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Elliptic Genera of Symmetric Products and Second Quantized Strings

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Abstract: In this note we prove an identity that equates the elliptic genus partition function of a supersymmetric sigma model on the N -fold symmetric product M^N/S_N of a manifold M to the partition function of a second quantized string theory on the space $M \times S^1$. The generating function of these elliptic genera is shown to be (almost) an automorphic form for $O(3, 2, \mathbf{Z})$. In the context of D-brane dynamics, this result gives a precise computation of the free energy of a gas of D-strings inside a higher-dimensional brane.

1. The Identity

Let M be a Kähler manifold. In this note we will consider the partition function of the supersymmetric sigma model defined on the N -fold symmetric product $S^N M$ of M , which is the orbifold space

$$S^N M = M^N / S_N \quad (1.1)$$

with S_N the symmetric group of N elements. The genus one partition function depends on the boundary conditions imposed on the fermionic fields. For definiteness, we will choose the boundary conditions such that the partition function $\chi(S^N M; q, y)$ coincides with the elliptic genus [1, 2], which is defined as the trace over the Ramond-Ramond sector of the sigma model of the evolution operator q^H times $(-1)^F y^{F_L}$. Here q and y are complex numbers and $F = F_L + F_R$ is the sum of the left- and right-moving fermion number. (See the Appendix for background.) In particular,

$$\chi(M; q, y) = \text{Tr}_{\mathcal{H}(M)} (-1)^F y^{F_L} q^H \quad (1.2)$$

with $H = L_0 - \frac{c}{24}$. Of the right-moving sector only the R-ground states contribute to the trace.

We will prove here an identity, conjectured in [3], that expresses the orbifold elliptic genera of the symmetric product manifolds in terms of that of M as follows:¹

$$\sum_{N=0}^{\infty} p^N \chi(S^N M; q, y) = \prod_{n>0, m \geq 0, \ell} \frac{1}{(1 - p^n q^m y^\ell)^{c(nm, \ell)}}, \quad (1.3)$$

where the coefficients $c(m, \ell)$ on the right-hand side are defined via the expansion

$$\chi(M; q, y) = \sum_{m \geq 0, \ell} c(m, \ell) q^m y^\ell. \quad (1.4)$$

The proof of this identity follows quite directly from borrowing standard results about orbifold conformal field theory [6], and generalizes the orbifold Euler number computation of [7] (see also [8]). Before presenting the proof, however, we will comment on the physical interpretation of this identity in terms of second quantized string theory.

1.1. String Theory Interpretation. Each term on the left-hand side with given N can be thought of as the left-moving partition sum of a single (non-critical) supersymmetric string with space-time $S^N M \times S^1 \times \mathbf{R}$. This string is wound once around the S^1 direction, and in the light-cone gauge its transversal fluctuations are described by the supersymmetric sigma-model on $S^N M$. The right-hand side, on the other hand, can be recognized as a partition function of a large Fock space, made up from bosonic and fermionic (depending on whether $c(nm, \ell)$ is positive or negative) creation operators $\alpha_{n,m,\ell}^I$ with $I = 1, 2, \dots, |c(nm, \ell)|$. This Fock space is identical to the one obtained by second quantization of the left-moving sector of the string theory on the space $M \times S^1$. In this correspondence, the oscillators $\alpha_{n,m,\ell}^I$ create string states with winding number n and momentum m around the S^1 . The number of such states is easily read off from the single string partition function (1.4). In the light-cone gauge we have the level matching condition

$$L_0 - \bar{L}_0 = mn, \quad (1.5)$$

and since $\bar{L}_0 = 0$, this condition implies that the left-moving conformal dimension is equal to $h = mn$. Therefore, according to (1.4) the number of single string states with winding n , momentum m and $F_L = \ell$ is indeed given by $|c(nm, \ell)|$. (Strictly speaking, the elliptic genus counts the number of bosonic minus fermionic states at each oscillator level. Because of the anti-periodic boundary condition in the time direction for the fermions, only the net number contributes in the space-time partition function (1.3).)

The central idea behind the proof of the above identity is that the partition function of a single string on the symmetric product $S^N M$ decomposes into several distinct topological sectors, corresponding to the various ways in which a once wound string on $S^N M \times S^1$ can be disentangled into separate strings that wind one or more times around $M \times S^1$. To visualize this correspondence, it is useful to think of the string on $S^N M \times S^1$ as a map that associates to each point on the S^1 a collection of N points in M . By following the path of these N points as we go around the S^1 , we obtain a collection of strings on $M \times S^1$ with total winding number N , that reconnect the N

¹ In case we have more than one conserved quantum number such as F_L , the index ℓ becomes a multi-index and the denominator on the RHS of (1.3) becomes a general product formula as appears in the work of Borchers [4], see also [5].

points with themselves. Since all permutations of the N points on M correspond to the same point in the symmetric product space, the strings can reconnect in different ways labeled by conjugacy classes $[g]$ of the permutation group S_N . The factorization of $[g]$ into a product of irreducible cyclic permutations (n) determines the decomposition into several strings of winding number n . (See Fig. 1). The combinatorial description of the

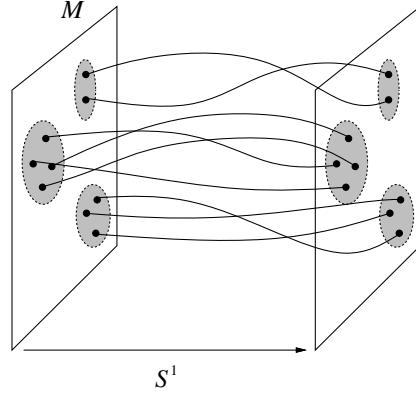


Fig. 1. The string configuration corresponding to a twisted sector by a given permutation $g \in S_N$. The string disentangles into separate strings according to the factorization of g into cyclic permutations

conjugacy classes, as well as the appropriate symmetrization of the wavefunctions, are both naturally accounted for in terms of a second quantized string theory.

2. The Proof

The Hilbert space of an orbifold field theory [6] is decomposed into twisted sectors \mathcal{H}_g , that are labelled by the conjugacy classes $[g]$ of the orbifold group, in our case the symmetric group S_N . Within each twisted sector, one only keeps the states invariant under the centralizer subgroup C_g of g . We will denote this C_g invariant subspace by $\mathcal{H}_g^{C_g}$. Thus the total orbifold Hilbert space takes the form

$$\mathcal{H}(S^N M) = \bigoplus_{[g]} \mathcal{H}_g^{C_g}. \quad (2.1)$$

For the symmetric group, the conjugacy classes $[g]$ are characterized by partitions $\{N_n\}$ of N

$$\sum_n n N_n = N, \quad (2.2)$$

where N_n denotes the multiplicity of the cyclic permutation (n) of n elements in the decomposition of g ,

$$[g] = (1)^{N_1} (2)^{N_2} \dots (s)^{N_s}. \quad (2.3)$$

The centralizer subgroup of a permutation g in this conjugacy class takes the form

$$C_g = S_{N_1} \times (S_{N_2} \times \mathbf{Z}_2^{N_2}) \times \dots \times (S_{N_s} \times \mathbf{Z}_s^{N_s}). \quad (2.4)$$

Here each subfactor S_{N_n} permutes the N_n cycles (n) , while each subfactor \mathbf{Z}_n acts within one particular cycle (n) .

Corresponding to the above decomposition of $[g]$ into irreducible cyclic permutations, we can decompose each twisted sector $\mathcal{H}_g^{C_g}$ into the product over the subfactors (n) of N_n -fold symmetric tensor products of appropriate smaller Hilbert spaces $\mathcal{H}_{(n)}^{\mathbf{Z}_n}$,

$$\mathcal{H}_g^{C_g} = \bigotimes_{n>0} S^{N_n} \mathcal{H}_{(n)}^{\mathbf{Z}_n}, \quad (2.5)$$

where we used the following notation for (graded) symmetric tensor products

$$S^N \mathcal{H} = \left(\underbrace{\mathcal{H} \otimes \dots \otimes \mathcal{H}}_{N \text{ times}} \right)^{S_N}. \quad (2.6)$$

Here the symmetrization is assumed to be compatible with the grading of \mathcal{H} . In particular for pure odd states S^N corresponds to the exterior product \bigwedge^N .

The Hilbert spaces $\mathcal{H}_{(n)}^{\mathbf{Z}_n}$ in (2.5) denote the \mathbf{Z}_n invariant subsector of the Hilbert space $\mathcal{H}_{(n)}$ of a single string on $M \times S^1$ with winding number n . We can represent $\mathcal{H}_{(n)}$ as the Hilbert space of the sigma model of n coordinate fields $X_i(\sigma) \in M$ with the cyclic boundary condition

$$X_i(\sigma + 2\pi) = X_{i+1}(\sigma), \quad i \in (1, \dots, n). \quad (2.7)$$

The group \mathbf{Z}_n , acting on the Hilbert space $\mathcal{H}_{(n)}$, is generated by the cyclic permutation

$$\omega : X_i \rightarrow X_{i+1}. \quad (2.8)$$

We can glue the n coordinate fields $X_i(\sigma)$ together into one single field $X(\sigma)$ defined on the interval $0 \leq \sigma \leq 2\pi n$. Hence, relative to the string with winding number one, the oscillators of the long string that generate $\mathcal{H}_{(n)}$ have a fractional $\frac{1}{n}$ moding. The \mathbf{Z}_n -invariant subspace $\mathcal{H}_{(n)}^{\mathbf{Z}_n}$ consists of those states in $\mathcal{H}_{(n)}$ for which the fractional oscillator numbers combined add up to an integer. We will make use of this observation in the next subsection.

2.1. Partition Function of a Single String. The elliptic genus of $S^N M$ can now be computed by taking the trace over the Hilbert space in the various twisted sectors. We introduce the following notation:

$$\chi(\mathcal{H}; q, y) = \text{Tr}_{\mathcal{H}} (-1)^F y^{F_L} q^H \quad (2.9)$$

for every (sub)Hilbert space \mathcal{H} of a supersymmetric sigma-model. Note that

$$\begin{aligned} \chi(\mathcal{H} \oplus \mathcal{H}'; q, y) &= \chi(\mathcal{H}; q, y) + \chi(\mathcal{H}'; q, y), \\ \chi(\mathcal{H} \otimes \mathcal{H}'; q, y) &= \chi(\mathcal{H}; q, y) \cdot \chi(\mathcal{H}'; q, y). \end{aligned} \quad (2.10)$$

These identities will be used repeatedly in the following.

As the first step we will now compute the elliptic genus of the twisted sector $\mathcal{H}_{(n)}$. This is the left-moving partition sum of a single string with winding n on $M \times S^1$. As we have explained, its elliptic genus can be simply related to that of a string with winding number one via a rescaling $q \rightarrow q^{\frac{1}{n}}$,

$$\chi(\mathcal{H}_{(n)}; q, y) = \chi(\mathcal{H}; q^{\frac{1}{n}}, y) = \sum_{m \geq 0, \ell} c(m, \ell) q^{\frac{m}{n}} y^{\ell}. \quad (2.11)$$

This rescaling accounts for the fractional $\frac{1}{n}$ moding of the string oscillation numbers.

The projection on the \mathbf{Z}_n invariant sector is implemented by insertion of the projection operator $P = \frac{1}{n} \sum_k \omega^k$, with ω as defined in (2.8),

$$\chi(\mathcal{H}_{(n)}^{\mathbf{Z}_n}; q, y) = \frac{1}{n} \sum_{k=0}^{n-1} \text{Tr}_{\mathcal{H}_{(n)}} \omega^k (-1)^F y^{F_L} q^H. \quad (2.12)$$

Since the boundary condition (2.7) on the Hilbert space $\mathcal{H}_{(n)}$ represents a \mathbf{Z}_n -twist by ω along the σ direction, the operator insertion of ω in the genus one partition sum can in fact be absorbed by performing a modular transformation $\tau \rightarrow \tau + 1$, which amounts to a redefinition $q^{\frac{1}{n}} \rightarrow q^{\frac{1}{n}} e^{\frac{2\pi i}{n}}$.² Thus we can write

$$\begin{aligned} \chi(\mathcal{H}_{(n)}^{\mathbf{Z}_n}; q, y) &= \frac{1}{n} \sum_{k=0}^{n-1} \chi(\mathcal{H}; q^{\frac{1}{n}} e^{\frac{2\pi i k}{n}}, y) \\ &= \sum_{m \geq 0, \ell} c(mn, \ell) q^m y^{\ell}. \end{aligned} \quad (2.13)$$

2.2. Symmetrized products. The next step is to consider the partition function for the symmetrized tensor products of the Hilbert spaces $\mathcal{H}_{(n)}^{\mathbf{Z}_n}$. We need the following result: If $\chi(\mathcal{H}; q, y)$ has the expansion

$$\chi(\mathcal{H}; q, y) = \sum_{m, \ell} d(m, \ell) q^m y^{\ell}, \quad (2.14)$$

then we want to show that the partition function of the symmetrized tensor products of \mathcal{H} is given by the generating function

$$\sum_{N \geq 0} p^N \chi(S^N \mathcal{H}; q, y) = \prod_{m, \ell} \frac{1}{(1 - pq^m y^{\ell})^{d(m, \ell)}}. \quad (2.15)$$

This identity is most easily understood in terms of second quantization. The sum over symmetrized products of \mathcal{H} is described by a Fock space with a generator for every state in \mathcal{H} , where states with negative ‘‘multiplicities’’ $d(m, \ell)$ are identified as fermions. The usual evaluation of the partition function in a Fock space then results in the RHS of equation (2.15).

² The redefinition $\tau \rightarrow \tau + 1$ means that the periodic boundary condition in the time direction is composed with a space-like translation $\sigma \rightarrow \sigma + 2\pi$. According to (2.7) and (2.8) this indeed results in an extra insertion of the operator ω into the trace.

In more detail, we can interpret the elliptic genus as computing the (super)dimension³ of vector spaces $V_{m,\ell}$

$$d(m, \ell) = \dim V_{m,\ell}. \quad (2.16)$$

We then evaluate

$$\begin{aligned} & \sum_{N \geq 0} p^N \chi(S^N \mathcal{H}; q, y) \\ &= \sum_{N \geq 0} p^N \sum_{\substack{m_1, \dots, m_N \\ \ell_1, \dots, \ell_N}} \dim \left(V_{m_1, \ell_1} \otimes \cdots \otimes V_{m_N, \ell_N} \right)^{S_N} q^{m_1 + \dots + m_N} y^{\ell_1 + \dots + \ell_N} \\ &= \sum_{N \geq 0} p^N \sum_{\substack{N_{m,\ell} \\ \sum N_{m,\ell} = N}} \prod_{m,\ell} (q^m y^\ell)^{N_{m,\ell}} \dim(S^{N_{m,\ell}} V_{m,\ell}) \\ &= \prod_{m,\ell} \sum_{N \geq 0} p^N (q^m y^\ell)^N \dim(S^N V_{m,\ell}). \end{aligned} \quad (2.17)$$

Using the identity

$$\dim(S^N V_{m,\ell}) = \binom{d(m,\ell) + N - 1}{N}, \quad (2.18)$$

where the RHS is defined as $(-1)^N \binom{|d(m,\ell)|}{N}$ for negative $d(m,\ell)$, gives the desired result.

2.3. Combining the ingredients. The proof of our main identity follows from combining the results of the previous two subsections. Our starting point has been the fact that the Hilbert space of the orbifold field theory has a decomposition in terms of twisted sectors as

$$\mathcal{H}(S^N M) = \bigoplus_{\sum n N_n = N} \bigotimes_{n > 0} S^{N_n} \mathcal{H}_{(n)}^{\mathbf{Z}_n}. \quad (2.19)$$

Physically speaking, the right-hand side describes the Hilbert space of a second quantized string theory with N_n the number of strings with winding number n .

With this form of the Hilbert space $\mathcal{H}(S^N M)$, we find for the partition function

$$\begin{aligned} \sum_{N \geq 0} p^N \chi(S^N M; q, y) &= \sum_{N \geq 0} p^N \sum_{\substack{N_n \\ \sum n N_n = N}} \prod_{n > 0} \chi(S^{N_n} \mathcal{H}_{(n)}^{\mathbf{Z}_n}; q, y) \\ &= \prod_{n > 0} \sum_{N \geq 0} p^{nN} \chi(S^N \mathcal{H}_{(n)}^{\mathbf{Z}_n}; q, y). \end{aligned} \quad (2.20)$$

Here we used repeatedly the identities (2.10). In order to evaluate the elliptic genera of the symmetric products, we apply the result (2.15) of the previous subsection to the Hilbert space $\mathcal{H}_{(n)}^{\mathbf{Z}_n}$, which gives

³ We define $\dim V = \text{Tr}_V(-1)^F = d^+ - d^-$, where d^\pm are the dimensions of the even and odd subspaces V^\pm in the decomposition $V = V^+ \oplus V^-$.

$$\sum_{N \geq 0} p^N \chi(S^N \mathcal{H}_{(n)}^{\mathbb{Z}_n}; q, y) = \prod_{m \geq 0, \ell} \frac{1}{(1 - pq^m y^\ell)^{c(mn, \ell)}}. \tag{2.21}$$

If we insert this into (2.20) we get our final identity

$$\sum_{N \geq 0} p^N \chi(S^N M; q, y) = \prod_{n > 0, m \geq 0, \ell} \frac{1}{(1 - p^n q^m y^\ell)^{c(mn, \ell)}}, \tag{2.22}$$

which concludes the proof.

3. One-Loop Free Energy

In this section we will discuss some properties of our identity. For convenience we will assume here that the space M is a Calabi-Yau manifold, so that the sigma-model defines a $N = 2$ superconformal field theory. For the elliptic genus this implies that it transforms as a modular form.

We have argued that the quantity on the right-hand side of (2.22)

$$\mathcal{Z}(p, q, y) = \prod_{n > 0, m, \ell} \frac{1}{(1 - p^n q^m y^\ell)^{c(mn, \ell)}} \tag{3.1}$$

has an interpretation as the partition function of a second quantized string theory with target space $M \times S^1$. This identification was based on the fact that \mathcal{Z} has the form of the trace over free field Fock space generators by oscillators $\alpha_{n, m, \ell}^I$ with $I = 1, \dots, |c(nm, \ell)|$, i.e. one oscillator for each first quantized string state. We will now comment on the path integral derivation of this expression.

Since we are dealing with a free string theory, we should be able to take the logarithm of the partition sum

$$\mathcal{F}(p, q, y) = \log \mathcal{Z}(p, q, y) \tag{3.2}$$

and obtain an interpretation of \mathcal{F} as the one-loop free energy of a single string. From a path-integral perspective, this free energy is obtained by summing over irreducible one-loop string amplitudes. The time coordinate of the target space is taken to be compactified (since the partition function is defined as a trace) and thus the irreducible one loop string amplitudes are described in terms of all possible maps of T^2 into the Euclidean target space-time $M \times T^2$. From this point of view the parameters p, q, y obtain the interpretation as moduli of the target space two-torus. We can introduce parameters ρ, σ, v via

$$p = e^{2\pi i \rho}, \quad q = e^{2\pi i \sigma}, \quad y = e^{2\pi i v}. \tag{3.3}$$

Here ρ and σ determine the complexified Kähler form and complex structure modulus of T^2 respectively, whereas v parametrizes the $U(1)$ bundle on T^2 corresponding to F_L .

3.1. Instanton sums and Hecke operators. We will now show that the logarithm \mathcal{F} of the partition function (3.1) indeed has the interpretation of a one-loop free energy for a string on $M \times T^2$. First we compute

$$\begin{aligned}
 \mathcal{F}(p, q, y) &= - \sum_{n>0, m, \ell} c(nm, \ell) \log(1 - p^n q^m y^\ell) \\
 &= \sum_{n>0, m, \ell, k>0} \frac{1}{k} c(nm, \ell) p^{kn} q^{km} y^{k\ell} \\
 &= \sum_{N>0} p^N \sum_{kn=N} \frac{1}{k} \sum_{m, \ell} c(nm, \ell) q^{km} y^{k\ell}.
 \end{aligned}
 \tag{3.4}$$

To write this expression in a more convenient form, it is useful to recall the definition of the Hecke operators T_N . (For more details on Hecke operators see e.g. [9].) In general, the Hecke operator T_N acting on a weak Jacobi form⁴ $\phi(\tau, z)$ of weight zero and index r produces a weak Jacobi form $T_N\phi$ of weight zero and index Nr , defined as follows

$$T_N\phi(\tau, z) = \sum_{\substack{ad=N \\ b \pmod d}} \frac{1}{N} \phi\left(\frac{a\tau + b}{d}, az\right).
 \tag{3.5}$$

Hence if $\phi(\tau, z)$ has a Fourier expansion

$$\phi(\tau, z) = \sum_{m \geq 0, \ell} c(m, \ell) q^m y^\ell,
 \tag{3.6}$$

then $T_N\phi(\tau, z)$ takes the form

$$T_N\phi(\tau, z) = \sum_{ad=N} \frac{1}{a} \sum_{m \geq 0, \ell} c(md, \ell) q^{am} y^{a\ell}.
 \tag{3.7}$$

Comparing with the expression (3.4) for the free energy \mathcal{F} , we thus observe that it can be rewritten as a sum of Hecke operators acting on the elliptic genus of M ,

$$\mathcal{F}(p, q, y) = \sum_{N>0} p^N T_N\chi(M; q, y).
 \tag{3.8}$$

(See also [4, 10] for similar expressions.)

This representation has a natural interpretation that arises from the geometric meaning of the Hecke operators T_N . The expression on the right-hand side of (3.7) that defined $T_N\phi$ can be reformulated as the sum of pullbacks for all holomorphic maps $f : T^2 \rightarrow T^2$ of degree N ,

$$T_N\phi = \frac{1}{N} \sum_f f^* \phi.
 \tag{3.9}$$

These maps f act as linear transformations on the two-torus and can be represented by the matrices

$$f = \begin{pmatrix} a & b \\ 0 & d \end{pmatrix},
 \tag{3.10}$$

⁴ See the Appendix for the definition of a Jacobi form.

where $ad = N$ and $0 \leq b \leq d - 1$. The factor $1/N$ in (3.9) is natural because of the automorphisms of the torus.

With this interpretation, the free energy is represented as a sum over holomorphic maps

$$\mathcal{F}(p, q, y) = \sum_{f: T^2 \rightarrow T^2} \frac{1}{N_f} p^{N_f} f^* \chi(M; q, y) \tag{3.11}$$

with N_f the degree of the map f . The right-hand side can be recognized as a summation over instanton sectors.

3.2. Automorphic properties. As suggested by its form, the above expression can indeed be reproduced from a standard string one-loop computation. To make this correspondence precise, we notice that the partition function \mathcal{Z} is in fact almost equal to an automorphic form for the group $SO(3, 2, \mathbf{Z})$ of the type discussed in [4].

The precise form of this automorphic function has been worked out in detail in [12]. It is defined by the product

$$\Phi(p, q, y) = p^a q^b y^c \prod_{(n,m,\ell) > 0} (1 - p^n q^m y^\ell)^{c(nm,\ell)}, \tag{3.12}$$

where the positivity condition means: $n, m \geq 0$ with $\ell > 0$ in the case $n = m = 0$. The ‘‘Weyl vector’’ (a, b, c) is defined by

$$a = b = \frac{1}{24} \chi(M), \quad c = \sum_{\ell} -\frac{|\ell|}{4} c(0, \ell). \tag{3.13}$$

One can then show that the expression Φ is an automorphic form of weight $c(0, 0)/2$ for the group $O(3, 2, \mathbf{Z})$ for a suitable quadratic form of signature $(3, 2)$, see [12].

The form Φ follows naturally from a standard one-loop string amplitude defined as an integral over the fundamental domain [5, 13]. The integrand consists of the genus one partition function of the string on $M \times T^2$ and has a manifest $O(3, 2, \mathbf{Z})$ T-duality invariance. We will not write down the explicit form of this partition function, but refer to [12] for the specific details. For our purpose it is sufficient to mention the final result of the integration

$$\mathcal{I} = -\log \left(Y^{c(0,0)/2} |\Phi(p, q, y)|^2 \right) \tag{3.14}$$

with $Y = \rho_2 \sigma_2 - \frac{1}{2} d v_2^2$, $d = \dim M$, in the notation (3.3). Since the integral \mathcal{I} is by construction invariant under the T-duality group $O(3, 2, \mathbf{Z})$, this determines the automorphic properties of Φ . The factor Y transforms with weight -1 , which fixes the weight of the form Φ to be $c(0, 0)/2$.

The holomorphic contribution in \mathcal{I} is recovered by taking the limit $\bar{p} \rightarrow 0$. In the sigma model this corresponds to the localization of the path-integral on holomorphic instantons and in this way one makes contact with the description of the free energy \mathcal{F} in the previous subsection. We note however that $\log \Phi$ contains extra terms that do not appear in \mathcal{F} . Apart from a $\log p$ contribution that arises from degree zero maps⁵ these terms are independent of p and have no straightforward interpretation in terms of instantons.

⁵ For degree zero the two-torus gets mapped to a point in M , and the moduli space of such maps is the product $M \times \mathcal{M}_1$, with \mathcal{M}_1 the moduli space of elliptic curves. Weighting this contribution by the appropriate characteristic class [11], we obtain $-\frac{\chi(M)}{24} \log p$, in accordance with (3.13).

4. Concluding Remarks

Our computation of the elliptic genus of the symmetric product space $S^N M$ can be seen as a refinement of the calculations in [7, 8] of the orbifold Euler number. In fact, if we restrict to $y = 1$, the elliptic genus reduces to the Euler number and our identity takes the simple form

$$\sum_{N \geq 0} p^N \chi(S^N M) = \prod_{n > 0} \frac{1}{(1 - p^n)^{\chi(M)}}. \quad (4.1)$$

Here the Kähler condition is not necessary. If M is an algebraic surface, it can be shown that this formula also computes the *topological* Euler characteristic of the Hilbert scheme $M^{[N]}$ of dimension zero subschemes of length n [15]. This space is a smooth resolution of the symmetric product $S^N M$. (In complex dimension greater than two the Hilbert scheme is unfortunately not smooth.) It is natural to conjecture that in the case of a two-dimensional Calabi-Yau space, *i.e.* a $K3$ or an abelian surface, the orbifold elliptic genus of the symmetric product also coincides with elliptic genus of the Hilbert scheme.

The left-hand side of our identity (1.3) can be seen to compute the superdimension of the infinite, graded vector space

$$\bigoplus_{N, m, \ell} V_{m, \ell}(S^N M), \quad (4.2)$$

where $V_{m, \ell}$ are the index bundles (A.12). Our result suggests that this space forms a natural representation of the oscillator algebra generated by string field theory creation operators $\alpha_{n, m, \ell}^I$. This statement is analogous to the assertion of Nakajima [16] (see also [17]) that the space $\bigoplus_N H^*(M^{[N]})$ forms a representation of the Heisenberg algebra generated by α_n^I , where I runs over a basis of $H^*(M)$.

It would be interesting to explore possible applications to gauge theories along the lines of [8]. On a $K3$ manifold the moduli space of Yang-Mills instantons takes (for certain instanton numbers) the form of a symmetric product of $K3$. This fact was used in [8] to relate the partition function of $N = 4$ Yang-Mills theory on $K3$ to the generating function of Euler numbers (4.1). Our formula gives an explicit expression for the elliptic genus of these instanton moduli spaces. It seems a natural conjecture that the analysis of [8] can be generalized to show that the generating function of the elliptic genera is the partition function of an appropriately twisted version of $N = 2$ Yang-Mills theory on $K3 \times T^2$. For some interesting recent work in this direction, see [18].

Finally, our calculation is likely to be relevant for understanding the quantum statistical properties of D-branes [19] and their bound states [20]. Particularly useful examples of such possible bound states are those between D-strings with one (or more) higher dimensional D-branes. In type II string compactifications on manifolds of the form $M \times S^1$, we can consider the configuration of a D-string wound N times around the S^1 bound to a $(\dim M + 1)$ -brane. (For the case where M is a $K3$ manifold, this situation was first considered by Vafa and Strominger [21] in their D-brane computation of the 5-dimensional black hole entropy.) As argued in [22, 21], the quantum mechanical degrees of freedom of this D-brane configuration are naturally encoded in terms of a two-dimensional sigma model on the N -fold symmetric tensor product of M , that describes the transversal fluctuations of the D-string. As was also pointed out in [23], this description implies that a multiply wound D-string can carry fractional oscillation numbers. Our result shows that the resulting quantum statistical description of these first

quantized “fractional” D-strings is in fact equivalent to a description in terms of second quantized “ordinary” strings. In this correspondence the extra degrees of freedom that arise from the fractional moding are used to assign to each individual string a momentum along the S^1 direction. This result may be a useful clue in explaining some of the miraculous non-perturbative dualities between strings and D-branes.

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A. Appendix: Elliptic Genus

We summarize some facts about the elliptic genus for a Kähler manifold M of complex dimension d [1, 2]. We start with an elliptic curve E with modulus τ and a line bundle labeled by $z \in Jac(E) \cong E$. We define $q = e^{2\pi i\tau}$, $y = e^{2\pi iz}$. The elliptic genus is defined as

$$\chi(M; q, y) = \text{Tr}_{\mathcal{H}(M)} (-1)^F y^{F_L} q^{L_0 - \frac{d}{8}} \bar{q}^{\bar{L}_0 - \frac{d}{8}}, \tag{A.1}$$

where $F = F_L + F_R$ and $\mathcal{H}(M)$ is the Hilbert space of the $N = 2$ supersymmetric field theory with target space M .

For a Calabi-Yau space the elliptic genus is a weak Jacobi form of weight zero and index $d/2$. Recall that a Jacobi form $\phi(\tau, z)$ of weight k and index r (possibly half-integer) transforms as [24]

$$\begin{aligned} \phi\left(\frac{a\tau + b}{c\tau + d}, \frac{z}{c\tau + d}\right) &= (c\tau + d)^k e^{\pi i \frac{r c z^2}{c\tau + d}} \phi(\tau, z), \\ \phi(\tau, z + m\tau + n) &= e^{-\pi i r(m^2\tau + 2mz)} \phi(\tau, z), \end{aligned} \tag{A.2}$$

and is called weak if it has a Fourier expansion of the form

$$\phi(\tau, z) = \sum_{m \geq 0, \ell} c(m, \ell) q^m y^\ell. \tag{A.3}$$

The coefficients of such a form depend only on $4rm - \ell^2$ and on $\ell \pmod{2r}$.

The elliptic genus has the following properties: First of all, it is a genus; that is, it satisfies the relations

$$\begin{aligned} \chi(M \sqcup M'; q, y) &= \chi(M; q, y) + \chi(M'; q, y), \\ \chi(M \times M'; q, y) &= \chi(M; q, y) \cdot \chi(M'; q, y), \\ \chi(M; q, y) &= 0, \quad \text{if } M = \partial N, \end{aligned} \tag{A.4}$$

where the last relation is in the sense of complex bordism. Furthermore, for $q = 0$ it reduces to a weighted sum over the Hodge numbers, which is essentially the Hirzebruch χ_y -genus,

$$\chi(M; 0, y) = \sum_{p, q} (-1)^{p+q} y^{p - \frac{d}{2}} h^{p, q}(M), \tag{A.5}$$

and for $y = 1$ its equals the Euler number of M ,

$$\chi(M; q, 1) = \chi(M). \quad (\text{A.6})$$

For smooth manifolds, the elliptic genus has an alternative definition in terms of characteristic classes, as follows. For any vector bundle V one defines the formal sums

$$\bigwedge_q V = \bigoplus_{k \geq 0} q^k \bigwedge^k V, \quad S_q V = \bigoplus_{k \geq 0} q^k S^k V, \quad (\text{A.7})$$

where \bigwedge^k and S^k denote the k^{th} exterior and symmetric product respectively. One then has an equivalent definition of the elliptic genus as

$$\chi(M; q, y) = \int_M ch(E_{q,y}) td(M) \quad (\text{A.8})$$

with

$$E_{q,y} = y^{-\frac{d}{2}} \bigotimes_{n \geq 1} \left(\bigwedge_{-yq^{n-1}} T_M \otimes \bigwedge_{-y^{-1}q^n} \bar{T}_M \otimes S_{q^n} T_M \otimes S_{q^n} \bar{T}_M \right), \quad (\text{A.9})$$

where T_M denotes the holomorphic tangent bundle of M . Expanding the bundle $E_{q,y}$ as

$$E_{q,y} = \bigoplus_{m,\ell} q^m y^\ell E_{m,\ell}, \quad (\text{A.10})$$

one can define the coefficients $c(m, \ell)$ as

$$c(m, \ell) = \text{index } \mathcal{D}_{E_{m,\ell}} \quad (\text{A.11})$$

with \mathcal{D}_E the Dirac operator twisted with the vector bundle E . So $c(m, \ell)$ computes the dimension of the virtual vector space

$$V_{m,\ell}(M) = \ker \mathcal{D}_{E_{m,\ell}} \ominus \text{cok } \mathcal{D}_{E_{m,\ell}}. \quad (\text{A.12})$$

References

1. Landweber, P.S. (ed.): Elliptic Curves and Modular Forms in Algebraic Topology. Berlin–Heidelberg–New York: Springer-Verlag, 1988; Witten, E.: Commun. Math. Phys. **109**, 525 (1987); Schellekens, A. and Warner, N.: Phys. Lett. **B177**, 317 (1986); Nucl. Phys. **B287**, 317 (1987)
2. Kawai, T., Yamada, Y. and Yang, S.-K.: Elliptic Genera and N=2 Superconformal Field Theory. Nucl. Phys. **B414**, 191-212 (1994); Eguchi, T., Ooguri, H., Taormina, A., Yang, S.-K.: Superconformal Algebras and String Compactification on Manifolds with $SU(N)$ Holonomy. Nucl.Phys. **B315**, 193 (1989)
3. Dijkgraaf, R., Verlinde, E. and Verlinde, H.: Counting Dyons in N=4 String Theory. hep-th/9607026; BPS Spectrum of the Five-Brane and Black Hole Entropy. hep-th/9603126
4. Borcherds, R. E.: Automorphic Forms on $O_{s+2,2}(R)$ and Infinite Products. Invent. Math. **120**, 161 (1995)
5. Harvey, J. and Moore, G.: Algebras, BPS States, and Strings. Nucl. Phys. **B463**, 315-368, (1996), hep-th/9510182
6. Dixon, L., Harvey, J., Vafa, C. and Witten, E. : Nucl. Phys. **B261**, 620 (1985); Nucl. Phys. **B274**, 285 (1986)

7. Hirzebruch, F. and Höfer, T.: On the Euler Number of an Orbifold. *Math. Ann.* **286**, 255 (1990)
8. Vafa, C. and Witten, E.; A Strong Coupling Test of S-Duality. *Nucl. Phys.* **B431**, 3-77 (1994), hep-th/9408074
9. Lang, S.: *Introduction to Modular Forms*. Berlin–Heidelberg–New York: Springer-Verlag, 1976
10. Gritsenko, V.A. and Nikulin, V.V.: Siegel Automorphic Form Corrections of Some Lorentzian Kac-Moody Algebras. alg-geom/9504006
11. Bershadsky, M., Cecotti, S., Ooguri, H. and Vafa, C.: Holomorphic Anomalies in Topological Field Theories. *Nucl. Phys.* **B405**, 279–304 (1993), hep-th/9302103
12. Neumann, C.D.D.: The Elliptic Genus of Calabi-Yau 3- and 4-Folds, Product Formulae and Generalized Kac-Moody Algebras. hep-th/9607029
13. Kawai, T.: $N = 2$ Heterotic String Threshold Correction, $K3$ Surface and Generalized Kac-Moody Superalgebra. hep-th/9512046
14. Dixon, L., Kaplunovsky, V. and Louis, J.: Moduli-dependence of string loop corrections to gauge coupling constants. *Nucl. Phys.* **B307**, 145 (1988)
15. Göttsche, L.: The Betti numbers of the Hilbert Scheme of Points on a Smooth Projective Surface. *Math. Ann.* **286**, 193–297 (1990)
16. Nakajima, H.: Heisenberg Algebra and Hilbert Schemes of Points on Projective Surfaces. alg-geom/9507012
17. Grojnowski, I.: Instantons and Affine Algebras I: The Hilbert Scheme and Vertex Operators. alg-geom/9506020
18. Nekrasov, N.: Princeton PhD-thesis, May 1996, unpublished
19. Polchinski, J.: Dirichlet-Branes and Ramond-Ramond Charges. hep-th/9510017; Polchinski, J., Chaudhuri, S. and Johnson, C.: Notes on D-Branes. hep-th/9602052
20. Witten, E.: Bound States Of Strings And p -Branes. hep-th/9510135
21. Strominger, A., Vafa, C.: Microscopic Origin of the Bekenstein-Hawking Entropy. hep-th/9601029
22. Vafa, C.: Gas of D-Branes and Hagedorn Density of BPS States. hep-th/9511026, Instantons on D-branes. hep-th/9512078; Bershadsky, M., Sadov, V. and Vafa, C.: D-Branes and Topological Field Theories. hep-th/9511222
23. Maldacena, J. and Susskind, L. : D-branes and Fat Black Holes. hep-th/9604042
24. Eichler, M. and Zagier, D.: *The Theory of Jacobi Forms*. Basel–Boston: Birkhäuser, 1985

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