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ELLIPTIC FUNCTIONS TO THE QUINTIC BASE

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We study two elliptic functions to the quintic base and find two nonlinear second order differential equations satisfied by them. We then derive two recurrence relations involving certain Eisenstein series associated with the group $\Gamma_0(5)$. These recurrence relations allow us to derive infinite families of identities involving the Eisenstein series and Dedekind η-products. An imaginary transformation for one of the elliptic functions is also derived.

1. Introduction

Let $q = e^{2\pi i\tau}$, where Im $\tau > 0$. The Dedekind η-function is defined by

$$\eta(\tau) = q^{1/24}(q;q)_\infty,$$

where

$$(a;q)_\infty = \prod_{n=1}^{\infty}(1-aq^{n-1}).$$

Srinivasa Ramanujan, on a page of his $\tau(n)$-$p(n)$ manuscript [Ramanujan 1988, p. 139; Berndt and Ono 1999], stated without proof that

$$\sum_{n=1}^{\infty}\left(\frac{n}{5}\right)\frac{q^n}{(1-q^n)^2} = \frac{\eta^5(5\tau)}{\eta(\tau)}, \tag{1–1}$$

where $\left(\frac{\cdot}{5}\right)$ is the Legendre symbol.

This identity is of great interest historically because Ramanujan deduced from it his famous partition identity

$$\sum_{n=0}^{\infty}p(5n+4)q^n = 5\frac{(q^5;q^5)_\infty^5}{(q;q)_\infty^6},$$

where $p(n)$ is the number of partitions of n. Proofs of (1–1) can be found in [Bailey 1952a; 1952b], and more recently in [Shen 1994, p. 329; Chan 1995].

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In an unpublished work, Z.-G. Liu and R. P. Lewis discovered that

$$\sum_{n=1}^{\infty}\frac{n^3(q^n-q^{2n}-q^{3n}+q^{4n})}{(1-q^{5n})}=\frac{\eta^5(5\tau)}{\eta(\tau)}A(\tau), \tag{1–2}$$

where

$$\begin{aligned} A(\tau)&=1+6\sum_{n\geq 1}\frac{nq^n}{1-q^n}-30\sum_{n\geq 1}\frac{nq^{5n}}{1-q^{5n}} \\ &=\frac{\eta^5(\tau)}{\eta(5\tau)}\sqrt{1+22\left(\frac{\eta(5\tau)}{\eta(\tau)}\right)^6+125\left(\frac{\eta(5\tau)}{\eta(\tau)}\right)^{12}}. \end{aligned} \tag{1–3}$$

The last identity relating $A(\tau)$ to η-products can be found in [Berndt et al. 2000, Lemma 3.6].

If we rewrite the left-hand side of (1–1) as

$$\sum_{n=1}^{\infty}\left(\frac{n}{5}\right)\frac{q^n}{(1-q^n)^2}=\sum_{n=1}^{\infty}\frac{n(q^n-q^{2n}-q^{3n}+q^{4n})}{(1-q^{5n})}$$

and set

$$L_k(q)=\sum_{n=1}^{\infty}\frac{n^k(q^n-q^{2n}-q^{3n}+q^{4n})}{(1-q^{5n})},$$

then we observe that (1–1) and (1–2) correspond to expressions in terms of the Dedekind η-products for $L_k(q)$ when $k=1$ and 3 respectively.

Motivated by our recent paper [Chan and Liu 2003], we propose to construct a recurrence satisfied by $L_{2k+1}(q)$, which would then give us expressions of $L_{2k+1}(q)$ in terms of η products for every integer $k\geq 0$. In order to achieve this, we first define

$$V(u|\tau)=4\sum_{k=0}^{\infty}(-1)^k\frac{(2u)^{2k+1}}{(2k+1)!}L_{2k+1}(q). \tag{1–4}$$

In Section 2, we will see that $V(u|\tau)$ is a elliptic function with periods π and $5\pi\tau$. In Sections 3 and 4, we construct two differential equations associated with $V(u|\tau)$ and

$$M(u|\tau):=\frac{4}{3}\left(A(\tau)+6\sum_{k=1}^{\infty}\frac{(-1)^k(2u)^{2k}}{(2k)!}S_{2k+1}(q)\right), \tag{1–5}$$

where $A(\tau)$ is given by (1–3) and

$$S_k(q)=\sum_{n\geq 1}\frac{n^k(q^n+q^{2n}+q^{3n}+q^{4n})}{1-q^{5n}}.$$

Using these differential equations, we obtain two difference equations satisfied by $S_{2k+1}(q)$ and $L_{2k+1}(q)$. These equations allow us to tabulate $S_{2k+1}(q)$ and $L_{2k+1}(q)$ in terms of the modular functions

$$x=\left(\frac{\eta(5\tau)}{\eta(\tau)}\right)^6 \quad \text{and} \quad z=\frac{\eta^5(\tau)}{\eta(5\tau)} \tag{1–6}$$

for any $k \geq 1$. In Section 5, we study the behavior of $V(u|\tau)$ under the imaginary transformation $\tau \mapsto -1/(5\tau)$. As a corollary, we indicate some computations of special values of Dirichlet L-series.

2. The Jacobi theta function $\vartheta_1(u|\tau)$ and $V(u|\tau)$

Recall that the Jacobi theta functions $\vartheta_1(u|\tau)$ and $\vartheta_4(u|\tau)$ are given by

$$\vartheta_1(u|\tau)=2q^{1/8}\sum_{n\geq 0}(-1)^n q^{n(n+1)/2}\sin(2n+1)u,$$

$$\vartheta_4(u|\tau)=1+2\sum_{n\geq 1}^{\infty}(-1)^n q^{n^2/2}\cos 2nu.$$

From [Whittaker and Watson 1927, p. 489] we know that

$$\frac{\vartheta_4'}{\vartheta_4}(u|\tau)=4\sum_{n\geq 1}\frac{q^{n/2}}{1-q^n}\sin 2nu.$$

Hence,

$$\begin{aligned}
\frac{\vartheta_4'}{\vartheta_4}&(u+\tfrac{3}{2}\pi\tau\,|5\tau)+\frac{\vartheta_4'}{\vartheta_4}(u-\tfrac{3}{2}\pi\tau\,|5\tau)\\
&=4\sum_{n\geq 1}\frac{q^{5n/2}}{1-q^{5n}}\big(\sin 2n(u+\tfrac{3}{2}\pi\tau)+\sin 2n(u-\tfrac{3}{2}\pi\tau)\big)\\
&=8\sum_{n\geq 1}\frac{q^{5n/2}}{1-q^{5n}}\cos 3n\pi\tau\,\sin 2nu\\
&=4\sum_{n\geq 1}\frac{q^{5n/2}(q^{3n/2}+q^{-3n/2})}{1-q^{5n}}\sin 2nu=4\sum_{n\geq 1}\frac{q^n+q^{4n}}{1-q^{5n}}\sin 2nu.
\end{aligned}$$

Similarly,

$$\frac{\vartheta_4'}{\vartheta_4}(u+\tfrac{1}{2}\pi\tau\,|5\tau)+\frac{\vartheta_4'}{\vartheta_4}(u-\tfrac{1}{2}\pi\tau\,|5\tau)=4\sum_{n\geq 1}\frac{q^{2n}+q^{3n}}{1-q^{5n}}\sin 2nu.$$

This implies that

$$V(u|\tau) = \frac{\vartheta_4'}{\vartheta_4}(u + \tfrac{3}{2}\pi\tau\,|\,5\tau) + \frac{\vartheta_4'}{\vartheta_4}(u - \tfrac{3}{2}\pi\tau\,|\,5\tau) - \frac{\vartheta_4'}{\vartheta_4}(u + \tfrac{1}{2}\pi\tau\,|\,5\tau) - \frac{\vartheta_4'}{\vartheta_4}(u - \tfrac{1}{2}\pi\tau\,|\,5\tau), \tag{2–1}$$

where $V(u|\tau)$ is given by (1–4). By using the transformation formulas

$$\frac{\vartheta_1'}{\vartheta_1}(u|\tau) = -i + \frac{\vartheta_4'}{\vartheta_4}(u - \tfrac{1}{2}\pi\tau|\tau), \qquad \frac{\vartheta_1'}{\vartheta_1}(u - \pi\tau|\tau) = \frac{\vartheta_1'}{\vartheta_1}(u|\tau) - 2i,$$

we find that

$$\begin{aligned} \frac{\vartheta_1'}{\vartheta_1}(u + \pi\tau\,|\,5\tau) + \frac{\vartheta_1'}{\vartheta_1}(u - \pi\tau\,|\,5\tau) &= \frac{\vartheta_4'}{\vartheta_4}(u + \tfrac{3}{2}\pi\tau\,|\,5\tau) + \frac{\vartheta_4'}{\vartheta_4}(u - \tfrac{3}{2}\pi\tau\,|\,5\tau) \\ &= 4\sum_{n\ge 1} \frac{q^n + q^{4n}}{1 - q^{5n}} \sin 2nu, \end{aligned} \tag{2–2}$$

$$\begin{aligned} \frac{\vartheta_1'}{\vartheta_1}(u + 2\pi\tau\,|\,5\tau) + \frac{\vartheta_1'}{\vartheta_1}(u - 2\pi\tau\,|\,5\tau) &= \frac{\vartheta_4'}{\vartheta_4}(u + \tfrac{1}{2}\pi\tau\,|\,5\tau) + \frac{\vartheta_4'}{\vartheta_4}(u - \tfrac{1}{2}\pi\tau\,|\,5\tau) \\ &= 4\sum_{n\ge 1} \frac{q^{2n} + q^{3n}}{1 - q^{5n}} \sin 2nu. \end{aligned} \tag{2–3}$$

Combining formulas (2–1) through (2–3), we conclude that

$$V(u|\tau) = \frac{\vartheta_1'}{\vartheta_1}(u{+}\pi\tau\,|\,5\tau) + \frac{\vartheta_1'}{\vartheta_1}(u{-}\pi\tau\,|\,5\tau) - \frac{\vartheta_1'}{\vartheta_1}(u{+}2\pi\tau\,|\,5\tau) - \frac{\vartheta_1'}{\vartheta_1}(u{-}2\pi\tau\,|\,5\tau). \tag{2–4}$$

3. Differential equations satisfied by $V(u|\tau)$ and $M(u|\tau)$

Before we proceed with the derivation of a differential equation satisfied by $V(u|\tau)$, we first define the classical Eisenstein series to be used in the sequel. For $\operatorname{Im}\tau > 0$, the Eisenstein series $E_2(\tau)$ is given by

$$E_2(\tau) = 1 - 24\sum_{n\ge 1} \frac{nq^n}{1 - q^n},\ q = e^{2\pi i\tau}. \tag{3–1}$$

For integer $k \ge 2$, we define

$$E_{2k}(\tau) = 1 + \frac{(2\pi i)^{2k}}{(2k-1)!\,\zeta(2k)} \sum_{n\ge 1} \frac{n^{2k-1}q^n}{1 - q^n}, \tag{3–2}$$

where

$$\zeta(s) = \sum_{n\ge 1} \frac{1}{n^s}, \qquad \operatorname{Re} s > 1.$$

We now construct a differential equation satisfied by $V(u|\tau)$. Define

$$f(v):=f(v|\tau)=\frac{\vartheta_1(v+u+\pi\tau\,|5\tau)\,\vartheta_1(v+u-\pi\tau\,|5\tau)\times\vartheta_1(v-u+2\pi\tau\,|5\tau)\vartheta_1(v-u-2\pi\tau\,|5\tau)}{\vartheta_1^4(v\,|5\tau)},$$

where $|u|<\pi|\tau|$. Our choice of $f(v|\tau)$ is clearly motivated by (2–4). Using the transformation formulas

$$\vartheta_1(t+\pi\,|\tau)=-\vartheta_1(t|\tau)\quad\text{and}\quad\vartheta_1(t+\pi\tau\,|\tau)=-q^{-1/2}e^{-2it}\vartheta_1(t|\tau)$$

from [Whittaker and Watson 1927, p. 463], we verify that $f(v)$ is an elliptic function with periods π and $5\pi\tau$. It has a pole of order 4 at $v=0$. Now, set $F(v)=v^4f(v)$ and

$$\phi(v)=\frac{F'(v)}{F(v)}.$$

By elementary calculations, we find that

$$\operatorname{res}(f(v);\,v=0)=\tfrac{1}{6}F(0)\left(\phi^3(0)+3\phi(0)\phi'(0)+\phi''(0)\right),$$

where $\operatorname{res}(f(v);\,v=p)$ denotes the residue of f at the pole p. Since the sum of residues of an elliptic function over a period parallelogram is zero and $F(0)\neq 0$, we conclude that

$$\phi^3(0)+3\phi(0)\phi'(0)+\phi''(0)=0. \tag{3–3}$$

Next, note that

$$\begin{aligned}\phi(v)&=\frac{4}{v}-4\frac{\vartheta_1'}{\vartheta_1}(u|5\tau)+\frac{\vartheta_1'}{\vartheta_1}(v+u+\pi\tau\,|5\tau)+\frac{\vartheta_1'}{\vartheta_1}(v+u-\pi\tau\,|5\tau)\\&\qquad+\frac{\vartheta_1'}{\vartheta_1}(v-u+2\pi\tau\,|5\tau)+\frac{\vartheta_1'}{\vartheta_1}(v-u-2\pi\tau\,|5\tau)\\&=\frac{4}{3}E_2(5\tau)v+\frac{\vartheta_1'}{\vartheta_1}(v+u+\pi\tau\,|5\tau)+\frac{\vartheta_1'}{\vartheta_1}(v+u-\pi\tau\,|5\tau)\\&\qquad+\frac{\vartheta_1'}{\vartheta_1}(v-u+2\pi\tau\,|5\tau)+\frac{\vartheta_1'}{\vartheta_1}(v-u-2\pi\tau\,|5\tau)+O(v^3),\end{aligned}\tag{3–4}$$

where we have used the expansion

$$\frac{\vartheta_1'}{\vartheta_1}(t|\tau)=\cot t+4\sum_{n\geq 1}\frac{q^n}{1-q^n}\sin 2nt \tag{3–5}$$

from [Whittaker and Watson 1927, p. 489] and the representation given in (3–1). From (3–4), we find that

$$\phi(0)=V(u|\tau).$$

Differentiating (3–4) with respect to v and setting $v=0$, we find that

$$\phi'(0)=\tfrac{4}{3}E_2(5\tau)+\left(\frac{\vartheta_1'}{\vartheta_1}\right)'(u+\pi\tau\,|\,5\tau)+\left(\frac{\vartheta_1'}{\vartheta_1}\right)'(u-\pi\tau\,|\,5\tau)+\left(\frac{\vartheta_1'}{\vartheta_1}\right)'(u+2\pi\tau\,|\,5\tau)+\left(\frac{\vartheta_1'}{\vartheta_1}\right)'(u-2\pi\tau\,|\,5\tau). \tag{3–6}$$

Differentiating both sides of (2–2) and (2–3), we conclude from (3–6) that

$$M(u|\tau)=\phi'(0),$$

where $M(u|\tau)$ is given by (1–5). Finally differentiating (3–4) twice and setting $v=0$, we conclude that

$$\phi''(0)=V''(u|\tau),$$

by (2–4). Substituting these expressions for $\phi(0)$, $\phi'(0)$ and $\phi''(0)$ into (3–3), we conclude:

Theorem 3.1. $$V''(u|\tau)+3M(u|\tau)V(u|\tau)+V^3(u|\tau)=0.$$

Corollary 3.2. *For integer $k\ge 1$,*

$$L_{2k+1}=AL_{2k-1}+6\sum_{l=1}^{k-1}\frac{(2k-1)!L_{2(k-l-1)+1}S_{2l+1}}{(2l)!\,(2(k-l)-1)!}-4\sum_{l=0}^{k-1}\sum_{m=0}^{k-l-2}\frac{(2k-1)!L_{2l+1}L_{2m+1}L_{2(k-l-m-2)+1}}{(2l+1)!\,(2m+1)!\,(2(k-l-m-2)+1)!},$$

where $L_k:=L_k(q)$ and $S_k:=S_k(q)$.

The recurrence in the corollary involves both L_k and S_k. In order to tabulate L_k, we need another difference equation expressing S_k in terms of L_s and S_t for $3\le s,t<k$. To achieve this, we need to obtain a second differential equation satisfied by $M:=M(u|\tau)$ and $V:=V(u|\tau)$. Using the method in the proof of Theorem 3.1 with f replaced by

$$g(v)=\frac{\vartheta_1(2v\,|\,5\tau)}{\vartheta_1^8(v\,|\,5\tau)}\vartheta_1(v+u+\pi\tau\,|\,5\tau)\,\vartheta_1(v+u-\pi\tau\,|\,5\tau)\times\vartheta_1(v-u+2\pi\tau\,|\,5\tau)\,\vartheta_1(v-u-2\pi\tau\,|\,5\tau),$$

we find that

$$V^6+14V^4M+60MV^{(2)}V+15M^3-\tfrac{128}{9}E_6(5\tau)+6V^{(4)}V+20V^{(2)}V^3+10(V'')^2+45M^2V^2+15\big(-\tfrac{16}{15}E_4(5\tau)+M^{(2)}\big)(M+V^2)=0,$$

where $E_4(\tau)$ and $E_6(\tau)$ are given by (3–2) and ${}^{(i)}$ denotes i-fold partial differentiation with respect to u.

It turns out that there is a differential equation simpler than the one just given:

$$M^{(2)}-\tfrac{8}{3}E_6(5\tau)+\tfrac{3}{2}M^2+3\big(M-\tfrac{4}{3}A\big)V^2+3V^{(2)}V+\tfrac{3}{2}(V')^2=0. \tag{3–7}$$

We will devote the next section to the proof this equation. In the meantime, here is a result that follows immediately from (3–7):

Corollary 3.3. *For $k\geq 2$,*

$$\begin{aligned}S_{2k+1}=AS_{2k-1}+6L_1L_{2k-1}+6\sum_{l=1}^{k-1}\frac{(2k-2)!L_{2l+1}L_{2(k-l-1)+1}}{(2l-1)!\,(2(k-l-1)+1)!}\\+3\sum_{l=1}^{k-2}\frac{(2k-2)!\,(S_{2(k-l-1)+1}S_{2l+1}+L_{2l+1}L_{2(k-l-1)+1})}{(2l)!\,(2(k-l-1))!}\\-12\sum_{l=0}^{k-3}\sum_{m=0}^{k-l-3}\frac{(2k-2)!L_{2l+1}L_{2m+1}S_{2(k-l-m-2)+1}}{(2l+1)!\,(2m+1)!\,(2(k-l-m-2))!}.\end{aligned}$$

Alternate applications of Corollaries 3.2 and 3.3 yield tables for L_{2k+1} and S_{2k+1} in terms of z and x given by (1–6). Here are first few terms derived from these recurrences:

$$\begin{aligned}L_1(q)&=zx, & S_3(q)&=z^2(x+13x^2),\\ L_3(q)&=zxA(\tau), & S_5(q)&=z^2A(\tau)(x+31x^2),\\ L_5(q)&=z^3x(1+40x+335x^2), & S_7(q)&=z^4x(1+143x+3255x^2+20345x^3).\end{aligned}$$

4. Sketch of the proof of (3–7)

We first note that for $k\geq 1$,

$$\begin{aligned}S_{2k+1}(q)&=\sum_{n\geq 1}\frac{n^{2k+1}q^{5n}}{1-q^{5n}}-\sum_{n\geq 1}\frac{n^{2k+1}q^n}{1-q^n}\\&=\frac{(-1)^{k+1}(2k+1)!\,\zeta(2k+2)(E_{2k+2}(5\tau)-E_{2k+2}(\tau))}{(2\pi)^{2k+2}},\end{aligned}\tag{4–1}$$

where $E_{2k}(\tau)$ is given by (3–2). Ramanujan [Berndt et al. 2000, Theorems 3.1, 3.2] offered the identities expressing $E_4(\tau)$, $E_4(5\tau)$, $E_6(\tau)$ and $E_6(5\tau)$ in terms of x, z and A, and since $E_{2k}(\tau)$ can be expressed in terms of $E_4(\tau)$ and $E_6(\tau)$ by [Serre 1973, Corollary 2, p. 89], we can compute $S_{2k+1}(q)$ in terms of x, z, and $A(\tau)$ for any positive integer k, and in particular for $1\leq k\leq 9$. Now, using Corollary 3.2, we compute $L_{2k+1}(q)$ in terms of x, z and A for $0\leq k\leq 8$ using the identity

$$A^2(\tau)=z^2(1+22x+125x^2),$$

which follows from (1–3) and the definitions of x and z in (1–6). These yield a total of 18 identities, which gives L_{2s+1} and S_{2t+1} in terms of x, z and $A(\tau)$ for $0 \le s \le 8$ and $1 \le t \le 9$.

The coefficients of u^{2k+1} in $V(u|\tau)$ and u^{2k} in $M(u|\tau)$ are constant multiples of L_{2k+1} and S_{2k+1} respectively, by (1–4) and (1–5). Hence, we may write

$$
\begin{aligned}
V(u|\tau) &= \sum_{k=0}^{8} F_k(x, z, A(\tau))u^{2k+1} + \mathrm{O}(u^{19}), \\
M(u|\tau) &= \frac{4}{3}A(\tau) + \sum_{k=1}^{9} G_k(x, z, A(\tau))u^{2k} + \mathrm{O}(u^{20}),
\end{aligned} \tag{4–2}
$$

where F_k and G_k are certain polynomials in x, z and $A(\tau)$.

Identity (3–7) is discovered by first assuming that there is a relation between $M^{(2)} + \alpha_1 E_6(5\tau)$,$M^2$,$(M + \alpha_2 A)\, V^2$, $V^{(2)}V$, and $(V')^2$, namely,

$$M^{(2)} + \alpha_1 E_6(5\tau) + \beta_1 M^2 + \beta_2\,(M + \alpha_2 A)\, V^2 + \beta_3 V^{(2)} V + \beta_4 (V')^2 = 0.$$

We then use the representations (4–2) to establish relations among β_i's by equating the coefficients of u^k. The determination of β_i's gives (3–7) immediately.

In order to show that (3–7) is valid, we note that M and V are both elliptic functions with the total number of poles being 8 and 4 respectively; see (2–4) and (3–6). Therefore the left-hand side $h(u)$ of (3–7) is an elliptic function with at most 16 poles. We know that an elliptic function has the same number of poles and zeros in a period parallelogram. Hence, $h(u)$ must have at most 16 zeros in a period parallelogram. This last step can be established by using the representations (4–2) and showing that $h(u) = \mathrm{O}(u^{17})$.

5. Behavior of $V(u|\tau)$ under the imaginary transformation $\tau \mapsto -\dfrac{1}{5\tau}$

Let

$$U(u|\tau) = \frac{\vartheta_1'}{\vartheta_1}\left(u + \frac{\pi}{5}\middle|\tau\right) + \frac{\vartheta_1'}{\vartheta_1}\left(u - \frac{\pi}{5}\middle|\tau\right) - \frac{\vartheta_1'}{\vartheta_1}\left(u + \frac{2\pi}{5}\middle|\tau\right) - \frac{\vartheta_1'}{\vartheta_1}\left(u - \frac{2\pi}{5}\middle|\tau\right).$$

Theorem 5.1. *The function $U(u|\tau)$ and $V(u|\tau)$ satisfy the transformation formula*

$$V\left(\frac{u}{\tau}\middle| -\frac{1}{5\tau}\right) = \tau U(u|\tau).$$

Proof. The function $\dfrac{\vartheta_1'}{\vartheta_1}(u|\tau)$ satisfies the relation [Chan and Liu 2003, (4.2)]

$$\frac{\vartheta_1'}{\vartheta_1}\left(\frac{u}{\tau}\middle| -\frac{1}{\tau}\right) = \frac{2iu}{\pi} + \tau \frac{\vartheta_1'}{\vartheta_1}(u|\tau). \tag{5–1}$$

Replacing u by $u+\frac{\pi}{5}$ and $u+\frac{2\pi}{5}$ in (5–1), we find that

$$\frac{\vartheta_1'}{\vartheta_1}\left(\frac{u}{\tau}+\frac{\pi}{5\tau}\middle|-\frac{1}{\tau}\right)=\frac{2iu}{\pi}+\frac{2i}{5}+\tau\frac{\vartheta_1'}{\vartheta_1}\left(u+\frac{\pi}{5}\middle|\tau\right) \quad \text{and}$$

$$\frac{\vartheta_1'}{\vartheta_1}\left(\frac{u}{\tau}+\frac{2\pi}{5\tau}\middle|-\frac{1}{\tau}\right)=\frac{2iu}{\pi}+\frac{4i}{5}+\tau\frac{\vartheta_1'}{\vartheta_1}\left(u+\frac{2\pi}{5}\middle|\tau\right).$$

Replacing u by $-u$ in these two equations and using the fact that $\frac{\vartheta_1'}{\vartheta_1}(u|\tau)$ is an odd function of u (see (3–5)), we find that

$$\frac{\vartheta_1'}{\vartheta_1}\left(\frac{u}{\tau}-\frac{\pi}{5\tau}\middle|-\frac{1}{\tau}\right)=\frac{2iu}{\pi}-\frac{2i}{5}+\tau\frac{\vartheta_1'}{\vartheta_1}\left(u-\frac{\pi}{5}\middle|\tau\right) \quad \text{and}$$

$$\frac{\vartheta_1'}{\vartheta_1}\left(\frac{u}{\tau}-\frac{2\pi}{5\tau}\middle|-\frac{1}{\tau}\right)=\frac{2iu}{\pi}-\frac{4i}{5}+\tau\frac{\vartheta_1'}{\vartheta_1}\left(u-\frac{2\pi}{5}\middle|\tau\right).$$

From these four equations we deduce Theorem 5.1 immediately. □

Let $\chi_5(n)=\left(\frac{n}{5}\right)$ and define

(5–2) $$E_{2r,\chi_5}(q)=1+(-1)^r\frac{\sqrt{5}}{(2r-1)!\,L(2r,\chi_5)}\left(\frac{2\pi}{5}\right)^{2r}\sum_{n\geq 1}\chi_5(n)\frac{n^{2r-1}q^n}{1-q^n},$$

where

$$L(r,\chi_5)=\sum_{n\geq 1}\frac{\chi_5(n)}{n^r}.$$

Our function $U(u|\tau)$ can be expressed in terms of $E_{2r,\chi_5}(q)$:

Theorem 5.2. $$U(u|\tau)=-2\sum_{r\geq 1}u^{2r-1}\left(\frac{5}{\pi}\right)^{2r}L(2r,\chi_5)E_{2r,\chi_5}(q).$$

Proof. Using (3–5), we find that

$$U(u|\tau)=S(u)+4\sum_{n\geq 1}S_n(u)\frac{q^n}{1-q^n},$$

where

$$S(u)=\cot\left(u+\frac{\pi}{5}\right)+\cot\left(u-\frac{\pi}{5}\right)-\cot\left(u+\frac{2\pi}{5}\right)-\cot\left(u-\frac{2\pi}{5}\right),$$

$$\begin{aligned}S_n(u)&=\sin\left(2n\left(u+\frac{\pi}{5}\right)\right)+\sin\left(2n\left(u-\frac{\pi}{5}\right)\right)\\&\qquad\qquad-\sin\left(2n\left(u+\frac{2\pi}{5}\right)\right)-\sin\left(2n\left(u-\frac{2\pi}{5}\right)\right)\\&=\sqrt{5}\chi_5(n)\sin 2nu.\end{aligned}$$

Here the last equality follows by elementary calculations using trigonometric identities.

From the expansion $\cot u = \sum_{m\geq 1}\left(\frac{1}{u-m\pi}+\frac{1}{u+(m-1)\pi}\right)$, we then find that

$$
\begin{aligned}
S(u) &= \sum_{k=1}^{4}\chi_5(k)\cot\left(u+\frac{k\pi}{5}\right) \\
&= \sum_{m\geq 1}\sum_{k=1}^{4}\chi_5(k)\left(\frac{1}{u+k\pi/5-m\pi}+\frac{1}{u+k\pi/5-(m-1)\pi}\right) \\
&= \sum_{r\geq 0}u^r\left(\frac{5}{\pi}\right)^{r+1}\sum_{m\geq 1}\sum_{k=1}^{4}\chi_5(k)\left(-\frac{1}{(5m-k)^{r+1}}+\frac{(-1)^r}{(5(m-1)+k)^{r+1}}\right).
\end{aligned}
\tag{5–3}
$$

Next, we find that

$$\sum_{m\geq 1}\sum_{k=1}^{4}\frac{\chi_5(k)}{(5m-k)^{r+1}} = \sum_{m\geq 1}\sum_{k=1}^{4}\frac{\chi_5(5m-k)}{(5m-k)^{r+1}} = \sum_{m\geq 1}\frac{\chi_5(n)}{n^{r+1}} = L(r+1,\chi_5)$$

and

$$\sum_{m\geq 1}\sum_{k=1}^{4}\frac{\chi_5(k)}{(5(m-1)+k)^{r+1}} = L(r+1,\chi_5).$$

Hence, we conclude from (5–3) that

$$S(u) = -2\sum_{r\geq 0}u^{2r+1}\left(\frac{5}{\pi}\right)^{2r+2}L(2r+2,\chi_5)$$

and this completes the proof of Theorem 5.2. □

Now, using Theorem 5.2 and equating the coefficients of powers of u in Theorem 5.1, we derive a transformation formula for L_{2k+1} and E_{2k+1,χ_5}:

Theorem 5.3.

$$L(2k+2,\chi_5)E_{2k+2,\chi_5}(e^{2\pi i\tau}) = \frac{(-1)^{k+1}}{(2k+1)!}\left(\frac{2\pi}{5}\right)^{2k+2}\tau^{-2k-2}L_{2k+1}\left(e^{-2\pi i/(5\tau)}\right).$$

This result shows that there is a "one-to-one correspondence" between identities satisfied by E_{2k+2,χ_5} and L_{2k+1} if we know the corresponding behavior of all the functions in the identities under the transformation $\tau\mapsto -1/(5\tau)$. We illustrate our observation in the case $k=0$. In this case,

$$L(2,\chi_5)E_{2,\chi_5}(e^{2\pi i\tau}) = -\left(\frac{2\pi}{5}\right)^{2}\tau^{-2}L_1\left(e^{-2\pi i/(5\tau)}\right). \tag{5–4}$$

From (1–1) and the transformation formula

$$\eta\left(-\frac{1}{\tau}\right) = \sqrt{-i\tau}\,\eta(\tau),$$

we find that

$$L_1\left(e^{-2\pi i/(5\tau)}\right) = -\frac{1}{\sqrt{5}}\tau^2 z(e^{2\pi i\tau}),$$

which upon substitution into (5–4) yields

$$L(2,\chi_5)E_{2,\chi_5}(e^{2\pi i\tau}) = \frac{4\pi^2}{25\sqrt{5}} z(e^{2\pi i\tau}). \tag{5–5}$$

Since the q-expansions of E_{2,χ_5} and $z(q)$ start with 1, we conclude from (5–5) that

$$L(2,\chi_5) = \frac{4\pi^2}{25\sqrt{5}}$$

and

$$E_{2,\chi_5}(q) = z(q). \tag{5–6}$$

The latter is another identity of Ramanujan [Ramanujan 1988, p. 139].

The equivalence of (5–6) and (1–1) was illustrated in [Chan 1996]. However, in that work, the special value $L(2,\chi_5)$ needed to be computed. Here, we obtain the value $L(2,\chi_5)$ as a consequence of the transformation formula of $\eta(\tau)$ and Theorem 5.3.

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