and for large* $|z_1|$ *admits the estimate*

$$|v_1(z_1)| \leqslant C|z_1|^\beta |\operatorname{Im} z_1|^{-\alpha}. \tag{5.32}$$

(ii) *If* s *is not an integer and* $s \notin \mathbf{S}$, *then* $v_1(z_1)$ *admits a finite expansion*

$$v_1(z_1) = v_{10}(z_1) + \sum_{\substack{\delta\in\Delta^\sigma, \operatorname{Re}\delta\leqslant s-1\\ 0\leqslant p\leqslant p^1_\delta}} v^1_{p\delta} \ln^p z_1 (z_1 + i)^{-\delta} + \sum_{\substack{\zeta\in Z_1\\ 1\leqslant q\leqslant q^1_\zeta}} r^1_{q\zeta}(z_1 - \zeta)^{-q}, \tag{5.33}$$

in the half-plane $\mathbf{C}^+$, *in which* v_{10} *is analytic in* $\mathbf{C}^+$, *belonging to the space* $\tilde{H}_{s-3/2}(\mathbf{R})$; Z_1 *is a finite set of points in* $\mathbf{C}^+$, p'_δ *and* q'_ζ *are finite, and* p *and* q *are integers.*

(iii) *We have*

$$\|v_{10}\|_{s-3/2}\leqslant C\Big(\sum_{l=1,2}(\|f_l\|_{s-m_l-1/2}+\|P_l\|)+\|\mathfrak{M}\|+|f^0|\Big), \tag{5.34}$$

and the constants $v'_{p\delta}$ *and* $r'_{q\zeta}$ *also admit this estimate.*

(iv) *The sets* Δ^σ *and* Z_1, *as well as the numbers* p'_δ *and* q'_ζ, *depend only on the symbols* A, B_1 *and* B_2. *Here*

$$\Delta^\sigma=\left(\frac{k}{2}+\frac{k}{2\nu^+}\right)\cup\left\{\bigcup_{l=1,2}\left(\frac{k}{2}+\frac{k}{2\nu^+}+\left(\frac{\mu_l}{\nu}\right)^+\right)\right\},\qquad k=0,\ 1,\ 2,\ \cdots. \tag{5.35}$$

Proof. In consequence of (5.4) and (5.14)

$$v_1=\frac{g_1}{S_1}-\frac{S_2}{S_1}v_{12}+v^*_{21}+T_1\mathfrak{M}. \tag{5.36}$$

But by Theorem 5.1 the function v_{12} satisfies conditions (5.32)–(5.35). It is clear that v^*_{21} also possesses similar properties. Let us now take into account the regularity of $\mathfrak{M}$ at ± 1 and the asymptotics (4.14). Then from (5.10) and Theorem A.1 it follows that v_1 also satisfies (5.32)–(5.35). Assertion (iv) is verified by direct computation. The theorem is proved.

Let us assume that problem (2.1) is strongly elliptic, s is not an integer and $s\notin \mathrm{S}$. Then from Theorem 5.2 and Theorem A.1 it follows that v_1 belongs to $\tilde{H}_{s-3/2}(\overline{\mathbf{R}}^+)$ if and only if the following system of conditions holds:

$$\begin{gathered} v^1_{p\delta}=0\ \text{ for }\ \delta\in\Delta^\sigma\ \text{ and }\ \operatorname{Re}\delta\leqslant s-1,\quad \text{if}\quad \delta\neq 1,2,\ \ldots\ \text{ or }\ 1\leqslant p\leqslant p^1_\delta,\\ r_{q\zeta}=0\ \text{ for }\ \zeta\in \mathbf{Z}_1\ \text{ and }\ 1\leqslant q\leqslant q^1_\zeta. \end{gathered} \tag{5.37}$$

Under these conditions, from (iii) of Theorem 5.2 and (ii) of Theorem A.1 we obtain the estimate

$$\|v_1\|_{s-3/2}\leqslant C\Big(\sum_{l=1,2}(\|f_l\|_{s-m_l-1/2}+\|P_l\|)+\|\mathfrak{M}\|+|f^0|\Big). \tag{5.38}$$

Evidently $v'_{p\delta}$ and $r'_{q\delta}$ are continuous linear functionals of g and $\mathfrak{M}$. In turn g depends linearly and continuously on f^0, f_l and P_l. Consequently (5.37) is equivalent to the system

$$L^{11}_j(f^0,P_1,P_2,\mathfrak{M})=R^{11}_j(f_1,f_2),\quad j=1,\ \ldots,\ N^{11}(s), \tag{5.39}$$

where L''_j and R''_j are continuous linear functionals of their arguments. Completely analogously the function $v_2=v_{21}+v_{22}$ belongs to $\tilde{H}_{s-3/2}(\overline{\mathbf{R}}^+)$ if and only if

$$L^{21}_j(f^0,P_1,P_2,\mathfrak{M})=R^{21}_j(f_1,f_2),\quad j=1,\ \ldots,\ N^{21}(s). \tag{5.40}$$

Let us state the main result of this section.

Theorem 5.3. *Suppose problem* (2.1) *is strongly elliptic,* s *is not an integer,* $s\notin\mathrm{S}$ *and* $f=0$. *Then*

(i) *A solution of equation* (2.13) *exists if and only if there is a meromorphic function* $\mathfrak{M}$ *on* $\mathbf{CP}^1$, *with poles at* $t \in \mathbf{CP}^1$ *of order no higher than* q_t, *satisfying conditions* (5.15), (5.39) *and* (5.40).

(ii) *The general solution of equation* (2.13) *has the form*

$$\begin{aligned}\tilde{v}_1^1 &= \frac{g_1}{S_1} - \frac{S_2}{S_1} v_{12} + v_{21}^* + T_1\mathfrak{M},\\ \tilde{v}_2^1 &= \frac{g_2}{S_2} - \frac{S_1}{S_2}(v_{21}^* + T_1\mathfrak{M}) + v_{12},\end{aligned} \tag{5.41}$$

where v_{12} *is determined from* (5.1) *and* v_{21}^* *from* (5.13).

(iii) *Under conditions* (5.39) *and* (5.40) *we have the estimate*

$$\|v_l^1\|_{s-3/2} \leqslant C\Big(\sum_{k=1,2}(\|f_k\|_{s-m_k-1/2} + \|P_k\|) + |f^0| + \|\mathfrak{M}\|\Big). \tag{5.42}$$

(iv) *Strong ellipticity of problem* (2.1) *is necessary in order that it be Noetherian in conjunction with the a priori estimate* (2.8).

Proof. Formulas (5.41) follow from (5.14), while (5.42) and assertion (iv) have been proved above.

§6. Investigation of the general solution of the boundary value problem in a quadrant

Recall that in §2 we reduced the investigation of the boundary value problem (2.1) to the system (2.5) of equations equivalent to it. In addition we have shown that it suffices to consider the case $f = 0$, and under this condition we have exhaustively studied equation (2.13). Since this equation was obtained by eliminating $\tilde{v}_l^0$ from the system (2.5), it is now possible to compute these functions: taking (ii) of Remark 1.1 into account, we obtain

$$\tilde{v}_l^0 = \frac{\tilde{f}_l' - P_{l1}\tilde{v}_l^1}{P_{l0}}. \tag{6.1}$$

Assuming that $v_l' \in \tilde{H}_{s-3/2}(\overline{\mathbf{R}}^+)$, let us clarify under what conditions $\tilde{v}_l^0$ belongs to $\tilde{H}_{s-1/2}(\overline{\mathbf{R}}^+)$.

In view of (6.1) and (5.41)

$$\begin{aligned}\tilde{v}_1^0 &= \frac{\tilde{f}_1' - P_{11}\left(\dfrac{g_1}{S_1} - \dfrac{S_2}{S_1}v_{12} + v_{21}^* + T_1\mathfrak{M}\right)}{P_{10}},\\ \tilde{v}_2^0 &= \frac{\tilde{f}_2' - P_{21}\left(\dfrac{g_2}{S_2} - \dfrac{S_1}{S_2}(v_{21}^* + T_2\mathfrak{M}) + v_{12}\right)}{P_{20}}.\end{aligned} \tag{6.2}$$

Taking (2.23) and (4.6) into account, we obtain

$$\widetilde{v}_1^0 = f_1 \frac{a_1^2 B_1^{h_1} + P_{11}(a_1^1 - iz_2 a_1^2)}{P_{10} a_1^2 B_1^{h_1}} + \mathfrak{M}_1^0,$$

where $\mathfrak{M}_1^0$ is a meromorphic function on W^+. But in consequence of (2.7) and (2.16)

$$B_1^{h_1} = P_{11}\left(-\frac{a_1^1}{a_1^2} + iz_2\right) + P_{10}.$$

Hence

$$\widetilde{v}_1^0 = \frac{\widetilde{f}_1}{B_1^{h_1}} + \mathfrak{M}_1^0.$$

By virtue of (5.10) it therefore follows that $\widetilde{v}_1^0$ (and $\widetilde{v}_2^0$) is locally summable on the real line under a finite number of conditions on f^0, f_1, f_2, P_1, P_2 and $\mathfrak{M}$. Let z_l^j be the zeros of the polynomial P_{l0} lying in $\mathbf{C}^+$ and k_l^j their multiplicity. Then $\widetilde{v}_l^0$, defined by (6.1), has poles only at z_l^j of orders no higher than k_l^j. From Theorem A.1 and conditions (2.11) and (2.12) it follows that $\widetilde{v}_l^0 \in \widetilde{H}_{s-1/2}$ if and only if all residues of $\widetilde{v}_l^0$ at the points z_l^j are zero. Let us write these conditions in a form similar to (5.39) and (5.40):

$$L_j^{l0}(f^0, P_1, P_2, \mathfrak{M}) = R_j^{l0}(f_1, f_2), \quad j = 1, \ldots, N^{l0}. \tag{6.3}$$

By Theorem A.1, under these conditions we have the estimate

$$\|v_l^0\|_{s-1/2} \leqslant C\Big(\sum_{k=1,2}(\|f_k\|_{s-m_k-1/2} + \|P_k\|) + |f^0| + \|\mathfrak{M}\|\Big). \tag{6.4}$$

Thus we have completely investigated problem (2.1) in the case $f = 0$. Let us state the result in the form of a theorem. For uniformity we write condition (5.15) in the form of a system

$$\mathcal{I}_j(\mathfrak{M}) = 0, \quad j = 1, \ldots, d,$$

where the $\mathcal{I}_j$ are linear functionals.

Theorem 6.1. *Suppose problem* (2.1) *is strongly elliptic, s is not an integer, $s \notin S$ and $f = 0$. Then*

(i) *Problem* (2.1) *is solvable if and only if the system of linear algebraic equations*

$$\begin{gathered} L_j^{l\beta}(f^0, P_1, P_1, \mathfrak{M}) = R_j^{l\beta}(f_1, f_2), \quad j = 1, \ldots, N^{l\beta}(s),\ l = 1, 2,\ \beta = 0, 1, \\ \mathcal{I}_j(\mathfrak{M}) = 0, \quad j = 1, \ldots, d, \\ L_j'(f^0, P_1, P_2, \mathfrak{M}) = R_j'(f_1, f_2), \quad j = 1, \ldots, 2\cdot[s], \end{gathered} \tag{6.5}$$

is solvable, where the functionals L_j' and R_j' are determined from the relations

$$L_j'(\ldots) - R_j'(\ldots) = L_j(f^0, \partial v_1(0), \partial v_2(0), 0).$$

(ii) *To each solution of the system* (6.5) *corresponds in a one-to-one manner a* *solution of the boundary value problem* (2.1) *according to the formula*

$$u = F^{-1} \frac{\tilde{v}_1^0 (a_1^1 - iz_2 a_1^2) + \tilde{v}_1^1 a_1^2 + \tilde{v}_2^0 (a_2^1 - iz_1 a_2^2) + \tilde{v}_2^1 a_2^2 - f^0}{A(z)}, \tag{6.6}$$

where the functions v_l^β *are determined from* (5.41) *and* (6.2).

(iii). *Let* $\epsilon < \min([s] + 1 - s, \rho(s))$, *where*

$$\rho(s) = \min_{\substack{0 < \delta^0 - s + 1 \\ \delta^0 \in \mathrm{Re}\Delta^\sigma}} (\delta^0 - s + 1). \tag{6.7}$$

If $u \in H_s(\overline{\mathbf{K}})$ *is a solution of* (2.1) *and* $f_l \in H_{s+\epsilon-m_l-1/2}(\overline{\mathbf{R}}^+)$ *for* $l = 1, 2$, *then* $u \in H_{s+\epsilon}(\overline{\mathbf{K}})$ *and we have the a priori estimate*

$$\|u\|_{s+\varepsilon} \leqslant C\Big(\sum_{l=1,2} \|f_l\|_{s+\varepsilon-m_l-1/2} + \|u\|_s\Big). \tag{6.8}$$

Proof. Assertion (i) has been proved above, and (ii) follows from (ii) of Theorem 1.2. To prove (iii) we note that the compatibility conditions (6.5) are the same for s and $s+\epsilon$, while the norms $|f^0|$, $\|P_1\|$, $\|P_2\|$ and $\|\mathfrak{M}\|$ can be estimated from above by $\|u\|_s$. Then reference to (ii) of Theorem 1.2 and the estimates (5.42) and (6.4) completes the proof.

Let us write the system (6.5) briefly in the form

$$\mathcal{L}_j(f^0, P_1, P_2, \mathfrak{M}) = \mathcal{R}_j(f_1, f_2), \quad 1 \leqslant j \leqslant N(s), \tag{6.9}$$

where $\mathfrak{L}_j$ and $\mathfrak{R}_j$ are continuous linear functionals of their arguments and $N(s)$ is the number of equations in the system (6.5): $N(s) = \Sigma N^{l\beta}(s) + d + 2\cdot[s]$. The number of unknowns in (6.5) is $1 + m_1 + m_2 + d$; we denote the last sum by M. Then the functionals generate a linear operator $\mathfrak{L}$ acting from $\mathbf{C}^M$ to $\mathbf{C}^{N(s)}$. The system (6.9) is solvable if and only if its right side is orthogonal to all solutions of the homogeneous adjoint system:

$$\mathcal{R}(f_1, f_2) \perp \operatorname{Ker} \mathcal{L}^*, \tag{6.10}$$

where $\mathfrak{R}(f_1, f_2)$ denotes the vector $(\mathfrak{R}_j(f_1, f_2))_{j=1,\dots,N(s)}$. Let $\{S^i\}_{i=1,\dots,l^*}$ be a basis in $\operatorname{Ker}\mathfrak{L}^*$. Then (6.10) is equivalent to the homogeneous linear system

$$\langle S^i, \mathcal{R}(f_1, f_2)\rangle = 0, \quad i = 1, \dots, l^*. \tag{6.11}$$

By Theorem 6.1 the conditions (6.11) are necessary and sufficient for the solvability of (2.1) in the case $f = 0$.

Let us turn, finally, to the general case when $f \neq 0$. Here the condition of solvability of (2.1) is equivalent to the system of equations

$$\langle \mathcal{S}^i, \mathcal{F}\rangle = \langle S^i, \mathcal{R}(f_1^0, f_2^0)\rangle = 0, \quad i = 1, \dots, l^*, \tag{6.12}$$

where $\mathfrak{F} = (f, f_1, f_2)$ and the f_l^0 are determined as in (2.20). We write formula (6.6) briefly in the form

$$u = R(f_1, f_2, f^0, P_1, P_2, \mathfrak{M}). \tag{6.13}$$

Then under condition (6.12) to each solution of the system $\mathfrak{L}(f^0, P_1, P_2, \mathfrak{M}) = \mathfrak{R}(f_1^0, f_2^0)$ corresponds in a one-to-one manner a solution of (2.1), namely

$$u = u_+ + R(f_1^0, f_2^0, f^0, P_1, P_2, \mathfrak{M}), \tag{6.14}$$

where u_+ is defined in (2.20). Let us state the final result.

Theorem 6.2. (i) *Suppose problem* (2.1) *is strongly elliptic, s is not an integer and $s \notin$* S. *Then*

1) *The kernel of the operator $\mathfrak{A}$ is isomorphic to the kernel of $\mathfrak{L}$ and* $\dim \operatorname{Ker} \mathfrak{A} = \dim \operatorname{Ker} \mathfrak{L} = M - \operatorname{rank} \mathfrak{L}$, *and its cokernel is isomorphic to the space spanned by the functionals $\mathfrak{S}_i$; moreover* $\dim \operatorname{Coker} \mathfrak{A} \leq l^* \leq N(s)$.

2) *A solution of* (2.1) *can be found by means of formula* (6.14). *If $u \in H_s(\overline{\mathbf{K}})$ is a solution of* (2.1) *and $\mathcal{F} \in \mathcal{H}_{(s+\epsilon)}(\overline{\mathbf{K}})$, then $u \in H_{s+\epsilon}(\overline{\mathbf{K}})$ for $0 < \epsilon < \min([s] + 1 - s, \rho(s))$ and we have the a priori estimate*

$$\|u\|_{s+\varepsilon} \leqslant C\left(\|f\|_{s+\varepsilon-2} + \sum_{l=1,2} (\|f_l\|_{s+\varepsilon-m_l-1/2}) + \|u\|_s\right). \tag{6.15}$$

(ii) *In order that the boundary value problem* (2.1) *be Noetherian and its solution admit the a priori estimate* (2.8) *it is necessary and sufficient that this problem be strongly elliptic.*

As is well known, a Noetherian operator can be modified via a finite-dimensional one to an invertible operator. Specifically, if $\mathfrak{A}: H_s \to \mathcal{H}_{(s)}$ is a Noetherian operator, then there exist nonnegative integers r and r^* and operators $\mathcal{K}$, T and J such that the operator

$$\mathfrak{A} = \begin{pmatrix} \mathcal{A} & \mathcal{K} \\ T & J \end{pmatrix} : \begin{matrix} H_s \\ \oplus \\ \mathbf{C}^{r^*} \end{matrix} \to \begin{matrix} \mathcal{H}_{(s)} \\ \oplus \\ \mathbf{C}^{r} \end{matrix} \tag{6.16}$$

is an isomorphism. In fact, for invertibility of $\mathfrak{A}$ it is necessary and sufficient that the operator

$$(\mathcal{A}, \mathcal{K}) : \begin{matrix} H_s \\ \oplus \\ \mathbf{C}^{r^*} \end{matrix} \to \mathcal{H}_{(s)} \tag{6.17}$$

be an epimorphism and the operator

$$(T, J) : \operatorname{Ker}(\mathcal{A}, \mathcal{K}) \to \mathbf{C}^r \tag{6.18}$$

be an isomorphism. Let $k^* = \dim \operatorname{Coker} \mathfrak{A}$ amd $k = \dim \operatorname{Ker} \mathfrak{A}$. Then the operator (6.17) can be an epimorphism when $r^* \geq k^*$, while (6.18) can be inveritble when $r - r^* = \operatorname{ind} \mathfrak{A}$, where $\operatorname{ind} \mathfrak{A} = k - k^*$. Note that $\mathcal{K} = (\mathcal{K}_1^1, \cdots, \mathcal{K}_1^{r^*})$, where $\mathcal{K}^j \in \mathcal{H}_{(s)}$.

We shall state the results of this section in the form in which they will be needed for studying boundary value problems on a manifold with piecewise smooth boundary in §8.

Theorem 6.3. *Suppose problem* (2.1) *is strongly elliptic, s is not an integer and $s \notin$* S. *Then*

(i) *For $r^* \geq k^*$ and $r - r^* = k - k^*$ there exist operators $\mathfrak{K}$, T and J such that the operator $\mathfrak{A}$ in* (6.16) *is an isomorphism.*

(ii) *Invertibility of $\mathfrak{A}$ is equivalent to the following system of conditions:*

$$\operatorname{rank} \langle \mathcal{S}^i, \mathcal{K}^j \rangle_{\substack{i=1,\dots,l \\ j=1,\dots,r^*}} = k^*, \qquad \det (T, J)_0 \neq 0, \tag{6.19}$$

where $(T, J)_0$ denotes the restriction of the operator (T, J) to the kernel of $(\mathfrak{A}, \mathfrak{K})$ in the space $H_s \oplus C^{r^*}$.

Proof. Assertion (i) follows from the fact that $\mathfrak{A}$ is Noetherian. The first condition in (6.19) is equivalent to the fact that the operator (6.17) is an epimorphism; the second to the invertibility of (6.18).

§7. Function spaces and classes of operators on manifolds with piecewise smooth boundary

Let $\mathfrak{M}$ be a smooth compact n-dimensional space. We assume that it is stratified, i.e. $\mathfrak{M} = \bigcup_{k=0}^n \mathfrak{M}_k$, where $\mathfrak{M}_k$ is a smooth manifold of dimension $n-k$ and $\mathfrak{M}_i \cap \mathfrak{M}_j = \varnothing$ for $i \neq j$. We shall assume that $\mathfrak{M}_k = \varnothing$ for $k > 2$. Then

$$\mathcal{M} = \mathcal{M}_0 \cup \mathcal{M}_1 \cup \mathcal{M}_2. \tag{7.1}$$

In addition we shall assume that $n \geq 3$.

Definition 7.1. The space $\mathfrak{M}$ has a *piecewise smooth boundary* if

(i) $\mathfrak{M}$ has the form (7.1).

(ii) Each point $x \in \mathfrak{M}_1$ has a neighborhood in $\mathfrak{M}$ diffeomorphic to a half-ball $\mathbf{B}^+$ in $\mathbf{R}^n$: $\mathbf{B}^+ = \{x \in \mathbf{R}^n \mid |x| < 1, x_1 \geq 0\}$.

(iii) Each point $x \in \mathfrak{M}_2$ has a neighborhood in $\mathfrak{M}$ diffeomorphic to a quarter-ball $\mathbf{B}^{++}$ in $\mathbf{R}^n$:

$$\mathbf{B}^{++} = \{x \in \mathbf{B}^+ \mid x_2 \geqslant 0\}. \tag{7.2}$$

Thus $\mathfrak{M}_0$ is the interior of $\mathfrak{M}$ and $\partial\mathfrak{M} = \mathfrak{M}_1 \cup \mathfrak{M}_2$. The stratum $\mathfrak{M}_1$ consists of regular points of the boundary and $\mathfrak{M}_2$ is the edge of codimension two in $\mathfrak{M}$.

The manifold $\mathfrak{M}_1$ is canonically diffeomorphic to the interior of a smooth space $\mathfrak{M}_1^\sigma$ determined up to a diffeomorphism. Namely, a boundary point of $\mathfrak{M}_1^\sigma$ consists of a point $x \in \partial\mathfrak{M}_1 = \mathfrak{M}_2$ and one of the normals to $\mathfrak{M}_2$ at x directed into $\mathfrak{M}_1$. From this definition it follows that $\mathfrak{E}(\mathfrak{M}_1^\sigma)$ is a space of smooth functions on $\mathfrak{M}_1$ having, together with all derivatives, discontinuities of the first kind on $\mathfrak{M}_2$. For convenience we set $\mathfrak{M}_0^\sigma = \mathfrak{M}$ and $\mathfrak{M}_2^\sigma = \mathfrak{M}_2$.

Let $s(x)$ be an arbitrary locally constant function on $\mathfrak{M}_k^\sigma$. We shall denote by $H_{s(x)}(\mathfrak{M}_k^\sigma)$ the Sobolev space of functions on $\mathfrak{M}_k^\sigma$ having locally constant "smoothness".

We shall consider differential operators on $\mathfrak{M}$ with smooth coefficients. In addition we need boundary differential operators acting from $\mathfrak{E}(\mathfrak{M})$ to $\mathfrak{E}(\mathfrak{M}_1^\sigma)$. Let x be local coordinates on $\mathfrak{M}$ defined in a neighborhood of a point of $\partial\mathfrak{M}$.

Definition 7.2. A boundary differential operator has the form

$$Bu(x) = B(x, D)\,u(x) = \sum_{|\alpha| \leqslant m(x)} B^{\alpha}(x)\,D^{\alpha}u(x), \quad x \in \mathcal{M}_1^{\sigma},$$

in local coordinates, where $u \in \mathcal{E}(\mathfrak{M})$ and $B^{\alpha}(x)$ is a function in $\mathcal{E}(\mathfrak{M}_1^{\sigma})$. The function $m(x)$ is assumed to be locally constant on $\mathfrak{M}_1^{\sigma}$ and is called the *order* of the operator B.

Lemma 7.1. (i) *The class of boundary differential operators is invariant with respect to change of variables.*

(ii) *A boundary differential operator of order $m(x)$ acts continuously from $H_{s(x)}(\mathfrak{M}_0)$ to $H_{s(x)-m(x)-1/2}(\mathfrak{M}_1^{\sigma})$ if $s(x) - m(x) - 1/2 > 0$ for $x \in \mathfrak{M}_1^{\sigma}$.*

Besides differential operators we introduce certain Fourier operators "concentrated on $\mathfrak{M}_2$". Let $\mathfrak{M}_{k_1}$ and $\mathfrak{M}_{k_2}$ be strata of $\mathfrak{M}$ and let P be a continuous linear operator acting from $\mathcal{E}(\mathfrak{M}_{k_1}^{\sigma})$ to $\mathcal{E}(\mathfrak{M}_{k_2}^{\sigma})$. Let ϑ_l, $l = 1, 2$, be smooth functions on $\mathfrak{M}$ such that coordinates of type (7.2) are defined in supp $\vartheta_1 \cup$ supp ϑ_2. Then $x_1 = x_2 = 0$ is the local equation for $\mathfrak{M}_2$, and $\mathfrak{M}_{k_l}$ coincides locally with $\mathfrak{M}_2 \times \mathbf{K}_l$, where $\mathbf{K}_l$ is a smooth submanifold in $\mathbf{R}^2$. Let $x' = (x_1, x_2)$ and $x'' = (x_3, \cdots, x_n)$. Evidently

$$\vartheta_2 P \vartheta_1 u(x) = \frac{1}{(2\pi)^{n''}} \int_{\mathbf{R}^{n''}} e^{-i\langle x'', z''\rangle} \hat{P}(x'', z'')\,\hat{u}(z'')\,dz'' \tag{7.3}$$

for $u \in \mathcal{E}(\mathfrak{M}_{k_1}^{\sigma})$ and $x \in$ supp ϑ_2. Here $n'' = \dim \mathfrak{M}_2 = n - 2$,

$$\hat{u}(z'') = \int_{\mathbf{R}^{n''}} e^{i\langle z'', y''\rangle} \vartheta_1(y)\,u(y)\,dy'',$$

and $\hat{P}(x'', z'')$ is a smooth function on $\mathbf{R}^{2n''}$ with values in the set of continuous linear operators from $\mathcal{E}(\mathbf{K}_1^{\sigma})$ to $\mathcal{E}(\mathbf{K}_2^{\sigma})$.

Let n_l' denote the codimension of $\mathfrak{M}_2$ in $\mathfrak{M}_{k_l}$: $n_l' = 2 - k_l$. In the following we shall assume that $n_1' \neq 1$. Consequently either $\mathbf{K}_1^{\sigma} = 0$ (when $n_1' = 0$) or $\mathbf{K}_1^{\sigma}$ is a quadrant of the plane (when $n_1' = 2$). Let $m(x)$ and $r(x)$ be locally constant real functions on $\mathfrak{M}_{k_2}^{\sigma}$. For a function $u \in \mathfrak{D}(\mathbf{K}_1^{\sigma})$ let lu be an arbitrary extension of u to a function in $\mathfrak{D}(\mathbf{R}^2)$.

Definition 7.3. An operator P: $\mathcal{E}(\mathfrak{M}_{k_1}^{\sigma}) \to \mathcal{E}(\mathfrak{M}_{k_2}^{\sigma})$ is called a *Fourier operator of codimension two, degree $m(x)$ and class $r(x)$* if it has the form (7.3) in local coordinates and for $u \in \mathfrak{D}(\mathbf{K}_1^{\sigma})$ and $x' \in \mathbf{K}_2^{\sigma}$

$$\hat{P}(x'', z'')\,u(x') = \frac{1}{(2\pi)^{n_1'}} \int_{\mathbf{R}^{n_1'}} P(x, z)\,\widetilde{lu}(z')\,dz', \tag{7.4}$$

where the function $P(x, z)$ has the following properties:

1) There exists a function $P^{\sigma}(x, y'; z)$ such that

$$P(x, z) = P^{\sigma}(x, x'; z), \tag{7.5}$$

where P^{σ} is a smooth function on $\mathfrak{M}^{\sigma}_{k_2} \times \mathbf{K}^{\sigma}_2 \times \mathbf{R}^{n1}$ analytic in z' in $-\mathbf{CK}^*_1$ and

2) for any N, β, α'_1, α'_2 and α'' there exists C such that

$$(1 + (1 + |z''|)|y'|)^N |D^{\beta}_x D^{\alpha'_2}_{y'} \partial^{\alpha'_1}_{z'} \partial^{\alpha''}_{z''} P^{\sigma}(x, y'; z)|$$

$$\leqslant C(1 + |z''|)^{m(x)+n_2-r(x)+|\alpha'_2|-|\alpha''|}(1 + |z|)^{r(x)-|\alpha'_1|} \tag{7.6}$$

for all $(x, y', z) \in \mathfrak{M}^{\sigma}_{k_2} \times \mathbf{K}^{\sigma}_2 \times (-\mathbf{CK}^*_1) \times \mathbf{R}^{n''}$. The function P^{σ} is called the symbol of the operator P in the coordinates x.

By the Paley-Wiener theorem it follows from 1) and 2) that the integral in (7.4) depends only on u and not on lu. We shall employ the notation

$$\hat{P}(x'', z'') = P^{\sigma}(x, x'; D', z'') \tag{7.7}$$

for the operator (7.4).

Theorem 7.1. (i) *The set of Fourier operators of codimension two, degree $m(x)$ and class $r(x)$ is invariant with respect to change of variables.*

(ii) *A Fourier operator of codimension two, degree $m(x)$ and class $r(x)$ acts continuously from the space $H_{s(x)}(\mathfrak{M}^{\sigma}_{k_1})$ to $H_{s(x)-m(x)-(n'_1+n'_2)/2}(\mathfrak{M}^{\sigma}_{k_2})$ for $s(x) - r(x) - n'_1/2 > 0$ if $n'_1 \neq 0$ and for any s if $n'_1 = 0$.*

(iii) *Under the same relations between $s(x)$ and $r(x)$ as in* (ii), *the operator $\vartheta_2 P \vartheta_1$ is infinitely smoothing if ϑ_1 or ϑ_2 is equal to zero in a neighborhood of $\mathfrak{M}_2$.*

We now consider Fourier operators of codimension two with homogeneous symbols.

Definition 7.4. A Fourier operator P of codimension two, degree $m(x)$ and class $r(x)$ is called *homogeneous in its principal part* if the symbol P^{σ} of P admits a decomposition $P^{\sigma} = P^{\sigma}_0 + P^{\sigma}_1$ in local coordinates, where the P^{σ}_i are symbols of Fourier operators of codimension two, degree $m_i(x)$ and class $r(x)$, with $m_0(x) = m(x)$, $m_1(x) \leq m(x) - 1$ and for $t \geq 1$ and $|z''| \geq 1$

$$P^{\sigma}_0(x, y'; z', tz'') = t^{m(x)+n_2} P^{\sigma}_0(x, ty'; z'/t, z''). \tag{7.8}$$

The function P^{σ}_0 is called the *principal homogeneous part* of the symbol P^{σ}.

Let $\zeta(z'')$ be a function in $\mathcal{E}(\mathbf{R}^{n''})$ such that

$$\zeta(z'') = \begin{cases} 0, & |z''| \leqslant 1/2, \\ 1, & |z''| \geqslant 1. \end{cases}$$

We denote by N^+_2 the bundle of inner normals to $\mathfrak{M}_2$ in $\overline{\mathfrak{M}}_{k_1}$ and by T^*_1 the restriction of the bundle of cotangent spaces to $\mathfrak{M}_{k_1}$ on $\mathfrak{M}_2$. Recall that $n \geq 3$.

Theorem 7.2. (i) *The set of Fourier operators P, homogeneous in their principal part, of codimension two, degree $m(x)$ and class $r(x)$ is invariant with respect to change of variables.*

(ii) *The function*

$$\vartheta_1^{-1}(x'')\vartheta_2^{-1}(x'')P_0^{\sigma}(x'', y'; z) \tag{7.9}$$

is invariantly defined on the bundle $N_2^+ \times T_1^*$ if we identify x'' and y' with coordinates in the base and fiber, respectively, of the bundle N_2^+, and z with the coordinate in the fiber of T_1^*.

(iii) *If the function $P_0^{\sigma}(x, y'; z', \omega')$ is defined on $\mathfrak{M}_{k_1}^{\sigma} \times \mathbf{K}_2^{\sigma} \times (-\mathbf{CK}_1^*) \times S^{n''-1}$ in local coordinates, is analytic in z' and for any α'', α_1', α_2', β and N admits the estimates*

$$|(1+|y'|)^N D_x^{\beta} D_{y'}^{\alpha_2'} \partial_{z'}^{\alpha_1'} \partial_{\omega''}^{\alpha''} P_0(x, y'; z', \omega'')| \leqslant C(1+|z'|)^{r(x)-|\alpha_1'|}, \tag{7.10}$$

then the Fourier operator of codimension two with symbol

$$P_0^{\sigma}(x, y'; z) = \zeta(z'')|z''|^{m(x)+n_2'} P_0^{\sigma}\left(x, |z''|y'; \frac{z}{|z''|}\right) \tag{7.11}$$

is homogeneous in its principal part and has degree $m(x)$ and class $r(x)$.

§8. Boundary value problems on manifolds with piecewise smooth boundary

In this section we consider the boundary value problem (0.1):

$$\begin{aligned} Au_0(x) + Ku_2(x) &= f_0(x), \quad x \in \mathcal{M}_0, \\ Bu_0(x) + Lu_2(x) &= f_1(x), \quad x \in \mathcal{M}_1, \\ Tu_0(x) + Ju_2(x) &= f_2(x), \quad x \in \mathcal{M}_2. \end{aligned} \tag{8.1}$$

For convenience we write this problem in the form

$$(\mathcal{A} + \mathfrak{T})\mathcal{U} = \mathcal{F}, \tag{8.2}$$

where $\mathfrak{U} = (u_0, u_2)$ and $\mathfrak{F} = (f_0, f_1, f_2)$. The operator (A, B) is denoted by $\mathfrak{A}$. We shall assume that

$$A = A(x, D) = \sum_{|\alpha| \leqslant 2} A^{\alpha}(x) D^{\alpha}, \tag{8.3}$$

where $A^{\alpha} \in \mathcal{E}(\mathfrak{M})$. B denotes a boundary differential operator of order $m_1(x)$:

$$B = B(x, D) = \sum_{|\alpha| \leqslant m_1(x)} B^{\alpha}(x) D^{\alpha}. \tag{8.4}$$

The operator $\mathfrak{T}$ is a matrix of Fourier operators, homogeneous in their principal part, of codimension two:

$$\mathfrak{T} = (\mathfrak{T}_{kl})_{\substack{k=0,1,2\\ l=0,2}}. \tag{8.5}$$

The operator $\mathfrak{T}_{kl}$ acts from $\mathfrak{S}^{r_l}(\mathfrak{M}_l^\sigma)$ to $\mathfrak{S}^{r_k}(\mathfrak{M}_k^\sigma)$, has matrix symbol $(\mathfrak{T}_{kl}^{ij})$, matrix degree $m_{kl} = (m_{kl}^{ij})$ and matrix class $r_{kl} = (r_{kl}^{ij})$, where $1 \leq j \leq r_l^*$ and $1 \leq i \leq r_k$.

For simplicity we restrict ourselves to the case when the edge $\mathfrak{M}_2$ is connected. We assume that $f_0 \in H_{t0}(\mathfrak{M})$ and $f_1 \in H_{t1(x)}(\mathfrak{M}_1^\sigma)$, while $f_2 = (f_2^1, \cdots, f_2^r)$ belongs to the space $H_{t_2}(\mathfrak{M}_2) = \bigoplus_{i=1}^r H_{t_2^i}(\mathfrak{M}_2)$, where t_2 is the vector $(t_2^1, \cdots, t_2^r)$. We seek the function u_0 in $H_{s_0}(\mathfrak{M})$, while u_2 belongs to $\bigoplus_{j=1}^{r^*} H_{s_2^j}(\mathfrak{M}_2)$. Thus $r_0^* = r_0 = r_1 = 1$, $r_2 = r$, $r_2^* = r^*$ and $r_1^* = 0$.

For brevity we denote the operator $\mathfrak{A} + \mathfrak{T}$ by $\mathfrak{A}$.

Theorem 8.1. *Let $s_0 - m_1(x) - ½ > 0$ and $s_0 - r_{20} - 1 > 0$. Then the operator $\mathfrak{A}$ acts continuously from $\mathcal{H}_s(\mathfrak{M}) = H_{s0}(\mathfrak{M}) \oplus H_{s_2}(\mathfrak{M}_2)$ to $\mathcal{H}_t(\mathfrak{M}) = H_{t0}(\mathfrak{M}) \oplus H_{t_1(x)}(\mathfrak{M}_1) \oplus H_{t2(x)}(\mathfrak{M}_2)$ if for $k, l = 0, 1, 2$*

$$t_k^i(x) = s_l^j(x) - m_{kl}^{ij}(x) - \frac{4-k-l}{2} \tag{8.6}$$

for all $x \in \mathfrak{M}_k^\sigma$, $1 \leq j \leq r_l^*$ and $1 \leq i \leq r_k$.

Proof. The assertion follows from (ii) of Theorem 7.1.

In the following we shall assume that the hypotheses of Theorem 8.1 are satisfied.

From classical results it follows that for the a priori estimate

$$\|u_0\|_{s_0+\varepsilon} \leqslant C(\|Au_0\|_{s_0+\varepsilon-2} + \|Bu_0\|_{s_0+\varepsilon-m_1(x)-1/2} + \|u_0\|_{s_0}) \tag{8.7}$$

to hold with $\epsilon > 0$ it is necessary that the ellipticity condition for A and the Šapiro-Lopatinskiĭ condition hold up to the boundary. From (iii) of Theorem 7.1 it follows that these same conditions are also necessary for the a priori estimate

$$\|u_0\|_{s_0+\varepsilon} \leqslant C(\|\mathfrak{A}u_0\|_{t+\varepsilon} + \|u_0\|_{s_0}). \tag{8.8}$$

Ellipticity of A means that for $(x, \xi) \in S^*\mathfrak{M}$

$$A_0(x, \xi) \neq 0, \tag{8.9}$$

where $A_0(x, \xi) = \sum_{|\alpha|=2} A^\alpha(x)\xi^\alpha$ and $S^*\mathfrak{M}$ is the bundle of cotangent spheres to $\mathfrak{M}$. We now state the Šapiro-Lopatinskiĭ condition. For our case, when the boundary is not smooth, it can be formulated in the following way. Let N_1^* be the bundle of inner conormals to $\mathfrak{M}_1^\sigma$ in $\mathfrak{M}$ and let $\pi^! N_1^*$ be a lifting of N_1^* to $S^*\mathfrak{M}_1^\sigma$ using the projection $\pi: S^*\mathfrak{M}_1^\sigma \to \mathfrak{M}_1^\sigma$. Let $B_0(x, \xi)$ be the principal part of the symbol of B:

$$B_0(x, \xi) = \sum_{|\alpha|=m(x)} B^\alpha(x)\xi^\alpha. \tag{8.10}$$

Then the Šapiro-Lopatinskiĭ condition consists in the fact that

$$B_0(x, \xi', z) \neq 0 \tag{8.11}$$

for $(x, \xi', z) \in \mathbf{C}\pi^! N_1^*$ if $A_0(x, \xi', z) = 0$ and $\operatorname{Im} z < 0$. Here $\mathbf{C}\pi^! N_1^*$ is the complexification of the bundle $\pi^! N_1^*$, (x, ξ', z) is a point of this bundle with projection $(x, \xi') \in S^*\mathfrak{M}_1^\sigma$ and z is the complex variable dual to the normal to $\mathfrak{M}_1^\sigma$ at the point x, directed into $\mathfrak{M}$.

Recall that in [13] inner and boundary symbols were defined for boundary value problems on manifolds with a smooth boundary. We now connect the biboundary symbol with problem (8.1). Let $\vartheta_l \in \mathfrak{D}(\mathfrak{M})$ for $l = 1, 2$ and let x be the coordinates in $\operatorname{supp}\vartheta_1 \cup \operatorname{supp}\vartheta_2$ used in Definition 7.3. The operator $\vartheta_2 \mathfrak{A} \vartheta_1$ admits a representation (7.6) with symbol $\hat{\mathfrak{A}}^{12}(x'', \xi'')$ that is a smooth function of $(x'', \xi'') \in T^*\mathfrak{M}_2$ with values in the set of boundary value problems in a quadrant:

$$\hat{\mathfrak{A}}^{12}(x'', \xi'') = \begin{pmatrix} A(x; D', \xi'') & K^\sigma(x, x'; \xi'') \\ B(x; D', \xi'') & L^\sigma(x, x'; \xi'') \\ T^\sigma(x''; D', \xi'') & J^\sigma(x''; \xi'') \end{pmatrix}. \tag{8.12}$$

Let $K\mathfrak{M}_2$ be the bundle of quadrants formed by the inner normals to $\mathfrak{M}_2$ in $\mathfrak{M}$. Let $K^!\mathfrak{M}_2$ be a lifting to $S^*\mathfrak{M}_2$ of the bundle $K\mathfrak{M}_2$, and let $\Phi K^!\mathfrak{M}_2$ be a bundle of operators acting in the bundle of functions on $K^!\mathfrak{M}_2$.

Lemma 8.1. *Under the identification of the quadrant $x_1 > 0$, $x_2 > 0$ with the fiber of the bundle $K^!\mathfrak{M}_2$ lying over $(x'', \xi'') \in S^*\mathfrak{M}_2$, the operator*

$$\hat{\mathfrak{A}}_0(x'', \xi'') = \vartheta_1^{-1}(x'')\,\vartheta_2^{-1}(x'')\,\hat{\mathfrak{A}}_0^{12}(x'', \xi'') \tag{8.13}$$

is invariantly defined by a cross-section of the bundle $\Phi K^!\mathfrak{M}_2$ if $\hat{\mathfrak{A}}_0^{12}(x'', \xi'')$ is the operator in the quadrant constructed from the principal parts of the symbols $A, \cdots, J^\sigma$ frozen at the origin of the quadrant:

$$\hat{\mathfrak{A}}_0^{12}(x'', \xi'') = \begin{pmatrix} A_0(x''; D', \xi'') & K_0^\sigma(x'', x'; \xi'') \\ B_0(x'' + 0x'; D', \xi'') & L_0^\sigma(x'' + 0x', x'; \xi'') \\ T_0^\sigma(x''; D', \xi'') & J_0^\sigma(x''; \xi'') \end{pmatrix}. \tag{8.14}$$

Proof. This lemma follows from (ii) of Theorem 7.2.

Definition 8.1. The *biboundary symbol* of problem (8.1) is the cross-section of the bundle $\Phi K^!\mathfrak{M}_2$ equal to $\hat{\mathfrak{A}}_0(x'', \xi'')$ at those points $(x'', \xi'') \in S^*\mathfrak{M}_2$ at which $\vartheta_1(x'')\,\vartheta_2(x'') \neq 0$.

We now note that the operators $\hat{K}_0$, $\hat{L}_0$, $\hat{J}_0$ and $\hat{T}_0$ are finite-dimensional. Consequently $\hat{\mathfrak{A}}_0$ differs from the operator

$$\hat{\mathcal{A}}(x'', \xi'') = \vartheta_1^{-1}(x'')\,\vartheta_2^{-1}(x'') \begin{pmatrix} A_0(x''; D', \xi'') \\ B_0(x'' + 0x'; D', \xi'') \end{pmatrix} \tag{8.15}$$

only by a finite-dimensional operator. But just such operators in a quadrant were considered in §§2–6 of this paper. Evidently $\hat{\mathfrak{A}}_0(x'', \xi'')$ is a strongly elliptic operator for $\xi'' \neq 0$ in the sense of Definition 5.1 if conditions (8.9) and (8.11) are satisfied. Consequently by Theorem 6.2 it is Noetherian if s_0 is not an integer and $s_0 \notin S(x'', \xi'')$. In accordance with (5.31)

$$\mathbf{S}(x'', \xi'') = \left\{ s = 1 + \frac{\mu_l^0(x'', \xi'') + k}{\nu^0(x'', \xi'')} \,\middle|\, l = 1, 2;\ k = 0, 1, \dots \right\}. \tag{8.16}$$

We state the definition of ellipticity for problem (8.1). Suppose s is not an integer. For convenience we introduce the notation

$$\hat{\mathcal{K}}_0^j(x'', \xi'') = \begin{pmatrix} \hat{K}_0^j(x'', \xi'') \\ \hat{L}_0^j(x'', \xi'') \end{pmatrix}, \quad 1 \leqslant j \leqslant r^*. \tag{8.17}$$

Definition 8.2. Problem (8.1) is called *elliptic* if the following conditions hold:

I. The symbol A_0 is elliptic: for $(x, \xi) \in S^*\mathfrak{M}$

$$A_0(x, \xi) \neq 0. \tag{8.18}$$

II. The symbols A_0 and B_0 satisfy the condition: for $(x, \xi') \in S^*\mathfrak{M}_1^\sigma$

$$B_0(x, \xi', z) \neq 0, \tag{8.19}$$

if $A_0(x, \xi', z) = 0$ and $\operatorname{Im} z < 0$.

III. For all $(x'', \xi'') \in S^*\mathfrak{M}_2$ the set $S(x'', \xi'')$ does not contain s_0 and

$$\operatorname*{rang}_{\substack{i=1,\dots,l^*(x'',\xi'') \\ j=1,\dots,r^*}} \langle \mathcal{S}^i(x'', \xi''), \hat{\mathcal{K}}_0^j(x'', \xi'') \rangle = k^*(x'', \xi''),$$

$$\det(\hat{T}_0(x'', \xi''), \hat{J}_0(x'', \xi''))_0 \neq 0, \tag{8.20}$$

where $\mathcal{S}^i(x'', \xi'')$ is a basis in the cokernel of the operator $\hat{\mathfrak{A}}_0(x'', \xi'')$ and $(\hat{T}_0, \hat{J}_0)_0$ is the restriction of $(\hat{T}_0, \hat{J}_0)$ to the kernel of $(\hat{\mathfrak{A}}_0, \hat{\mathfrak{H}}_0)$ in the space $\mathcal{H}_s(\overline{\mathbf{K}})$:

$$\mathcal{H}_s(\overline{\mathbf{K}}) = H_{s_0}(\overline{\mathbf{K}}) \oplus H_{s_1(+0,0)}(\overline{\mathbf{R}}^+) \oplus H_{s_1(0,+0)}(\overline{\mathbf{R}}^+) \oplus \mathbf{C}^{r^*}. \tag{8.21}$$

Conditions I and II are equivalent to invertibility of the inner and boundary symbols, respectively, of problem (8.1). They are algebraic conditions on the symbols. Condition III is equivalent to invertibility of the biboundary symbol of the problem under consideration. It is effective by virtue of the results of the second part of this paper. In fact, formula (6.12) permits us to construct a basis in the cokernel of the operator $\hat{\mathfrak{A}}_0(x'', \xi'')$. Verification of (8.20) reduces to solving a finite system of linear equations in a finite number of unknowns, since we constructed the operator inverse to $\hat{\mathfrak{A}}_0(x'', \xi'')$ in §6 (formulas (6.14), (6.6), (6.2) and (5.41)).

Let us state the main result of this paper.

Theorem 8.2. *Suppose the hypotheses of Theorem* 8.1 *are satisfied. If* s_0 *is not an integer and problem* (8.1) *is elliptic, then*

(i) *this problem is Noetherian*;

(ii) *if* $\mathfrak{U} \in \mathcal{H}_s(\mathfrak{M})$ *satisfies problem* (8.1), *then, for some* $\epsilon > 0$ *depending on* s_0, $\mathfrak{U} \in \mathcal{H}_{s+\epsilon}(\mathfrak{M})$ *follows from* $\mathfrak{F} \in \mathcal{H}_{t+\epsilon}(\mathfrak{M})$, *and the a priori estimate*

$$\| \mathcal{U} \|_{s+\varepsilon} \leqslant C(\| \mathcal{F} \|_{t+\varepsilon} + \| \mathcal{U} \|_s) \tag{8.22}$$

holds.

Proof. Using the method of freezing coefficients it is possible to construct the parametrix for problem (8.1). In this connection assertion (ii) of Theorem 7.2 plays the role of commutation lemma. In addition, it is essential that the principal parts of the symbols of all operators in (8.1) be homogeneous.

Thus we have found a condition that problem (8.1) be well posed. We now set ourselves the problem of when there exist, for given operators A and B, Fourier operators of codimension two completing problem (0.2) to the elliptic problem (0.1). An answer to this question can be stated in terms of the K-functor similarly to the way this was done in [9] and [14] for the case of a smooth boundary. But to do this we need Fourier operators of codimension two acting in cross-sections of vector bundles. Let $\hat{\mathfrak{A}}_0(x'', \xi'')$ denote the biboundary symbol corresponding to problem (0.2). Suppose conditions I and II of Definition 8.2 are satisfied. We assume that s_0 is sufficiently large, not an integer and $s_0 \notin S(x'', \xi'')$ for $(x'', \xi'') \in S^*\mathfrak{M}_2$. Then by Theorem 6.2 the symbol $\hat{\mathfrak{A}}_0$ defines a Fredholm complex on $S^*\mathfrak{M}_2$. The Euler characteristic of this complex is an element of the ring $K(\mathfrak{M}_2)$ and is denoted by $\operatorname{ind} \hat{\mathfrak{A}}_0$. Let π be the projection of $S^*\mathfrak{M}_2$ onto $\mathfrak{M}_2$, and let $\pi^!$ be the induced ring homomorphism $K(\mathfrak{M}_2) \to K(S^*\mathfrak{M}_2)$.

Now let the $\mathfrak{T}_{kl}$ in (8.5) be the Fourier operators of codimension two acting in the Sobolev spaces of cross-sections of finite-dimensional vector bundles.

Theorem 8.3. *If the operators A and B and the number s_0 satisfy the requirements enumerated above, then we have an elliptic boundary value problem of the type* (8.1) *if and only if*

$$\operatorname{ind} \hat{\mathcal{A}}_0 \in \pi^! K(\mathcal{M}_2).$$

Proof. This theorem is proved similarly to Theorem 5.14 of [14], using Theorem 6.3 and (iii) of Theorem 7.2.

In the form we have stated it, problem (8.1) corresponds to the case when the operators $\mathfrak{T}_{kl}$ act in cross-sections of trivial bundles. Then the existence condition for a well-posed problem is formulated similarly to that in [9]: we have an elliptic boundary value problem (8.1) if and only if for some integral nonnegative κ

$$\operatorname{ind} \hat{\mathcal{A}}_0 = \pm \mathbf{C}^{\varkappa} \times \mathcal{M}_2.$$

Appendix

Recall that in §5 we repeatedly used the following theorem:

Theorem A.1. (i) *If $g \in H_s(\overline{\mathbf{R}}^+)$, where s is not a half-integer and $s > -\frac{1}{2}$, then*

$$\tilde{g}(z) = \tilde{g}_0(z) + \sum_{-s+3/2 \leqslant k \leqslant -1} g^k (z+i)^k,$$

with $g_0 \in \overset{\circ}{H}_s(\overline{\mathbf{R}}^+)$ and k an integer. In addition

$$\|g_0\|_{\overset{\circ}{H}_s} + \sum_{-s+3/2 \leqslant k \leqslant 1} |g^k| \leqslant C_s \|g\|_{H_s}.$$

(ii) *Conversely, if* $\tilde{g}$ *has an expansion* (A.1), *then* $g \in H_s(\overline{\mathbf{R}}^+)$ *and*

$$\|g\|_{H_s} \leqslant C_s\left(\|g_0\|_{\overset{\circ}{H}_s} + \sum_{-s+3/2\leqslant k\leqslant 1} |g^k|\right).$$

Proof. This theorem follows from the fact that the Mellin transform of a function in $H_s(\overline{\mathbf{R}}^+)$ is meromorphic in the strip $-1/2 < \operatorname{Im} z < s - 1/2$ and the transform of a function in $\overset{\circ}{H}_s(\overline{\mathbf{R}}^+)$ is holomorphic in the same strip.

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