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Elliptic equations and rearrangements

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Elliptic Equations and Rearrangements.

GIORGIO TALENTI (*)

dedicated to Jean Leray

1. – Introduction.

We are concerned with linear elliptic second-order partial differential equations in divergence form. The equations we deal with are uniformly elliptic and have real-valued coefficients. Typically we consider equations of the following form

$$-\sum_{i,k=1}^{m} \frac{\partial}{\partial x_i}\left(a_{ik}(x)\frac{\partial u}{\partial x_k}\right) + c(x)u = f(x)\,, \tag{1}$$

where $x=(x_1,\dots,x_m)$ is a point in the m-dimensional euclidean space R^m, the matrix $\big(a_{ik}(x)\big)_{i,k=1,\dots,m}$ of the leading coefficients is uniformly positive definite and $c(x)$ is everywhere nonnegative. Except for the last section, where eigenvalues are considered, the symmetry of the matrix $\big(a_{ik}(x)\big)_{i,k=1,\dots,m}$ is not needed. For convenience, we normalize the coefficients in such a way that: $1=$ the lower ellipticity constant. Thus our assumptions read

$$\sum_{i,k=1}^{m} a_{ik}(x)\,\xi_i\xi_k \geqslant \xi_1^2+\dots+\xi_m^2\,, \qquad c(x)\geqslant 0\,. \tag{2}$$

We consider solutions of Dirichlet problems with zero boundary data. Thus a basic ingredient is

$$G = \text{an open subset of } R^m\,, \tag{3}$$

and the solutions of interest are real-valued functions verifying the equa-

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tion (1) in G and the condition

$$u = 0 \quad \text{on the boundary } \partial G \text{ of } G. \tag{4}$$

Our aim is to establish some sharp estimates of solutions to problems (1) ... (4).

Roughly speaking, a good estimate of some norm of a solution to any differential problem is a number depending in a readable way from the information we have, or we are willing to use (e.g. smoothness of coefficients, smoothness of the ground domain, size of the right-hand side, etc.), on the data. Thus, in a sense the simplest estimates are obtained when the relevant information on the data amounts to a minimum (or, which is the same, if the maximum generality is allowed in the assumptions). Because of this, we seek estimates of solutions to problems (1) ... (4) which can be derived from the ellipticity condition (2a) and the positiveness (2b) of $c(x)$ only, and involve nothing but L^p-norms of the right-hand side $f(x)$ and the measure of the domain G (incidentally, for technical reasons it is convenient for us to use also Lorentz norms of $f(x)$).

Thus, we do not need any smoothness of the coefficients $a_{ik}(x)$ and $c(x)$ in (1), nor of the right-hand side, nor of the domain G: $a_{ik}(x)$ and $c(x)$ are assumed to be merely measurable functions verifying the conditions (2) only, G is any open set, $f(x)$ is any function from some L^p-space (or from some Lorentz space).

A sharp estimate comes very often from a variational planning: fix a set of data (which possibly consists of a kind of cartesian product of a set of equations, a set of ground domains, a set of boundary data or other things) and look at the less favourable ones, namely those for which the quantity under estimation takes its maximum value. Accordingly, we consider the set of all pairs whose elements are: (i) the differential operator

$$-\sum_{i,k=1}^{m} \frac{\partial}{\partial x_i}\left(a_{ik}\frac{\partial}{\partial x_k}\ \cdot\right) + c \tag{5}$$

at the left-hand side of (1), restricted by the conditions (2); (ii) a domain G, with an arbitrary but fixed measure. We seek the pair on which some functionals of the following type

$$\text{supremum of the ratio } \frac{\text{a norm of } u}{\text{another norm of } f} \tag{6}$$

attain their maximum value. Here f runs over a dense set in such a way that a weak solution u of (1) ... (4) exists.

For precision of speech, a weak solution in an open set G to the equation (1) is a locally integrable function u, whose first derivatives are *square integrable* over G, such that:

$$\int_G \Big[\sum_{i,k=1}^m a_{ik} u_{x_i} \varphi_{x_k} + cu\varphi - f\varphi\Big]\, dx = 0 \qquad \text{for all } \varphi \text{ in } C_0^\infty(G)$$

(= the class of infinitely differentiable functions vanishing outside a compact subset of G). A weak solution u in G verifies the boundary condition (4) if a sequence φ_n exists in $C_0^\infty(G)$ such that:

$$\int_G |\mathrm{grad}(u - \varphi_n)|^2\, dx \to 0\,.$$

The main result of the present paper can be stated as follows. *Let the norms in* (6) *be any L^p-norm of f, and any L^q-norm of u or the L^q-norm (with $q \leqslant 2$) of the gradient of u. Then the differential operator and the domain, which maximize* (6), *are the Laplace operator and a ball.*

An analogous result for the ratio $\max|u| \cdot \big(\int |f|^p dx\big)^{-1/p}$ was proved by Weinberger [21].

The theorem above enables us to obtain the best form of some estimates of solutions to elliptic equations in divergence form, see theorem 2 below. Of course, our results on the last matter are nothing more than refinements of well-known ones (see Stampacchia [17] [18], Ladyzenskaja-Ural'ceva [8], Miranda [9]).

In the last section we consider the eigenvalue problem:

$$(7) \qquad -\sum_{i,k=1}^m \frac{\partial}{\partial x_i}\left(a_{ik}(x)\frac{\partial u}{\partial x_k}\right) + c(x)u = \lambda u \ \text{ in } G\,, \qquad u = 0 \ \text{ on } \partial G\,.$$

Thanks to the ellipticity condition (2a) and the positiveness (2b) of $c(x)$, the smallest eigenvalue λ of the problem (7) is greater than the smallest eigenvalue of the Laplace operator $-\Delta$ in the same set G with the same boundary conditions. Hence λ is greater than

$$(8) \qquad \pi\{\Gamma(1 + m/2)\,\mathrm{meas}\, G\}^{-2/m} j^2_{(m/2)-1}\,,$$

the smallest eigenvalue of the Laplace operator in a ball with the same measure as G. Here $j_{(m/2)-1}$ is the smallest positive zero of the Bessel function $J_{(m/2)-1}$. The latter estimates comes from the well-known Faber-Krahn inequality, see [15] and also [13]. We give a simple answer to the following problem: can the deviation of λ from (8) be estimated from below in terms of L^p-norms of $c(x)$?

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2. – Main results.

The precise statement and the proof of our main theorem require a few of words on the rearrangements of functions in the sense of Hardy and Littlewood.

Let u be a real-valued measurable function, defined in a measurable (say open) subset G of R^m. We can form: (i) the distribution function of u; (ii) the decreasing rearrangement of u into $[0, +\infty]$; (iii) the spherically symmetric rearrangement of u.

As is well-known, the distribution function of u (or of $|u|$) is

$$\mu(t) = \text{meas}\,\{x \in G : |u(x)| > t\}\,, \tag{9}$$

a right-continuous function of t, decreasing from $\mu(0) = \text{meas}$ (support of u) to $\mu(+\infty) = 0$ as t increases from 0 to $+\infty$ and jumping at every value t which is assumed by $|u|$ on a set of positive measure:

$$\mu(t-) - \mu(t) = \text{meas}\,\{x \in G : |u(x)| = t\}\,.$$

The decreasing rearrangement of u into $[0, +\infty]$ will be denoted by u^*. In the case where μ decreases strictly, u^* is the decreasing function which extends to the whole of the half-line $[0, +\infty]$ the inverse function of μ. In any case, u^* can be defined as the smallest decreasing function from $[0, +\infty]$ into $[0, +\infty]$ such that $u^*(\mu(t)) \geqslant t$ for every t. Hence $u^*(s)$ is the endpoint of the interval $\mu^{-1}(s) = \{t \geqslant 0 : \mu(t) = s\}$ if s lies in the range of μ; if $s \geqslant 0$ is not a value of μ, then either $u^*(s) = 0$ or $u^*(s) = t$ according as $s > \mu(0)$ or $\mu(t) < s \leqslant \mu(t-)$. In particular, u^* is constant in every connected component of the complementary set of the range of μ. More concisely

$$u^*(s) = \inf\{t \geqslant 0 : \mu(t) < s\}\,. \tag{10}$$

One of the most important properties of u^* is the following: u and u^* have the same distribution function. Indeed the level set $\{s \geqslant 0 : u^*(s) > t\}$ is exactly the interval with extreme points in 0 and $\mu(t)$. Consequently

$$\int_G |u(x)|^p\,dx = \int_0^{+\infty} u^*(t)^p\,dt$$

for every $p > 0$, since both sides equal $\int_0^{+\infty} t^p\, d(-\mu(t))$. Clearly: ess.sup.$|u| = u^*(0+)$. Incidentally, from (10) we infer at once that u^* is left-continuous.

The spherically symmetric rearrangement of u is a function $u^\star$ from R^m into $[0, +\infty]$ whose level sets $\{x \in R^m : u^\star(x) > t\}$ are concentric balls with the same measure as the level sets $\{x \in G : |u(x)| > t\}$ of $|u|$. In other words, $u^\star$ is a positive spherically symmetric function with the same distribution function as u. Consequently

$$\int_G |u(x)|^p\, dx = \int_{R^m} u^\star(x)^p\, dx$$

for every $p > 0$, and

$$\text{ess.sup.}|u| = \text{ess.sup.}\, u^\star .$$

The precise definition of $u^\star$ is

$$u^\star(x) = u^*(C_m|x|^m) \equiv \inf\{t \geqslant 0 : \mu(t) < C_m|x|^m\}, \tag{11}$$

where

$$C_m = \frac{\pi^{m/2}}{\Gamma(1 + m/2)} \tag{12}$$

is the measure of the m-dimensional unit ball.

Rearrangements bring out integral properties of functions. For example, a theorem of Hardy and Littlewood tells us the following. If u and v are real-valued measurable functions, defined in a measurable subset G of R^m, then

$$\int_G |uv|\, dx \leqslant \int_0^{+\infty} u^*(s) v^*(s)\, ds = \int_{R^m} u^\star(x) v^\star(x)\, dx, \tag{13}$$

and in particular

$$\int_A |u|\, dx \leqslant \int_0^{\text{meas}\, A} u^*(s)\, ds \tag{14}$$

whenever A is a measurable subset of G.

On the other hand, there are connections between rearrangements and smoothness properties of functions. In a sense, the spherically symmetric rearrangement is a smoothing process, when it acts on functions whose level sets do not meet the boundary. In fact, if u is continuous in the closure $\bar{G}$ of an open bounded set G and vanishes on the boundary ∂G, then $u^\star$ is continuous and its modulus of continuity is smaller than the modulus of continuity of u, see [7]. If u is Lipschitz continuous in G and vanishes on ∂G, then the Lipschitz constant of $u^\star$ does not exceed that of u, see [4] and also [20]. Moreover, an extremely remarkable principle of Polya and Szegö (see e.g. [15]) tells us that the Dirichlet integral of functions vanishing on the boundary decreases under the spherically symmetric rearrangement. More precisely, if u is a function from any Sobolev space $W^{2,p}(G)$ and u is zero (in the appropriate sense) on ∂G, then $u^\star$ is in $W^{1,p}(R^\star)$ and the inequality holds

$$\int_G |\operatorname{grad} u|^p\, dx \geqslant \int_{R^m} |\operatorname{grad} u^\star|^p\, dx\,. \tag{15}$$

See e.g. [20] for a proof. ■

Now we are in position to state our main theorem.

THEOREM 1. *Assumption* :

(i) *the differential operator* (5) *satisfies the constraints* (2);

(ii) G *is an open subset of* R^m (*if* $m = 2$, *we require*: $\operatorname{meas} G < \infty$);

(iii) f *is in* $L^p(G)$; *here* $p = 2m/(m+2)$ *if* $m > 2$, $p > 1$ *if* $m = 2$.

If

Δ *is the Laplace operator*;

$G^\star$ *is the ball* (centered at the origin) *with the same measure as* G (*or the whole of* R^m *if* $\operatorname{meas} G = \infty$);

$f^\star$ *is the spherically symmetric rearrangement of* f;

u *is the weak solution on the set* G *to the equation* (1), *satisfying the boundary condition* (4); v *is the weak solution to the Dirichlet problem*

$$-\Delta v = f^\star \quad \text{in } G^\star, \qquad v = 0 \quad \text{on } \partial G^\star\,; \tag{16}$$

then

(iv) $v \geqslant u^\star$ *pointwise. Here* $u^\star$ *is the spherically symmetric rearrange-*

ment of u. Therefore

$$\text{ess.sup.}\, v \geqslant \text{ess.sup.}|u|,$$

$$\int_{G^\star} v^q\,dx \geqslant \int_G |u|^q\,dx \ \textit{for every positive exponent } q\,.$$

$$\text{(v)}\quad \int_{G^\star} |\operatorname{grad} v|^q\,dx \geqslant \int_G \Big[\sum_{i,k=1}^m a_{ik} u_{x_i} u_{x_k}\Big]^{q/2} dx\,,\ \textit{if}\ 0 < q \leqslant 2\,.$$

Let us mention that a theorem, on connections between the rearrangement of f and the rearrangement of solutions u to the Poisson equation $\Delta u = f$, was proved by R. O'Neil (private communication).

THEOREM 2. *Let u be a weak solution in an open set G to the equation* (1) *and let u satisfy the boundary condition* (4). *For the sake of simplicity, we restrict ourselves to the case: $m =$ number of dimensions $\geqslant 3$. Then u verifies the inequalities listed below, provided the ellipticity conditions* (2) *hold and the appropriate power of the right-hand side f is integrable over G. All these inequalities are the best possible.*

$$\text{(i)}\quad \int_G \Big(\sum_{i,k=1}^m a_{ik} u_{x_i} u_{x_k}\Big)^{q/2} dx \leqslant A^q \Big(\int_G |f|^p\,dx\Big)^{q/p}.$$

Here $1 < p \leqslant 2m/(m+2)$, $q = mp/(m-p)$, and A is the following constant

$$A = \frac{q^{-1/q}}{m\sqrt{\pi}}\Big(\frac{p}{p-1}\Big)^{1/p}\Big\{\frac{\Gamma(m)\Gamma(m/2)}{2\Gamma(m/p)\Gamma(m-m/p)}\Big\}^{1/m}.$$

$$\text{(ii)}\quad \text{ess. sup.}\,|u| \leqslant B\Big(\int_G |f|^p\,dx\Big)^{1/p},$$

$$\Big(\int_G |u|^{p'}\,dx\Big)^{1/p'} \leqslant B\int_G |f|\,dx\,.$$

Here $p > m/2$, $p' = p/(p-1)$ and B is the following constant

$$B = (\text{meas } G)^{(2/m)-(1/p)}\,\frac{\Gamma(1+m/2)^{2/m}}{m(m-2)\pi}\Big\{\frac{\Gamma(1+p')\Gamma\big((m/(m-2))-p'\big)}{\Gamma\big(m/(m-2)\big)}\Big\}^{1/p'}.$$

$$\text{(iii)}\qquad \Big(\int_G |u|^q\,dx\Big)^{1/q} \leqslant C\Big(\int_0^{+\infty} \bar{f}(s)^p\,ds\Big)^{1/p}.$$

Here $1 < p < m/2$, $q = mp/(m-2p)$, *C is the following constant*

$$C = (1/m^2\pi)\, q^{1-1/q}\left(1-\frac{1}{p}\right)^{1-1/p}\left\{\frac{\Gamma(m/2)^2}{\Gamma(m/2q)\,\Gamma\big((m/2)-(m/2q)\big)}\right\}^{2/m}$$

and $\bar{f}$ *is the maximal function of Hardy and Littlewood associated to the decreasing rearrangement of f into* $[0, +\infty]$. *More explicitely*

$$\bar{f}(s) = \frac{1}{s}\int_0^s f^*(s')\,ds'\,, \tag{18a}$$

$$\bar{f}(s) = \frac{1}{s}\sup\left(\int_A |f(x)|\,dx\colon A\subset G,\ \text{meas}\, A = s\right). \tag{18b}$$

Moreover

$$\|u\|_{q,k} \leqslant \frac{q^2\Gamma(1+m/2)^{2/m}}{(q-1)\,m^2\pi}\,\|f\|_{p,k}\,,$$

where p and q are as before, $1 \leqslant k \leqslant +\infty$ *and*

$$\|f\|_{p,k} = \left\{\int_0^{+\infty}\left(\bar{f}(s)\,s^{1/p}\right)^k\frac{ds}{s}\right\}^{1/k}$$

(with the usual modification in the case $k = +\infty$) *is a norm in the Lorentz space* $L(p,k)$.

REMARK 1. The first inequality in (ii) is due to Weinberger [21]. The second inequality in (ii) might be derived from the first via a duality argument. In particular, we get from (ii)

$$\text{ess. sup.}\,|u| \leqslant \frac{1}{2m\pi}\,[\Gamma(1+m/2)\ \text{meas}\ G]^{2/m}\,\text{ess. sup.}\,|f|\,,$$
$$\int_G |u|\,dx \leqslant (\text{same constant})\int_G |f|\,dx\,.$$

From the inequalities in (iii) we get in particular

$$\left(\int_G |u|^q\,dx\right)^{1/q} \leqslant \frac{p}{p-1}\,C\left(\int_G |f|^p\,dx\right)^{1/p}$$

$$\text{meas}\,\{x\in G\colon |u(x)| > t\} \leqslant \left\{\frac{q^2\Gamma(1+m/2)^{2/m}}{(q-1)\,m^2\pi}\right\}^q t^{-1+2p/m}\left(\int_G |f|^p\,dx\right)^{q/p}$$

where $1 < p < m/2$, $q = mp/(m-2p)$ (here it is not claimed that the constants are the best possible). In fact, the Lorentz norms previously introduced have the following properties (see [11], [12]):

$$\left(\int_G |f|^p\,dx\right)^{1/p} \leqslant \|f\|_{p,p} = \left(\int_0^{+\infty} \bar{f}(s)^p\,ds\right)^{1/p} \leqslant \frac{p}{p-1}\left(\int_G |f|^p\,dx\right)^{1/p}$$

if $p > 1$ and

$$\sup_{t\geqslant 0} t\,[\operatorname{meas}\{x\in G\colon |f(x)|>t\}]^{1/p} = \|f\|_{p,\infty} \leqslant \left(\int_G |f|^p\,dx\right)^{1/p}.$$

REMARK 2. The exponent q in (i) lies in the range: $m' = m/(m-1) < q \leqslant 2$. It must be pointed out that an inequality of the following form

$$\int_G |\operatorname{grad} u|^q\,dx \leqslant (\text{constant independent of } f)\left(\int_G |f|^p\,dx\right)^{q/p}$$

for functions u, endowed with first derivatives in $L^q(G)$, which verify (in the appropriate generalized sense) equations of the form (1) in a set G and vanish on the boundary, is impossible no matter what p and f are, if $q < m'$ and no smoothness assumptions are made on the coefficients, even if the ellipticity conditions (2) are retained and G is arbitrarily smooth. The following example is an elaboration of one of Serrin [16]. The homogeneous equation

$$\sum_{i,k=1}^{m} \frac{\partial}{\partial x_i}\left[\left(\delta_{ik} + (a-1)\frac{x_i x_k}{|x|^2}\right)\frac{\partial u}{\partial x_k}\right] = 0\,,$$

where δ_{ik} is the Kronecker delta and a is a constant greater than 1, is uniformly elliptic. It has the following non-zero solutions

$$u(x) = \sum_{k=0}^{n} \left(|x|^{-\lambda_k} - |x|^{\mu_k}\right) \sum_{j=1}^{N(k,m)} u_{kj}\, Y_{kj}(x/|x|)\,,$$

where $\{Y_{kj}\}_{j=1,\dots,N(k,m)}$ is an orthonormal complete sequence of spherical harmonics of degree k, u_{kj} are arbitrary constants and

$$\lambda_k = \frac{1}{2}(m-2) + \sqrt{\frac{1}{4}(m-2)^2 + \frac{1}{a}k(m-2+k)}$$

$$\mu_k = \lambda_k - (m-2)\,.$$

Let G be the unit ball in R^m. Then all these solutions vanish on ∂G. Moreover they are generalized solutions in G, with first derivatives in $L^q(G)$, provided n verifies

$$n < -\frac{1}{2}(m-2) + \sqrt{\frac{1}{4}(m-2)^2 + aq^{-2}(m-q)\left(\frac{m}{m-1} - q\right)},$$

a condition consistent with $n \geqslant 0$ if $q < m/(m-1)$. ■

Theorem 2 is a natural consequence of the previous one and its proof is straightforward. In fact, thanks to theorem 1 the estimates there listed come automatically from analogous estimates of the solutions to the much more simple problem (16).

Let us sketch a proof. We have to write down a formula for the solution v to the problem (16). This is a trivial task, for the right-hand side is a spherically symmetric function and consequently so is the solution, since the domain is a ball. Using the notations previously introduced and remembering that the radius of $G^\star$ equals $C_m^{-1/m}(\text{meas}\, G)^{1/m}$, we find

$$v(x) = \frac{1}{m^2 C_m^{2/m}} \int_{C_m|x|^m}^{\text{meas}\, G} r^{-2+2/m}\, dr \int_0^r f^*(s)\, ds\,. \tag{20}$$

In particular

$$\text{ess. sup.}\, v = v(0) = \int_0^{\text{meas}\, G} \frac{s^{-1+2/m} - (\text{meas}\, G)^{-1+2/m}}{m(m-2)\, C_m^{2/m}} f^*(s)\, ds\,, \tag{21}$$

$$|\text{grad}\, v(x)| = -\frac{\partial v}{\partial |x|}(x) = \frac{1}{m C_m |x|^{m-1}} \int_0^{C_m|x|^m} f^*(s)\, ds\,; \tag{22}$$

of course, the kernel in (20) must be replaced by $1/2\pi \ln(1/s\, \text{meas}\, G)$ if $m = 2$.

Estimates of the supremum of v in terms of L^p-norms of f, or estimates of L^q-norms of v in terms of the L^1-norm of f, are easily deduced from (21) via Hölder inequality, or from (20). Estimates of L^q-norms of the gradient of v in terms of L^p-norms of f can be obtained from (22) via Bliss inequality. For the convenience of the reader, we quote the following lemma.

LEMMA. *If $\varphi(r)$ is positive for $0 < r < +\infty$ and $q > p > 1$, then*

$$\int_0^\infty \left(\frac{1}{r}\int_0^r \varphi(s)\, ds\right)^q r^{-1+q/p}\, dr \leqslant A \left(\int_0^\infty \varphi(r)^p\, dr\right)^{q/p}, \tag{23a}$$

$$\int_0^\infty \left(\int_r^\infty \varphi(s)\, ds\right)^q dr \leqslant B \left(\int_0^\infty \varphi(r)^p r^{-1+p+p/q}\, dr\right)^{q/p}, \tag{24a}$$

where

$$A = \frac{p}{q(p-1)}\left\{\frac{\Gamma(pq/(q-p))}{\Gamma(q/(q-p))\Gamma(p(q-1)/(q-p))}\right\}^{(q/p)-1}, \tag{23b}$$

$$B = A\big(q(1-1/p)\big)^{q-(q/p)+1}. \tag{24b}$$

The equality holds in (23*a*) *if and only if* $\varphi(r)$ *is the following function*

$$\varphi(r) = a(1+br^{-1+q/p})^{-q/(q-p)} \qquad (a,\ b = \text{positive constants});$$

the equality holds in (24*a*) *if and only if*

$$\varphi(r) = ar^{-1-p'/q}(1+br^{p'-1-p'/q})^{-q/(q-p)}$$

$$(p'=p/(p-1);\ a,\ b = \text{positive constants}).$$

The inequality (23) is the Bliss inequality [1]. The inequality (24) derives from (23) via a change of variables.

The formula (20) and the inequality (24) give at once estimates of L^q-norms of v in terms of Lorentz norms of f. To see this, it is enough to rewrite (20) in the following form

$$v(x) = \frac{1}{m^2 C_m^{2/m}}\int_{C_m|x|^m}^{\text{meas}\,G} r^{-1+2/m}\bar{f}(r)\,dr, \tag{25}$$

where $\bar{f}$ is given by (18*a*).

Finally, estimates of Lorentz norms of v are easily derived from (25). In fact, it is apparent from (25) that the decreasing rearrangement of v into $[0, +\infty]$ is

$$v^*(s) = \frac{1}{m^2 C_m^{2/m}}\int_s^{\text{meas}\,G} r^{-1+2/m}\bar{f}(r)\,dr,$$

hence

$$m^2 C_m^{2/m}\bar{v}(r) = \int_0^{\text{meas}\,G} \frac{t^{2/m}\bar{f}(t)}{\max(r,t)}\,dt,$$

a formula which is easily handled with the theorem 319 of [6].

3. – An approach to theorem 1.

There is a transparent proof of (iv)-theorem 1 in the case where the coefficients $a_{ik}(x)$ and $c(x)$ of equation (1), the right-hand side $f(x)$ and the ground domain G are smooth—and consequently so is the solution u of (1) ... (4).

To describe such a proof, it is convenient (although not really necessary) to assume $f(x) \geqslant 0$. This is not a loss of generality, because if we replace $f(x)$ with $|f(x)|$, the new solution is greater than the absolute value of the old solution. Hence, by the maximum principle u is positive in G.

Let us integrate both sides of equation (1) over the level set

$$\{x \in G : u(x) > t\} . \tag{27}$$

Clearly, (27) is a relatively compact subdomain of G for all $t > 0$; furthermore the boundary of (27) is

$$\{x \in G : u(x) = t\} \tag{28}$$

for almost every $t > 0$ and the inner normal to this boundary at a point x is exactly $Du(x)/|Du(x)|$. Here we denote in short Du the gradient of u. In fact, thanks to the boundary condition (4), the distance of (27) from ∂G is greater than t/L, where L is a Lipschitz constant for u. On the other hand, the set of all levels t, for which (28) contains critical points of u, has (1-dimensional) measure zero, by Sard's theorem, see e.g. [19].

By Gauss theorem

$$-\int\limits_{u(x)>t} \sum_{i,k=1}^{m} \frac{\partial}{\partial x_i}\left(a_{ik}\frac{\partial u}{\partial x_k}\right) dx = \int\limits_{u(x)=t} \sum_{i,k=1}^{m} a_{ik} u_{x_k}(u_{x_i}/|Du|) H_{m-1}(dx) , \tag{29}$$

provided t is positive and the level surface (28) does not contain critical points of u. Hence from equation (1) and ellipticity conditions (2) we get

$$\int\limits_{u(x)=t} |Du| H_{m-1}(dx) \leqslant \int\limits_{u(x)>t} f(x)\, dx \tag{30}$$

for almost every $t > 0$. Here H_{m-1} stands for the $(m-1)$-dimensional measure.

Consider now the distribution function (9). It is an easy matter to find the following formula for the derivative of μ:

$$(31)\qquad -\mu'(t)=\int\limits_{u(x)=t}\frac{1}{|Du|}H_{m-1}(dx)\,.$$

The equality certainly holds in (31) if t is such that no critical point of u is in (28); thus (31) holds for almost every $t>0$.

Formula (31) and the Schwartz inequality give

$$H_{m-1}\{x\in G\colon u(x)=t\}\leqslant\Bigl[-\mu'(t)\cdot\int\limits_{u(x)=t}|Du|H_{m-1}(dx)\Bigr]^{\frac{1}{2}};$$

on the other hand, the isoperimetric inequality gives

$$H_{m-1}\{x\in G\colon u(x)=t\}\geqslant mC_m^{1/m}\mu(t)^{1-1/m}\,.$$

Thus we obtain the following inequality

$$(32)\qquad \int\limits_{u(x)=t}|Du|H_{m-1}(dx)\geqslant m^2C_m^{2/m}\,\mu(t)^{2-2/m}\,\frac{-1}{\mu'(t)}$$

for almost every $t>0$. Here C_m is the measure (12) of the m-dimensional unit ball. We emphasize the necessity of the boundary condition (4) for the inequality (32) to hold true.

The (32) gives an estimate from below of the left-hand side of (30): such an estimate involves u through the distribution function μ only. We have to bound the right-hand side of (30) with an estimate enjoying the same property. This point is crucial, of course. The ad hoc tool is the theorem (14), which gives

$$(33)\qquad \int\limits_{u(x)>t}f(x)\,dx\leqslant\int\limits_0^{\mu(t)}f^*(s)\,ds\,,$$

where f^* is the decreasing rearrangement of f into $[0,+\infty]$.

From (30), (32), (33) we get

$$(34)\qquad 1\leqslant\frac{1}{m^2C_m^{2/m}}\bigl(-\mu'(t)\bigr)\mu(t)^{-2+2/m}\int\limits_0^{\mu(t)}f^*(s)\,ds$$

for almost every $t>0$.

Note that the right-hand side of (34) is the derivative of an increasing function of t. Hence, integrating both sides of (34) between 0 and t gives

$$t \leqslant \frac{1}{m^2 C_m^{2/m}} \int\limits_{\mu(t)}^{\operatorname{meas} G} dr\, r^{-2+2/m} \int\limits_0^r f^*(s)\, ds\,. \tag{35}$$

By the definition of the decreasing rearrangement u^* of u, (35) implies

$$u^*(s) \leqslant \frac{1}{m^2 C_m^{2/m}} \int\limits_s^{\operatorname{meas} G} \mathrm{d}r\, r^{-2+2/m} \int\limits_0^r f^*(s')\, ds'\,. \tag{36}$$

By the definition (11) of spherically symmetric rearrangements, the latter inequality can be rewritten in this way

$$u^{\star}(x) \leqslant v(x) = \frac{1}{m^2 C_m^{2/m}} \int\limits_{C_m|x|^m}^{\operatorname{meas} G} dr\, r^{-2+2/m} \int\limits_0^r f^*(s)\, ds\,. \tag{37}$$

Thus we hit the mark, for v is just the solution of the Dirichlet problem (16).

4. – Proof of theorem 1.

A complete proof of theorem 1 might be as follows.

4.1.. As is well-known, the hypotheses (ii) and (iii) guarantee the existence in the Sobolev space $W_0^{1,2}(G)$ of a solution u to equation (1). Such a solution is just what we call the weak solution in the set G, subject to the boundary condition (4). Here $W_0^{1,2}(G)$ is the completion of $C_0^\infty(G)$ with respect to the norm: $u \to \left(\int\limits_G |Du|^2\, dx\right)^{\frac{1}{2}}$. In the case where the matrix of the leading coefficients $a_{ik}(x)$ is symmetric, u is the function which minimizes in $W_0^{1,2}(G)$ the convex functional

$$J(u) = \int\limits_G \left(\sum_{i,k=1}^m a_{ik} u_{x_i} u_{x_k} + cu^2 - 2uf \right) dx\,.$$

Note that, if the number m of dimensions is $\geqslant 3$ and $p = 2m/(m+2)$, the conjugate exponent p' is given by $p' = 2m/(m-2)$. Then the Sobolev inequality [20] gives

$$\|u\|_{L^{p'}(G)} \leqslant \gamma\Big(\int_G |Du|^2\,dx\Big)^{\frac{1}{2}}$$

for every u in $W_0^{1,2}(G)$, where

$$\gamma = 2^{1-1/m}\Gamma\Big(\frac{m+1}{2}\Big)^{2/m}[m(m-2)\pi^{1+1/m}]^{-\frac{1}{2}}.$$

Hence, using Hölder inequality and ellipticity conditions (2) and assuming f in $L^p(G)$, we see at once that the functional J is bounded from below and the appropriate level sets of J are bounded in $W_0^{1,2}(G)$, namely

$$J(u) \geqslant -\gamma^2\|f\|^2_{L^p(G)},$$
$$\Big(\int_G |Du|^2\,dx\Big)^{\frac{1}{2}} \leqslant \gamma\|f\|_{L^p(G)} + \sqrt{\gamma^2\|f\|^2_{L^p(G)} + J(u)},$$

for every u in $W_0^{1,2}(G)$. Since J is sequentially lower semicontinuous with respect to the weak convergence in $W_0^{1,2}(G)$ (as a refinement of the previous arguments shows), the existence of the minimum follows.

4.2. Let u be any function from $W_0^{1,2}(G)$, and let μ be its distribution function (9). Then the following inequality

$$m^2 C_m^{2/m} \leqslant \mu(t)^{-2+2/m}\big(-\mu'(t)\big)\Big(-\frac{d}{dt}\int_{|u(x)|>t}|Du|^2\,dx\Big) \tag{40}$$

holds for almost ewery $t > 0$. Moreover, if $0 < q \leqslant 2$ we have

$$\int_G |Du|^q\,dx \leqslant \int_0^{+\infty}\Big\{-\frac{\mu(t)^{-1+1/m}}{mC_m^{1/m}}\frac{d}{dt}\int_{|u(x)|>t}|Du|^2\,dx\Big\}^q\big(-d\mu(t)\big). \tag{41}$$

Here C_m is the measure (12) of the m-dimensional unit ball and Du is the gradient of u. In the inequalities (40) and (41) $|Du|$ can be replaced by

$$\Big(\sum_{i,k=1}^m a_{ik}u_{x_i}u_{x_k}\Big)^{\frac{1}{2}},$$

where (a_{ik}) is any $m\times m$ matrix verifying the condition (2a).

PROOF. We can suppose $G = R^m$ without loss of generality, for any function from $W_0^{1,2}(G)$ is a function belonging to $W_0^{1,2}(R^m)$ and vanishing outside G.

From the formula of Fleming and Rishel [3] we easily infer

$$\int_{|u(x)|>t} |Du|\, dx = \int_t^{+\infty} P\{x\colon |u(x)| > \xi\}\, d\xi \tag{42}$$

for every $t > 0$. Here P stands for the perimeter in the sense of De Giorgi [5]. Roughly speaking, the perimeter of a subset E of R^m is the $(m-1)$-dimensional measure of that part of the boundary of E where a normal vector can be defined. More precisely, $P(E)$ is the total variation of the characteristic function of E, namely

$$P(E) = \sup\left\{ \int_E \sum_{k=1}^m \frac{\partial \varphi_k}{\partial x_k}\, dx\colon \varphi_k \in C_0^\infty(R^m),\ \max \sum_{k=1}^m \varphi_k(x)^2 \leqslant 1 \right\}.$$

The formula (42) gives

$$-\frac{d}{dt} \int_{|u(x)|>t} |Du|\, dx = P\{x\colon |u(x| > t\}$$

for almost every $t > 0$. On the other hand, the isoperimetric theorem, in the form of De Giorgi [5], gives

$$P\{x\colon |u(x)| > t\} \geqslant m C_m^{1/m} \mu(t)^{1-1/m}$$

Hence

$$-\frac{d}{dt} \int_{|u(x)|>t} |Du|\, dx \geqslant m C_m^{1/m} \mu(t)^{1-1/m}\,. \tag{43}$$

Forming differential quotients and using the Schwartz inequality gives

$$-\frac{d}{dt} \int_{|u(x)|>t} |Du|\, dx \leqslant (-\mu'(t))^{\frac12} \left(-\frac{d}{dt} \int_{|u(x)|>t} |Du|^2\, dx \right)^{\frac12}. \tag{44}$$

From (43) and (44), follows (40).

The inequality (41) is a straightforward consequence of (40) and of the following inequality

$$\int\limits_{\operatorname{sprt} u}\varphi(x)\,dx \leqslant \int\limits_0^{+\infty}\left\{\frac{1}{\mu'(t)}\frac{d}{dt}\int\limits_{|u(x)|>t}\varphi(x)^p\,dx\right\}^{1/p}(-d\mu(t))\,. \tag{45}$$

In (45) $\operatorname{sprt} u$ denotes the support of u, p is any exponent $\geqslant 1$, φ is any nonnegative function from $L^p(R^m)$ such that $]0,+\infty[\ni t\to\Phi(t)=\int\limits_{|u(x)|>t}\varphi(x)\,dx$ is absolutely continuous.

A proof of (45) might be as follows.

$$\int\limits_{\operatorname{sprt} u}\varphi(x)\,dx=\Phi(0+)=-\int\limits_0^{+\infty}\Phi'(t)\,dt\,,\qquad -\Phi'(t)=\lim_{h\to 0}\frac{1}{h}\int\limits_{t<|u(x)|\leqslant t+h}\varphi(x)\,dx\leqslant$$

$$\leqslant(-\mu'(t))^{1-1/p}\left(-\frac{d}{dt}\int\limits_{|u(x)|>t}\varphi(x)^p\,dx\right)^{1/p},\qquad \int\limits_0^{+\infty}\ldots(-\mu'(t))\,dt\leqslant\int\limits_0^{+\infty}\ldots(-d\mu(t))\,.$$

To derive (41) from (45) and (40), put $\varphi=|Du|^q$, $p=2/q$. Note that $0<t\to\int\limits_{|u(x)|>t}|Du|^q\,dx$ actually is an absolutely continuous function of t. This depends on the fact that either $\{x\colon |u(x)|=t\}$ has m-dimensional measure zero or $|Du|$ vanishes almost everywhere on the same set. A formal proof of this absolute continuity comes at once from (42) in the case $0<q\leqslant 1$ and might be derived from the Fleming-Rishel result [3] if $1<q\leqslant 2$ as well. In the case where u is the solution from $W_0^{1,2}(G)$ to the equation (1), the absolute continuity in question can be checked also as in the subsection 4.3 below.

4.3. Let u be the solution from $W_0^{1,2}(G)$ to the equation (1). Then

$$\Phi(t)=\int\limits_{|u(x)|>t}\sum_{i,k=1}^m a_{ik}u_{x_i}u_{x_k}\,dx \tag{47}$$

is a decreasing Lipschitz continuous function of t in $[0,+\infty[$, whose derivative satisfies

$$0\leqslant-\Phi'(t)\leqslant\int\limits_0^{\mu(t)}f^*(s)\,ds\,. \tag{48}$$

Here μ is the distribution function (9) of u and f^* is the decreasing rearrangement into $[0, +\infty]$ of the right-hand side f.

PROOF. From the definition of weak solution we get

$$\int_G \Big(\sum_{i,k=1}^m a_{ik} u_{x_i} \varphi_{x_k} + cu\varphi \Big) dx = \int_G \varphi f \, dx \tag{49}$$

for every φ belonging to $W_0^{1,2}(G)$. In the case where the matrix of the a_{ik} is symmetric, (49) expresses the vanishing of the Fréchet derivative of the functional J at u. We choose test functions φ defined thus:

$$\varphi(x) = \begin{cases} \big(|u(x)| - t\big) \operatorname{sgn} u(x) & \text{if } x \text{ is such that } |u(x)| > t\,, \\ 0 & \text{otherwise}\,, \end{cases} \tag{50}$$

where t is any positive number. The lemmas 1.1 and 1.2 of [17] tell us that these φ are in $W_0^{1,2}(G)$. Hence (49) and (50) give the formula

$$\Phi(t) = \int_{|u(x)|>t} \big(|u| - t\big)\big(f \operatorname{sgn} u - c|u|\big) \, dx \,.$$

Consequently

$$0 \leqslant -\frac{\Phi(t+h) - \Phi(t)}{h} = \int_{|u(x)|>t} \big(f \operatorname{sgn} u - c|u|\big) \, dx + \text{a remainder}\,,$$

where the remainder is less than the integral of $|f|$ over the set $\{x \in G\colon t + \min(0, h) < |u(x)| \leqslant t + \max(0, h)\}$. Thus the remainder term is bounded from above and tends to zero as $h \to 0$ if t is a point of continuity for the distribution function μ.

Thus we have proved the Lipschitz continuity of Φ and the inequality

$$0 \leqslant -\Phi'(t) \leqslant \int_{|u(x)|>t} |f(x)| \, dx - \int_{|u(x)|>t} c|u| \, dx \,,$$

which obviously implies (48) because of the positiveness of $c(x)$ and the Hardy-Littlewood theorem (14).

4.4. From (40), (47) and (48) we get

$$1 \leqslant \frac{1}{m^2 C_m^{2/m}} \mu(t)^{-2+2/m} \big(-\mu'(t)\big) \int_0^{\mu(t)} f^*(s) \, ds \,,$$

namely the inequality (34) we have obtained in section 3 under different hypotheses. As in section 3, we conclude: $u^\star \leqslant v$, where u is the solution from $W_0^{1,2}(G)$ to the equation (1) v is the solution from $W_0^{1,2}(G^\star)$ to the equation $-\Delta v = f^\star$.

Recall that v and its gradient are given by formulas (20) and (22). Thus from (41), (47) and (48) we get

$$\int_G |Du|^q\,dx \leqslant \int_0^{\operatorname{meas} G} \left\{ \frac{r^{-1+1/m}}{mC_m^{1/m}} \int_0^r f^*(s)\,ds \right\}^q dr = \int_{G^\star} |Dv^\star|^q\,dx\,.$$

As already remarked, the length $|Du|$ on left-hand of the last formula can be replaced by

$$\Big(\sum_{i,k=1}^m a_{ik} u_{x_i} u_{x_k}\Big)^{\frac{1}{2}}\,.$$

Thus the proof is complete.

5. – A lower bound for eigenvalues.

In this section we consider the eigenvalue problem (7) under the following hypotheses: (i) the matrix $\big(a_{ik}(x)\big)$ of the leading coefficients is symmetric; (ii) the ellipticity conditions (2) hold; (iii) the domain G is bounded; (iv) the coefficient $c(x)$ is bounded.

We put

$$\alpha = \operatorname*{ess.\,sup.}_{x\in G} c(x)\,, \qquad \beta = \int_G c(x)\,dx\,. \tag{55}$$

THEOREM 3. *The smallest eigenvalue λ of the problem* (7) *is greater than* (or equal to) *the smallest eigenvalue ϱ of the following problem*

$$-\Delta v + h(x)v = \varrho v \quad in\ G^\star, \qquad v = 0 \quad on\ \partial G^\star. \tag{56}$$

Here $G^\star$ is the ball of the same measure as G, and $h(x)$ is the step function defined by the following rule: $h(x)$ vanishes in a ball (concentric and) *interior to $G^\star$, $h(x)$ takes a positive constant value outside that ball, the positive value of $h(x)$ being α and the radius of that ball being such that*

$$\int_{G^\star} h(x)\,dx = \beta\,.$$

PROOF. As is well-known

$$\lambda = \min \left\{ \int_G \left[\sum_{i,k=1}^{m} a_{ik} u_{x_i} u_{x_k} + cu^2 \right] dx : u \in W_0^{1,2}(G), \int_G u^2 \, dx = 1 \right\}. \tag{58}$$

Let u be the function which realizes the minimum in (58), i.e. the first (normalized) eigensolution of the problem (7).

From the ellipticity condition (2a) and the Polya-Szegö inequality (15) we infer

$$\int_G \sum_{i,k=1}^{m} a_{ik} u_{x_i} u_{x_k} dx \geqslant \int_{R^m} |\operatorname{grad} u^\star|^2 \, dx \,, \tag{59}$$

where $u^\star$ is the spherically symmetric rearrangement of u.

From a theorem of Hardy-Littlewood (which is a corollary of the theorem (13)) we infer

$$\int_G cu^2 \, dx \geqslant \int_0^{\text{meas}\, G} c^*(\text{meas}\, G - s) \, u^*(s)^2 \, dx \,, \tag{60}$$

where u^* is the decreasing rearrangement of u into $[0, +\infty]$. Note that $[0, \text{meas}\, G] \ni s \to c^*(\text{meas}\, G - s)$ is the *increasing* rearrangement of c into $[0, +\infty]$.

Let us recall the Steffensen inequality, see [10]. If $\varphi(s)$ is a positive bounded function defined in an interval $(0, L)$ and $\Phi(s)$ is positive decreasing in $(0, L)$, then

$$\int_0^L \varphi(s) \Phi(s) \, ds \geqslant (\sup. \varphi) \int_\xi^L \Phi(s) \, ds \,, \qquad \text{where } \xi = L - \frac{\int_0^L \varphi(s) \, ds}{\sup. \varphi} \,.$$

If the Steffensen inequality is applied to the right-hand side of (60), inequality (60) gives

$$\int_G cu^2 \, dx \geqslant \alpha \int_{\text{meas}G - (\beta/\alpha)}^{\text{meas}\, G} u^*(s)^2 \, ds \,,$$

where α, β are defined in (55). Hence we obtain

$$\int_G cu^2 \, dx \geqslant \int_{R^m} h(x) (u^\star)^2 \, dx \,, \tag{61}$$

where $h(x)$ is defined as in the statement of the theorem.

From (59) and (61) we get

$$\lambda \geqslant \int_{R^m} \left[|\operatorname{grad} u^\star|^2 + h(x)(u^\star)^2 \right] dx \,. \tag{62}$$

As the support of $u^\star$ is contained in $G^\star$ and $\int_{R^m} (u^\star)^2 \, dx = 1$, right-hand of (62) is $\geqslant \varrho$, therefore $\lambda \geqslant \varrho$.

REMARK. The Payne-Rayner inequality [14] can be easily extended to the solutions of the eigenvalue problem (7), no matter what the coefficients are, as long as the ellipticity conditions (2) hold.

Let us consider for example the bidimensional case: $m = 2$. Then any weak solution u to the problem (7) verifies

$$\lambda \Big(\int_G |u| \, dx \Big)^2 \geqslant 4\pi \int_G u^2 \, dx \,.$$

Indeed, the arguments of section 4 (the inequality (51) in particular) show

$$4\pi \mu(t) \leqslant \lambda\big(-\mu'(t)\big) \int_0^{\mu(t)} u^*(s) \, ds \,,$$

where μ is the distribution function of u and u^* has the customary meaning. Multiplying both sides of (63) by $2t$, then integrating over $(0, +\infty)$, we get

$$\begin{aligned} 4\pi \int_G u^2 \, dx &= 8\pi \int_0^{+\infty} t\mu(t) \, dt \leqslant 2\lambda \int_0^{+\infty} t \Big(\int_0^{\mu(t)} u^*(s) \, ds \Big) \big(-\mu'(t)\big) \, dt \leqslant \\ &\leqslant 2\lambda \int_0^{+\infty} u^*(\mu(t)) \Big(\int_0^{\mu(t)} u^*(s) \, ds \Big) \big(-d\mu(t)\big) \leqslant \lambda \int^{\operatorname{meas} G} \frac{d}{ds} \Big(\int_0^{s} u^*(t) \, dt \Big)^2 ds = \\ &= \lambda \Big(\int_0^{\operatorname{meas} G} u^*(t) \, dt \Big)^2 = \lambda \Big(\int_G |u| \, dx \Big)^2 . \end{aligned}$$

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