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ELLIPTIC PROBLEMS IN VARIABLE EXPONENT SPACES

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In this paper we study a nonlinear elliptic equation involving $p(x)$-growth conditions on a bounded domain having cylindrical symmetry. We establish existence and multiplicity results using as main tools the mountain pass theorem of Ambosetti and Rabinowitz and Ekeland's variational principle.

1. INTRODUCTION

The study of partial differential equations and variational problems involving $p(x)$-growth conditions has captured a special attention in the last decade. This is a consequence of the fact that such equations can be used to model phenomena which arise in mathematical physics. Elastic mechanics and electrorheological fluids (sometimes referred to as "smart fluids") are two classical examples of physical fields which benefit from such kind of studies. In that context we refer to Acerbi and Mingione [**1**], Diening [**7**], Halsey [**13**], Ruzicka [**17, 18**], Winslow [**21**], Zhikov [**22**], and the references therein. In what concerns the investigation of existence and multiplicity of solutions for equations with $p(x)$-growth conditions we refer to the recent papers by Alves and Souto [**2**], Chabrowski and Fu [**6**], Fan and Zhang [**12**], Mihăilescu and Rădulescu [**15**], where different techniques of finding solutions are illustrated.

The goal of this paper is to establish the existence of solutions for problems of the type

$$(1) \qquad \begin{cases} -\operatorname{div}\left(|\nabla u|^{p(x)-2}\nabla u\right) = g(x,u), & \text{if } \ x \in \Omega \\ u = 0, & \text{if } \ x \in \partial\Omega\,, \end{cases}$$

where $\Omega \subset \mathbb{R}^N$ is a bounded domain having cylindrical symmetry, $p(x)$ is a continuous function on $\overline{\Omega}$, with $1 < p(x)$ for all $x \in \overline{\Omega}$ and $g(x,u)$ is a real-valued function which will be specified later. We investigate problem (1) in the case when $\Omega = \Omega_1 \times \Omega_2$, where $\Omega_1 \subset \mathbb{R}^m$ is a bounded regular domain, and Ω_2 is a k dimensional ball of radius R, centred in the origin, and consequently $m + k = N$.

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2. Preliminary results

In this section we recall some background facts concerning the generalised Lebesgue–Sobolev spaces $L^{p(x)}(\Omega)$ and $W_0^{1,p(x)}(\Omega)$, where Ω is a bounded domain in $\mathbb{R}^N$. We refer the reader to the book by Musielak [**16**] and the papers by Edmunds, Lang and Nekvinda [**8**], Edmunds and Rákosník [**9, 10**], Kovacik and Rákosník [**14**], and Samko and Vakulov [**19**].

Throughout this paper we assume that $p(x) > 1$, $p(x) \in C(\overline{\Omega})$.

Set

$$C_+(\overline{\Omega}) = \left\{h;\ h \in C(\overline{\Omega}),\ h(x) > 1 \text{ for all } x \in \overline{\Omega}\right\}.$$

For any $h \in C_+(\overline{\Omega})$ we define

$$h^+ = \sup_{x\in\Omega} h(x) \quad \text{and} \quad h^- = \inf_{x\in\Omega} h(x).$$

For any $p(x) \in C_+(\overline{\Omega})$, we define the variable exponent Lebesgue space

$$L^{p(x)}(\Omega) = \left\{u;\ u \text{ is a measurable real-valued function such that } \int_\Omega |u(x)|^{p(x)}\,dx < \infty\right\}.$$

We define a norm, the so-called *Luxemburg norm*, on this space by the formula

$$|u|_{p(x)} = \inf\left\{\mu > 0;\ \int_\Omega \left|\frac{u(x)}{\mu}\right|^{p(x)} dx \leqslant 1\right\}.$$

We remember that the variable exponent Lebesgue spaces are separable and reflexive Banach spaces. If $0 < |\Omega| < \infty$ and p_1, p_2 are variable exponents such that $p_1(x) \leqslant p_2(x)$ almost everywhere in Ω then there exists the continuous embedding $L^{p_2(x)}(\Omega) \hookrightarrow L^{p_1(x)}(\Omega)$.

We denote by $L^{p'(x)}(\Omega)$ the conjugate space of $L^{p(x)}(\Omega)$, where $1/p(x) + 1/p'(x) = 1$. For any $u \in L^{p(x)}(\Omega)$ and $v \in L^{p'(x)}(\Omega)$ the Hölder type inequality

$$\left|\int_\Omega uv\,dx\right| \leqslant \left(\frac{1}{p^-} + \frac{1}{p'^-}\right)|u|_{p(x)}|v|_{p'(x)} \tag{2}$$

holds true.

An important role in manipulating the generalised Lebesgue-Sobolev spaces is played by the *modular* of the $L^{p(x)}(\Omega)$ space, which is the mapping $\rho_{p(x)} : L^{p(x)}(\Omega) \to \mathbb{R}$ defined by

$$\rho_{p(x)}(u) = \int_\Omega |u|^{p(x)}\,dx.$$

If (u_n), $u \in L^{p(x)}(\Omega)$ then the following relations hold true

$$|u|_{p(x)} > 1 \Rightarrow |u|_{p(x)}^{p^-} \leqslant \rho_{p(x)}(u) \leqslant |u|_{p(x)}^{p^+} \tag{3}$$

$$|u|_{p(x)} < 1 \Rightarrow |u|_{p(x)}^{p^+} \leqslant \rho_{p(x)}(u) \leqslant |u|_{p(x)}^{p^-} \tag{4}$$

$$|u_n - u|_{p(x)} \to 0 \Leftrightarrow \rho_{p(x)}(u_n - u) \to 0. \tag{5}$$

Next, we consider the weighted variable exponent Lebesgue spaces. Let $a : \Omega \to \mathbb{R}$ be a measurable real function such that $a(x) > 0$ almost everywhere $x \in \Omega$. We define

$$L_{a(x)}^{p(x)}(\Omega) = \Big\{u;\ u \text{ is a measurable real-valued function such that } \int_\Omega a(x)\big|u(x)\big|^{p(x)}\,dx < \infty\Big\}$$

with the norm

$$|u|_{L_{a(x)}^{p(x)}(\Omega)} = |u|_{(p(x),a(x))} = \inf\Big\{\mu > 0;\ \int_\Omega a(x)\Big|\frac{u(x)}{\mu}\Big|^{p(x)}\,dx \leqslant 1\Big\}.$$

The space $L_{a(x)}^{p(x)}(\Omega)$ endowed with the above norm is a Banach space which has similar properties with the usual variable exponent Lebesgue spaces.

Finally, we define $W_0^{1,p(x)}(\Omega)$ as the closure of $C_0^\infty(\Omega)$ under the norm

$$\|u\| = |\nabla u|_{p(x)}.$$

The space $\big(W_0^{1,p(x)}(\Omega), \|\cdot\|\big)$ is a separable and reflexive Banach space. We note that if $s(x) \in C_+(\overline{\Omega})$ and $s(x) < p^\star(x)$ for all $x \in \overline{\Omega}$ then the embedding $W_0^{1,p(x)}(\Omega) \hookrightarrow L^{s(x)}(\Omega)$ is compact and continuous, where $p^\star(x) = \big(Np(x)/N - p(x)\big)$ if $p(x) < N$ or $p^\star(x) = +\infty$ if $p(x) \geqslant N$.

3. The main results

In what follows we establish two existence results for some problems of type (1). We are seeking solutions for this kind of equations in the space

$$W_{0,s}^{1,p(x)}(\Omega) = \Big\{u \in W_0^{1,p(x)}(\Omega);\ u(x_1,x_2) = u\big(x_1,|x_2|\big),\ \forall\ (x_1,x_2) \in \Omega\Big\},$$

which is a closed subspace of $W_0^{1,p(x)}(\Omega)$.

First, we study problem (1) in the case when

$$g(x,u) = h(x)|u|^{q(x)-2}u\,,$$

where $q(x) \in C_+(\overline{\Omega})$ and $h(x) = |x_2|^l$, for all $x = (x_1,x_2) \in \Omega_1 \times \Omega_2$, with $l > 0$ a real number.

Thus, problem (1) becomes

$$\begin{cases} -\operatorname{div}\big(|\nabla u|^{p(x)-2}\nabla u\big) = h(x)|u|^{q(x)-2}u, & \text{if } x \in \Omega \\ u = 0, & \text{if } x \in \partial\Omega\,. \end{cases} \tag{6}$$

We say that $u \in W_{0,s}^{1,p(x)}(\Omega)$ is a weak solution for problem (6) if

$$\int_\Omega |\nabla u|^{p(x)-2}\nabla u \nabla v \, dx - \int_\Omega h(x)|u|^{q(x)-2}uv \, dx = 0\,,$$

for all $v \in W_{0,s}^{1,p(x)}(\Omega)$.

We prove

Theorem 1. *Assume that $m \geqslant p^-$, $k \geqslant 1 + p^-$, $p^+ < N$, $p^+ < q^-$ and $q^+ < (p^-)^\star + \delta$, where $\delta = p^-/(N-p^-)\min\Big\{\big(p^-(k-p^-)/m\big), l\Big\}$. Then problem (6) has at least a nontrivial weak solution.*

Next, we consider the perturbed problem

$$\text{(7)} \qquad \begin{cases} -\operatorname{div}\big(|\nabla u|^{p(x)-2}\nabla u\big) = h(x)|u|^{q(x)-2}u + f(x), & \text{if } \ x \in \Omega \\ u = 0, & \text{if } \ x \in \partial\Omega\,, \end{cases}$$

where f is a function which belongs to the dual space of $W_{0,s}^{1,p(x)}(\Omega)$, denoted by $\big(W_{0,s}^{1,p(x)}(\Omega)\big)^{-1}$.

We say that $u \in W_{0,s}^{1,p(x)}(\Omega)$ is a weak solution for problem (7) if

$$\int_\Omega |\nabla u|^{p(x)-2}\nabla u \nabla v \, dx - \int_\Omega h(x)|u|^{q(x)-2}uv \, dx - \int_\Omega f(x)v \, dx = 0\,,$$

for all $v \in W_{0,s}^{1,p(x)}(\Omega)$.

We show

Theorem 2. *Assume that the hypotheses of Theorem 1 are satisfied. Then there exists $\varepsilon_0 > 0$ such that for any $f \in \big(W_{0,s}^{1,p(x)}(\Omega)\big)^{-1}$, $f \neq 0$, with $\|f\|_{-1} < \varepsilon_0$ problem (7) has at least two weak solutions.*

Remark. We point out the fact that, in general, the existence of solutions for problems involving $p(x)$-growth conditions is studied in the sub-critical case when

$$q(x) < p^\star(x), \quad \forall\, x \in \overline{\Omega}.$$

The conditions assumed in the hypotheses of Theorems 1 and 2 allow a relaxation of the above inequality since it is possible to have

$$p^+ < (p^-)^\star < q^- \leqslant q^+ < (p^-)^\star + \delta < (p^+)^\star.$$

That situation is linked to the special geometry of the domain Ω in our study. More exactly, since Ω is a cylindrically symmetric domain we can state some compactness results which will lead to the above quoted relaxation. Originally, this idea was illustrated in the case of elliptic boundary value problems in the papers by Brezis and Nirenberg [5] and Bahri and Coron [4], and recently in the paper by Wang [20].

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4. A compactness result

Wang in [20, Theorem 2.4] asserts that if $\dim \Omega_1 = m \geqslant 2$ and $\dim \Omega_2 = k \geqslant 3$ then the embedding

$$W^{1,2}_{0,s}(\Omega) \hookrightarrow L^q_{h(x)}(\Omega)$$

is compact for any real number $q \in \bigl(1, (2N/N-2) + \tau\bigr)$, where

$$\tau = \frac{2}{N-2} \min\Bigl\{\frac{2(k-2)}{m}, l\Bigr\}.$$

A careful analysis of the proof of that theorem shows that similar arguments enable the extension of the above result in the following way: assuming that $p \in (1, N)$ is a real number, $\dim \Omega_1 = m \geqslant p$ and $\dim \Omega_2 = k \geqslant p+1$ then the embedding

$$W^{1,p}_{0,s}(\Omega) \hookrightarrow L^q_{h(x)}(\Omega)$$

is compact for any real number $q \in \bigl(1, (Np/N-p) + \tau\bigr)$, where

$$\tau = \frac{p}{N-p} \min\Bigl\{\frac{p(k-p)}{m}, l\Bigr\}.$$

The above information, combined with the facts that $L^{q^+}_{h(x)}(\Omega)$ is continuously embedded in $L^{q(x)}_{h(x)}(\Omega)$ and $W^{1,p(x)}_{0,s}(\Omega)$ is continuously embedded in $W^{1,p^-}_{0,s}(\Omega)$, leads to the main result of this section.

Theorem 3. *Assuming that $p(x), q(x) \in C_+(\overline{\Omega})$ then the embedding*

$$W^{1,p(x)}_{0,s}(\Omega) \hookrightarrow L^{q(x)}_{h(x)}(\Omega)$$

is compact, providing that $1 < q^+ < (Np^-/N-p^-) + \tau$ with

$$\tau = \frac{p^-}{N-p^-} \min\Bigl\{\frac{p^-(k-p^-)}{m}, l\Bigr\}.$$

5. Proof of the main results

In this section we shall concentrate our efforts in order to prove Theorem 2. The proof of Theorem 1 will follow using a part of the arguments used in the proof of Theorem 2 that will be specified of the end of this section.

Let E denote the generalised Sobolev space $W^{1,p(x)}_{0,s}(\Omega)$ and E^{-1} the dual space, $\bigl(W^{1,p(x)}_{0,s}(\Omega)\bigr)^{-1}$.

The energy functional corresponding to problem (7) is defined as $J : E \to \mathbb{R}$,

$$J(u) = \int_\Omega \bigl(1/p(x)\bigr)|\nabla u|^{p(x)}\,dx - \int_\Omega h(x)\bigl(1/q(x)\bigr)|u|^{q(x)}\,dx - \int_\Omega f(x)u\,dx.$$

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Standard arguments imply that $J \in C^1(E, \mathbb{R})$ and

$$\langle J'(u), v\rangle = \int_\Omega |\nabla u|^{p(x)-2}\nabla u \nabla v \, dx - \int_\Omega h(x)|u|^{q(x)-2}uv \, dx - \int_\Omega f(x)v \, dx,$$

for all u, $v \in E$. Thus the weak solutions of (7) are exactly the critical points of J.

We establish some properties of the functional J.

Lemma 1. *There exist, $\varepsilon_0 > 0$ $\eta > 0$ and $\alpha > 0$ such that $J(u) \geqslant \alpha > 0$ for any $u \in E$ with $\|u\| = \eta$, providing that $0 < \|f\|_{-1} < \varepsilon_0$.*

Proof: First, we point out that for any $u \in E$ we have

$$|u(x)|^{q(x)} \leqslant |u(x)|^{q^-} + |u(x)|^{q^+}, \quad \forall x \in \overline{\Omega}. \tag{8}$$

Since $p^+ < q^- \leqslant q^+ < (p^-)^\star + \delta$ it follows by Theorem 3 that E is continuously embedded in $L^{q^-}_{h(x)}(\Omega)$ and $L^{q^+}_{h(x)}(\Omega)$. Thus, there exist two positive constants C_1 and C_2 such that

$$C_1 \cdot |u|_{(q^-,h(x))} \leqslant \|u\|, \quad C_2 \cdot |u|_{(q^+,h(x))} \leqslant \|u\|, \quad \forall u \in E. \tag{9}$$

Next, we focus our attention on the case when $u \in E$ with $\|u\| < 1$. For such a u by relation (4) we obtain

$$\int_\Omega |\nabla u|^{p(x)} \, dx \geqslant \|u\|^{p^+}. \tag{10}$$

On the other hand, for any $\varepsilon > 0$ there exists a constant $C_\varepsilon > 0$ such that we have

$$\left|\int_\Omega f(x)u \, dx\right| \leqslant \|f\|_{-1} \cdot \|u\| \leqslant \frac{\varepsilon}{p^+} \cdot \|u\|^{p^+} + C_\varepsilon \cdot \|f\|_{-1}, \tag{11}$$

for any $u \in E$.

Relations (4), (8), (9), (10), and (11) imply

$$\begin{aligned} J(u) &\geqslant \frac{1}{p^+} \cdot \|u\|^{p^+} - \frac{1}{q^-} \cdot \left[\left(\frac{1}{C_1} \cdot \|u\|\right)^{q^-} + \left(\frac{1}{C_2} \cdot \|u\|\right)^{q^+}\right] - \|f\|_{-1} \cdot \|u\| \\ &\geqslant \left(\frac{1}{p^+} - \frac{\varepsilon}{p^+}\right) \cdot \|u\|^{p^+} - \beta_2 \cdot \|u\|^{q^-} - \beta_3 \cdot \|u\|^{q^+} - C_\varepsilon \cdot \|f\|_{-1}, \end{aligned}$$

for any $u \in E$ with $\|u\| < 1$ and any $\varepsilon > 0$, where β_2 and β_3 are positive constants. Fixing $\varepsilon \in (0,1)$ we can easily find $\eta \in (0,1)$, $\varepsilon_0 > 0$ and $\alpha > 0$ such that the conclusion of Lemma 1 holds true. ∎

Lemma 2. *There exists $e \in E$ such that $\|e\| > \eta$ and $J(e) < 0$ (where η is given in Lemma 1).*

PROOF: We consider a function $w_0 \in C_0^\infty(\Omega) \cap E$, such that $\int_\Omega h(x)|w_0(x)|^{q(x)}\,dx > 0$. Then for any $t > 1$ we obtain

$$J(t\cdot w_0) = \int_\Omega \frac{t^{p(x)}}{p(x)}|\nabla w_0|^{p(x)}\,dx - \int_\Omega h(x)\frac{t^{q(x)}}{q(x)}|w_0|^{q(x)}\,dx - \int_\Omega tf(x)w_0\,dx$$
$$\leqslant \frac{t^{p^+}}{p^-}\cdot\int_\Omega |\nabla w_0|^{p(x)}\,dx - \frac{t^{q^-}}{q^+}\cdot\int_\Omega h(x)|w_0|^{q(x)}\,dx - t\cdot\int_\Omega f(x)w_0\,dx.$$

Taking into account that $q^- > p^+$ we conclude that $\lim_{t\to\infty} J(t\cdot w_0) = -\infty$ and thus Lemma 2 holds true. □

PROOF OF THEOREM 2. We set

$$\Gamma = \Big\{\gamma \in C\big([0,1], E\big);\ \gamma(0) = 0,\ \gamma(1) = e\Big\},$$

where $e \in E$ is given by Lemma 2, and

$$c = \inf_{\gamma\in\Gamma}\max_{t\in[0,1]} J\big(\gamma(t)\big).$$

According to Lemma 2 we know that $\|e\| > \eta$ so every path $\gamma \in \Gamma$ intersects the sphere $|x| = \eta$. Then Lemma 1 implies

$$c \geqslant \inf_{\|u\|=\eta} J(u) \geqslant \alpha,$$

with the constant $\alpha > 0$ given in Lemma 1, thus $c > 0$.

The mountain-pass theorem (see, for example, [3]) implies the existence of a sequence $\{u_n\} \subset E$ such that

$$J(u_n) \to c \quad \text{and} \quad J'(u_n) \to 0. \tag{12}$$

First, we show that $\{u_n\}$ is bounded in E. Assume by contradiction the contrary. Then, passing eventually to a subsequence, still denoted by $\{u_n\}$, we may assume that $\|u_n\| \to \infty$ as $n \to \infty$. Thus we may consider that $\|u_n\| > 1$ for any integer n.

By relation (12) it results that there exists a positive constant M such that for any n large enough we have

$$M \geqslant J(u_n) - \frac{1}{q^-}\langle J'(u_n), u_n\rangle$$
$$\geqslant \Big(\frac{1}{p^+} - \frac{1}{q^-}\Big)\int_\Omega |\nabla u_n|^{p(x)}\,dx - \int_\Omega h(x)\Big(\frac{1}{q(x)} - \frac{1}{q^-}\Big)|u_n|^{q(x)}\,dx - \Big(1 - \frac{1}{q^-}\Big)\int_\Omega f(x)u_n\,dx$$
$$\geqslant \Big(\frac{1}{p^+} - \frac{1}{q^-}\Big)\|u_n\|^{p^-} - \Big(1 - \frac{1}{q^-}\Big)\cdot\|f\|_{-1}\cdot\|u_n\|.$$

Since $q^- > p^+ > 1$ letting $n \to \infty$ in the above inequality, we obtain a contradiction. It follows that $\{u_n\}$ is bounded in E.

Since $\{u_n\}$ is bounded in E we deduce that there exists a subsequence, again denoted by $\{u_n\}$, and $u_0 \in E$ such that $\{u_n\}$ converges weakly to u_0 in E. We prove that $\{u_n\}$ converges strongly to u_0 in E.

First, we point out that the above information and relation (12) imply

$$\langle J'(u_n) - J'(u_0), u_n - u_0 \rangle \to 0 \quad \text{as } n \to \infty. \tag{13}$$

Next, applying Theorem 3 we obtain that $\{u_n\}$ converges strongly to u_0 in $L^{q(x)}_{h(x)}(\Omega)$. Using that fact and inequality (2) we can deduce that

$$\lim_{n\to\infty} \int_\Omega h(x)\big(|u_n|^{q(x)-2}u_n - |u_0|^{q(x)-2}u_0\big)(u_n - u_0)\, dx = 0. \tag{14}$$

Relations (13) and (14) yield

$$\lim_{n\to\infty} \int_\Omega \big(|\nabla u_n|^{p(x)-2}\nabla u_n - |\nabla u_0|^{p(x)-2}\nabla u_0\big)\nabla(u_n - u_0)\, dx = 0. \tag{15}$$

Relation (15) and the fact that $\{u_n\}$ converges weakly to u_0 in E enable us to apply Theorem 3.1 in Fan and Zhang [**12**] in order to obtain that $\{u_n\}$ converges strongly to u_0 in E. So, by (12),

$$J(u_0) = c > 0 \quad \text{and} \quad J'(u_0) = 0. \tag{16}$$

We conclude that u_0 is a weak solution for problem (7). □

We prove now that there exists a second weak solution $v_0 \in E$ such that $v_0 \neq u_0$. Consider

$$d := \inf\{J(u);\ u \in E \text{ and } \|u\| \leqslant \eta\},$$

where $\eta > 0$ is given by Lemma 1. Since $f \not\equiv 0$ it follows that $d < J(0) = 0$. The set

$$\overline{B}_\eta(0) := \{u \in E;\ \|u\| \leqslant \eta\},$$

is a complete metric space with respect to the distance

$$\operatorname{dist}(u, v) := \|u - v\|, \quad \forall\, u,\ v \in \overline{B}_\eta(0).$$

On the other hand, J is lower semi-continuous and bounded from below in $\overline{B}_\eta(0)$. Thus, letting $0 < \varepsilon < \inf_{\partial B_1(0)} J - \inf_{B_1(0)} J$, and applying Ekeland's Variational Principle for functional $J : \overline{B}_\eta(0) \to \mathbb{R}$, (see [**11**]), there exists $u_\varepsilon \in \overline{B}_\eta(0)$ such that

$$J(u_\varepsilon) < \inf_{\overline{B}_\eta(0)} J + \varepsilon$$
$$J(u_\varepsilon) < J(u) + \varepsilon \cdot \|u - u_\varepsilon\|, \quad u \neq u_\varepsilon.$$

Since

$$J(u_\varepsilon) \leqslant \inf_{\overline{B}_\eta(0)} J + \varepsilon \leqslant \inf_{B_\eta(0)} J + \varepsilon < \inf_{\partial B_\eta(0)} J$$

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it follows that $u_\varepsilon \in B_\eta(0)$. Now, we define $I : \overline{B}_\eta(0) \to \mathbf{R}$ by $I(u) = J(u) + \varepsilon \cdot \|u - u_\varepsilon\|$. It is clear that u_ε is a minimum point of I and thus

$$\frac{I(u_\varepsilon + t \cdot v) - I(u_\varepsilon)}{t} \geqslant 0$$

for a small $t > 0$ and $v \in B_\eta(0)$. The above relation yields

$$\frac{J(u_\varepsilon + t \cdot v) - J(u_\varepsilon)}{t} + \varepsilon \cdot \|v\| \geqslant 0.$$

Letting $t \to 0$ it follows that $\langle J'(u_\varepsilon), v\rangle + \varepsilon \cdot \|v\| > 0$ and we infer that $\|J'(u_\varepsilon)\| \leqslant \varepsilon$.

We deduce that there exists a sequence $\{v_n\} \subset B_\eta(0)$ such that

$$J(v_n) \to d \quad \text{and} \quad J'(v_n) \to 0. \tag{17}$$

Using the same arguments as in the case of solution u_0 we can prove that $\{v_n\}$ converges strongly to v_0 in E. Moreover, since $J \in C^1(E, \mathbf{R})$, by relation (17) it follows that

$$J(v_0) = d \quad \text{and} \quad J'(v_0) = 0. \tag{18}$$

Thus, v_0 is also a weak solution for problem (7).

Finally, we point out the fact that $v_0 \neq u_0$ since

$$J(u_0) = c > 0 > d = J(v_0).$$

The proof of Theorem 2 is complete. ∎

REMARK. The proof of Theorem 1 can be carried out from the first part of the proof of Theorem 2. Moreover, the solution of problem (6) obtained in this way is not trivial, since a similar relation to (16) will hold true.

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