Elliptic quantum curves of class $\mathcal{S}_k$

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ABSTRACT: Quantum curves arise from Seiberg-Witten curves associated to 4d $\mathcal{N} = 2$ gauge theories by promoting coordinates to non-commutative operators. In this way the algebraic equation of the curve is interpreted as an operator equation where a Hamiltonian acts on a wave-function with zero eigenvalue. We find that this structure generalises when one considers torus-compactified 6d $\mathcal{N} = (1,0)$ SCFTs. The corresponding quantum curves are elliptic in nature and hence the associated eigenvectors/eigenvalues can be expressed in terms of Jacobi forms. In this paper we focus on the class of 6d SCFTs arising from M5 branes transverse to a $\mathbb{C}^2/\mathbb{Z}_k$ singularity. In the limit where the compactified 2-torus has zero size, the corresponding 4d $\mathcal{N} = 2$ theories are known as class $\mathcal{S}_k$. We explicitly show that the eigenvectors associated to the quantum curve are expectation values of codimension 2 surface operators, while the corresponding eigenvalues are codimension 4 Wilson surface expectation values.

Contents

1   Introduction

Since their classification [1, 2], 6d superconformal field theories (SCFTs) with 8 super-charges have played a prominent role in constructing lower dimensional quantum field theories. In particular, it appears that 5d SCFTs arise as compactifications of such 6d theories with Wilson line expectation values for background flavour fields turned on [3, 4], while 5d theories of KK type admitting an affine quiver description can be understood as twisted compactifications of 6d SCFTs [5–7]. Moreover, 4d $\mathcal{N} = 1$ SCFTs can be understood as compactifications on Riemann surfaces with fluxes [8–14].

     In this paper we focus on 6d SCFTs arising from N M5 branes probing $\mathbb{C}^2/\mathbb{Z}_k$ singularities. When compactified on a 2-torus $\mathbb{T}^2$, BPS partition functions of such theories have been computed in [15, 16] ($k = 1$) and [17] ($k > 1$). As it turns out, a crucial property of these partition functions is that they can be expressed in terms of an infinite sum over elliptic genera of BPS strings wrapping the torus. These elliptic genera are Jacobi forms

with modular parameter τ, being the complex structure of the torus, and several elliptic parameters arising from gauge, flavour, and R-symmetry chemical potentials. Using the correspondence described in the first paragraph, the torus-compactified theory can be equally understood as a circle compactification of a 5d gauge theory whose moduli space of vacua also carries this elliptic structure [18]. In particular, the corresponding Seiberg-Witten curve can be expressed in terms of a polynomial in a variable t whose coefficients are Jacobi forms v_l of an elliptic parameter z:

$$H(w,z) = t^N + v_1(z)t^{N-1} + \ldots + v_l(z)t^{N-l} + \ldots v_N(z) = 0, \quad t = e^{2\pi i w}. \tag{1.1}$$

A central question is about the interpretation of this curve as a *quantum curve*. To this end, the variables w and z are promoted to operators satisfying a non-trivial commutation relation

$$[\hat{w}, \hat{z}] \sim \hbar. \tag{1.2}$$

Interpreting $\hat{z}$ as a position operator, by the above commutation relation $\hat{w}$ becomes a momentum operator and $Y \equiv e^{-\hat{w}}$ will be a shift operator. In this framework the algebraic curve equation (1.1) becomes a difference equation in the sense that the operator $\widehat{H}(\hat{w}, \hat{z})$ acts on a wave-function with zero eigenvalue. This notion of a quantum curve is intimately related to partition functions arising from surface defects in gauge theories [19]. In this interpretation the wave-function annihilated by the operator $\widehat{H}(\hat{w}, \hat{z})$ is the expectation value of a codimension 2 defect operator. In the context of our 6d SCFT such defect operators arise from half BPS operators extended over $\mathbb{T}^2 \times \mathbb{R}^2$ and localised at a point on the remaining $\mathbb{R}^2$. Localisation is done by turning on the Omega-background $\mathbb{R}^4_{\epsilon_1, \epsilon_2} \times \mathbb{T}^2$ [15, 20] and $\hbar$ is identified with ϵ_1, while ϵ_2 is sent to zero in the Nekrasov-Shatashvili limit [21]. The theory living on the defect flows in the IR to a 4d $\mathcal{N} = 1$ SCFT and in some instances the defect partition function in the NS-limit can be understood as the superconformal index of this SCFT on $S^1 \times S^3$ [22]. In this correspondence, the S^3 is understood as a Hopf-fibration of a circle over a two-sphere such that the two circles are identified with $\mathbb{T}^2$ and the two-sphere is identified with a compactification of $\mathbb{R}^2$.

From a more geometric point of view, 6d SCFTs can be engineered by compactifying F-theory on an elliptic Calabi-Yau manifold. Performing F-theory/M-theory duality, one observes that the BPS partition function of the theory on $\mathbb{T}^2 \times \mathbb{R}^4_{\epsilon_1, \epsilon_2}$ corresponds to the refined topological string partition function of the Calabi-Yau manifold [15, 17]. In this picture, the surface defect arises from an M5 brane wrapping a Lagrangian cycle inside the Calabi-Yau threefold and extended over $S^1 \times \mathbb{R}^2$ transverse to the Calabi-Yau. The theory living on such a defect is expected to flow to a 3d SCFT with four supercharges coupled to the parent 5d gauge theory. Using the 3d/3d correspondence of [23], the partition function of the 3d SCFT is equivalent to the partition function of $\mathrm{SL}(2, \mathbb{C})$ Chern-Simons theory on a three-manifold which is a knot complement. As is well-known, the moduli space of flat $\mathrm{SL}(2, \mathbb{C})$ connections on the knot complement is characterised by the so-called A-polynomial $A(z, w)$ where z and w characterise holonomies around the two cycles of the boundary torus. The equation $A(z, w) = 0$ then describes the subspace of those holonomies which can be extended to the entire three-manifold. The partition function of

SL$(2, \mathbb{C})$ Chern-Simons theory on the knot complement satisfies a difference equation which arises from the quantisation of the A-polynomial [24–26]. By the 3d/3d correspondence, the partition function of the 3d SCFT then satisfies the same difference equation. In the case of our 3d defect, the 3d SCFT is coupled to a 5d gauge theory and the A-polynomial receives a Q-deformation [27–30] where by Q we collectively denote the moduli of the 5d theory. The quantised Q-deformed A-polynomial can then be identified with our difference operator $H(\hat{w}, \hat{z})$. As our 5d theory arises from a 6d SCFT, we find that the difference operator is elliptic with elliptic modulus $Q = e^{2\pi i \tau}$.

The concrete example, on which we focus in this paper, is the 6d SCFT arising from 2 M5 branes probing a $\mathbb{Z}_k$ singularity. In this case, compactification on a two-torus leads to the following Seiberg-Witten curve [18]

$$t + q_\phi \prod_{l=1}^{2k} \vartheta_1(z - \mu_l)\, t^{-1} - (1 + q_\phi) \prod_{l=1}^{k} \vartheta_1(z - z_l) = 0, \tag{1.3}$$

where $q_\phi \equiv e^{2\pi i \phi}$ with ϕ being the tensor branch parameter of the 6d theory, the μ_l denote collectively the flavour chemical potentials, and z_l are complicated functions of gauge chemical potentials. For the definition of the theta functions ϑ_1 we refer to appendix A.2.1. A central result of the present paper is that the defect partition function Ψ of the torus-compactified (or equivalently the circle-compactified 5d affine quiver gauge theory) satisfies the following difference equation corresponding to the quantisation of the above algebraic curve

$$\left[Y^{-1} + q_\phi \prod_{l=1}^{2k} \vartheta_1(z - \mu_l) \cdot Y - \langle \mathcal{W} \rangle \right] \Psi = 0, \tag{1.4}$$

where we have identified

$$\langle \mathcal{W} \rangle \equiv (1 + q_\phi) \prod_{l=1}^{k} \vartheta_1(z + \epsilon_1 - z_l), \tag{1.5}$$

with $\langle \mathcal{W} \rangle$ the Wilson surface expectation value of a codimension 4 operator wrapping the torus to be further specified in the main text.

The remainder of this paper is organised as follows: after reviewing the 6d $\mathcal{N} = (1, 0)$ theory and its partition function, section 2 details the inclusion of codimension 2 and 4 defects. For both cases, the partition functions are derived and evaluated up to order q_ϕ^2. Thereafter, the difference equation is derived in section 3. In detail, starting from a path integral representation for the partition function of the codimension 2 defect, the corresponding saddle point equation naturally leads to a difference equation. Crucially, one contribution of the difference equation is identified with the partition function of the codimension 4 defect. The 6d theories originating from 2 M5 branes on a $\mathbb{C}^2/\mathbb{Z}_k$ family have 8 supercharges for $k > 1$, but 16 supercharged for $k = 1$. The analysis of this enhanced $\mathcal{N} = (2, 0)$ case is presented in section 4, and compared to the dual 5d $\mathcal{N} = 2$ theory. Finally, section 5 provides a conclusion and outlook. Appendix A contains definitions and conventions used in the evaluation of the various partition functions as well as computational results. As a remark, most computational details are delayed to appendix A in order to ease the readability of the main text.

2 Defects for M5 branes on A-type singularity

The 6d $\mathcal{N} = (1,0)$ SCFTs originating from N M5 branes on a A-type singularity $\mathbb{C}^2/\mathbb{Z}_k$ are naturally labeled by two integers (N,k). For $k = 1$, the 6d world-volume theories have enhanced supersymmetry and are known as the A_{N-1} $\mathcal{N} = (2,0)$ theories [31, 32], whose 4d descendants are the A_{N-1} $\mathcal{N} = 2$ theories of class $\mathcal{S}$ [33]. For $k > 1$, the resulting $\mathcal{N} = (1,0)$ world-volume theories are well-studied [34–37] and their 4d descendants are the $\mathcal{N} = 1$ theories of class $\mathcal{S}_k$ [8]. In this section, the set-up is reviewed and, thereafter, defects of codimension 2 and 4 are introduced.

2.1 2 M5 branes on A-type singularity

In this work, the focus is placed on 6d $\mathcal{N} = (1,0)$ SCFTs for $N = 2$. The M-theory set-up admits a dual realisation in Type IIA superstring theory. The 2 M5 branes become NS5 branes filling the space-time directions $x^0, x^1, \ldots, x^5$ and being points in the transverse directions. The A-type ALE space $\mathbb{C}^2/\mathbb{Z}_k$ dualises into a stack of k D6 branes filling space-time directions $x^0, x^1, \ldots, x^6$, which are transverse to the original singularity. The set-up is summarised in table 1. The 6d $\mathcal{N} = (1,0)$ low-energy effective theory living on the world-volume of the D6 branes is composed of hypermultiplets and vector multiplets encoded in the following quiver diagram for 8 supercharges:

$$
\begin{array}{ccc}
\begin{array}{c}
\mathrm{U}(1)_b \\
\mathrm{SU}(k)_m\ \square\!\!\!-\!\!\!\overset{-b}{}\!\!\!\underset{}{\bigcirc}\!\!\!\overset{b}{}\!\!\!-\!\!\!\square\ \mathrm{SU}(k)_n \\
\mathrm{SU}(k)_a
\end{array}
&
\cong
&
\begin{array}{c}
\mathrm{SU}(2k)_y \\
\square \\
| \\
\bigcirc \\
\mathrm{SU}(k)_a
\end{array}
\end{array}
\tag{2.1}
$$

and one tensor multiplet. The global symmetry $\mathrm{SU}(2k)_y$ can be decomposed into $\mathrm{SU}(k)_{m,n}$, from the two stacks of semi-infinite D6 branes for $x^6 \to \pm\infty$, and $\mathrm{U}(1)_b$, which is the $\mathbb{C}^2/\mathbb{Z}_k$ isometry.

As a remark, the general family, i.e. N M5 branes on a $\mathbb{C}^2/\mathbb{Z}_k$ or N NS5s intersected by k D6 branes in Type IIA, leads to a 6d $\mathcal{N} = (1,0)$ quiver gauge theory on the tensor branch with global symmetry $\mathrm{SU}(k)_m \times \mathrm{U}(1) \times \mathrm{SU}(k)_n$. For $N = 2$ there exists an accidental enhancement $\mathrm{SU}(k)_m \times \mathrm{U}(1) \times \mathrm{SU}(k)_n \subset \mathrm{SU}(2k)$ as indicated in (2.1). For $N = k = 2$, the $\mathrm{SU}(4)$ global symmetry is further enhanced to $\mathrm{SO}(7)$ at the fixed point and to $\mathrm{SO}(8)$ on the tensor branch [38–40].

Partition function. In order to evaluate the partition function, the 6d theory is placed on $\mathbb{T}^2 \times \mathbb{R}^4_{\epsilon_1, \epsilon_2}$, where the 2-torus is along $x^{0,1}$ and the 4d Omega background $\mathbb{R}^4_{\epsilon_1, \epsilon_2}$ fills directions $x^2, \ldots, x^4$, see table 1. The two parameters ϵ_1 and ϵ_2 denote rotations in the $x^{2,3}$ and $x^{4,5}$ planes, respectively. The full partition function is composed of two contributions

$$
Z_{6d} = Z_{pert} \cdot Z_{str}
\tag{2.2}
$$

denoting the perturbative contributions Z_{pert} and the non-perturbative contributions Z_{str}. The perturbative part is fully determined by the 6d supermultiplets in (2.1) plus a single

M-theory

M-theory	$\mathbb{T}^2$		$\mathbb{R}^4_{\epsilon_1,\epsilon_2}$					TN_k			
	0	1	2	3	4	5	6	7	8	9	10
2 M5	•	•	•	•	•	•					
l M2	•	•					•				
1 $\widetilde{M5}$	•	•	•	•				•			•
1 M5′	•	•						•	•	•	•

reduction $S^1 \subset \mathbb{R}_1$ — reduction $S^1 \subset \mathbb{R}_{10}$

IIA (reduction on x^1)

IIA	S^1	$\mathbb{R}^4_{\epsilon_1,\epsilon_2}$					TN_k			
	0	2	3	4	5	6	7	8	9	10
2 D4	•	•	•	•	•					
l F1	•					•				
1 $\widetilde{D4}$	•	•	•				•			•
1 D4′	•						•	•	•	•

IIA (reduction on x^{10})

IIA	$\mathbb{T}^2$		$\mathbb{R}^4_{\epsilon_1,\epsilon_2}$					7	8	9
	0	1	2	3	4	5	6			
2 NS5	•	•	•	•	•	•				
l D2	•	•					•			
1 $\widetilde{D4}$	•	•	•	•				•		
1 D4′	•	•						•	•	•
k D6	•	•	•	•	•	•	•			

T-dual on $\mathbb{R}_{10}$ — T-dual on $\mathbb{R}_1$

IIB

IIB	S^1	$\mathbb{R}^4_{\epsilon_1,\epsilon_2}$					S^1			
	0	2	3	4	5	6	7	8	9	10
2 D5	•	•	•	•	•					•
l F1	•					•				
1 $\widetilde{D3}$	•	•	•				•			
1 D3′	•						•	•	•	
k NS5	•	•	•	•	•	•				

IIB

IIB	S^1	S^1	$\mathbb{R}^4_{\epsilon_1,\epsilon_2}$					7	8	9
	0	1	2	3	4	5	6			
2 NS5	•	•	•	•	•	•				
l D1	•						•			
1 $\widetilde{D3}$	•		•	•				•		
1 D3′	•							•	•	•
k D5	•		•	•	•	•	•			

S-dual and $x^{10} \leftrightarrow x^1$

Table 1. Brane set-up for codimension 2 and codimension 4 defect. The theory without defect is realised via 2 M5 branes transverse to a Taub-NUT space TN_k, which is a resolution of the $\mathbb{C}^2/\mathbb{Z}_k$ singularity. The isometries of TN_k can be identified with the admissible $\epsilon_{1,2}$ twists defining the Omega background and the $U(1)_b$ symmetry. The codimension 2 defect is realised via an additional $\widetilde{M5}$ brane, while the codimension 4 defect corresponds to an M5′ brane. Assuming the M-theory circle is either along x^1 or x^{10} direction, one arrives at two different Type IIA realisation. The reduction on x^1 leads to branes on a non-trivial TN_k background, while the reduction on x^{10} results in an intersecting brane configuration on a flat background. Further employing T-duality on either x^{10} or x^1, respectively, leads to two 5-brane web configurations, which are S-dual to one another.

tensor multiplet. In contrast, the non-perturbative parts originate from the 2d $\mathcal{N} = (0,4)$ world-volume theories of D2 branes filling x^0, x^1, x^6 directions, see table 1. The instanton string partition function can be written as sum of elliptic genera of the 2d theories:

$$Z_{\text{str}} = 1 + \sum_{l=1}^{\infty} e^{2\pi i \cdot l \phi} Z_l \equiv \sum_{l=0}^{\infty} q_\phi^l Z_l \quad \text{with} \quad q_\phi = e^{2\pi i \cdot \phi}, \tag{2.3}$$

where ϕ is the vacuum expectation value of the scalar field in the tensor multiplet. The BPS partition functions have been computed for $k = 1$ in [15, 16] and for $k > 1$ in [17]. In this work, the partition function of the 6d $\mathcal{N} = (1,0)$ without defect is required for the computation of the normalised partition function in the presence of defects, see appendix A.3.1. For completeness and concreteness, the details of Z_{pert} and Z_{str} are discussed in turn in the following subsections.

2.1.1 Perturbative contribution

Following [41], the perturbative single-letter contribution of the 6d supermultiplets are given as follows:

$$I_{\text{tensor}} = -\frac{p+q}{(1-p)(1-q)} \tag{2.4a}$$

$$I_{\text{vector}} = -\frac{(1+p\cdot q)}{(1-p)(1-q)} \left(\sum_{i,j=1}^{k} e^{a_i - a_j} - 1 \right)$$

$$= -\frac{(1+p\cdot q)}{(1-p)(1-q)} \left((k-1) + \sum_{1 \le j < i \le k} \left(e^{a_i - a_j} + e^{a_j - a_i} \right) \right) \tag{2.4b}$$

$$I_{\text{hyper}} = \frac{\sqrt{p \cdot q}}{(1-p)(1-q)} \sum_{i=1}^{k} \left\{ \sum_{l=1}^{k} \left(e^{a_i - m_l + b} + e^{m_l - b - a_i} \right) + \sum_{l=1}^{k} \left(e^{a_i - n_l - b} + e^{n_l + b - a_i} \right) \right\} \tag{2.4c}$$

where the $\{a_i\}$ gauge as well as the $\{m_l\}$, $\{n_l\}$, and b flavour charges of the hypermultiplets are derived from (2.1). The $\text{SU}(k)_a$ gauge as well as the $\text{SU}(k)_{n,m}$ flavour fugacities need to satisfy

$$\prod_{i=1}^{k} e^{a_i} = \prod_{l=1}^{k} e^{m_l} = \prod_{l'=1}^{k} e^{n_{l'}} = 1 \quad \Leftrightarrow \quad \sum_{i=1}^{k} a_i = \sum_{l=1}^{k} m_l = \sum_{l'=1}^{k} n_{l'} = 0. \tag{2.5}$$

Moreover, $p = e^{2\pi i \epsilon_1}$, $q = e^{2\pi i \epsilon_2}$ denote the Cartan generators of the rotation symmetries of the Omega background $\mathbb{R}^4_{\epsilon_1,\epsilon_2}$. The total perturbative contribution becomes

$$Z_{\text{pert}} = \text{PE} \left[(I_{\text{tensor}} + I_{\text{vector}} + I_{\text{hyper}}) \cdot \left(\frac{Q}{1-Q} + \frac{1}{2} \right) \right] \tag{2.6}$$

which includes the contributions of the KK-modes generated by $\sum_{n=1}^{\infty} Q^n = \frac{Q}{1-Q}$, with $Q = e^{2\pi i \tau}$.

2.1.2 Elliptic genus

To compute the l-th instanton string partition function Z_l, one can add l D2 branes along the x^0, x^1, x^2 directions, see table 1. The D2 world-volume theory is a 2d $\mathcal{N} = (0,4)$ effective theory, whose elliptic genera encode the Z_l partition functions.

Considering the NS5-D6-D2 brane system in table 1, the space-time symmetry is broken to

$$\mathrm{SO}(1,9) \to \mathrm{SO}(1,1) \times \mathrm{SO}(4)_{2345} \times \mathrm{SO}(3)_{789}\,,$$
$$\text{with} \quad \mathrm{SO}(4)_{2345} \cong \mathrm{SU}(2)_l \times \mathrm{SU}(2)_r \quad \text{and} \quad \mathrm{SO}(3)_{789} \cong \mathrm{SU}(2)_I\,. \tag{2.7}$$

The 16 supersymmetries can be decomposed in representations of $(\mathrm{SU}(2)_l, \mathrm{SU}(2)_r, \mathrm{SU}(2)_I)_{\pm\pm}$, where the two "$\pm$" label the chirality of world-sheet x^0, x^1 and space along x^6. The supersymmetries preserved by the NS5-D6-D2 brane system transform as $(\mathbf{1}, \mathbf{2}, \mathbf{2})_{-+}$, such that the D2 world-volume theory is a 2d $\mathcal{N} = (0,4)$ quiver theory, see for instance [42]. The brane configuration allows one to read off the field content and the charges of the supermultiplets with respect to $\mathrm{SU}(2)_l \times \mathrm{SU}(2)_r \times \mathrm{SU}(2)_I$, labelled as $(\alpha, \dot{\alpha}, A)$. One finds:

- The D2-D2 open strings give rise to the $\mathcal{N} = (0,4)$ vector $(A_\mu, \lambda^{\dot\alpha A})$ and a hyper-multiplet $(\phi^{\alpha\dot\beta}, \chi^{\alpha A})$ in the adjoint representation of $\mathrm{U}(l)$ group.
- The D2-D6 open strings, which do not cross a NS5, provide a $\mathcal{N} = (0,4)$ hypermultiplet $(q^{\dot\alpha}, \psi^A)$ in the bi-fundamental representation of $\mathrm{U}(l) \times \mathrm{SU}(k)$.
- The D2-D6 open strings, which cross a NS5 brane, provide two additional $\mathcal{N} = (0,4)$ Fermi multiplets Ψ and Ψ' in the bi-fundamental representation of $\mathrm{U}(l) \times \mathrm{SU}(k)$.

All these $\mathcal{N} = (0,4)$ multiplets can be decomposed into $\mathcal{N} = (0,2)$ multiplets as follows:

$$\text{vector } (A_\mu, \lambda^{\dot\alpha A}) \longrightarrow \text{vector } V\ (A_\mu, \lambda^{\dot{1}1}, \lambda^{\dot{2}2})\ +\ \text{Fermi } \Lambda\ (\lambda^{\dot{1}2})\,, \tag{2.8a}$$

$$\text{hyper } (\varphi^{\alpha\dot\beta}, \chi^{\alpha A}) \longrightarrow \text{chiral } B\ (\varphi^{1\dot{1}}, \chi^{12})\ +\ \text{chiral } \tilde{B}^\dagger\ (\varphi^{1\dot{2}}, \chi^{11})\,, \tag{2.8b}$$

$$\text{hyper } (q^{\dot\alpha}, \psi^A) \longrightarrow \text{chiral } q\ (q^{\dot{1}}, \psi^2)\ +\ \text{chiral } \tilde{q}^\dagger\ (q^{\dot{2}}, \psi^1)\,, \tag{2.8c}$$

$$\text{Fermi } \Psi,\ \Psi' \longrightarrow \text{Fermi } \Psi,\ \Psi'\,. \tag{2.8d}$$

From the decomposition, one can read off the charges of these $\mathcal{N} = (0,2)$ multiplets as summarised in table 2. This 2d quiver gauge theory is known from [42] and reduces to the $\mathcal{N} = (0,4)$ gauge theory description for M-strings introduced in [17] for case $k = 1$. For completeness, the 2d quiver gauge theory with multiplets (2.8) can be written as

$$\tag{2.9}$$

$\mathcal{N}=(0,2)$ multiplets			J_l	J_r	J_I	U(l)	U(k)$_a$	U(k)$_m$	U(k)$_n$	U(1)$_b$	U(1)$_x$	U(1)$_z$
D2-D2	vector	V	0	0	0	adj.	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{1}$	0	0	0
	Fermi	Λ	0	$\frac{1}{2}$	$-\frac{1}{2}$	adj.	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{1}$	0	0	0
	chiral	B	$\frac{1}{2}$	$\frac{1}{2}$	0	adj.	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{1}$	0	0	0
	chiral	$\tilde{B}$	$-\frac{1}{2}$	$\frac{1}{2}$	0	adj.	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{1}$	0	0	0
D2-D6	chiral	q	0	$\frac{1}{2}$	0	$\mathbf{l}$	$\overline{\mathbf{k}}$	$\mathbf{1}$	$\mathbf{1}$	0	0	0
	chiral	$\tilde{q}$	0	$\frac{1}{2}$	0	$\bar{\mathbf{l}}$	$\mathbf{k}$	$\mathbf{1}$	$\mathbf{1}$	0	0	0
	Fermi	Ψ	0	0	0	$\mathbf{l}$	$\mathbf{1}$	$\overline{\mathbf{k}}$	$\mathbf{1}$	1	0	0
	Fermi	Ψ'	0	0	0	$\bar{\mathbf{l}}$	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{k}$	1	0	0
D2-$\widetilde{\text{D4}}$	chiral	σ	0	0	0	$\mathbf{l}$	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{1}$	0	-1	0
	Fermi	Ξ	$-\frac{1}{2}$	$\frac{1}{2}$	0	$\mathbf{l}$	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{1}$	0	-1	0
D2-D4$'$	chiral	ϕ	0	0	$\frac{1}{2}$	$\mathbf{l}$	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{1}$	0	0	-1
	chiral	$\tilde{\phi}$	0	0	$\frac{1}{2}$	$\bar{\mathbf{l}}$	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{1}$	0	0	1
	Fermi	Γ_1	$\frac{1}{2}$	0	0	$\mathbf{l}$	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{1}$	0	0	-1
	Fermi	$\Gamma_2^{\dagger}$	$\frac{1}{2}$	0	0	$\bar{\mathbf{l}}$	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{1}$	0	0	1
D6-D4$'$	Fermi	ρ	0	0	0	$\mathbf{1}$	$\overline{\mathbf{k}}$	$\mathbf{1}$	$\mathbf{1}$	0	0	1

Table 2. Charge assignments of the fields in the 2d world-volume theory from the D2-D6-NS5 system with or without the presence of a $\widetilde{\text{D4}}$ or D4$'$ defect, see table 1. Here J_l, J_r, and J_I denote the Cartans of SU(2)$_l$, SU(2)$_r$, and SU(2)$_I$ respectively. U(l) is the 2d gauge group on the D2 world-volume. The U(k)$_{a,m,n}$ denote the 6d gauge and flavour symmetries, whose fugacities need to be subjected to the constraint (2.5) in order to reduce to SU(k)$_{a,m,n}$. U(1)$_b$ is part of the 6d global symmetry. The U(1)$_{x,z}$ denote the defect groups for the $\widetilde{\text{D4}}$ and D4$'$ defects, respectively.

with the conventions: circles $\circ$ denote $\mathcal{N}=(0,4)$ or $\mathcal{N}=(0,2)$ vector multiplets, and squares $\square$ are flavour nodes. In addition, for lines without/with arrows: solid lines denote hypermultiplets / chiral multiplets, and dashes lines denote Fermi multiplets, respectively. The arrow in $\mathcal{N}=(0,2)$ bifundamental matter fields points towards that node under which the field transforms in the fundamental representation.

For a fixed number l of D2 branes, the partition function of the 2d $\mathcal{N}=(0,4)$ theory placed on a torus $\mathbb{T}^2$, with complex structure τ, is known to coincide with the elliptic genus [43, 44]. The non-perturbative contributions are then encoded in the elliptic genera for all $l \geq 1$. The elliptic genus Z_l for the 2d theory with gauge group U(l) on torus $\mathbb{T}^2$ is computed by picking up $\mathcal{N}=(0,2)$ supercharges $\mathcal{Q} \equiv Q_-^{1\dot{1}}$ and $\mathcal{Q}^{\dagger} \equiv Q_-^{2\dot{2}}$, and evaluating

$$Z_l = \text{Tr}\left[(-1)^F Q^{H_L} \bar{Q}^{H_R} e^{2\pi \mathrm{i}\,\epsilon_-(2J_l)} e^{2\pi \mathrm{i}\,\epsilon_+ 2(J_r - J_I)} e^{2\pi \mathrm{i}\,bF} \prod_{j}^{k} e^{2\pi \mathrm{i}\,m_j F_j} e^{2\pi \mathrm{i}\,n_j F_j'} \prod_{i}^{k} e^{2\pi \mathrm{i}\,a_i G_i} \right]. \tag{2.10}$$

Here $Q = e^{2\pi \mathrm{i}\tau}$, and $\epsilon_{\pm} \equiv \frac{1}{2}(\epsilon_1 \pm \epsilon_2)$, such that $2\epsilon_- J_l + 2\epsilon_+ J_r = \epsilon_1 J_{23} + \epsilon_2 J_{45}$ with $J_{r,l} = \frac{1}{2}(J_{23} \pm J_{45})$ are the Cartan generators of SU(2)$_l \times$ SU(2)$_r \simeq$ SO(4)$_{2345}$.

Based on a path integral representation for the elliptic genus, a generic prescription for the elliptic genera via supersymmetric localisation has been derived [43, 44]. To briefly

summarise, the first step involves identifying compact zero modes $\{u_p\}$ originating from flat connections on $\mathbb{T}^2$. Keeping the zero modes fixed, the next step requires an integration over massive fluctuations, which results in a 1-loop determinant for each multiplet. According to [43, 44], the contributions of the different multiplets in (2.8) can be summarised as follows:

$$Z_{\text{vec}} = \left(\frac{2\pi\eta^2}{i}\right)^l \prod_{\alpha \in \text{root}} \frac{\theta_1\left(\alpha(u)\right)}{i\eta} = \left(\frac{2\pi\eta^2}{i}\right)^l \prod_{1 \le p < q \le k} \frac{\theta_1\left(\pm(u_p - u_q)\right)}{(i\eta)^2} \,, \tag{2.11a}$$

$$Z_{\text{chiral}} = \left(\prod_{p,q=1}^{l} \frac{(i\eta)^2}{\theta_1(\epsilon_{1,2} + u_p - u_q)}\right) \left(\prod_{p=1}^{l} \prod_{i=1}^{k} \frac{(i\eta)^2}{\theta_1(\epsilon_+ \pm (u_p - a_i))}\right) \,, \tag{2.11b}$$

$$Z_{\text{Fermi}} = \left(\prod_{p,q=1}^{l} \frac{\theta_1(2\epsilon_+ + u_{pq})}{i\eta}\right) \left(\prod_{p=1}^{l} \prod_{l=1}^{k} \frac{\theta_1(u_p - m_l + b)\,\theta_1(-u_p + n_l + b)}{(i\eta)^2}\right) \,, \tag{2.11c}$$

where the definitions of the Dedekind eta function η and the Theta function $\theta_1(z) \equiv \theta_1(\tau|z)$ are recalled in (A.11) and (A.14), respectively. As customary in the literature, the convention

$$\theta_1(\epsilon_+ \pm (u_p - a_i)) \equiv \theta_1(\epsilon_+ + (u_p - a_i)) \cdot \theta_1(\epsilon_+ - (u_p - a_i)) \tag{2.12}$$

etc. is used. Note that the $\text{SU}(k)_{m,n,a}$ fugacities need to satisfy (2.5). Collecting all the individual contributions leads to the expression

$$Z_{1-\text{loop}}(k, l) := Z_{\text{vec}} \cdot Z_{\text{chiral}} \cdot Z_{\text{Fermi}} \equiv \left(\frac{2\pi\,\eta^3\theta_1(2\epsilon_+)}{\theta_1(\epsilon_1)\,\theta_1(\epsilon_2)}\right)^l \prod_{\substack{p,q=1 \\ p \ne q}}^{l} D(u_p - u_q) \cdot \prod_{p=1}^{l} Q(u_p)\,, \tag{2.13}$$

where, inspired from [45, 46], the following conventions have been used:

$$D(u_p - u_q) := \frac{\theta_1(u_p - u_q)\theta_1(u_p - u_q + \epsilon_1 + \epsilon_2)}{\theta_1(u_p - u_q + \epsilon_1)\theta_1(u_p - u_q + \epsilon_2)}$$
$$= \frac{\vartheta_1(u_p - u_q)\vartheta_1(u_p - u_q + \epsilon_1 + \epsilon_2)}{\vartheta_1(u_p - u_q + \epsilon_1)\vartheta_1(u_p - u_q + \epsilon_2)} \,, \tag{2.14a}$$

$$Q(u) := \frac{\prod_{l=1}^{k} \theta_1(u - m_l + b)\theta_1(-u + n_l + b)}{\prod_{i=1}^{k} \theta_1(\epsilon_+ + (u - a_i))\theta_1(\epsilon_+ - (u - a_i))}$$
$$= \frac{\prod_{l=1}^{k} \vartheta_1(u - m_l + b)\vartheta_1(u - n_l - b)}{\prod_{i=1}^{k} \vartheta_1(u - a_i + \epsilon_+)\vartheta_1(u - a_i - \epsilon_+)} =: \frac{M(u)}{P_0(u)P_0(u + \epsilon_1 + \epsilon_2)} \,, \tag{2.14b}$$

with $\quad M(u) := \prod_{l=1}^{k} \vartheta_1(u - m_l + b)\vartheta_1(u - n_l - b) \,, \tag{2.14c}$

$$P_0(u) := \prod_{i=1}^{k} \vartheta_1(u - a_i - \epsilon_+) \quad \text{such that} \quad P_0(u + \epsilon_1 + \epsilon_2) = \prod_{i=1}^{k} \vartheta_1(u - a_i + \epsilon_+) \,. \tag{2.14d}$$

Note, in particular, the change to $\vartheta_1(\tau|z)$ defined in (A.13), which is more convenient than the Theta function $\theta_1(\tau|z)$. Lastly, one needs to integrate the several 1-loop determinants (2.13) over the zero modes $\{u_p\}$. As shown in [43, 44], this integral becomes a

contour integral. The contour integration needs to be performed with care, as the choice of integration contour determines whether the results yields the partition function or not. A consistent choice of contour is given by the Jeffrey-Kirwan residue prescription [47]. The expression becomes

$$Z_l = \frac{1}{l!} \oint \frac{\mathrm{d}^l u}{(2\pi\mathrm{i})^l} Z_{1-\mathrm{loop}}(k,l) = \frac{1}{l!} \sum_{u_\star} \mathrm{JK} - \mathrm{Res}_{u_\star} Z_{1-\mathrm{loop}}(k,l) \tag{2.15}$$

where the sum is taken over existing poles $u_\star$ in the integrand $Z_{1-\mathrm{loop}}$. For details on the computational aspects of the JK residue, the reader is referred to [43, 44]. The following conventions are useful for the residue calculus of the elliptic genera:

$$P_0^\vee(a_i \pm \epsilon_+) := \prod_{\substack{j=1 \\ j \neq i}}^k \vartheta_1(u - a_j - \epsilon_+)\Big|_{u=a_i\pm\epsilon_+} , \tag{2.16a}$$

$$Q^\vee(a_i - \epsilon_+) := \frac{M(a_i - \epsilon_+)}{P_0(a_i - \epsilon_+)P_0^\vee(a_i + \epsilon_+)} . \tag{2.16b}$$

For the Nekrasov-Shatashvili limit, the following abbreviations are used:

$$L(u) := \frac{\vartheta_1'(u)}{\vartheta_1(u)} , \qquad K(u) := \frac{\vartheta_1''(u)}{\vartheta_1(u)} , \tag{2.17}$$

where $\vartheta_1'(u) \equiv \frac{\partial}{\partial u}\vartheta_1(u)$ and $\vartheta_1''(u) \equiv \frac{\partial^2}{\partial u^2}\vartheta_1(u)$. For later purposes, the $l = 1, 2$ genera are computed.

1-string. The $l = 1$ elliptic genus reads

$$Z_1 = \frac{\vartheta_1(2\epsilon_+)}{\vartheta_1(\epsilon_1)\,\vartheta_1(\epsilon_2)} \sum_{i=1}^k Q^\vee(a_i - \epsilon_+) , \tag{2.18}$$

and the details are presented in appendix A.4.1.

2-string. The $l = 2$ elliptic genus reads

$$Z_2 = \left(\frac{\vartheta_1(2\epsilon_+)}{\vartheta_1(\epsilon_1)\vartheta_1(\epsilon_2)}\right)^2 \sum_{1 \leq i < j \leq k} D(a_i - a_j)D(a_j - a_i)Q^\vee(a_i - \epsilon_+)Q^\vee(a_j - \epsilon_+) \tag{2.19}$$

$$+ \frac{\vartheta_1(2\epsilon_+)}{\vartheta_1(2\epsilon_-)} \sum_{m=1}^k Q^\vee(a_m - \epsilon_+)\left[\frac{\vartheta_1(\epsilon_1 + 2\epsilon_+)}{\vartheta_1(\epsilon_2)\vartheta_1(2\epsilon_1)}Q(a_m - \epsilon_+ - \epsilon_1)\right.$$

$$\left. - \frac{\vartheta_1(\epsilon_2 + 2\epsilon_+)}{\vartheta_1(\epsilon_1)\vartheta_1(2\epsilon_2)}Q(a_m - \epsilon_+ - \epsilon_2)\right]$$

and the derivation is summarised in appendix A.4.2.

2.1.3 Enhancement of global symmetry

For the case of two NS5 branes, one needs to recover the global symmetry enhancement to $\mathrm{SU}(2k)$, as indicated in (2.1).

Perturbative part. The perturbative contribution of the 6d $\mathcal{N} = (1,0)$ hypermultiplets can be rewritten as

$$I_{\text{hyper}} = \frac{\sqrt{p \cdot q}}{(1-p)(1-q)} \sum_{i=1}^{k} \left\{ \sum_{l=1}^{k} \left(e^{a_i - m_l + b} + e^{a_i - n_l - b} \right) + \sum_{l=1}^{k} \left(e^{m_l - b - a_i} + e^{n_l + b - a_i} \right) \right\}$$

$$= \frac{\sqrt{p \cdot q}}{(1-p)(1-q)} \sum_{i=1}^{k} \sum_{l=1}^{2k} \left(e^{a_i - y_l} + e^{y_l - a_i} \right) \tag{2.20}$$

$$\text{with} \quad y_l = \begin{cases} m_l - b & , l = 1, \ldots, k \\ n_l + b & , l = k+1, \ldots, 2k \end{cases} \tag{2.21}$$

and one verifies that y_l are SU(2k) fugacities via

$$\prod_{l=1}^{2k} e^{y_l} = \prod_{l=1}^{k} e^{m_l} \cdot \prod_{l'=1}^{k} e^{n'_l} \cdot \prod_{l''=1}^{k} e^{b-b} = 1 \tag{2.22}$$

using (2.5).

Non-perturbative part. For the 2d elliptic genus (2.15), the Ψ, Ψ' Fermi multiplet contributions can also be rearranged

$$Z_{\text{Fermi}} \supset \prod_{p=1}^{l} \prod_{l=1}^{k} \frac{\theta_1(u_p - m_l + b)\, \theta_1(u_p - n_l - b)}{(\mathrm{i}\,\eta)^2}$$

$$= \prod_{p=1}^{l} \prod_{l=1}^{k} \frac{\theta_1(u_p - (m_l - b))\, \theta_1(u_p - (n_l + b))}{(\mathrm{i}\,\eta)^2}$$

$$= \prod_{p=1}^{l} \prod_{l=1}^{2k} \frac{\theta_1(u_p - y_l)}{\mathrm{i}\,\eta} \tag{2.23}$$

with y_l fugacities as defined in (2.21).

Enhancement for $k = 2$. The symmetry enhancement for the case $k = 2$, has been discussed via various techniques [38–40]. The global symmetry at the origin of the tensor branch is enhanced from SU(4) to SO(7), while on the tensor branch it is further enhanced to SO(8) [38, appendix A]. It has been shown in [39, section 9.2] that the elliptic genera can be expanded in SO(7) characters.

2.2 Higgs mechanism in partition functions

For later purposes, in which a codimension 2 defect is introduced via a position dependent vacuum expectation value (VEV), this section reviews the standard Higgs mechanism. To begin with, consider the Higgsing of the 6d gauge theory on the tensor branch:

$$\mathrm{SU}(k+1)\,, \ N_f = 2k+2 \quad \longrightarrow \quad \mathrm{SU}(k)\,, \ N_f = 2k\,. \tag{2.24}$$

The first task is to find a suitable VEV assignment for a gauge invariant operator and then derive a condition in terms of fugacities for the gauge invariant operator that realises the Higgs mechanism on the level of partition functions.

2.2.1 Standard Higgsing

Consider the field theoretical description of the mesonic Higgs branch deformation (2.24), but seen as $\mathrm{SU}(k+1)_a$ gauge theory with $\mathrm{SU}(k+1)_m \times \mathrm{U}(1)_b \times \mathrm{SU}(k+1)_n$ global symmetry. In other words, there are the flavour hypermultiplets $(Q, \widetilde{Q})$ of $\mathrm{SU}(k+1)_m \times \mathrm{U}(1)_b \times \mathrm{SU}(k+1)_a$ and $(Q', \widetilde{Q}')$ of $\mathrm{SU}(k+1)_n \times \mathrm{U}(1)_b \times \mathrm{SU}(k+1)_a$. Since each hypermultiplet in (2.1) has charge $(\frac{1}{2}, \frac{1}{2})$ under the Cartan generators $J_{r,l}$ of $\mathrm{SU}(2)_l \times \mathrm{SU}(2)_r \cong \mathrm{SO}(4)_{3456}$, the fugacity contributions for each chiral are

$$Q_l^i \in \overline{(\mathbf{k+1})}_a \otimes \left((\mathbf{k+1})_m \otimes \mathbf{1}_n\right)^{-1} \qquad \rightarrow \qquad \sqrt{pq}\, e^{-a_i + m_l - b}\,, \qquad (2.25a)$$

$$\widetilde{Q}_i^l \in (\mathbf{k+1})_a \otimes \left(\overline{(\mathbf{k+1})}_m \otimes \mathbf{1}_n\right)^{+1} \qquad \rightarrow \qquad \sqrt{pq}\, e^{a_i - m_l + b}\,, \qquad (2.25b)$$

$$Q_l^{\prime i} \in \overline{(\mathbf{k+1})}_a \otimes \left(\mathbf{1}_m \otimes (\mathbf{k+1})_n\right)^{+1} \qquad \rightarrow \qquad \sqrt{pq}\, e^{-a_i + n_l + b}\,, \qquad (2.25c)$$

$$\widetilde{Q}_i^{\prime l} \in (\mathbf{k+1})_a \otimes \left(\mathbf{1}_m \otimes \overline{(\mathbf{k+1})}_n\right)^{-1} \qquad \rightarrow \qquad \sqrt{pq}\, e^{a_i - n_l - b}\,, \qquad (2.25d)$$

with $i = 1, \ldots, k+1$ for $\mathrm{SU}(k+1)_a$ and $l, l' = 1, \ldots, k+1$ for $\mathrm{SU}(k+1)_{m,n}$ respectively. The exponent $(\ldots)^{\pm 1}$ denotes the $\mathrm{U}(1)_b$ charge. There are two possibilities for meson operators

$$\mathcal{M}_l^{l'} = \sum_{i=1}^{k+1} Q_l^i \widetilde{Q}_i^{\prime l'} \qquad \rightarrow \qquad \left(\sqrt{pq}\, e^{m_l - b}\right) \cdot \left(\sqrt{pq}\, e^{-n_{l'} - b}\right) = pq\, e^{m_l - n_{l'} - 2b}\,, \qquad (2.26a)$$

$$\widetilde{\mathcal{M}}_{l'}^l = \sum_{i=1}^{k+1} \widetilde{Q}_i^l Q_{l'}^{\prime i} \qquad \rightarrow \qquad \left(\sqrt{pq}\, e^{-m_l + b}\right) \cdot \left(\sqrt{pq}\, e^{n_{l'} + b}\right) = pq\, e^{-m_l + n_{l'} + 2b}\,, \qquad (2.26b)$$

and one can consider assigning a VEV to the $(k+1, k+1)$ meson components. A gauge transformation is sufficient to see that one only needs to assign VEVs to the following components

$$\mathcal{M}_{k+1}^{k+1} = \sum_{i=1}^{k+1} Q_{k+1}^i \widetilde{Q}_i^{\prime k+1} \cong Q_{k+1}^{k+1} \widetilde{Q}_{k+1}^{\prime k+1} \qquad \text{and} \qquad \widetilde{\mathcal{M}}_{k+1}^{k+1} \cong \widetilde{Q}_{k+1}^{k+1} Q_{k+1}^{\prime k+1}\,. \qquad (2.27)$$

Following the prescription of [48], see also [19, section 2], Higgsing is achieved in a partition function via choosing the pole corresponding to the operator acquiring a VEV, i.e.

$$\langle \mathcal{M}_{k+1}^{k+1}\rangle \neq 0 \quad \Leftrightarrow \quad pq\, e^{m_{k+1} - n_{k+1} - 2b} = 1 \quad \Leftrightarrow \quad \begin{cases} n_{k+1} &= m_{k+1} - 2b + 2\epsilon_+ \\ a_{k+1} &= m_{k+1} - b + \epsilon_+ \end{cases}, \qquad (2.28a)$$

$$\langle \widetilde{\mathcal{M}}_{k+1}^{k+1}\rangle \neq 0 \quad \Leftrightarrow \quad pq\, e^{-m_{k+1} + n_{k+1} + 2b} = 1 \quad \Leftrightarrow \quad \begin{cases} n_{k+1} &= m_{k+1} - 2b - 2\epsilon_+ \\ a_{k+1} &= m_{k+1} - b - \epsilon_+ \end{cases}, \qquad (2.28b)$$

and eliminating the contributions of the flat directions as well as any appearing Goldstone modes. Note that the condition for a_{k+1} in (2.28) is derived by requiring that the fugacity of the chiral Q_{k+1}^{k+1} or $\widetilde{Q}_{k+1}^{k+1}$ equals unity, respectively.

In the Type IIA brane configuration, the mesonic Higgsing is realised via aligning a semi-infinite flavour D6 brane on the left and right hand side with a gauge D6 such

that a single D6 is free to move along the Higgs branch directions $x^{7,8,9}$, see figure 1. The codimension 2 defect is introduced via a $\widetilde{D4}$ brane that connects the remaining brane configuration with the single D6 on the Higgs branch. Moving the D6 to infinity in figure 1, leads to the natural connection between *defect via Higgsing* and *defect via additional branes*, see also section 2.3.

Perturbative contribution. Consider the perturbative partition function for 6d SU($k+$ 1) theory with $N_f = 2k + 2$ flavours

$$Z_{\text{pert}}^{k+1} = \text{PE}\left[\frac{1}{(1-p)(1-q)}\left(\frac{Q}{1-Q}+\frac{1}{2}\right)\left\{-(p+q)-(1+pq)\left(\sum_{i,j=1}^{k+1}e^{a_i-a_j}-1\right)\right.\right.$$
$$\left.\left. + \sqrt{pq}\sum_{i=1}^{k+1}\sum_{l=1}^{k+1}\left(e^{a_i}(e^{-m_l+b}+e^{-n_l-b})+e^{-a_i}(e^{m_l-b}+e^{n_l+b})\right)\right\}\right]. \quad (2.29)$$

The Higgsing (2.28b) takes the form

$$e^{a_{k+1}} = \frac{1}{\sqrt{pq}}\cdot e^{m_{k+1}-b} \qquad e^{n_{k+1}} = \frac{1}{pq}\cdot e^{m_{k+1}-2b} = \frac{1}{\sqrt{pq}}\cdot e^{a_{k+1}-b}. \quad (2.30)$$

A straightforward computation, see appendix A.1.1, shows that the Higgsing (2.28b) leads to the expected result

$$Z_{\text{pert}}^{k+1} = Z_{\text{pert}}^k \cdot Z_{\text{G}} \quad (2.31)$$

where the Goldstone modes for the breaking of the global symmetry

$$\text{SU}(k+1)_m \times \text{U}(1) \times \text{SU}(k+1)_n \to \text{SU}(k)_m \times \text{U}(1) \times \text{SU}(k)_n \quad (2.32)$$

contribute as

$$Z_{\text{G}} = \text{PE}\left[\frac{\sqrt{pq}}{(1-p)(1-q)}\left(\frac{Q}{1-Q}+\frac{1}{2}\right)\right.$$
$$\times\left\{\left(\frac{1}{\sqrt{pq}}+\sqrt{pq}\right)+\sum_{l=1}^{k}\left(\frac{1}{\sqrt{pq}}e^{m_{k+1}-m_l}+\sqrt{pq}e^{m_l-m_{k+1}}\right)\right.$$
$$\left.\left. + \sum_{l=1}^{k}\left(\frac{1}{\sqrt{pq}}e^{m_{k+1}-n_l-2b}+\sqrt{pq}e^{n_l-m_{k+1}+2b}\right)\right\}\right], \quad (2.33)$$

such that there are $4k + 2$ massless chiral fields. Considering the Higgsing (2.24), one computes that the sub-space of the Higgs branch, where the theory is broken to SU(k), has complex dimension $4k + 2$, which matches the degrees of freedom in (2.33). Taking the closure of this sub-space, the $2k + 1$ quaternionic degrees of freedom parametrise the closure of the minimal nilpotent orbit of SU($2k + 2$), see [49].

Elliptic genus. Consider the elliptic genus (2.15) for the theory without defect. Suppose one aims to realise the Higgs mechanism (2.24) on the level of the elliptic genus, then

starting from $k+1$ one factorises (2.15) as follows:

$$
\begin{aligned}
Z_l^{k+1} &= \frac{1}{l!} \oint \frac{\mathrm{d}^l u}{(2\pi\mathrm{i})^l} \, Z_{1-\text{loop}}(k+1, l) \\
&= \frac{1}{l!} \oint \frac{\mathrm{d}^l u}{(2\pi\mathrm{i})^l} \left(\frac{2\pi\,\eta^3 \theta_1(2\epsilon_+)}{\theta_1(\epsilon_1)\,\theta_1(\epsilon_2)} \right)^l \cdot \prod_{\substack{p,q=1 \\ p\neq q}}^{l} D(u_p - u_q) \\
&\quad \cdot \prod_{p=1}^{l} \left(\frac{\prod_{l=1}^{k+1} \theta_1(u_p - m_l + b)\,\theta_1(u_p - n_l - b)}{\prod_{i=1}^{k+1} \theta_1(u_p - a_i + \epsilon_+)\,\theta_1(u_p - a_i - \epsilon_+)} \right) \\
&= \frac{1}{l!} \oint \frac{\mathrm{d}^l u}{(2\pi\mathrm{i})^l} Z_{1-\text{loop}}(k, l) \cdot \left(\prod_{p=1}^{l} \frac{\theta_1(u_p - m_{k+1} + b)\theta_1(u_p - n_{k+1} - b)}{\theta_1(u_p - a_{k+1} + \epsilon_+)\,\theta_1(u_p - a_{k+1} - \epsilon_+)} \right) . \quad (2.34)
\end{aligned}
$$

Since the Higgsing process should reduce $Z_l^{k+1} \to Z_l^k$, the last fraction is expected to be equal to one upon any of the fugacity assignments of (2.28). Explicitly, for (2.28b) one verifies that

$$
\prod_{p=1}^{l} \frac{\theta_1(u_p - m_{N+1} + b)\theta_1(u_p - n_{N+1} - b)}{\theta_1(u_p - a_{N+1} + \epsilon_+)\theta_1(u_p - a_{N+1} - \epsilon_+)} \bigg|_{(2.28b)}
$$

$$
= \prod_{p=1}^{l} \frac{\theta_1(u_p - m_{N+1} + b)\theta_1(u_p - m_{N+1} + 2b + 2\epsilon_+ - b)}{\theta_1(u_p - m_{N+1} + b + \epsilon_+ + \epsilon_+)\theta_1(u_p - m_{N+1} + b + \epsilon_+ - \epsilon_+)} = 1 \quad (2.35)
$$

holds. Therefore, the elliptic genus is compatible with the fugacity assignment (2.28) derived for the Higgs mechanism.

2.2.2 Higgsing to defects

Building on (2.28), a surface defect of type (r, s) can be introduced via a position dependent VEV [19, 48, 50] which is related to a pole at

$$
\langle \mathcal{M}_{k+1}^{k+1} \rangle = \text{fct.} \quad \Leftrightarrow \quad p^r q^s \cdot pq e^{m_{k+1} - n_{k+1} - 2b} = 1 \quad \Leftrightarrow \quad \begin{cases} n_{k+1} = m_{k+1} - 2b + 2\epsilon_+ + r\epsilon_1 + s\epsilon_2 \\ a_{k+1} = m_{k+1} - b + \epsilon_+ \end{cases},
$$
$$(2.36a)$$

$$
\langle \widetilde{\mathcal{M}}_{k+1}^{k+1} \rangle = \text{fct.} \quad \Leftrightarrow \quad p^r q^s \cdot pq e^{-m_{k+1} + n_{k+1} + 2b} = 1 \quad \Leftrightarrow \quad \begin{cases} n_{k+1} = m_{k+1} - 2b - 2\epsilon_+ - r\epsilon_1 - s\epsilon_2 \\ a_{k+1} = m_{k+1} - b - \epsilon_+ \end{cases},
$$
$$(2.36b)$$

such that the condition for the 6d gauge fugacity remains unchanged.

Without loss of generality, one can restrict to one choice of mesonic VEV. For this note, consider $\langle \widetilde{\mathcal{M}}_{k+1}^{k+1} \rangle$ such that (2.28b) and (2.36b) are relevant. If the defect is of type $(r, 0)$ then the codimension 2 defect occupies $\mathbb{R}^2_{\epsilon_2}$ while being a point on $\mathbb{R}^2_{\epsilon_1}$ inside the 4d Omega background; whereas an $(0, s)$ defect occupies $\mathbb{R}^2_{\epsilon_1}$ inside $\mathbb{R}^4_{\epsilon_1, \epsilon_2}$ and is point-like in $\mathbb{R}^2_{\epsilon_2}$.
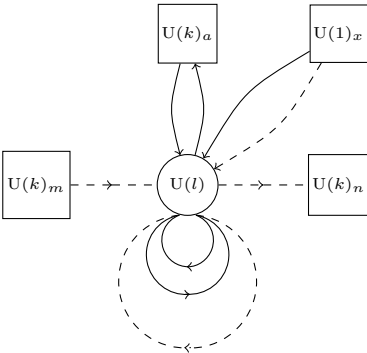
2.3 Codimension 2 defect

There are multiple ways to introduce a codimension 2 defect. For instance, one may either employ a position dependent vacuum expectation value (2.36) as in [19, 48, 50] or one may include an additional $\widetilde{D4}$ brane in the Type IIA brane configuration as in table 1, see also [51, 52] for surface defects in 4d theories. In the original M-theory setting of table 1, the defect introduced via the $\widetilde{D4}$ brane corresponds to another $\widetilde{M5}$ brane filling $(x^0, x^1, x^2, x^3, x^7, x^{10})$, as studied in [53]. Further, codimension 2 defects in 6d $\mathcal{N} = (1,0)$ SU(N) theories with adjoint matter are studied in [54].

2.3.1 Defect via D4 brane

One way to add a codimension 2 defect into the 6d theory is given by including additional D-branes. In the Type IIA brane configuration, this can be realised by introducing an additional $\widetilde{D4}$ brane with world-volume $(x^0, x^1, x^2, x^3, x^7)$ ending on a NS5 brane, see table 1. This $\widetilde{D4}$ is, indeed, of codimension 2 for the 6d world-volume theory on the D6 branes. One notes that this space-time occupancy of the branes breaks supersymmetry further to 4 supercharges. Moreover, the $\widetilde{D4}$ brane breaks the space-time symmetry (2.7) to

$$
\begin{aligned}
\mathrm{SO}(1,9) &\to \mathrm{SO}(1,1) \times \mathrm{SO}(4)_{2345} \times \mathrm{SO}(3)_{789} \\
&\to \mathrm{SO}(1,1) \times \mathrm{SO}(2)_{23} \times \mathrm{SO}(2)_{45} \times \mathrm{SO}(3)_{789} \, .
\end{aligned}
\tag{2.37}
$$

The world-volume theory on the D2 branes, which now only has $\mathcal{N} = (0,2)$ supersymmetry, is read off from the open string modes as above. The open strings between the D2-D6-NS5 branes induce the multiplets (2.8) from the original set-up. In addition, the D2-$\widetilde{D4}$ open strings give rise to a pair (σ, Ξ) of $\mathcal{N} = (0,2)$ bosonic and fermionic multiplets charged under gauge group U(l) of the world-sheet theory, such that the supersymmetry is broken from $\mathcal{N} = (0,4)$ to $\mathcal{N} = (0,2)$. The charges are summarised in table 2 and the resulting 2d quiver gauge theory is given by

$$\tag{2.38}$$

where the difference compare to (2.9) is given by the U(1)$_x$ defect flavour node of the additional bosonic and fermionic multiplets. Considering the elliptic genus, the 2d multiplets from the theory without defect contribute the 1-loop determinants (2.13), while
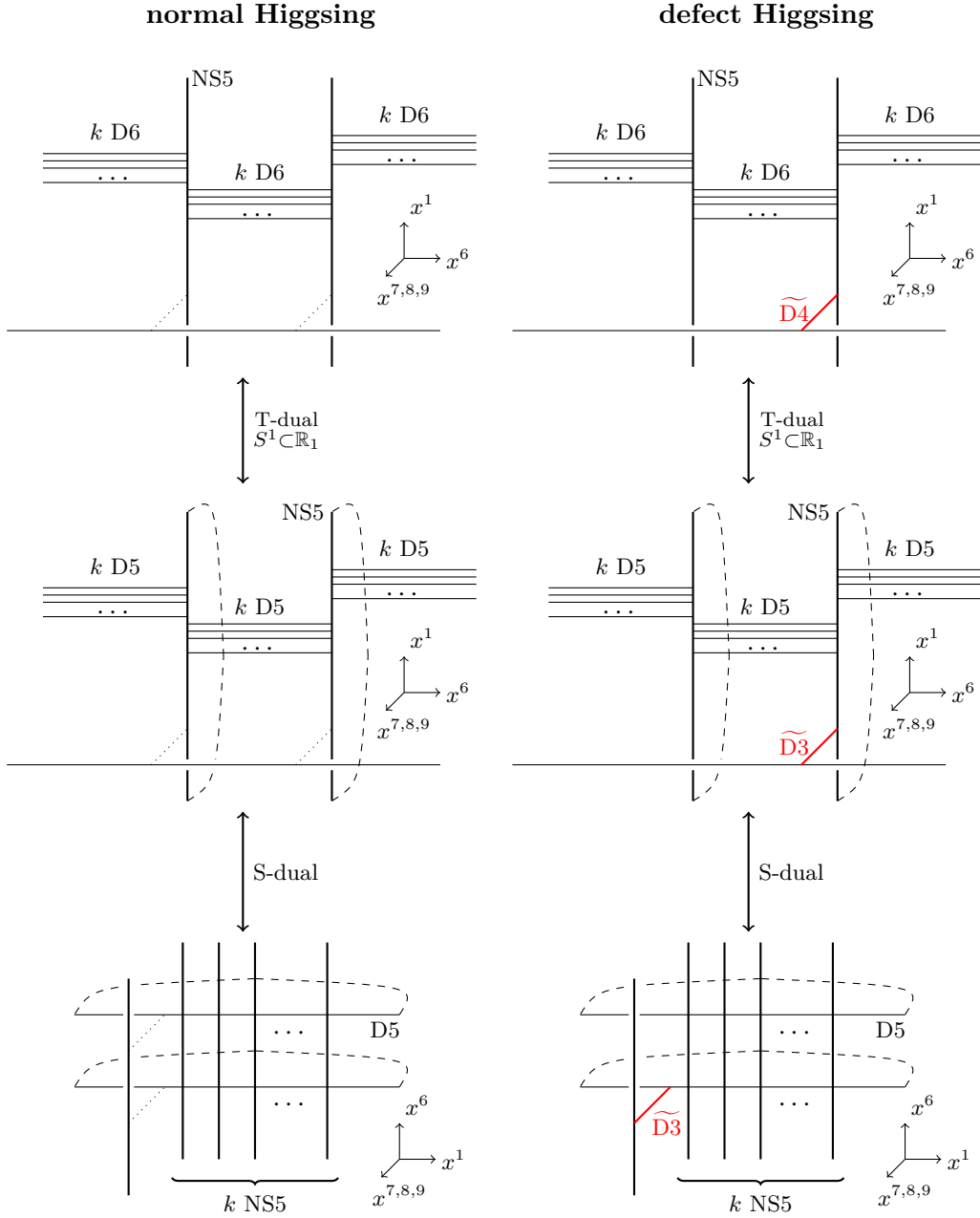
Figure 1. Higgsing in the brane configuration of table 1. The mesonic Higgsing of the 6d theory $SU(k+1)$ with $N_f = 2k+2$ to $SU(k)$ with $N_f = 2k$, is realised by moving one D6 away along the $x^{7,8,9}$ direction. For the dual 5d theory, the corresponding baryonic Higgsing of the affine A_k quiver to the affine A_{k-1} quiver is realised by moving one NS5 brane along $x^{7,8,9}$. A codimension 2 defect for the 6d brane configuration is introduced via a $\widetilde{D4}$ brane that is attached to the D6 brane which is moved along $x^{7,8,9}$. In the dual 5d system this becomes a $\widetilde{D3}$ brane suspended between the 5-brane web and the NS5 that is displaced in $x^{7,8,9}$ direction.

the additional multiplets σ and Ξ have determinants

$$Z_{\text{chiral}}^{\widetilde{\text{D4}}} = \prod_{p=1}^{l} \frac{\mathrm{i}\,\eta}{\theta_1(u_p - x)}\,,\tag{2.39a}$$

$$Z_{\text{Fermi}}^{\widetilde{\text{D4}}} = \prod_{p=1}^{l} \frac{\theta_1(u_p - x + \epsilon_2)}{\mathrm{i}\,\eta}\,,\tag{2.39b}$$

where x denotes the fugacity of the U(1) symmetry. The ϵ_2 charge of the new multiplets follows because the $\widetilde{\text{D4}}$ occupies $\mathbb{R}_{23}^2$. Collecting the determinants (2.11) and (2.39), one finds

$$Z_{1-\text{loop}}^{\widetilde{\text{D4}}}(k,l) := Z_{\text{vec}}Z_{\text{chiral}}Z_{\text{Fermi}} \cdot Z_{\text{chiral}}^{\widetilde{\text{D4}}}Z_{\text{Fermi}}^{\widetilde{\text{D4}}} \equiv Z_{1-\text{loop}}(k,l) \cdot Z_{\text{chiral}}^{\widetilde{\text{D4}}}Z_{\text{Fermi}}^{\widetilde{\text{D4}}}\,.\tag{2.40}$$

The claim is that the additional $\widetilde{\text{D4}}$ brane induces a $(r,s) = (0,1)$ defect in the sense of (2.36), see also [19, 48, 50]. As a remark, a $(r,s) = (1,0)$ defect can be constructed via a $\widetilde{\text{D4}}$ brane that extends along $(x^0, x^1, x^4, x^5, x^7)$, such that the 2d defect multiplets are charged under ϵ_1 instead.

Consequently, one may label the resulting 2d elliptic genera as follows:

$$Z_l^{(0,1)\text{def}} = \frac{1}{l!} \oint \frac{\mathrm{d}^l u}{(2\pi\mathrm{i})^l} Z_{1-\text{loop}}^{\widetilde{\text{D4}}}(k,l)\,.\tag{2.41}$$

As a next step, the result (2.39) is re-derived and generalised to a (r,s) defect via a position-dependent vacuum expectation values, as in section 2.2.

2.3.2 Defect via Higgsing: perturbative contribution

Here, the chosen approach is to modify the standard Higgsing (2.28) such that the VEV becomes dependent on one $\mathbb{R}^2$ plane of the $\mathbb{R}_{\epsilon_1,\epsilon_2}^4$ as in (2.36). For later purposes, one defines the defect fugacity x in (2.36b) as follows:

$$a_{k+1} = m_{k+1} - \epsilon_+ - b = x + \epsilon_+\,,$$
$$n_{k+1} = m_{k+1} - 2\epsilon_+ - 2b - r\epsilon_1 - s\epsilon_2 \equiv x - b - r\epsilon_1 - s\epsilon_2\,,\tag{2.42}$$
$$\text{with}\qquad x \equiv m_{k+1} - 2\epsilon_+ - b\,.$$

For the exponentiated fugacities, the Higgsing (2.36b) takes the form

$$e^{a_{k+1}} = \sqrt{pq}X\,,\quad e^{n_{k+1}} = \frac{X}{Bp^r q^s}\,,\quad e^{m_{k+1}} \equiv pqXB\,,\quad \text{with}\quad X \equiv e^x\,,\, B \equiv e^b\tag{2.43}$$

using the definition of the defect fugacity (2.42). As detailed in appendix A.1.2, this Higgsing results in

$$Z_{\text{pert}}^{k+1}\bigg|_{(2.36b)} = Z_{\text{pert}}^{k} \cdot Z_G \cdot Z_{\text{pert}}^{(r,s)\text{def}}\tag{2.44}$$

$$Z_{\text{pert}}^{(r,s)\text{def}} = \text{PE}\Bigg[\frac{(1-p^r q^s)}{(1-p)(1-q)}\left(\frac{Q}{1-Q} + \frac{1}{2}\right)$$
$$\times \left\{\frac{(1-p^{r+1}q^{s+1})}{p^r q^s} + \sqrt{pq}\sum_{i=1}^{k}\left(-\frac{e^{a_i}}{X} + \frac{1}{p^r q^s}\frac{X}{e^{a_i}}\right)\right\}\Bigg]\tag{2.45}$$

and (2.45) contains the additional contributions from the codimension 2 defect, i.e.

$$Z_{\text{pert}}^{k+(r,s)\text{def}} = Z_{\text{pert}}^k \cdot Z_{\text{pert}}^{(r,s)\text{def}}, \tag{2.46}$$

where the Goldstone mode contribution have been removed. Note that $Z_{\text{pert}}^{(r,s)\text{def}} = 1$ for $(r,s) = (0,0)$. To be specific, specialising (2.45) to $(r,s) = (0,s)$ yields

$$
\begin{aligned}
Z_{\text{pert}}^{(0,s)\text{def}} &= \text{PE}\left[\frac{(1-q^s)}{(1-p)(1-q)}\left(\frac{Q}{1-Q}+\frac{1}{2}\right)\left\{\frac{(1-pq^{s+1})}{q^s}+\sqrt{pq}\sum_{i=1}^k\left(\frac{1}{q^s}\frac{X}{e^{a_i}}-\frac{e^{a_i}}{X}\right)\right\}\right] \\
&= \text{PE}\left[\frac{\sum_{l=0}^{s-1}q^l}{(1-p)}\left(\frac{Q}{1-Q}+\frac{1}{2}\right)\left\{\frac{(1-pq^{s+1})}{q^s}+\sqrt{pq}\sum_{i=1}^k\left(\frac{1}{q^s}\frac{X}{e^{a_i}}-\frac{e^{a_i}}{X}\right)\right\}\right]. \tag{2.47}
\end{aligned}
$$

In the NS limit $q \to 1$, one obtains

$$
\begin{aligned}
\lim_{\epsilon_2 \to 0} Z_{\text{pert}}^{(0,s)\text{def}} &= \text{PE}\left[\frac{s}{(1-p)}\left(\frac{Q}{1-Q}+\frac{1}{2}\right)\left\{(1-p)+\sqrt{p}\sum_{i=1}^k\left(\frac{X}{e^{a_i}}-\frac{e^{a_i}}{X}\right)\right\}\right] \tag{2.48} \\
&= \left(\lim_{\epsilon_2 \to 0} Z_{\text{pert}}^{(0,1)\text{def}}\right)^s.
\end{aligned}
$$

Thus, the contribution of a $(0,s)$ defect factorises into s copies of a $(0,1)$ defect in the NS limit.

2.3.3 Defect via Higgsing: elliptic genus

The fugacity assignment for a Higgsing with a position dependent VEV has been derived in (2.36b). Inserting the fugacity assignment into (2.34) yields the following:

$$
\begin{aligned}
Z_l^{k+1}\Big|_{(2.36\text{b})} &= \frac{1}{l!}\oint \frac{d^l u}{(2\pi i)^l} Z_{1-\text{loop}}(k,l) \cdot \prod_{p=1}^l \frac{\theta_1(u_p - x - 2\epsilon_+)\theta_1(u_p - x + r\epsilon_1 + s\epsilon_2)}{\theta_1(u_p - x)\,\theta_1(u_p - x - 2\epsilon_+)} \\
&= \frac{1}{l!}\oint \frac{d^l u}{(2\pi i)^l} Z_{1-\text{loop}}(k,l) \cdot \prod_{p=1}^l \frac{\theta_1(u_p - x + r\epsilon_1 + s\epsilon_2)}{\theta_1(u_p - x)} \tag{2.49} \\
&\equiv \frac{1}{l!}\oint \frac{d^l u}{(2\pi i)^l} Z_{1-\text{loop}}(k,l) \cdot \prod_{p=1}^l V_{(r,s)}(u_p)
\end{aligned}
$$

with the definition

$$V_{(r,s)}(u) := \frac{\theta_1(u - x + r\epsilon_1 + s\epsilon_2)}{\theta_1(u - x)} = \frac{\vartheta_1(u - x + r\epsilon_1 + s\epsilon_2)}{\vartheta_1(u - x)}, \tag{2.50}$$

which corresponds to the contribution of an (r,s) defect. In other words, (2.50) are the 1-loop determinants of the Fermi and chiral multiplet that define the defect. In particular, for $(r,s) = (0,1)$ the defect contribution reduces to the results (2.39) of the defect introduced by the $\widetilde{\text{D4}}$ brane.

The resulting 1-loop determinant and elliptic genus are then defined as follows:

$$Z_{1-\text{loop}}^{(r,s)\text{def}}(k,l) \coloneqq Z_{1-\text{loop}}(k,l) \cdot \prod_{p=1}^{l} V_{(r,s)}(u_p) \,, \tag{2.51}$$

$$Z_l^{(r,s)\text{def}} = \frac{1}{l!} \oint \frac{\mathrm{d}^l u}{(2\pi \mathrm{i})^l} Z_{1-\text{loop}}^{(r,s)\text{def}}(k,l) \,, \tag{2.52}$$

employing the definitions (2.13) and (2.50).

1-string. Performing the integration for $l = 1$ yields:

$$Z_1^{(0,s)\text{def}} = \frac{\vartheta_1(2\epsilon_+)}{\vartheta_1(\epsilon_1)\,\vartheta_1(\epsilon_2)} \left[\sum_{i=1}^{k} \left(Q^\vee(a_i - \epsilon_+) \cdot V_{(0,s)}(a_i - \epsilon_+) \right) + \vartheta_1(s\epsilon_2) \cdot Q(x) \right] . \tag{2.53}$$

The normalised 1-string contribution in the NS-limit [21] reads

$$\widetilde{Z}_1^{(0,s)\text{def}} = Z_1^{(0,s)\text{def}} - Z_1$$

$$\lim_{\epsilon_2 \to 0} \widetilde{Z}_1^{(0,s)\text{def}} = \frac{s}{\vartheta_1'(0)} \sum_{i=1}^{k} Q_{(0)}^\vee \left(a_i - \frac{1}{2}\epsilon_1 \right) \cdot L \left(a_i - x - \frac{1}{2}\epsilon_1 \right) + s \cdot Q_{(0)}(x) \tag{2.54}$$

$$= s \cdot \lim_{\epsilon_2 \to 0} \widetilde{Z}_1^{(0,1)\text{def}}$$

and the $(0,s)$ defect part is the product of s copies of the $(0,1)$ defect contribution. $L(\cdot)$ is defined in (2.17). The detailed derivation of (2.53) and (2.54) is provided in appendix A.5.1.

2-string. The $l = 2$ case yields the following elliptic genus:

$$Z_2^{(0,s)\text{def}} = \left(\frac{\vartheta_1(2\epsilon_+)}{\vartheta_1(\epsilon_1)\vartheta_1(\epsilon_2)} \right)^2 \sum_{1 \leq i < j \leq k} D(a_i - a_j)D(a_j - a_i) \tag{2.55}$$

$$\cdot\, Q^\vee(a_i - \epsilon_+)Q^\vee(a_j - \epsilon_+)V_{(0,s)}(a_i - \epsilon_+)V_{(0,s)}(a_j - \epsilon_+)$$

$$+ \frac{\vartheta_1(2\epsilon_+)}{\vartheta_1(2\epsilon_-)} \sum_{m=1}^{k} Q^\vee(a_m - \epsilon_+)V_{(0,s)}(a_m - \epsilon_+)$$

$$\cdot \left[\frac{\vartheta_1(\epsilon_1 + 2\epsilon_+)}{\vartheta_1(\epsilon_2)\vartheta_1(2\epsilon_1)} Q(a_m - \epsilon_+ - \epsilon_1)V_{(0,s)}(a_m - \epsilon_+ - \epsilon_1) \right.$$

$$\left. - \frac{\vartheta_1(\epsilon_2 + 2\epsilon_+)}{\vartheta_1(\epsilon_1)\vartheta_1(2\epsilon_2)} Q(a_m - \epsilon_+ - \epsilon_2)V_{(0,s)}(a_m - \epsilon_+ - \epsilon_2) \right]$$

$$+ \left(\frac{\vartheta_1(2\epsilon_+)}{\vartheta_1(\epsilon_1)\vartheta_1(\epsilon_2)} \right)^2 \vartheta_1(s\epsilon_2) \sum_{m=1}^{k} D(a_m - x - \epsilon_+)D(x + \epsilon_+ - a_m)$$

$$\cdot\, Q^\vee(a_m - \epsilon_+)Q(x)V_{(0,s)}(a_m - \epsilon+)$$

$$+ \frac{\vartheta_1(2\epsilon_+)}{\vartheta_1(2\epsilon_-)} \cdot Q(x)\vartheta_1(s\epsilon_2) \cdot \left[\frac{\vartheta_1(\epsilon_1 + 2\epsilon_+)}{\vartheta_1(\epsilon_2)\vartheta_1(2\epsilon_1)} Q(x - \epsilon_1)V_{(0,s)}(x - \epsilon_1) \right.$$

$$\left. - \frac{\vartheta_1(\epsilon_2 + 2\epsilon_+)}{\vartheta_1(\epsilon_1)\vartheta_1(2\epsilon_2)} Q(x - \epsilon_2)V_{(0,s)}(x - \epsilon_2) \right] .$$

Consider the normalised 2-string elliptic genus

$$\widetilde{Z}_2^{(0,s)\mathrm{def}} = Z_2^{(0,s)\mathrm{def}} - Z_2 - Z_1 \left(Z_1^{(0,s)\mathrm{def}} - Z_1 \right), \qquad (2.56)$$

see appendix A.3.1. The full normalised 2-string elliptic genus for the codimension 2 defect in the NS-limit is given by

$$
\begin{aligned}
\widetilde{Z}_{l=2}^{(0,s)\mathrm{def}} = {}& -\frac{s}{2} \sum_{j=1}^{k} \left(\frac{Q_{(0)}^{\vee}(a_j - \frac{\epsilon_1}{2})}{\vartheta_1'(0)} \right)^2 K\left(a_j - x - \frac{\epsilon_1}{2} \right) \qquad\qquad (2.57) \\
& + \sum_{j=1}^{k} \left(\frac{Q_{(0)}^{\vee}(a_j - \frac{\epsilon_1}{2})}{\vartheta_1'(0)} \right)^2 \left\{ \frac{s(s+1)}{2} L\left(a_j - x - \frac{\epsilon_1}{2} \right)^2 + 2s \cdot L(\epsilon_1) L\left(a_j - x - \frac{\epsilon_1}{2} \right) \right. \\
& \qquad + sL\left(a_j - x - \frac{\epsilon_1}{2} \right) \Big[\sum_{i=1}^{k} L(a_j - a_i - \epsilon_1) + \sum_{\substack{i=1 \\ i\neq j}}^{k} L(a_j - a_i) \\
& \qquad\qquad\qquad - \sum_{i=1}^{k} \left(L\left(a_j - \frac{\epsilon_1}{2} - m_i + b \right) + L\left(a_j - \frac{\epsilon_1}{2} - n_i - b \right) \right) \Big] \Big\} \\
& + \sum_{\substack{i,j=1 \\ i\neq j}}^{k} \frac{Q_{(0)}^{\vee}(a_i - \frac{\epsilon_1}{2})}{\vartheta_1'(0)} \frac{Q_{(0)}^{\vee}(a_j - \frac{\epsilon_1}{2})}{\vartheta_1'(0)} \left\{ \frac{s^2}{2} L\left(a_i - x - \frac{\epsilon_1}{2} \right) L\left(a_j - x - \frac{\epsilon_1}{2} \right) \right. \\
& \qquad + sL\left(a_i - x - \frac{\epsilon_1}{2} \right) \Big[L(a_i - a_j + \epsilon_1) - L(a_i - a_j) + L(a_j - a_i + \epsilon_1) - L(a_j - a_i) \Big] \Big\} \\
& + s \sum_{j=1}^{k} \frac{Q_{(0)}^{\vee}(a_j - \frac{\epsilon_1}{2})}{\vartheta_1'(0)} Q_{(0)}\left(a_j - \frac{3\epsilon_1}{2} \right) \left[L\left(a_j - x - \frac{\epsilon_1}{2} \right) + L\left(a_j - x - \frac{3\epsilon_1}{2} \right) \right] \\
& + s \cdot Q_{(0)}(x) \sum_{j=1}^{k} \frac{Q_{(0)}^{\vee}(a_j - \frac{\epsilon_1}{2})}{\vartheta_1'(0)} \Big[L\left(a_j - x + \frac{\epsilon_1}{2} \right) - L\left(a_j - x - \frac{\epsilon_1}{2} \right) \\
& \qquad\qquad + L\left(x - a_j + \frac{3\epsilon_1}{2} \right) - L\left(x - a_j + \frac{\epsilon_1}{2} \right) + sL\left(a_j - x - \frac{\epsilon_1}{2} \right) \Big] \\
& + s \cdot Q_{(0)}(x) \left(Q_{(0)}(x - \epsilon_1) - \frac{1-s}{2} Q_{(0)}(x) \right),
\end{aligned}
$$

with $L(\cdot)$ and $K(\cdot)$ as defined in (2.17). The computational details of (2.55) and (2.57) are presented in appendix A.5.2.

Full defect partition function. The 6d partition function in the presence of the codimension 2 defect is then denoted as

$$Z_{6\mathrm{d}}^{(r,s)\mathrm{def}} := Z_{\mathrm{pert}}^{(r,s)\mathrm{def}} \cdot Z_{\mathrm{str}}^{(r,s)\mathrm{def}} \qquad\qquad (2.58)$$

in the rest of this paper.

2.4 Codimension 4 defect

A natural candidate for a codimension 4 defect is a Wilson surface Σ [55, 56] that acquires a vacuum expectation value. The VEV of a Wilson surface in representation $\mathcal{R}$ can formally

be expressed in terms of the two-form potential $B_{\mu\nu}$ and the associated supersymmetric strings as,

$$\mathcal{W}_{\mathcal{R}}[\Sigma] = \mathrm{Tr}_{\mathcal{R}} \left(\mathcal{P} \; e^{i \int_{\Sigma} d\sigma^{\mu\nu}(B_{\mu\nu}+...)} \right) , \tag{2.59}$$

where $\ldots$ denotes the necessary supersymmetric partners of $B_{\mu\nu}$. There exists another type of codimension 4 BPS defects that couples to the 6d gauge symmetry. One can consider a 2d chiral fermion field ψ localised at the origin of $\mathbb{R}^4$ that couples to the bulk 6d gauge group through the following action:

$$S^{2\mathrm{d}} = \int \mathrm{d}^2 x \, \bar{\psi}_{-}(D_0 + D_1)\psi_{-} , \tag{2.60}$$

where $D_i = \partial_i + iA_i$ with $i = 0, 1$ and A_i is the bulk $\mathrm{SU}(k)$ gauge field. Adding this action to the path integral introduces a codimension 4 defect preserving half the supersymmetries. This defect is a 6d generalisation of the Wilson loop generating function in a 5d gauge theory that can also be called the 6d qq-character [57, 58]. The codimension 4 defect that is discussed below is a product of these two types (2.59) and (2.60) which is called the Wilson surface defect from now on. Consequently, the Wilson surface defect carries both tensor and gauge charges.

In practice, because of the lack of a field theoretical formulation of 6d SCFTs, one has to resort to string theory to formulate the Wilson surface defect and compute it. Wilson surface defects have, for example, been considered on the Ω-deformed $\mathbb{R}^4 \times \mathbb{T}^2$ in [57–60], see also [61]. Following [60], a Wilson surface defect in the 6d $\mathcal{N} = (1, 0)$ A_1 SCFTs can be realised in the Type IIA brane construction via an additional D4$'$ brane filling the x^0, x^1, x^7, x^8, x^9 space-time directions, see table 1. As the D4$'$ occupies different space-time directions as the $\widetilde{\mathrm{D4}}$ brane of section 2.3, the codimension 4 defect differs from the codimension 2 defect. In contrast to the $\widetilde{\mathrm{D4}}$ brane, the addition of the D4$'$ brane to the D2-D6-NS5 brane preserves the broken space-time symmetry (2.7) of the original set-up. As a consequence, the 2d world-volume theory is composed of the multiplets (2.8) of the D2-D6-NS5 system which are then supplemented by additional multiplets that originate from the presence of the D4$'$ brane. These new multiplets originate from the following:

- The D2-D4$'$ open string modes give rise to an additional $\mathcal{N} = (0, 4)$ twisted hyper ϕ^A and a Fermi multiplet Γ_α, which do not break the $\mathcal{N} = (0, 4)$ supersymmetry of the resulting 2d quiver theory.
- The D6-D4$'$ open strings introduce an additional Fermi multiplet ρ, which is a singlet under the 2d gauge group as well as the $\mathrm{SO}(4)_R$ R-symmetry.

Decomposing $\mathcal{N} = (0, 4)$ multiplets into $\mathcal{N} = (0, 2)$ multiplets, yields the field content from the original theory (2.8) plus the additional $\mathcal{N} = (0, 2)$ multiplets due to the additional D4$'$ brane. For the latter, one finds [60]

$$\text{twist hyper } (\phi^A, \eta^{\dot\alpha}) \longrightarrow \text{chiral } \phi \; (\phi^1, \eta^{\dot 1}) \; + \; \text{chiral } \tilde\phi^{\dagger} \; (\phi^{\dot 2}, \eta^2) \tag{2.61a}$$

$$\text{Fermi } \Gamma_\alpha, \rho \longrightarrow \text{Fermi } \Gamma_\alpha, \rho . \tag{2.61b}$$

and the charges are detailed in table 2. The resulting 2d quiver gauge theory can be encoded in

(2.62)
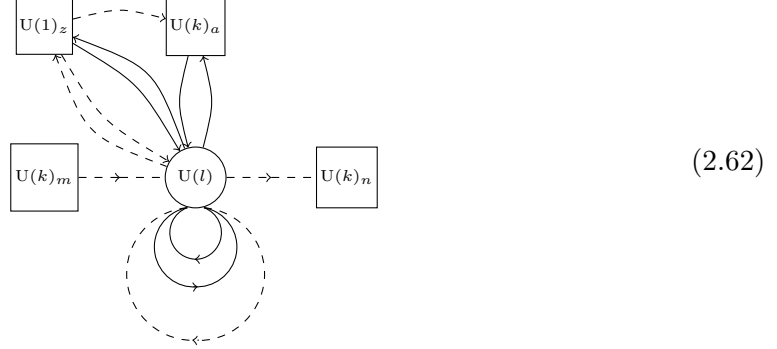
where the changes due to the D4′ brane are manifest in the additional $U(1)_z$ defect flavour node compared to (2.9).

Analogously to the elliptic genus (2.15) of the theory without defect, the 1-loop determinant contributions from the 2d multiplets include the terms (2.11) form the original theory plus the following defect parts:

$$Z_{\text{chiral}}^{D4'} = \prod_{p=1}^{l} \frac{(\mathrm{i}\,\eta)^2}{\theta_1(-\epsilon_+ \pm (u_p - z))} \,, \tag{2.63a}$$

$$Z_{\text{fermi}}^{D4'} = \prod_{p=1}^{l} \frac{\theta_1(\epsilon_- \pm (u_p - z))}{(\mathrm{i}\,\eta)^2} \cdot \prod_{j=1}^{k} \frac{\theta_1(z - a_j)}{\mathrm{i}\,\eta} \,, \tag{2.63b}$$

where the z-fugacity labels the $U(1)$ charge of the additional twisted hyper multiplet ϕ^A and Fermi multiplets Γ_α, ρ due to the D4′ brane. Collecting all the contributions from (2.11) and (2.63), one obtains

$$\begin{aligned} Z_{1-\text{loop}}^{D4'}(k,l) &\coloneqq Z_{1-\text{loop}}(k,l) \cdot Z_{\text{chiral}}^{D4'} Z_{\text{fermi}}^{D4'} \\ &= W_{\text{pert}} \cdot \left(\frac{2\pi\,\eta^3\theta_1(2\epsilon_+)}{\theta_1(\epsilon_1)\,\theta_1(\epsilon_2)} \right)^l \cdot \prod_{\substack{p,q=1 \\ p \neq q}}^{l} D(u_p - u_q) \cdot \prod_{p=1}^{l} Q(u_p)W(u_p) \\ &\equiv W_{\text{pert}} \cdot Z_{1-\text{loop}}^{\text{Wilson}}(k,l) \,, \end{aligned} \tag{2.64}$$

where the following definitions have been used

$$W(u) \coloneqq \frac{\theta_1(\epsilon_- \pm (u - z))}{\theta_1(-\epsilon_+ \pm (u - z))} = \frac{\theta_1(u - z \pm \epsilon_-)}{\theta_1(u - z \pm \epsilon_+)} = \frac{\vartheta_1(u - z \pm \epsilon_-)}{\vartheta_1(u - z \pm \epsilon_+)} \,, \tag{2.65a}$$

$$W_{\text{pert}} \coloneqq \prod_{j=1}^{k} \frac{\theta_1(z - a_j)}{\mathrm{i}\,\eta} = \prod_{j=1}^{k} \frac{\vartheta_1(z - a_j)}{\mathrm{i}\,Q^{-\frac{1}{12}}} \,. \tag{2.65b}$$

Note that W_{pert} is independent of the 2d gauge fugacities such that its contribution in the contour integral reduces to an identical prefactor for all elliptic genera. Hence, one may define

$$Z_l^{\text{Wilson}} = \frac{1}{l!} \oint \frac{\mathrm{d}^l u}{(2\pi\mathrm{i})^l} Z_{1-\text{loop}}^{\text{Wilson}}(k,l) \,, \tag{2.66}$$

using the definitions (2.64)–(2.65). Therefore, the partition function of the theory in the presence of a Wilson surface is given by

$$
Z_{\text{6d}}^{\text{Wilson}} = Z_{\text{pert}} \cdot W_{\text{pert}} \cdot \left(1 + \sum_{l=1}^{\infty} q_{\phi}^l \, Z_l^{\text{Wilson}} \right) ,
\tag{2.67}
$$

where Z_{pert} is the perturbative contribution (2.6) of the theory without defect, see also [60, section 3.3]. Since the interest is placed on the Wilson surface expectation value, one has to normalise the partition function with respect to the partition function of the theory without codimension 4 defect. Therefore, the expectation value of Wilson surface is given by

$$
\langle \mathcal{W} \rangle = W_{\text{pert}} \cdot \frac{\left(1 + \sum_{l=1}^{\infty} q_{\phi}^l \, Z_l^{\text{Wilson}} \right)}{\left(1 + \sum_{l'=1}^{\infty} q_{\phi}^{l'} \, Z_{l'} \right)} = W_{\text{pert}} \cdot \left[1 + \left(Z_1^{\text{Wilson}} - Z_1 \right) q_{\phi} + \mathcal{O}(q_{\phi}^2) \right] ,
\tag{2.68}
$$

see also appendix A.3.1. Before turning to the computation details, one may wonder in which representation $\mathcal{R}$ the Wilson surface transforms. As argued in [60], the codimension 4 defect of a single D4′ brane introduces a Wilson surface in the fundamental representation.

2.4.1 Wilson surface: perturbative contribution

The perturbative contribution acts as a multiplicative factor. The explicit contribution is

$$
W_{\text{pert}} = (-\mathrm{i})^k Q^{\frac{k}{12}} P_0(z + \epsilon_+) .
\tag{2.69}
$$

2.4.2 Wilson surface: elliptic genus

For the non-perturbative contributions of the Wilson surface expectation value, the 1-string and 2-string contributions are computed in this section.

1-string. Similar to the codimension 2 defect computation (2.53), one finds for the $l = 1$ case of the contour integral (2.66) the following result

$$
Z_1^{\text{Wilson}} = \frac{\vartheta_1(2\epsilon_+)}{\vartheta_1(\epsilon_1)\,\vartheta_1(\epsilon_2)} \sum_{i=1}^{k} Q^{\vee}(a_i - \epsilon_+) \cdot W(a_i - \epsilon_+) + Q(z + \epsilon_+) .
\tag{2.70}
$$

The normalised 1-string contribution in the NS-limit becomes

$$
\widetilde{Z}_1^{\text{Wilson}} = Z_1^{\text{Wilson}} - Z_1 ,
\tag{2.71}
$$

$$
\lim_{\epsilon_2 \to 0} \widetilde{Z}_1^{\text{Wilson}} = \frac{1}{\vartheta_1'(0)} \sum_{i=1}^{k} Q_{(0)}^{\vee} \left(a_i - \frac{1}{2}\epsilon_1 \right) \cdot \left[L \left(a_i - z - \epsilon_1 \right) - L \left(a_i - z \right) \right] + Q_{(0)} \left(z + \frac{1}{2}\epsilon_1 \right) .
$$

The detailed computations that lead to (2.70) and (2.71) are summarised in appendix A.7.1.

2-string. Consider the $l = 2$ elliptic genus (2.66), a computation yields

$$Z_2^{\text{Wilson}} = \left(\frac{\vartheta_1(2\epsilon_+)}{\vartheta_1(\epsilon_1)\vartheta_1(\epsilon_2)}\right)^2 \sum_{1 \le i < j \le k} D(a_i - a_j)D(a_j - a_i) \tag{2.72}$$

$$\cdot Q^\vee(a_i - \epsilon_+)Q^\vee(a_j - \epsilon_+)W(a_i - \epsilon_+)W(a_j - \epsilon_+)$$

$$+ \frac{\vartheta_1(2\epsilon_+)}{\vartheta_1(2\epsilon_-)} \sum_{j=1}^k Q^\vee(a_j - \epsilon_+)W(a_j - \epsilon_+)$$

$$\cdot \left[\frac{\vartheta_1(\epsilon_1 + 2\epsilon_+)}{\vartheta_1(\epsilon_2)\vartheta_1(2\epsilon_1)}Q(a_j - \epsilon_+ - \epsilon_1)W(a_j - \epsilon_+ - \epsilon_1)\right.$$

$$\left. - \frac{\vartheta_1(\epsilon_2 + 2\epsilon_+)}{\vartheta_1(\epsilon_1)\vartheta_1(2\epsilon_2)}Q(a_j - \epsilon_+ - \epsilon_2)W(a_j - \epsilon_+ - \epsilon_2)\right]$$

$$+ \frac{\vartheta_1(2\epsilon_+)}{\vartheta_1(\epsilon_1)\vartheta_1(\epsilon_2)} \sum_{j=1}^k D(a_j - z - 2\epsilon_+)D(z + 2\epsilon_+ - a_j)$$

$$\cdot Q^\vee(a_j - \epsilon_+)Q(z + \epsilon_+)W(a_j - \epsilon_+).$$

The normalised 2-string elliptic genus for the codimension 4 defect reads

$$\widetilde{Z}_2^{\text{Wilson}} = \sum_{\substack{i,j=1 \\ i \ne j}}^k \frac{Q_{(0)}^\vee(a_i - \frac{\epsilon_1}{2})}{\vartheta_1'(0)} \frac{Q_{(0)}^\vee(a_j - \frac{\epsilon_1}{2})}{\vartheta_1'(0)} \left[\frac{1}{2}L(a_i - z)L(a_j - z)\right.$$

$$\left. - L(a_i - z)L(a_j - z - \epsilon_1) + \frac{1}{2}L(a_i - z - \epsilon_1)L(a_j - z - \epsilon_1)\right]$$

$$+ \sum_{\substack{i,j=1 \\ i \ne j}}^k \frac{Q_{(0)}^\vee(a_i - \frac{\epsilon_1}{2})}{\vartheta_1'(0)} \frac{Q_{(0)}^\vee(a_j - \frac{\epsilon_1}{2})}{\vartheta_1'(0)} [L(a_i - z - \epsilon_1) - L(a_i - z)]$$

$$\cdot \left[L(a_i - a_j + \epsilon_1) - L(a_i - a_j) + L(a_j - a_i + \epsilon_1) - L(a_j - a_i)\right]$$

$$+ \frac{1}{2}\sum_{j=1}^k \left(\frac{Q_{(0)}^\vee(a_j - \frac{\epsilon_1}{2})}{\vartheta_1'(0)}\right)^2 [K(a_j - z) - K(a_j - z - \epsilon_1)]$$

$$+ \sum_{j=1}^k \left(\frac{Q_{(0)}^\vee(a_j - \frac{\epsilon_1}{2})}{\vartheta_1'(0)}\right)^2 \left(L(a_j - z - \epsilon_1) - L(a_j - z)\right)\left[\frac{1}{2}L(a_j - z - \epsilon_1) + 2L(\epsilon_1)\right.$$

$$+ \sum_{i=1}^k L(a_j - a_i - \epsilon_1) + \sum_{\substack{i=1 \\ i \ne j}}^k L(a_j - a_i - \epsilon_1)$$

$$\left. - \sum_{i=1}^N \left(L\left(a_j - \frac{\epsilon_1}{2} - m_i + b\right) + L\left(a_j - \frac{\epsilon_1}{2} - n_i - b\right)\right)\right]$$

$$+ \sum_{j=1}^k \frac{Q_{(0)}^\vee(a_j - \frac{\epsilon_1}{2})}{\vartheta_1'(0)}Q_{(0)}\left(a_j - \frac{3\epsilon_1}{2}\right)[L(a_j - z - 2\epsilon_1) - L(a_j - z)]$$

$$+ Q_{(0)}\left(z + \frac{\epsilon_1}{2}\right)\sum_{j=1}^k \frac{Q_{(0)}^\vee(a_j - \frac{\epsilon_1}{2})}{\vartheta_1'(0)}[L(z - a_j + 2\epsilon_1) - L(z - a_j + \epsilon_1)], \tag{2.73}$$

with $L(\cdot)$, $K(\cdot)$ as in (2.17). The derivation of (2.72) and (2.73) is detailed in appendix A.7.2.

3 Difference equation

In section 2, several partition functions have been discussed. Focusing on the defects introduced by a *single* $\widetilde{\mathrm{D}4}$ and *single* D4′, the partition functions are related as follows:

$$Z^{6\mathrm{d}}_{k+1} \xrightarrow{\text{normal Higgs (2.28)}} Z_G \cdot Z^{6\mathrm{d}}_k\,, \tag{3.1a}$$

$$Z^{6\mathrm{d}}_{k+1} \xrightarrow{(0,1)\text{-defect Higgs (2.36)}} Z_G \cdot Z^{6\mathrm{d}/4\mathrm{d}}_k(x)\,, \tag{3.1b}$$

where $Z^{6\mathrm{d}/4\mathrm{d}}_k := Z^{(0,1)\mathrm{def}}_{6\mathrm{d}}/Z_{6\mathrm{d}}$ denotes the normalised partition function in the presence of a codimension 2 defect. Consequently, $Z^{6\mathrm{d}/4\mathrm{d}}_k$ depends on the defect fugacity x. In addition, one may introduce a codimension 4 defect to the 6d theory, which in terms of partition functions means

$$Z^{6\mathrm{d}}_k \xrightarrow{\text{codim 4 defect}} Z^{6\mathrm{d}/2\mathrm{d}}_k(z)\,, \tag{3.2}$$

where $Z^{6\mathrm{d}/2\mathrm{d}}_k(z) := Z^{\mathrm{Wilson}}_{6\mathrm{d}}/Z_{6\mathrm{d}}$ is the normalised partition function in the presence of the codimension 4 defect. This codimension 4 defect is characterised by another defect fugacity z.

The aim of this section is to derive a difference operator $\mathcal{D}$, which acts via shifts on the codimension 2 defect fugacity x, and, similarly to [8, 22, 48, 50, 62, 63], is expected to generate the partition functions for the 6d theory in the presence of both, the codimension 2 and the codimension 4 defect, i.e.

$$\mathcal{D}Z^{6\mathrm{d}/4\mathrm{d}}(x) = Z^{6\mathrm{d}/4\mathrm{d}/2\mathrm{d}}(x)\,. \tag{3.3}$$

Clearly, since $Z^{6\mathrm{d}/4\mathrm{d}}$ only depends on the defect fugacity x, and the flavour and gauge fugacities inherited from the pure 6d theory, the generated $Z^{6\mathrm{d}/4\mathrm{d}/2\mathrm{d}}$ cannot depend on z. In the NS-limit [21], one expects a factorisation of the latter

$$Z^{6\mathrm{d}/4\mathrm{d}/2\mathrm{d}}(x) \xrightarrow{\text{NS}} \langle \mathcal{W} \rangle(x) \cdot Z^{6\mathrm{d}/4\mathrm{d}}(x) \tag{3.4}$$

with $\langle \mathcal{W} \rangle$ being the Wilson surface expectation value of the 6d theory. In other words, $\langle \mathcal{W} \rangle \cong Z^{6\mathrm{d}/2\mathrm{d}}$ for a *suitable identification* of the defect fugacity z. As a consequence, the $Z^{6\mathrm{d}/4\mathrm{d}}$ partition function is annihilated by the following operator in the NS-limit:

$$\mathcal{D} - \langle \mathcal{W} \rangle \quad \cong \quad \text{quantised SW-curve} \tag{3.5}$$

which, in the spirit of [19, 64], is expected to yield a quantisation of the Seiberg-Witten curve of the 6d $\mathcal{N} = (1,0)$ A_1 theory. The defect fugacity x becomes the coordinate of the SW-curve.

3.1 Path integral representation

As a first step towards the quantised SW-curve, one may try to express the non-perturbative parts of the partition function with codimension 2 defect as a path integral. Following the approach of [65], one may write the elliptic genus contributions via (2.14) and (2.50) as follows:

$$Z_{\text{str}}^{(r,s)\text{def}} = \sum_{l=0}^{\infty} \frac{1}{l!} q_\phi^l \oint \left(\prod_{p=1}^{l} \frac{\mathrm{d}u_p}{2\pi\mathrm{i}} \right) \left(\frac{2\pi\eta^3 \theta_1(\epsilon_1 + \epsilon_2)}{\theta_1(\epsilon_1)\theta_1(\epsilon_2)} \right)^l \prod_{\substack{p,q=1 \\ p\neq q}}^{l} D(u_p - u_q) \prod_{p=1}^{l} Q(u_p) \prod_{p=1}^{l} V_{(r,s)}(u_p). \tag{3.6}$$

For all specific considerations, the defect is specialised to $(r,s) = (0,s)$. Next, introduce the density

$$\bar{\rho}(u) = \sum_{p=1}^{l} (\#)^{-1} \cdot \delta(u - u_p) \qquad \text{with} \qquad \# := \frac{2\pi\eta^3 \theta_1(\epsilon_1 + \epsilon_2)}{\theta_1(\epsilon_1)\theta_1(\epsilon_2)} \tag{3.7}$$

and rewrite the partition function

$$Z_{\text{str}}^{(r,s)\text{def}} = \sum_{l=0}^{\infty} \frac{1}{l!} q_\phi^l \oint \left(\prod_{p=1}^{l} \frac{\mathrm{d}u_p}{2\pi\mathrm{i}} \right) \cdot (\#)^l \int \mathcal{D}\rho(u) \delta \left(\rho(u) - \sum_{p=1}^{l} (\#)^{-1} \cdot \delta(u - u_p) \right)$$

$$\cdot \exp \left[\int \mathrm{d}u \, \mathrm{d}u' (\#)^2 \rho(u) \log(D(u - u')) \rho(u') \right.$$

$$\left. + \int \mathrm{d}u \, \# \, \rho(u) (\log(Q(u)) + \log(V_{(r,s)}(u))) \right]. \tag{3.8}$$

With the Fourier representation

$$\delta(\rho(u) - \bar{\rho}) = \int \mathcal{D}\lambda \exp \left[\mathrm{i} \int \mathrm{d}u \, \lambda(u) \, (\rho(u) - \bar{\rho}) \right] \tag{3.9}$$

of the Delta function, one obtains

$$Z_{\text{str}}^{(r,s)\text{def}} = \sum_{l=0}^{\infty} \frac{1}{l!} q_\phi^l \int \left(\prod_{p=1}^{l} \frac{\mathrm{d}u_p}{2\pi\mathrm{i}} \right) \cdot (\#)^l \int \mathcal{D}\rho(u) \int \mathcal{D}\lambda(u) \prod_{p=1}^{l} e^{-\mathrm{i} \int \mathrm{d}u (\#)^{-1} \delta(u-u_p)\lambda(u)}$$

$$\cdot \exp \left[\int \mathrm{d}u \, \mathrm{d}u' (\#)^2 \rho(u) \log(D(u - u')) \rho(u') \right.$$

$$\left. + \int \mathrm{d}u \left(\mathrm{i}\,\lambda(u) \cdot \rho(u) + \# \, \rho(u) \left[\log(Q(u)) + \log(V_{(r,s)}(u)) \right] \right) \right]$$

$$= \sum_{l=0}^{\infty} \frac{1}{l!} q_\phi^l \int \left(\prod_{p=1}^{l} \frac{\mathrm{d}u_p}{2\pi\mathrm{i}} \right) \cdot (\#)^l \int \mathcal{D}\rho(u) \int \mathcal{D}\lambda(u) \prod_{p=1}^{l} e^{-\mathrm{i}\,(\#)^{-1}\lambda(u_p)}$$

$$\cdot \exp \left[\int \mathrm{d}u \, \mathrm{d}u' (\#)^2 \rho(u) \log(D(u - u')) \rho(u') \right.$$

$$\left. + \int \mathrm{d}u \left(\mathrm{i}\,\lambda(u) \cdot \rho(u) + \# \, \rho(u) \left[\log(Q(u)) + \log(V_{(r,s)}(u)) \right] \right) \right]. \tag{3.10}$$

The sum over l can be evaluated

$$\sum_{l=0}^{\infty} \frac{1}{l!} q_\phi^l \int \left(\prod_{p=1}^{l} \frac{\mathrm{d}u_p}{2\pi\mathrm{i}} \right) \cdot (\#)^l \prod_{p=1}^{l} e^{-\mathrm{i}\,(\#)^{-1}\lambda(u_p)} = \sum_{l=0}^{\infty} \frac{1}{l!} \left(q_\phi \# \int \frac{\mathrm{d}u}{2\pi\,\mathrm{i}} e^{-\mathrm{i}\,(\#)^{-1}\lambda(u)} \right)^l$$

$$= \exp\left[q_\phi \# \int \frac{\mathrm{d}u}{2\pi\,\mathrm{i}} e^{-\mathrm{i}\,(\#)^{-1}\lambda(u)} \right] \quad (3.11)$$

such that

$$Z_{\mathrm{str}}^{(r,s)\mathrm{def}} = \int \mathcal{D}\rho(u) \int \mathcal{D}\lambda(u) \exp\left[\int \mathrm{d}u\,\mathrm{d}u'(\#)^2 \rho(u) \log(D(u-u'))\rho(u') \right. \quad (3.12)$$

$$\left. + \int \mathrm{d}u \left(\mathrm{i}\,\lambda(u)\cdot\rho(u) + \#\,\rho(u)\left[\log(Q(u)) + \log(V_{(r,s)}(u))\right] + \frac{\#\,q_\phi}{2\pi\,\mathrm{i}} e^{-\mathrm{i}\,(\#)^{-1}\lambda(u)} \right) \right].$$

Analogous to [65], one may employ a shift in the auxiliary variable[1]

$$\lambda(u) = \lambda'(u) - \mathrm{i}\,\#\,\log(-q_\phi) \qquad \text{such that} \qquad e^{-\mathrm{i}\,(\#)^{-1}\lambda(u)} = -\frac{1}{q_\phi} e^{-\mathrm{i}\,(\#)^{-1}\lambda'(u)} \quad (3.13)$$

which yields

$$Z_{\mathrm{str}}^{(r,s)\mathrm{def}} = \int \mathcal{D}\rho(u) \int \mathcal{D}\lambda'(u) \exp\left[\int \mathrm{d}u\,\mathrm{d}u'(\#)^2 \rho(u) \log(D(u-u'))\rho(u') \right. \quad (3.14)$$

$$\left. + \int \mathrm{d}u \left(\mathrm{i}\,\lambda'(u)\cdot\rho(u) + \# \left[\rho(u)\log\left(-q_\phi Q(u)V_{(r,s)}(u)\right) - \frac{1}{2\pi\,i} e^{-\mathrm{i}\,(\#)^{-1}\lambda'(u)} \right] \right) \right].$$

This represents a path integral representation of the elliptic genera for the theory with codimension 2 defect. For the theory without defect, one simply puts $(r,s) = (0,0)$ because $V_{(0,0)}(u) = 1$.

3.1.1 Leading and next-to-leading order

Following [65], consider the behaviour as $\epsilon_2 \to 0$. One computes the following expansions:

$$\# = \frac{1}{\epsilon_2} + L(\epsilon_1) + \mathcal{O}(\epsilon_2), \quad (3.15\mathrm{a})$$

where the abbreviation $L(\cdot)$ is defined in (2.17). For the $D(u - u')$-terms one considers

$$\int \mathrm{d}u\,\mathrm{d}u'(\#)^2 \rho(u) \log(D(u-u'))\rho(u')$$

$$= (\#)^2 \int \mathrm{d}u \int_{-\infty}^{u} \mathrm{d}u'\rho(u) \left[\log(D(u-u')) + \log(D(u'-u)) \right] \rho(u'),$$

such that the ϵ_2-expansion leads to

$$\log(D(u-u')) + \log(D(u'-u)) = G_1(u-u')\cdot\epsilon_2 + G_2(u-u')\cdot\epsilon_2^2 + \mathcal{O}(\epsilon_2^3), \quad (3.15\mathrm{b})$$

$$G_1(u-u') = L(u-u'+\epsilon_1) - L(u-u'-\epsilon_1),$$

$$G_2(u-u') = \left[L(u-u')^2 - K(u-u') + \frac{1}{2}K(u-u'+\epsilon_1) - \frac{1}{2}L(u-u'+\epsilon_1)^2 \right.$$

$$\left. + \frac{1}{2}K(u-u'-\epsilon_1) - \frac{1}{2}L(u-u'-\epsilon_1)^2 \right],$$

---

[1]The shift here differs from the 4d case in [65] by a minus sign in front of q_ϕ. The alteration seems necessary as the quantised 6d SW-curve derived in this way passes nontrivial consistency checks, see section 3.4.

using the $L(\cdot)$, $K(\cdot)$ notation (2.17). Similarly, for the $Q(u)$-terms the ϵ_2 expansion yields

$$\log Q(u) = \mathcal{Q}_0 + \mathcal{Q}_1 \cdot \epsilon_2 + \mathcal{O}(\epsilon_2^2)\,, \tag{3.15c}$$

$$\mathcal{Q}_0 = \log Q(u)|_{\epsilon_2 = 0}\,,$$

$$\mathcal{Q}_1 = -\sum_{i=1}^{k}\left[L\left(\frac{\epsilon_1}{2} - (u - a_i)\right) + L\left(\frac{\epsilon_1}{2} + (u - a_i)\right)\right]\,,$$

and similarly for the $(0,s)$ defect terms $V_{(0,s)}(u)$ one finds

$$\log V_{(0,s)}(u) = \mathcal{V}_1^{(0,s)} \cdot \epsilon_2 + \mathcal{O}(\epsilon_2^2)\,, \tag{3.15d}$$

$$\mathcal{V}_1^{(0,s)} = s \cdot L(u - x) \equiv s \cdot \mathcal{V}_1^{(0,1)}\,.$$

The ϵ_2 expansion of the path integral for $\epsilon_2 \ll 1$ becomes

$$Z_{\mathrm{str}}^{(0,s)\mathrm{def}} = \int \mathcal{D}\rho(u) \int \mathcal{D}\lambda'(u)\, \exp\left[\frac{1}{\epsilon_2}\int \mathrm{d}u\, \mathrm{d}u'\frac{1}{2}\rho(u)G_1(u - u')\rho(u') \right. \tag{3.16}$$

$$+ \frac{1}{\epsilon_2}\int \mathrm{d}u\left(\rho(u)\left(\log(-q_\phi) + \mathcal{Q}_0\right) - \frac{1}{2\pi\,i}e^{-i\epsilon_2\lambda'(u)}\right)$$

$$+ \int \mathrm{d}u\, \mathrm{d}u'\rho(u)\left(\frac{1}{2}G_2(u - u') + L(\epsilon_1)G_1(u - u')\right)\rho(u')$$

$$+ \int \mathrm{d}u\left(\rho(u)\left(i\,\lambda'(u) + \mathcal{Q}_1 + \mathcal{V}_1^{(0,s)} + L(\epsilon_1)\left(\log(-q_\phi) + \mathcal{Q}_0\right)\right) - L(\epsilon_1)\frac{1}{2\pi\,i}e^{-i\epsilon_2\lambda'(u)}\right)$$

$$\left. + \mathcal{O}(\epsilon_2)\right]$$

and the expression for $Z_{\mathrm{str}} \equiv Z_{\mathrm{str}}^{(0,0)\mathrm{def}}$ is obtained by setting all the codimension 2 defect contributions $\mathcal{V}_n$ to zero, i.e. $r = s = 0$.

3.1.2 Saddle point analysis

Considering (3.16), the saddle point contribution comes from the leading order term

$$\frac{\delta}{\delta\rho(u)}Z_{\mathrm{str}}^{(0,s)\mathrm{def}} \sim Z_{\mathrm{str}}^{(0,s)\mathrm{def}} \cdot \frac{1}{\epsilon_2}\left(\int \mathrm{d}u'G_1(u - u')\rho(u') + \log(-q_\phi) + \mathcal{Q}_0(u)\right) \tag{3.17}$$

such that the saddle point equation is

$$\int \mathrm{d}u'G_1(u - u')\rho(u') + \log(-q_\phi) + \mathcal{Q}_0(u) = 0\,, \tag{3.18}$$

which then defines a critical density ρ_*. Inspired by [46], define the following objects:

$$\mathcal{Y}(u) \coloneqq \exp\left[-\int \mathrm{d}u'\,\rho(u')\frac{\mathrm{d}}{\mathrm{d}u}\log(\vartheta_1(u - u'))\right]\,, \tag{3.19}$$

$$\omega(u) \coloneqq \frac{\mathcal{Y}(u - \epsilon_1)}{\mathcal{Y}(u)P_0(u)}\,, \tag{3.20}$$

and observe that

$$\int \mathrm{d}u' \rho(u') \mathcal{G}_1(u - u') = \log \frac{\mathcal{Y}(u - \epsilon_1)}{\mathcal{Y}(u + \epsilon_1)}, \tag{3.21a}$$

$$\mathcal{Q}_0(u) = \log(M(u)) - \log(P_0(u)) - \log(P_0(u + \epsilon_1))$$

$$= \log \frac{\mathcal{Y}(u + \epsilon_1)}{\mathcal{Y}(u - \epsilon_1)} + \log \left(M(u)\omega(u)\omega(u + \epsilon_1) \right). \tag{3.21b}$$

The saddle point equation (3.18) becomes

$$\log\left(-q_\phi M(u_*)\omega(u_*)\omega(u_* + \epsilon_1)\right) = 0 \quad \Leftrightarrow \quad 1 + q_\phi M(u_*)\omega(u_*)\omega(u_* + \epsilon_1) = 0, \tag{3.22}$$

for some points u_*. Next, define the following function:

$$f(u) := \frac{1 + q_\phi M(u - \epsilon_1)\omega(u)\omega(u - \epsilon_1)}{\omega(u)} \tag{3.23}$$

the properties of f indicate that it can be written as a product of k Theta functions

$$f(u) \equiv P(u) = \prod_{l=1}^{k} \vartheta_1(u - e_l) \tag{3.24}$$

with roots e_l to be determined. The saddle point equation (3.22) becomes equivalent to

$$-q_\phi M(u - \epsilon_1)\omega(u)\omega(u - \epsilon_1) + \omega(u)P(u) - 1 = 0. \tag{3.25}$$

From (3.25) one can now derive a difference equation for the defect partition function.

3.2 Shift operator

Having derived a path integral expression (3.14), which is dominated by the contribution of the saddle point (3.25), the next step is to define a shift operator. For this, the (exponentiated) defect fugacity X is promoted to a non-commutative parameter together with conjugate coordinate Y such that

$$YX = \frac{1}{p}XY \tag{3.26}$$

i.e. $Yf(x) = f(x - \epsilon_1)$. Now, one can act with the shift operator Y on the two parts of the partition function. For the perturbative part, one proceeds with the natural expressions; while the Y-action on the non-perturbative part is greatly simplified by the path integral representation.

Perturbative contribution. The normalised perturbative part (2.48) for an $(0, s)$ defect can be written as

$$\widetilde{Z}_{\mathrm{pert}}^{(0,s)\mathrm{def}} = \mathrm{PE}\left[\frac{s}{2(1-p)}\left(\frac{1+Q}{1-Q}\right)\left\{(1-p) + \sqrt{p}\sum_{i=1}^{k}\left(\frac{X}{A_i} - \frac{A_i}{X}\right)\right\}\right], \tag{3.27}$$

for $A_i = e^{a_i}$. A direction computation, see appendix A.6.1, shows that the action of Y is given by

$$Y \widetilde{Z}_{\text{pert}}^{(0,s)\text{def}} = \left[\sqrt{\left(\prod_{i=1}^{k} \frac{1}{\vartheta_1(a_i - x + \frac{1}{2}\epsilon_1, \tau)}\right)^2}\right]^s \widetilde{Z}_{\text{pert}}^{(0,s)\text{def}} = \left[\frac{1}{P_0(x)}\right]^s \widetilde{Z}_{\text{pert}}^{(0,s)\text{def}}. \quad (3.28)$$

Note that *the sign of the argument of the theta function can be flipped without any consequence.*

Elliptic genus. Consider the defect contribution (2.50), (3.15d), which one may write as

$$\widetilde{Z}_{\text{str}}^{(0,s)\text{def}} \supset \exp\left[\int \mathrm{d}u \rho_*(u) \mathcal{V}_1^{(0,s)}\right] = \exp\left[s \cdot \int \mathrm{d}u \rho_*(u) \partial_u \log \theta_1(u - x)\right] \quad (3.29)$$

$$= \exp\left[-s \cdot \int \mathrm{d}u \rho_*(u) \partial_x \log \theta_1(u - x)\right] = (\mathcal{Y}(x))^s.$$

The shift operator acting on the *normalised* instanton-strings partition function yields

$$Y \widetilde{Z}_{\text{str}}^{(0,s)\text{def}} \sim Y \int \mathcal{D}\rho \exp\left[\int \mathrm{d}u \rho(u) \mathcal{V}_1^{(0,s)}(u, x)\right]$$

$$\sim Y \exp\left[\int \mathrm{d}u \rho_*(u) \mathcal{V}_1^{(0,s)}(u, x)\right]$$

$$\sim \exp\left[\int \mathrm{d}u \rho_*(u) \mathcal{V}_1^{(0,s)}(u, x - \epsilon_1)\right]. \quad (3.30)$$

Consequently, one arrives at

$$Y Z_{\text{str}}^{(0,s)\text{def}} = \left(\frac{\mathcal{Y}(x - \epsilon_1)}{\mathcal{Y}(x)}\right)^s \cdot \widetilde{Z}_{\text{str}}^{(0,s)\text{def}} \qquad \text{in leading order}$$

$$= (\omega(x) P_0(x))^s \cdot \widetilde{Z}_{\text{str}}^{(0,s)\text{def}} \qquad \text{using (3.20)}. \quad (3.31)$$

Alternatively, a direct computation on the defect contribution (2.50) leads to the same conclusion as in (3.31), as detailed in appendix A.6.2.

Full partition function. Combining (3.31) and (3.28) implies that

$$Y\left(\widetilde{Z}_{\text{pert}}^{(0,s)\text{def}} \cdot \widetilde{Z}_{\text{str}}^{\text{def}}\right) = Y \widetilde{Z}_{\text{pert}}^{(0,s)\text{def}} \cdot Y \widetilde{Z}_{\text{str}}^{(0,s)\text{def}} = \left[\frac{1}{P_0(x)}\right]^s \widetilde{Z}_{\text{pert}}^{(0,s)\text{def}} \cdot [\omega(x) P_0(x)]^s \cdot \widetilde{Z}_{\text{str}}^{(0,s)\text{def}}$$

$$= [\omega(x)]^s \cdot \left(\widetilde{Z}_{\text{pert}}^{(0,s)\text{def}} \cdot \widetilde{Z}_{\text{str}}^{(0,s)\text{def}}\right), \quad (3.32)$$

where one notes the cancellation of the contribution from the perturbative part. In particular, notice the ratio

$$\frac{\widetilde{Z}_{\text{tot}}^{(0,1)\text{def}}(x - \epsilon_1)}{\widetilde{Z}_{\text{tot}}^{\text{def}}(x)} \equiv \frac{Y\left(\widetilde{Z}_{\text{pert}}^{(0,1)\text{def}} \cdot \widetilde{Z}_{\text{str}}^{(0,1)\text{def}}\right)}{\widetilde{Z}_{\text{pert}}^{(0,1)\text{def}} \cdot \widetilde{Z}_{\text{str}}^{(0,1)\text{def}}} = \omega(x), \quad (3.33)$$

which is reminiscent of [46, eq. (55)].

3.3 Difference equation

Finally, following the logic of [46, 66], the saddle point equation can be used to derive a difference equation on the level of the normalised codimension 2 partition function. For this, one starts from the saddle point equation (3.25) and performs the following manipulations:

$$
\begin{aligned}
0 &= -q_\phi M(x - \epsilon_1)\omega(x)\omega(x - \epsilon_1) + \omega(x)P(x) - 1 \\
&= -q_\phi M(x)\omega(x + \epsilon_1)\omega(x) + \omega(x + \epsilon_1)P(x + \epsilon_1) - 1 \qquad \text{by shifting } x \to x + \epsilon_1 \\
&= -q_\phi M(x)\frac{\widetilde{Z}^{(0,1)\mathrm{def}}(x)}{\widetilde{Z}^{(0,1)\mathrm{def}}(x + \epsilon_1)}\frac{\widetilde{Z}^{(0,1)\mathrm{def}}(x - \epsilon_1)}{\widetilde{Z}^{(0,1)\mathrm{def}}(x)} + P(x + \epsilon_1)\frac{\widetilde{Z}^{(0,1)\mathrm{def}}(x)}{\widetilde{Z}^{(0,1)\mathrm{def}}(x + \epsilon_1)} - 1 \qquad \text{using (3.33)} \\
&= -q_\phi M(x)\cdot\frac{\widetilde{Z}^{(0,1)\mathrm{def}}(x - \epsilon_1)}{\widetilde{Z}^{(0,1)\mathrm{def}}(x + \epsilon_1)} + P(x + \epsilon_1)\frac{\widetilde{Z}^{(0,1)\mathrm{def}}(x)}{\widetilde{Z}^{(0,1)\mathrm{def}}(x + \epsilon_1)} - 1 \\
&= -q_\phi M(x)\cdot\widetilde{Z}^{(0,1)\mathrm{def}}(x - \epsilon_1) + P(x + \epsilon_1)\widetilde{Z}^{(0,1)\mathrm{def}}(x) - \widetilde{Z}^{(0,1)\mathrm{def}}(x + \epsilon_1) \\
&= \left[-q_\phi M(x)\cdot Y + P(x + \epsilon_1) - Y^{-1}\right]\widetilde{Z}^{(0,1)\mathrm{def}}(x).
\end{aligned}
\tag{3.34}
$$

Hence, (3.34) shows the existence of an operator that annihilates the codimension 2 defect partition function. Nevertheless, the expression needs to be considered with care. Comparing to the results of [17], the form is already suggestive of the Seiberg-Witten curve. In order to consolidate this further, one can equivalently rewrite (3.34) as

$$
\left[q_\phi M(x)\cdot Y + Y^{-1}\right]\widetilde{Z}^{(0,1)\mathrm{def}}(x) = P(x + \epsilon_1)\cdot\widetilde{Z}^{(0,1)\mathrm{def}}(x)
\tag{3.35}
$$

where the left-hand-side contains expressions that are fully known, while the right-hand-side contains the degree k modular form $P(u)$ of (3.24), whose existence follows from the saddle point analysis. Therefore, the purpose of the remainder of this section is to establish a physical interpretation of $P(x + \epsilon_1)$. As it turns out, the codimension 4 defect in form of the VEV of a Wilson surface is a suitable object to consider.

3.4 Comparison to Wilson surface

The strategy for determining the physical meaning of $P(x + \epsilon_1)$ has two steps:

(i) Starting from (3.35), together with the *known* normalised codimension 2 defect partition function $\widetilde{Z}^{(0,1)\mathrm{def}}(x)$, one can compute $P(x + \epsilon_1)$ order by order in q_ϕ.

(ii) The *predictions* for $P(x + \epsilon_1)$ are compared to the normalised codimension 4 defect partition function $\widetilde{Z}_2^{\mathrm{Wilson}}(z)$, i.e. the Wilson surface VEV. This determines z as a function of x.

To begin with, consider the difference equation (3.34) or (3.35) together with the q_ϕ-expansions

$$
\widetilde{Z}^{(0,1)\mathrm{def}}(x) = \widetilde{Z}_0^{(0,1)\mathrm{def}}(x)\left(1 + \sum_{l=1}^{\infty} q_\phi^l\,\widetilde{Z}_l^{(0,1)\mathrm{def}}(x)\right), \qquad P(x) = P_0(x)\left(1 + \sum_{l=1}^{\infty} q_\phi^l\,P_l(x)\right),
\tag{3.36}
$$

such that (3.34) becomes

$$
0 = \Bigg[P_0(x+\epsilon_1) - Y^{-1} + q_\phi \left(P_0(x+\epsilon_1)P_1(x+\epsilon_1) - M(x)Y \right)
$$
$$
+ \sum_{l=2}^{\infty} q_\phi^l P_0(x+\epsilon_1)P_l(x+\epsilon_1) \Bigg] \widetilde{Z}_0^{(0,1)\mathrm{def}}(x) \left(1 + \sum_{j=1}^{\infty} q_\phi^j \widetilde{Z}_j^{(0,1)\mathrm{def}}(x) \right) . \tag{3.37}
$$

Next, one can try to match the predictions for $P_l(x+\epsilon_1)$ with the results from the Wilson surface. Based on the explicit computations detailed below, the *claim* is that

$$
P_l(x+\epsilon_1) = \widetilde{Z}_l^{\mathrm{Wilson}}(z) \qquad \forall l \qquad \Leftrightarrow \qquad z = x + \frac{1}{2}\epsilon_1 , \tag{3.38}
$$

i.e. the fugacities x and z are suitably identified.

3.4.1 Perturbative level

The lowest order in the q_ϕ expansion reads

$$
0 = \left[P_0(x+\epsilon_1) - Y^{-1} \right] \widetilde{Z}_0^{(0,1)\mathrm{def}}(x) \tag{3.39}
$$

and one finds

$$
P_0(x+\epsilon_1) = \frac{Y^{-1}\widetilde{Z}_0^{(0,1)\mathrm{def}}(x)}{\widetilde{Z}_0^{(0,1)\mathrm{def}}(x)} . \tag{3.40}
$$

Comparing to W_{part} in the NS-limit yields

$$
P_0(x+\epsilon_1) = \frac{W_{\mathrm{part}}(z)}{(-\mathrm{i})^N Q^{\frac{k}{12}}} \quad \Leftrightarrow \quad z = x + \frac{\epsilon_1}{2} . \tag{3.41}
$$

3.4.2 1-string level

Next, the linear q_ϕ order reads

$$
0 = \left[P_0(x+\epsilon_1) - Y^{-1} \right] \widetilde{Z}_0^{(0,1)\mathrm{def}}(x)\widetilde{Z}_1^{(0,1)\mathrm{def}}(x)
$$
$$
+ \left[P_0(x+\epsilon_1)P_1(x+\epsilon_1) - M(x)Y \right] \widetilde{Z}_0^{(0,1)\mathrm{def}}(x) , \tag{3.42}
$$

and, using (3.39), one finds

$$
P_1(x+\epsilon_1) = Q_{(0)}(x) + Y^{-1}\widetilde{Z}_1^{(0,1)\mathrm{def}}(x) - \widetilde{Z}_1^{(0,1)\mathrm{def}}(x) . \tag{3.43}
$$

Using the results from above, one computes the prediction (3.43) to be

$$
P_1(x+\epsilon_1) = Q_{(0)}(x+\epsilon_1) + \frac{1}{\vartheta_1'(0)} \sum_{i=1}^{k} Q_{(0)}^\vee \left(a_i - \frac{1}{2}\epsilon_1 \right) \cdot \left[L \left(a_i - x - \frac{3}{2}\epsilon_1 \right) - L \left(a_i - x - \frac{1}{2}\epsilon_1 \right) \right] ,
$$
$$
\tag{3.44}
$$

see appendix A.8.1 for details. Comparing to the Wilson surface result (2.71), one finds

$$
P_1(x+\epsilon_1) = \widetilde{Z}_1^{\mathrm{Wilson}}(z) \quad \Leftrightarrow \quad z = x + \frac{\epsilon_1}{2} . \tag{3.45}
$$

3.4.3 2-string level

Lastly, the quadratic q_ϕ order in the expansion reads

$$
\begin{aligned}
0 = & \left[P_0(x+\epsilon_1) - Y^{-1} \right] \widetilde{Z}_0^{(0,1)\mathrm{def}}(x) \widetilde{Z}_2^{(0,1)\mathrm{def}}(x) \\
& + \left[P_0(x+\epsilon_1) P_1(x+\epsilon_1) - M(x)Y \right] \widetilde{Z}_0^{(0,1)\mathrm{def}}(x) \widetilde{Z}_1^{(0,1)\mathrm{def}}(x) \\
& + P_0(x+\epsilon_1) P_2(x+\epsilon_1) \widetilde{Z}_0^{(0,1)\mathrm{def}}(x) \,,
\end{aligned}
\tag{3.46}
$$

and using (3.39) and (3.42) one finds

$$
\begin{aligned}
P_2(x+\epsilon) = & \, Q_{(0)}(x) \left[Y - 1 \right] \widetilde{Z}_1^{(0,1)\mathrm{def}}(x) + \left[Y^{-1} - 1 \right] \widetilde{Z}_2^{(0,1)\mathrm{def}}(x) \\
& - \widetilde{Z}_1^{(0,1)\mathrm{def}}(x) \left[Y^{-1} - 1 \right] \widetilde{Z}_1^{(0,1)\mathrm{def}}(x) \,.
\end{aligned}
\tag{3.47}
$$

Using the results from above, one compute the prediction (3.47) to be

$$
\begin{aligned}
P_2(x+\epsilon_1) = & -\frac{1}{2} \sum_{j=1}^{k} \left(\frac{Q_{(0)}^{\vee}\left(a_j - \frac{1}{2}\epsilon_1\right)}{\vartheta_1'(0)} \right)^2 \left[K\left(a_j - x - \frac{3}{2}\epsilon_1\right) - K\left(a_j - x - \frac{1}{2}\epsilon_1\right) \right] \\
& + \sum_{j=1}^{k} \left(\frac{Q_{(0)}^{\vee}\left(a_j - \frac{1}{2}\epsilon_1\right)}{\vartheta_1'(0)} \right)^2 \\
& \times \left\{ L\left(a_j - x - \frac{3}{2}\epsilon_1\right) \left[L\left(a_j - x - \frac{3}{2}\epsilon_1\right) - L\left(a_j - x - \frac{1}{2}\epsilon_1\right) \right] \right. \\
& \left. \qquad + 2L(\epsilon_1) \left[L\left(a_j - x - \frac{3}{2}\epsilon_1\right) - L\left(a_j - x - \frac{1}{2}\epsilon_1\right) \right] \right\} \\
& + \sum_{\substack{i,j=1 \\ i \neq j}}^{k} \frac{Q_{(0)}^{\vee}\left(a_i - \frac{1}{2}\epsilon_1\right)}{\vartheta_1'(0)} \frac{Q_{(0)}^{\vee}\left(a_j - \frac{1}{2}\epsilon_1\right)}{\vartheta_1'(0)} \left[L\left(a_i - x - \frac{3}{2}\epsilon_1\right) - L\left(a_i - x - \frac{1}{2}\epsilon_1\right) \right] \\
& \cdot \left\{ L(a_i - a_j - \epsilon_1) + L(a_j - a_i - \epsilon_1) - L(a_i - a_j) - L(a_j - a_i) \right\} \\
& + \sum_{\substack{i,j=1 \\ i \neq j}}^{k} \frac{Q_{(0)}^{\vee}\left(a_i - \frac{1}{2}\epsilon_1\right)}{\vartheta_1'(0)} \frac{Q_{(0)}^{\vee}\left(a_j - \frac{1}{2}\epsilon_1\right)}{\vartheta_1'(0)} \left[\frac{1}{2} L\left(a_i - x - \frac{3}{2}\epsilon_1\right) L\left(a_j - x - \frac{3}{2}\epsilon_1\right) \right. \\
& \left. - L\left(a_i - x - \frac{1}{2}\epsilon_1\right) L\left(a_j - x - \frac{3}{2}\epsilon_1\right) + \frac{1}{2} L\left(a_i - x - \frac{1}{2}\epsilon_1\right) L\left(a_j - x - \frac{1}{2}\epsilon_1\right) \right] \\
& + \sum_{j=1}^{k} \frac{Q_{(0)}^{\vee}\left(a_j - \frac{1}{2}\epsilon_1\right)}{\vartheta_1'(0)} Q_{(0)}\left(a_j - \frac{3}{2}\epsilon_1\right) \left[L\left(a_j - x - \frac{5}{2}\epsilon_1\right) - L\left(a_j - x - \frac{1}{2}\epsilon_1\right) \right] \\
& + Q_{(0)}(x+\epsilon_1) \sum_{j=1}^{k} \frac{Q_{(0)}^{\vee}\left(a_j - \frac{1}{2}\epsilon_1\right)}{\vartheta_1'(0)} \left[L\left(x + \frac{5}{2}\epsilon_1 - a_j\right) - L\left(x + \frac{3}{2}\epsilon_1 - a_j\right) \right] \,,
\end{aligned}
\tag{3.48}
$$

see appendix A.8.2 for details. Comparing to the Wilson surface result (2.73), one finds

$$
P_2(x+\epsilon_1) = \widetilde{Z}_2^{\mathrm{Wilson}}(z) \quad \Leftrightarrow \quad z = x + \frac{\epsilon_1}{2} \,.
\tag{3.49}
$$

3.4.4 Implications

The results of sections 3.4.1–3.4.3 provide evidence that the claim (3.38) is correct. Thus, the difference equation (3.35) can be re-written

$$
\begin{aligned}
\mathcal{D}_{\mathrm{NS}} \widetilde{Z}^{(0,1)\mathrm{def}}(x) &\equiv \left[q_\phi M(x) \cdot Y + Y^{-1} \right] \widetilde{Z}^{(0,1)\mathrm{def}}(x) \\
&= P(x + \epsilon_1) \cdot \widetilde{Z}^{(0,1)\mathrm{def}}(x) \equiv \langle \mathcal{W} \rangle \left(z = x + \frac{\epsilon_1}{2} \right) \cdot \widetilde{Z}^{(0,1)\mathrm{def}}(x), \quad (3.50)
\end{aligned}
$$

which identifies the operator $\mathcal{D}$ of (3.3) in the NS limit. In addition, the degree k modular form P of (3.24) has been identified with the expectation value of the Wilson surface defect.

As a comment, the found identification (3.38) is a qualitatively new feature of the $\mathcal{N} = (1,0)$ theories in contrast to the $\mathcal{N} = (2,0)$ case discussed in the next section. As shown in the 6d $\mathcal{N} = (2,0)$ A_1 case [60], the Wilson surface expectation value is independent of the defect fugacity z; similarly, the dual 5d picture has been considered in [64], where the Wilson loop expectation values also has no dependence on the defect fugacity.

4   2 M5 branes: matching 6d and 5d with enhanced SUSY

In this section, the methods developed in the above sections are applied to the simplest 6d $\mathcal{N} = (1,0)$ theory with SU(2) gauge group and 4 flavours. The interest in this model comes because Higgsing the SU(2) gauge group as above leads to a theory with no gauge theory left. Put differently, in the Type IIA brane construction the Higgsing is realised by removing a D6 brane, see figure 1. Starting from the 2 D6 branes for the SU(2) theory and removing one of them, leads to a single D6 which is dual to $\mathbb{C}^2/\mathbb{Z}_1 \cong \mathbb{C}^2$, i.e. the A_0 singularity in the original M-theory setup. Thus, the Higgsing leads to a system of M5 branes which preserve 16 supercharges instead of the 8 supersymmetries of the generic case with an A_k singularity.

Building on section 2.2, one can study the $\mathcal{N} = (2,0)$ A_1 theory in the presence of a codimension 2 defect by Higgsing the $\mathcal{N} = (1,0)$ SU(2) theory with a position dependent VEV. In addition, the path integral formalism developed in previous section allows one to derive the quantised Seiberg-Witten curve therein. As a consistency check, it is verify in this section that the established SW-curve matches the result obtained from the 5d/3d perspective by compactifying the 6d $\mathcal{N} = (2,0)$ A_1 theory onto S^1 [64].

4.1   Defects in 6d $\mathcal{N} = (2,0)$ A_1 theory

To begin with, one computes the partition function for the 6d case. In order to find agreement with the 5d result of section 4.2, the derivation is repeated in a slightly different manner compared to sections 2 and 3.

4.1.1   Elliptic genus

Firstly, the saddle point approach is used to derive the difference equation of the non-perturbative part of the partition function, analogously to section 3.1. The SU(2) gauge and SU(4) $\subset$ SO(8) flavour fugacities are labeled in terms of α, μ_i and t as

$$
\alpha = e^a, \qquad e^{m_1} = t\mu_1, \qquad e^{m_2} = \frac{t}{\mu_1}, \qquad e^{m_3} = \frac{\mu_2}{t}, \qquad \text{and} \qquad e^{m_4} = \frac{1}{t\mu_2}. \quad (4.1)
$$

The instanton partition function Z_k for $\mathcal{N} = (1,0)$ SU(2) with 4 flavours is thus given by the l-th elliptic genus, contributing to the non-perturbative partition function (2.3)

$$Z_l = \frac{1}{l!} \oint \prod_{I=1}^{l} \frac{\mathrm{d}\phi_I}{2\pi\mathrm{i}} \prod_{I,J=1}^{l} \frac{\vartheta_1^\vee(\phi_{IJ})\vartheta_1(\phi_{IJ} + 2\epsilon_+)}{\vartheta_1(\phi_{IJ} + \epsilon_{1,2})} \cdot \prod_{I=1}^{l} \frac{\prod_{i=1}^{4} \vartheta_1(\phi_I - m_i)}{\vartheta_1(\phi_I \pm a \pm \epsilon_+)}, \qquad (4.2)$$

with $\phi_{IJ} := \phi_I - \phi_J$. Here, $\vartheta_1^\vee(\phi_{IJ})$ means that those terms in $\vartheta_1(\phi_{IJ})$ with $\phi_I = \phi_J$ are replaced by $\vartheta_1'(0)$. Next, a $(0,1)$ codimension 2 defect is introduced via the Higgsing (2.36b), which becomes

$$a = m_3 - \epsilon_+ \qquad \text{and} \qquad m_4 = m_3 - 2\epsilon_+ - \epsilon_2 \equiv x - \epsilon_2, \qquad (4.3\mathrm{a})$$

$$\text{or} \qquad \alpha = \frac{\sqrt{q}}{t} \qquad \text{and} \qquad \mu_2 = \sqrt{pq^2}, \qquad (4.3\mathrm{b})$$

such that the elliptic genus of the $\mathcal{N} = (2,0)$ theory with defect is given by

$$Z_l^{\mathrm{def}} = \frac{1}{l!} \oint \prod_{I=1}^{l} \frac{\mathrm{d}\phi_I}{2\pi\mathrm{i}} \prod_{I,J=1}^{l} \frac{\vartheta_1^\vee(\phi_{IJ})\vartheta_1(\phi_{IJ} + 2\epsilon_+)}{\vartheta_1(\phi_{IJ} + \epsilon_{1,2})}$$

$$\cdot \prod_{I=1}^{l} \frac{\vartheta_1(\phi_I - m_1)\vartheta_1(\phi_I - m_2)\vartheta_1(\phi_I - m_3 + 2\epsilon_+ + \epsilon_2)}{\vartheta_1(\phi_I + m_3)\vartheta_1(\phi_I + m_3 - 2\epsilon_+)\vartheta_1(\phi_I - m_3 + 2\epsilon_+)}. \qquad (4.4)$$

Further notice that

$$e^{m_1 + m_3} = \mu_1 \mu_2 \qquad \text{and} \qquad e^{m_2 + m_3} = \mu_1^{-1}\mu_2. \qquad (4.5)$$

For convenience, one defines $\mu_1 \equiv e^{m - \epsilon_2/2}$, and, additionally, shifts the 2d gauge variables as

$$\phi_I \mapsto \phi_I - m_3 + \epsilon_+. \qquad (4.6)$$

Finally, one ends up with the instanton partition function for the theory with a codimension 2 defect of type $(0,1)$, which is given by

$$Z_l^{\mathrm{def}} = \frac{1}{l!} \oint \prod_{I=1}^{l} \frac{\mathrm{d}\phi_I}{2\pi\mathrm{i}} \prod_{I,J=1}^{l} \frac{\vartheta_1(\phi_{IJ})\vartheta_1(\phi_{IJ} + 2\epsilon_+)}{\vartheta_1(\phi_{IJ} + \epsilon_{1,2})} \cdot \prod_{I=1}^{l} \frac{\vartheta_1(\phi_I - m)\vartheta_1(\phi_I + m - \epsilon_2)}{\vartheta_1(\phi_I \pm \epsilon_+)}$$

$$\cdot \prod_{I=1}^{l} \frac{\vartheta_1(\phi_I - 2x - \epsilon_-)}{\vartheta_1(\phi_I - 2x - \epsilon_+)}, \qquad (4.7)$$

and, following the definitions (2.14) of section 2, one defines

$$D(u) = \frac{\vartheta_1(u)\vartheta_1(u + 2\epsilon_+)}{\vartheta_1(u + \epsilon_1)\vartheta_1(u + \epsilon_2)}, \qquad (4.8\mathrm{a})$$

$$Q(u) = \frac{\vartheta_1(u - m)\vartheta_1(u + m - \epsilon_2)}{\vartheta_1(u + \epsilon_+)\vartheta_1(u - \epsilon_+)}, \qquad (4.8\mathrm{b})$$

$$V(u) = \frac{\vartheta_1(u - 2x - \epsilon_-)}{\vartheta_1(u - 2x - \epsilon_+)}. \qquad (4.8\mathrm{c})$$

Having set-up the notation, one recasts the instanton partition function in a path integral, analogous to section 3.1, as follows:

$$Z_{\text{str}}^{\text{def}} \sim \int \mathcal{D}\rho(u) \exp\left[\frac{1}{\epsilon_2} \int \mathrm{d}u\mathrm{d}u' \frac{1}{2}\rho(u)G_1(u-u')\rho(u') + \frac{1}{\epsilon_2} \int \mathrm{d}u \log(-q_\phi Q_0(u)) + \mathcal{O}(\epsilon_2^0)\right],$$
(4.9)

with the expansion coefficients

$$G_1(u-u') = L(u-u'+\epsilon_1) - L(u-u'-\epsilon_1), \quad \text{and} \quad Q_0(u) = Q(u)|_{\epsilon_2=0} = \frac{\vartheta_1(u \pm m)}{\vartheta_1(u \pm \frac{\epsilon_1}{2})}, \quad (4.10)$$

with $L(\cdot)$ introduced in (2.17). As in (3.19), one may define

$$\mathcal{Y}(u) = \exp\left[-\int \mathrm{d}u'\rho(u') \frac{\vartheta_1'(u-u')}{\vartheta_1(u-u')}\right], \quad (4.11)$$

such that the saddle point equation can be written as

$$\log\left(-q_\phi \frac{\mathcal{Y}(u_* - \epsilon_1)}{\mathcal{Y}(u_* + \epsilon_1)} Q_0(u_*)\right) = 0, \quad \text{or} \quad 1 + q_\phi \frac{\mathcal{Y}(u_* - \epsilon_1)}{\mathcal{Y}(u_* + \epsilon_1)} Q_0(u_*) = 0 \quad (4.12)$$

for certain specified solutions ρ_* and u_*.

On the other hand, one can apply the saddle point equation to the normalised $Z_{\text{str}}^{\text{def}}$ and take the NS-limit, $\epsilon_2 \to 0$ and $q \to 1$,

$$\widetilde{Z}_{\text{str}}^{\text{def}}(x) \equiv \lim_{\epsilon_2 \to 0} \frac{Z_{\text{str}}^{\text{def}}}{Z_{\text{str}}} = \exp\left[\int \mathrm{d}u\rho_*(u)\mathcal{V}_1(u)\right], \quad (4.13)$$

with
$$\mathcal{V}_1(u) = \log V(u)|_{\mathcal{O}(\epsilon_2^1)} = L\left(u - 2x - \frac{\epsilon_1}{2}\right) = -L\left(2x + \frac{\epsilon_1}{2} - u\right). \quad (4.14)$$

Therefore, by the virtue of (4.11), one finds

$$\widetilde{Z}_{\text{str}}^{\text{def}}(x) = \mathcal{Y}\left(2x + \frac{\epsilon_1}{2}\right) \quad (4.15)$$

For a shift operator $Y : x \mapsto x - \epsilon_1$, the action on the partition function is

$$Y \cdot \widetilde{Z}_{\text{str}}^{\text{def}}(x) = \mathcal{Y}\left(2x - \frac{3\epsilon_1}{2}\right) = \frac{\mathcal{Y}\left(2x - \frac{3\epsilon_1}{2}\right)}{\mathcal{Y}\left(2x + \frac{\epsilon_1}{2}\right)} \widetilde{Z}_{\text{str}}^{\text{def}}(x). \quad (4.16)$$

Next, consider the left-hand-side of (4.12) for arbitrary values of u, i.e.

$$\mathcal{L} \equiv 1 + q_\phi \frac{\mathcal{Y}(u - \epsilon_1)}{\mathcal{Y}(u + \epsilon_1)} Q_0(u). \quad (4.17)$$

whose purpose is clarified shortly. For $u = 2x - \frac{\epsilon_1}{2}$, one has

$$Q_0\left(2x - \frac{\epsilon_1}{2}\right) = \frac{\vartheta_1(2x \pm m - \epsilon_1/2)}{\vartheta_1(2x)\vartheta_1(2x - \epsilon_1)} \equiv \frac{\widetilde{\theta}_1(p^{-1}X\eta^{-1})\widetilde{\theta}_1(p^{-2}X\eta)}{\widetilde{\theta}_1(p^{-1}X)\widetilde{\theta}_1(p^{-2}X)}, \quad (4.18)$$

due to (A.14). To compare with the results in [64], one defines the following variables

$$X \equiv e^{2x+\epsilon_1} = t^{-2} \quad \text{and} \quad \eta \equiv e^{m+\epsilon_1/2} = \sqrt{pq}\mu_1. \quad (4.19)$$

Therefore, using (4.16), one finds

$$\mathcal{L} = 1 + q_\phi \frac{\widetilde{\theta}_1(p^{-1}X\eta^{-1})\widetilde{\theta}_1(p^{-2}X\eta)}{\widetilde{\theta}_1(p^{-1}X)\widetilde{\theta}_1(p^{-2}X)} \cdot \frac{\mathcal{Y}(2x - \frac{3\epsilon_1}{2})}{\mathcal{Y}(2x + \frac{\epsilon_1}{2})}$$

$$= 1 + q_\phi \frac{\widetilde{\theta}_1(p^{-1}X\eta^{-1})\widetilde{\theta}_1(p^{-2}X\eta)}{\widetilde{\theta}_1(p^{-1}X)\widetilde{\theta}_1(p^{-2}X)} \frac{Y \cdot \widetilde{Z}_{str}^{def}(x)}{\widetilde{Z}_{str}^{def}(x)} . \tag{4.20}$$

Notice that $YX = p^{-2}XY$, for convenience, one defines

$$Y_X X := p^{-1}XY_X \qquad \text{meaning} \qquad Y_X^2 = Y , \tag{4.21a}$$

$$\widetilde{Z}(X) := \widetilde{Z}_{str}^{def}(x) , \qquad \text{such that} \qquad \widetilde{Z}(p^{-1}X) = Y_X \widetilde{Z}_{str}^{def}(x) . \tag{4.21b}$$

Now (4.20) can be recast as

$$Y_X^{-1} \cdot \widetilde{Z}(p^{-1}X) + q_\phi \frac{\widetilde{\theta}_1(p^{-1}X\eta^{-1})\widetilde{\theta}_1(p^{-2}X\eta)}{\widetilde{\theta}_1(p^{-1}X)\widetilde{\theta}_1(p^{-2}X)} Y_X \cdot \widetilde{Z}(p^{-1}X) = \mathcal{L} \, Y_X^{-1} \cdot \widetilde{Z}(p^{-1}X) . \tag{4.22}$$

Lastly, one shifts $X \to pX$ and re-defines the right-hand-side of (4.22) to be

$$Y_X^{-1} \cdot \widetilde{Z}(X) + q_\phi \frac{\widetilde{\theta}_1(X\eta^{-1})\widetilde{\theta}_1(p^{-1}X\eta)}{\widetilde{\theta}_1(X)\widetilde{\theta}_1(p^{-1}X)} Y_X \cdot \widetilde{Z}(X) =: \mathcal{W}(X) \cdot \widetilde{Z}(X) , \tag{4.23}$$

where $\mathcal{W}$ is identified with the 6d partition function of the codimension 4 defect, i.e. the Wilson surface, in section 4.1.3. Therefore, (4.23) is exactly the difference equation obtained from 5d/3d perspective in [64].

4.1.2 Perturbative part

Next, the difference equation for the perturbative part of the $\mathcal{N} = (2,0)$ A_1 theory is derived. As above, the starting point is the 6d $\mathcal{N} = (1,0)$ SU(2) theory with 4 flavours, whose perturbative contributions to the partition function are given by

$$Z_{\mathcal{N}=(1,0)\,A_1}^{\text{pert}} = \text{PE}[I_t + I_v + I_h] , \tag{4.24a}$$

$$\text{with} \qquad I_t = -\frac{p+q}{(1-p)(1-q)} \frac{Q}{1-Q} , \tag{4.24b}$$

$$I_v = -\frac{1+pq}{(1-p)(1-q)(1-Q)} \left(\alpha^2 + \alpha^{-2}Q + Q \right) , \tag{4.24c}$$

$$I_h = \frac{\sqrt{pq}}{(1-p)(1-q)(1-Q)} \left(\alpha + \alpha^{-1}Q \right) \left(t + t^{-1} \right) \left(\mu_1 + \mu_1^{-1} + \mu_2 + \mu_2^{-1} \right) , \tag{4.24d}$$

where the contributions of the tensor, vector, and hyper multiplets I_t, I_v and I_h, respectively, have been *flopped* compared to (2.4), for comparison with the 5d result.

Before introducing the codimension 2 defect, one first computes the contribution of Goldstone bosons from the usual Higgsing procedure by assigning

$$\alpha = t^{-1} \qquad \text{and} \qquad \mu_2 = \sqrt{pq} . \tag{4.25}$$

The Goldstone boson part is given by

$$
\begin{aligned}
Z_G &= \mathrm{PE}\left[\frac{\sqrt{pq}}{(1-p)(1-q)(1-Q)} \left(\alpha + \alpha^{-1}Q \right) \left(t + t^{-1} \right) \left(\mu_1 + \mu_1^{-1} \right) \right]\Bigg|_{\alpha = t^{-1}} \\
&= \mathrm{PE}\left[\frac{\sqrt{pq}}{(1-p)(1-q)(1-Q)} \left(t^{-1} + tQ \right) \left(t + t^{-1} \right) \left(\mu_1 + \mu_1^{-1} \right) \right]. \quad (4.26)
\end{aligned}
$$

With this preparation, one can introduce a $(0,1)$ codimension 2 defect as in (4.3). The partition function $Z^{\mathrm{pert}}_{\mathcal{N}=(1,0)\ A_1}$ can be factorised as

$$
Z^{\mathrm{pert}}_{\mathcal{N}=(1,0)\ A_1} = Z^{\mathrm{pert}}_{\mathcal{N}=(2,0)\ A_1} \cdot Z_G \cdot Z^{\mathrm{def}}_{\mathrm{pert}}(X), \quad (4.27)
$$

where only $Z^{\mathrm{def}}_{\mathrm{pert}}(X)$ is a function of the defect parameter X. Using (4.3), computing I_v and the μ_2-dependent part of I_h leads to

$$
\begin{aligned}
Z_1 &= \mathrm{PE}\left[I_v + \frac{\sqrt{pq}}{(1-p)(1-q)(1-Q)} \left(\alpha + \alpha^{-1}Q \right) \left(t + t^{-1} \right) \left(\mu_2 + \mu_2^{-1} \right) \right]\Bigg|_{\substack{\alpha = \sqrt{q}t^{-1} \\ \mu_2 = \sqrt{pq^2}}} \\
&= \mathrm{PE}\left[\frac{1}{(1-p)(1-Q)} \left(t^{-2} - t^2 p\, Q \right) \right] + \text{etc.} \\
&= \prod_{i=0}^{\infty} \frac{1}{\widetilde{\theta}_1(Xp^i)} + \text{etc.} \quad (4.28)
\end{aligned}
$$

using (4.19) in the last line. Further, all irrelevant terms independent of the defect parameter X have been omitted.

On the other hand, one also needs to extract additional $Z^{\mathrm{def}}_{\mathrm{pert}}(X)$ contributions,[2] which are μ_1-dependent, from the Goldstone part Z_G. In detail

$$
Z_2 = \mathrm{PE}\left[\frac{\sqrt{pq}}{(1-p)(1-q)(1-Q)} \left(\alpha + \alpha^{-1}Q \right) \left(t + t^{-1} \right) \left(\mu_1 + \mu_1^{-1} \right) \right]\Bigg|_{\substack{\alpha = \sqrt{q}t^{-1} \\ \mu_2 = \sqrt{pq^2} \\ \mu_1 \to \sqrt{q}\mu_1}} \cdot \frac{1}{Z_G}, \quad (4.29)
$$

where one has shifted $\mu_1 \to \sqrt{q}\mu_1$ in order to compare with the contribution of Goldstone bosons. With some algebra, apart from some irrelevant terms, one finds

$$
\begin{aligned}
Z_2 &= \mathrm{PE}\left[\frac{1}{(1-p)(1-Q)} \left(-X\eta + (X\eta)^{-1}p\, Q \right) \right] \\
&= \prod_{i=0}^{\infty} \widetilde{\theta}_1(X\eta\, p^i), \quad (4.30)
\end{aligned}
$$

using (4.19). Hence, combining the various parts, one arrives at

$$
Z^{\mathrm{def}}_{\mathrm{pert}}(X) = Z_1 \cdot Z_2 = \prod_{i=0}^{\infty} \frac{\widetilde{\theta}_1(X\eta\, p^i)}{\widetilde{\theta}_1(Xp^i)}. \quad (4.31)
$$

---

[2]Different from the generic $\mathcal{N}=(1,0)$ case, there is an additional contribution depending on the defect parameter X and flavour fugacity η. Because the $(2,0)$ A_1 theory contains no vector multiplet, the new piece thus originates from the term depending on the a_1 gauge fugacity and the left flavour fugacity η. Since the SU(2) fugacities satisfy $a_1 + a_2 = 0$, both a_1 *and* a_2 have been replaced by the defect parameter X after Higgsing.

By acting with Y_X on it, one finds the following difference equation

$$Y_X \cdot Z^{\text{def}}_{\text{pert}}(X) = \frac{\theta(p^{-1}X\eta)}{\theta(p^{-1}X)} Z^{\text{def}}_{\text{pert}}(X) \,. \tag{4.32}$$

Therefore, the full partition function

$$Z(X) = Z^{\text{def}}_{\text{pert}}(X) \cdot \widetilde{Z}(X) \,, \tag{4.33}$$

satisfies the following difference equation in the NS-limit $q \to 1$:

$$\frac{\widetilde{\theta}_1(X\eta)}{\widetilde{\theta}_1(X)} Y_X^{-1} \cdot Z(X) + q_\phi \frac{\widetilde{\theta}_1(X\eta^{-1})}{\widetilde{\theta}_1(X)} Y_X \cdot Z(X) =: \mathcal{W}(X) \cdot Z(X)$$

$$\Leftrightarrow \qquad \left[\frac{\widetilde{\theta}_1(X\eta)}{\widetilde{\theta}_1(X)} Y_X^{-1} + q_\phi \frac{\widetilde{\theta}_1(X\eta^{-1})}{\widetilde{\theta}_1(X)} Y_X - \mathcal{W}(X) \right] Z(X) = 0 \,, \tag{4.34}$$

where the last line already bears resemblance to (4.50). As in (4.23), one still has to provide an interpretation of $\mathcal{W}$, which is the subject of the next section.

4.1.3 Wilson surface

In this subsection, $\mathcal{W}$ is identified with the Wilson surface from 6d perspective as discussed above for the generic 6d $\mathcal{N} = (1,0)$ case. As in section 3.4, the identification proceeds in two steps:

(i) Computation of the prediction for $\mathcal{W}$ from the difference equation (4.23).

(ii) Direct evaluation of the Wilson surface expectation value.

Firstly, one computes $\mathcal{W}$ from (4.23) up to one-instanton order. A computation shows that

$$\begin{aligned}
\mathcal{W} &= \frac{Y_X^{-1}\widetilde{Z}(X)}{\widetilde{Z}(X)} + q_\phi \frac{\widetilde{\theta}_1(X\eta^{-1})\widetilde{\theta}_1(p^{-1}X\eta)}{\widetilde{\theta}_1(X)\widetilde{\theta}_1(p^{-1}X)} \frac{Y_X \widetilde{Z}(X)}{\widetilde{Z}(X)} \\
&= 1 + q_\phi \left(\widetilde{Z}_1(pX) - \widetilde{Z}_1(X) + \frac{\widetilde{\theta}_1(X\eta^{-1})\widetilde{\theta}_1(p^{-1}X\eta)}{\widetilde{\theta}_1(X)\widetilde{\theta}_1(p^{-1}X)} \right) + \mathcal{O}(q_\phi^2) \\
&= 1 + q_\phi \left(\frac{\widetilde{\theta}_1(pX\eta^{-1})\widetilde{\theta}_1(X\eta)}{\widetilde{\theta}_1(X)\widetilde{\theta}_1(pX)} + pX \frac{\widetilde{\theta}_1(\eta)\widetilde{\theta}_1(p\eta^{-1})\widetilde{\theta}_1'(pX)}{\widetilde{\theta}_1'(1)\widetilde{\theta}_1(p)\widetilde{\theta}_1(pX)} - X \frac{\widetilde{\theta}_1(\eta)\widetilde{\theta}_1(p\eta^{-1})\widetilde{\theta}_1'(X)}{\widetilde{\theta}_1'(1)\widetilde{\theta}_1(p)\widetilde{\theta}_1(X)} \right) \\
&\quad + \mathcal{O}(q_\phi^2) \,, \tag{4.35}
\end{aligned}$$

where $\widetilde{\theta}_1'(X)$ denotes the derivative of $\widetilde{\theta}_1(X)$. As it turns out, the $\mathcal{W}$ expression is independent on X, as can be verified by expanding (4.35) with respect to Q, i.e.

$$\begin{aligned}
\mathcal{W} &= 1 + q_\phi \left(1 + \frac{(1-\eta)^2(1-p^{-1}\eta)^2}{p^{-1}\eta^2} Q + \frac{(1-\eta)^2(1-p^{-1}\eta)^2(p^{-2}+4p^{-1}+1)}{p^{-2}\eta^2} Q^2 + \mathcal{O}(Q^3) \right) \\
&\quad + \mathcal{O}(q_\phi^2) \,. \tag{4.36}
\end{aligned}$$

In fact, $q_\phi^{-1/2}\mathcal{W}$ coincides with the Wilson line $\mathcal{W}_{\text{SU}(2)}$ computed from the 5d SU(2) SYM via compactifying the 6d theory on a circle as in [60, 64].

Secondly, one can directly compute the expectation value of the 6d Wilson surface in the 6d $\mathcal{N} = (2,0)$ A_1 theory, as studied in [60]. For a Wilson surface in a minuscule representation, for instance the fundamental representation, one finds either from [60] or section 2.4 that

$$\mathcal{W}^{(2,0)} = \sum_{l=0}^{\infty} q_\phi^l W_l \tag{4.37}$$

$$W_l = \frac{1}{l!} \int \prod_{I=1}^{l} \frac{\mathrm{d}\phi_I}{2\pi i} \prod_{I,J=1}^{l} \frac{\vartheta_1(\phi_{IJ})\vartheta_1(\phi_{IJ}+2\epsilon_+)}{\vartheta_1(\phi_{IJ}+\epsilon_{1,2})} \prod_{I=1}^{l} \frac{\vartheta_1(m\pm\phi_I)}{\vartheta_1(\epsilon_+\pm\phi_I)} \prod_{I=1}^{l} \frac{\vartheta_1(\epsilon_-\pm(\phi_I-z))}{\vartheta_1(-\epsilon_+\pm(\phi_I-z))},$$

where z denotes the U(1) fugacity from D4′ brane, see table 1. Up to one-instanton order, one finds

$$\mathcal{W}^{(2,0)} = 1 + q_\phi \left(\frac{\widetilde{\theta}_1(pZ\eta^{-1})\widetilde{\theta}_1(Z\eta)}{\widetilde{\theta}_1(pZ)\widetilde{\theta}_1(Z)} + pZ\frac{\widetilde{\theta}_1(\eta)\widetilde{\theta}_1(p\eta^{-1})\widetilde{\theta}_1'(pZ)}{\widetilde{\theta}_1(p)\widetilde{\theta}_1(pZ)\widetilde{\theta}_1'(1)} - Z\frac{\widetilde{\theta}_1(\eta)\widetilde{\theta}_1(p\eta^{-1})\widetilde{\theta}_1'(Z)}{\widetilde{\theta}_1(q)\widetilde{\theta}_1(Z)\widetilde{\theta}_1'(1)} \right)$$
$$+ \mathcal{O}(q_\phi^2), \tag{4.38}$$

which is the same as (4.35) by replacing $Z \equiv e^z$ with X. However, as shown in (4.36), $\mathcal{W}^{(2,0)}$ is independent of Z or X. As a consequence, the direct 6d computation of the Wilson surface, which coincides with the 5d Wilson loop result [60], also verifies the quantised SW-curve (4.23) proposed in the subsection above for the 6d $\mathcal{N} = (2,0)$ A_1 case.

4.2 Codimension 2 defect in 5d $\mathcal{N} = 2$ SU(2) SYM

A circle compactification of the 6d $\mathcal{N} = (2,0)$ A_1 theory gives rise to the 5d $\mathcal{N} = 2$ maximal supersymmetric Yang-Mills theory with gauge group SU(2). In fact, the instanton states in this 5d theory capture the Kaluza-Klein momentum modes. Therefore, the 5d SU(2) maximal SYM theory at strong coupling is conjectured to be dual to the 6d $\mathcal{N} = (2,0)$ A_1 theory [67–69].

A codimension 2 defect preserving half of the supersymmetries in the 5d $\mathcal{N} = 2$ SU(2) gauge theory has been studied in [64]. This defect was introduced as a monodromy defect. However, the same defect can also be introduced by Higgsing the SU(2) × SU(2) affine quiver theory with two bi-fundamental hypermultiplets with position dependent VEV of a baryonic operators formed by one of the bi-fundamental hypermultiplets. In terms of a 8 supercharges quiver, the Higgsing is summarised as follows:

Equivalently, the Higgsing of a 5d $\mathcal{N} = 1$ affine A_k quiver gauge theory with a constant or position dependent VEV is realised in Type IIB superstring theory as shown in figure 1. From the 6d viewpoint, this corresponds to a mesonic Higgsing of the SU(2) gauge theory with 4 flavours towards the $\mathcal{N} = (2,0)$ A_1 theory with a codimension 2 defect. The duality between the 5d and 6d description can be verified on the level of partition functions.

Partition function before Higgsing. Let us start with the partition function of the 5d $SU(2) \times SU(2)$ affine quiver gauge theory on $\mathbb{R}^4_{\epsilon_1,\epsilon_2} \times S^1$. The perturbative partition function can be written as

$$Z^{\text{5d,pert}}_{SU(2)^2} = \text{PE}\left[-\frac{1+pq}{(1-p)(1-q)}(A_1^2+A_2^2) + \frac{\sqrt{pq}}{(1-p)(1-q)}A_1(A_2+A_2^{-1})(\mu_1+\mu_2+\mu_1^{-1}+\mu_2^{-1})\right], \tag{4.40}$$

where $A_{1,2} \equiv e^{a_{1,2}}$ are the gauge fugacities for two $SU(2)$ gauge groups in (4.39) and $\mu_1 \equiv e^{M_1}$, $\mu_2 \equiv e^{M_2}$ are the fugacities for the bi-fundamental flavours. The instanton partition function can be evaluated from a 1d gauged quantum mechanics and is given by

$$Z^{\text{5d,inst}}_{SU(2)^2} = \sum_{k_1,k_2=0}^{\infty} y^{k_1}\left(\frac{Q}{y}\right)^{k_2} Z^{5d}_{k_1,k_2} \tag{4.41}$$

$$\begin{aligned}
Z^{5d}_{k_1,k_2} = &\frac{1}{k_1!k_2!} \oint \left(\prod_{I=1}^{k_1}\frac{\mathrm{d}\phi_I}{2\pi\mathrm{i}}\right)\left(\prod_{J=1}^{k_2}\frac{\mathrm{d}\tilde{\phi}_J}{2\pi\mathrm{i}}\right) \frac{\prod_{I\neq J}^{k_1}\text{sh}(\phi_{IJ})\prod_{I,J}^{k_1}\text{sh}(\phi_{IJ}+2\epsilon_+)}{\prod_{I,J}^{k_1}\text{sh}(\phi_{IJ}+\epsilon_{1,2})} \\
&\cdot \frac{\prod_{I\neq J}^{k_2}\text{sh}(\tilde{\phi}_{IJ})\prod_{I,J}^{k_2}\text{sh}(\tilde{\phi}_{IJ}+2\epsilon_+)}{\prod_{I,J}^{k_2}\text{sh}(\tilde{\phi}_{IJ}+\epsilon_{1,2})} \\
&\cdot \prod_{I=1}^{k_1}\prod_{J=1}^{k_2} \frac{\text{sh}(\phi_I\pm a_2+M_{1,2})\text{sh}(\tilde{\phi}_J\pm a_1-M_{1,2})}{\text{sh}(\phi_I\pm a_1\pm\epsilon_+)\text{sh}(\tilde{\phi}_J\pm a_2\pm\epsilon_+)} \cdot \frac{\text{sh}(\phi_I-\tilde{\phi}_J+M_{1,2}\pm\epsilon_-)}{\text{sh}(\phi_I-\tilde{\phi}_J+M_{1,2}\pm\epsilon_+)},
\end{aligned}$$

where y and Q/y are the instanton fugacities for the $SU(2)$ gauge groups, respectively, and $\text{sh}(x) \equiv 2\sinh(\frac{x}{2})$ as well as $\phi_{IJ} = \phi_I - \phi_J$, $\tilde{\phi}_{IJ} = \tilde{\phi}_I - \tilde{\phi}_J$. The contour integral (4.41) at each instanton sector can again be evaluated by using the JK-prescription [70]. As expected from the duality between the 5d $SU(2)\times SU(2)$ affine quiver gauge theory and the 6d SCFT for 2 M5-branes on A_1 singularity, the full partition function for the 5d $SU(2)\times SU(2)$ affine quiver theory coincides with the partition function of the 6d SCFT given in section 2.1. Namely,

$$\begin{aligned}
Z^{\text{6d}}_{\mathcal{N}=(1,0)\ A_1} &= Z^{\text{5d}}_{SU(2)^2} \cdot Z_{\text{extra}} \\
Z^{\text{6d}}_{\mathcal{N}=(1,0)\ A_1} &\equiv Z^{\text{pert}}_{\mathcal{N}=(1,0)\ A_1} \cdot \sum_{l=0}^{\infty} q_\phi^l Z_l, \quad Z^{\text{5d}}_{SU(2)^2} \equiv Z^{\text{5d,pert}}_{SU(2)^2}\cdot Z^{\text{5d,inst}}_{SU(2)^2},
\end{aligned} \tag{4.42}$$

with the identification of the 5d/6d fugacities as

$$(A_1, A_2, y, \mu_{1,2})^{5d} = (q_\phi^{1/2}\alpha^{-1}t, q_\phi^{1/2}, t^2, \mu_{1,2})^{6d}. \tag{4.43}$$

Here, $Z^{\text{pert}}_{\mathcal{N}=(1,0)\ A_1}$ and Z_l are given in (4.24) and (4.2), respectively; and Z_{extra} is an extra factor independent of dynamical fugacities defined as

$$Z_{\text{extra}} = \text{PE}\left[-\frac{(1+p)(1+q)Q}{(1-p)(1-q)(1-Q)} - \frac{\left(\frac{t^2\mu_1}{\mu_2}+\frac{\mu_2}{t^2\mu_1}Q+\frac{pqt^2\mu_2}{\mu_1}+\frac{pq\mu_1}{t^2\mu_2}Q\right)}{(1-p)(1-q)(1-Q)}\right]. \tag{4.44}$$

One can check the equality (4.42) by expanding both sides in terms of Q and q_ϕ.

Higgsing. Higgsing (4.39) to the 5d $\mathcal{N} = 2$ SU(2) gauge theory can be performed by tuning the fugacities in the partition function as

$$A_1 \to A_2, \quad \mu_2 \to \frac{1}{\sqrt{pq}}. \tag{4.45}$$

This leads to the partition function of the 5d $\mathcal{N} = 2$ SU(2) gauge theory as

$$Z_{\mathrm{SU(2)}^2}^{5d}\bigg|_{\substack{A_1 \to A_2 \\ \mu_2 \to 1/\sqrt{pq}}} = Z_{\mathcal{N}=2 \; \mathrm{SU(2)}}^{5d} \cdot Z_{\mathrm{extra'}}, \quad Z_{\mathrm{extra'}} = \mathrm{PE}\left[-\frac{(1-pq)\left(1 - \frac{pq}{\mu_1^2}\right)\mu_1}{\sqrt{pq}(1-p)(1-q)}y\right], \tag{4.46}$$

up to the extra factor $Z_{\mathrm{extra'}}$ independent of the dynamical fugacity A_2. After the Higgsing, A_2 becomes the fugacity for the SU(2) gauge symmetry and μ_1 becomes the fugacity for the SU(2) $\subset$ SO(5) flavour symmetry.

Next consider the Higgsing with a position dependent VEV that introduces a codimension 2 defect in the 5d $\mathcal{N} = 2$ SU(2) theory. The Higgsing can achieve by the following fugacity assignment:

$$A_1 \to A_2\sqrt{q}, \quad \mu_2 \to \frac{1}{\sqrt{pq^2}}, \quad \mu_1 \to \mu_1\sqrt{q}. \tag{4.47}$$

With this specialisation of the fugacities, the partition function reduces to that of the 5d $\mathcal{N} = 2$ SU(2) theory in the presence of the monodromy defect, called $\mathcal{Z}_{[1,1]}$, introduced in [64]:

$$Z_{\mathrm{SU(2)}^2}^{5d}\bigg|_{\substack{A_1 \to A_2\sqrt{q} \\ \mu_2 \to 1/\sqrt{pq^2} \\ \mu_1 \to \mu_1\sqrt{q}}} = \mathcal{Z}_{[1,1]} \cdot Z_{\mathrm{extra'}}. \tag{4.48}$$

This shows that the codimension 2 defect introduced by the Higgsing is identical to the monodromy defect considered in [64]. The instanton part of the codimension 2 defect partition function is expanded in terms of y and Q/y, and the first few terms are given by

$$\mathcal{Z}_{[1,1]}^{\mathrm{inst}} = 1 - \frac{(\eta-1)(\eta - A_2^2)}{(1-p)(1 - A_2^2/p)\eta}y - \frac{(\eta-1)(1 - \eta q A_2^2)p}{(1-p)(1 - pq A_2^2)\eta}\frac{Q}{y} + \dots, \tag{4.49}$$

with η defined in (4.19).

In the NS limit $q \to 1$, the codimension 2 defect partition function satisfies the following difference equation [64]:

$$\left[A_2^{-1}\frac{\tilde{\theta}(y\eta)}{\tilde{\theta}_1(y)}Y_y^{-1} + A_2\frac{\tilde{\theta}_1(y/\eta)}{\tilde{\theta}_1(y)}Y_y - \langle W_{\mathrm{SU(2)}}\rangle\right]\lim_{q\to 1}\mathcal{Z}_{[1,1]}^{\mathrm{inst}} = 0, \tag{4.50}$$

where $Y_y y = py Y_y$ and $\langle W_{\mathrm{SU(2)}}\rangle$ is the SU(2) fundamental Wilson loop expectation value in the 5d maximal SYM discussed in [59, 64]. This is the difference equation of the two-body elliptic Ruijsenaars-Schneider integrable system. Here, the Wilson loop expectation value $\langle W_{\mathrm{SU(2)}}\rangle$ of the 5d theory is related to the VEV of Wilson surface $\mathcal{W}^{(2,0)}$ in the 6d (2,0) A_1 theory in [60] as

$$\mathcal{W}^{(2,0)} = q_\phi^{1/2}\langle W_{\mathrm{SU(2)}}\rangle. \tag{4.51}$$

One can verify that, by replacing

$$X = y^{-1} \quad \text{and} \quad Y_X = \eta^{-1} Y_y^{-1}, \tag{4.52}$$

eq. (4.34) becomes (4.50). Hence, the difference equations agree.

5 Conclusions

In this paper we explored elliptic difference equations arising from quantisation of Seiberg-Witten curves of compactified 6d A-type $\mathcal{N} = (1,0)$ SCFTs. In order to obtain a 4d $\mathcal{N} = 2$ supersymmetric theory, the 6d theory is compactified on a two-torus together with an Omega-background. This allows to compute the BPS partition function of the theory together with expectation values of various defect operators using localisation. We explicitly showed, using a matrix-model approach, that the corresponding quantum curves annihilate expectation values of codimension 2 surface defects inside the 6d theory. Moreover, we found that our difference equations can be rewritten as eigenvalue equations with eigenvectors being our codimension 2 defects and eigenvalues corresponding to expectation values of codimension 4 defects arising from Wilson surfaces wrapping the two-torus.

One important insight of our analysis is the fact that our difference operator equally well applies to the 5d dual of the 6d SCFT. This duality, as for example recently explored in [5–7], results in a 5d supersymmetric gauge theory admitting an affine quiver description. In our case, this is an affine A-type quiver with SU(N) gauge nodes [17]. BPS partition functions of the circle-compactified 5d theory are then equal to the torus-compactified 6d partition function. The codimension 2 defect of the 6d theory is mapped to a codimension 2 defect inside the 5d theory giving rise to a coupled 3d/5d system. Difference operators for such systems are not easy to obtain, but our approach via the dual 6d theory gives a recipe to construct such operators from first principles.

Another direction, particularly interesting for future research, is the realisation of 4d $\mathcal{N} = 1$ SCFTs as surface defects inside a 6d SCFT. Indeed, our codimension 2 defect is itself such a 4d theory extended over $\mathbb{T}^2 \times_{\epsilon_2} \mathbb{R}^2$. The expectation value of the defect operator on such a geometry is related to the supersymmetric index of the corresponding 4d $\mathcal{N} = 1$ SCFT and, thus, it is expected that such indices satisfy similar difference equation. For instance, the superconformal indices for the 4d class $\mathcal{S}_k$ theories with surface defects have been computed via the action of a difference operator in [8, 62, 63]. The rough expectation is as follows: the 6d partition function with codimension 2 defect corresponds to the 4d index of the 4d theory obtained via compactifiying the pure 6d theory on a Riemann surface (with punctures). The addition of a surface defect in the 4d theory is realised via a difference operator acting on the 4d index. Similarly, the codimension 4 defect in the 6d theory is introduced via a difference operator, which yields a quantisation of the SW-curve in the NS-limit. Thus, one expects an identification between the 4d difference operator and the 6d difference operator, as the codimension 4 defect in 6d reduces to a surface defect in 4d. In fact, the same codimension 2 defect was interpreted as a flux leading to a minimal puncture on the Riemann surface in the context of the 4d class $\mathcal{S}_k$ theories in [71]. We expect our difference operator is related to a difference operator acting on the flavor

fugacity associated to the minimal puncture in the 4d class $\mathcal{S}_k$ theory. However, further detailed analysis is require for a precise statement.

From this point of view, it would be interesting to ask whether the knowledge of the difference operator is enough to reconstruct the index of the corresponding 4d SCFT. First steps in this direction have been taken in [8, 22, 48]. In particular, in [22] the authors give a detailed derivation of the difference operator associated to $\mathcal{N} = 1$ compactifications of E-string theory. It would be interesting to extend these results by applying our techniques to the torus-compactified E-string theory. The corresponding difference operator should in this case arise from the quantisation of the SW-curve derived in [72]. We leave this and the derivation of quantum curves for a wider class of 6d SCFTs for future work. Likewise, the difference equations of other 6d SCFTs and their relation to integrable models, as for example considered in [54], are interesting future directions.

Acknowledgments

A Details of partition functions

The computational details of the various partition functions are provided in this appendix.

A.1 Perturbative contribution

The perturbtative part of the partition function is composed of the single letter contributions (2.4) for the 6d $\mathcal{N} = (1,0)$ multiplets.

A.1.1 Higgsing: constant VEV

The perturbative part can be written as

$$
\begin{aligned}
Z_{\text{pert}}^{k+1} = Z_{\text{pert}}^k \cdot \text{PE}\Big[& \frac{1}{(1-p)(1-q)} \left(\frac{Q}{1-Q} + \frac{1}{2} \right) \\
& \times \bigg\{ -(1+pq) \sum_{i=1}^{k} \left(e^{a_i - a_{k+1}} + e^{a_{k+1} - a_i} \right) - (1+pq) \\
& + \sqrt{pq} \sum_{l=1}^{k} \left(e^{a_{k+1}}(e^{-m_l+b} + e^{-n_l-b}) + e^{-a_{k+1}}(e^{m_l-b} + e^{n_l+b}) \right) \\
& + \sqrt{pq} \sum_{i=1}^{k} \left(e^{a_i}(e^{-m_{k+1}+b} + e^{-n_{k+1}-b}) + e^{-a_i}(e^{m_{k+1}-b} + e^{n_{k+1}+b}) \right) \\
& + \sqrt{pq} \left(e^{a_{k+1}}(e^{-m_{k+1}+b} + e^{-n_{k+1}-b}) + e^{-a_{k+1}}(e^{m_{k+1}-b} + e^{n_{k+1}+b}) \right) \bigg\} \Big] \quad \text{(A.1)}
\end{aligned}
$$

such that Higgsing (2.28b) yields for the different parts

$$
-(1+pq)\sum_{i=1}^{k}\left(e^{a_i-a_{k+1}}+e^{a_{k+1}-a_i}\right)
$$

$$
=-(1+pq)\sum_{i=1}^{k}\left(\sqrt{pq}\,e^{a_i-m_{k+1}+b}+\frac{1}{\sqrt{pq}}e^{m_{k+1}-b-a_i}\right) \tag{A.2a}
$$

$$
\sqrt{pq}\sum_{l=1}^{k}\left(e^{a_{k+1}}(e^{-m_l+b}+e^{-n_l-b})+e^{-a_{k+1}}(e^{m_l-b}+e^{n_l+b})\right)
$$

$$
=\sqrt{pq}\sum_{l=1}^{k}\left(\frac{1}{\sqrt{pq}}e^{m_{k+1}-b}(e^{-m_l+b}+e^{-n_l-b})+\sqrt{pq}\,e^{-m_{k+1}+b}(e^{m_l-b}+e^{n_l+b})\right) \tag{A.2b}
$$

$$
\sqrt{pq}\sum_{i=1}^{k}\left(e^{a_i}(e^{-m_{k+1}+b}+e^{-n_{k+1}-b})+e^{-a_i}(e^{m_{k+1}-b}+e^{n_{k+1}+b})\right)
$$

$$
=\sqrt{pq}\sum_{i=1}^{k}\left(e^{a_i}(e^{-m_{k+1}+b}+pqe^{-m_{k+1}+b})+e^{-a_i}\left(e^{m_{k+1}-b}+\frac{1}{pq}e^{m_{k+1}-b}\right)\right)
$$

$$
=(1+pq)\sum_{i=1}^{k}\left(\sqrt{pq}\,e^{a_i-m_{k+1}+b}+\frac{1}{\sqrt{pq}}e^{-a_i+m_{k+1}-b}\right) \tag{A.2c}
$$

$$
\sqrt{pq}\left(e^{a_{k+1}}(e^{-m_{k+1}+b}+e^{-n_{k+1}-b})+e^{-a_{k+1}}(e^{m_{k+1}-b}+e^{n_{k+1}+b})\right)
$$

$$
=\sqrt{pq}\left(\frac{1}{\sqrt{pq}}e^{m_{k+1}-b}(e^{-m_{k+1}+b}+pqe^{-m_{k+1}+b})+\sqrt{pq}\,e^{-m_{k+1}+b}\left(e^{m_{k+1}-b}+\frac{1}{pq}e^{m_{k+1}-b}\right)\right)
$$

$$
=2(1+pq) \tag{A.2d}
$$

and collecting all the pieces leads to

$$
Z_{\text{pert}}^{k+1}=Z_{\text{pert}}^{k}\cdot\text{PE}\left[\frac{1}{(1-p)(1-q)}\left(\frac{Q}{1-Q}+\frac{1}{2}\right)\right. \tag{A.3}
$$

$$
\times\left\{-(1+pq)\sum_{i=1}^{k}\left(\sqrt{pq}\,e^{a_i-m_{k+1}+b}+\frac{1}{\sqrt{pq}}e^{m_{k+1}-b-a_i}\right)-(1+pq)\right.
$$

$$
+(1+pq)\sum_{i=1}^{k}\left(\sqrt{pq}\,e^{a_i-m_{k+1}+b}+\frac{1}{\sqrt{pq}}e^{m_{k+1}-b-a_i}\right)+2(1+pq)
$$

$$
\left.\left.+\sqrt{pq}\sum_{l=1}^{k}\left(\frac{1}{\sqrt{pq}}e^{m_{k+1}-b}(e^{-m_l+b}+e^{-n_l-b})+\sqrt{pq}\,e^{-m_{k+1}+b}(e^{m_l-b}+e^{n_l+b})\right)\right\}\right]
$$

$$
=Z_{\text{pert}}^{k}\cdot\text{PE}\left[\frac{1}{(1-p)(1-q)}\left(\frac{Q}{1-Q}+\frac{1}{2}\right)\left\{(1+pq)\right.\right. \tag{A.4}
$$

$$
+\sqrt{pq}\sum_{l=1}^{k}\left(\frac{1}{\sqrt{pq}}e^{m_{k+1}-m_l}+\sqrt{pq}\,e^{m_l-m_{k+1}}\right)
$$

$$
\left.\left.+\sqrt{pq}\sum_{l=1}^{k}\left(\frac{1}{\sqrt{pq}}e^{m_{k+1}-n_l-2b}+\sqrt{pq}\,e^{n_l-m_{k+1}+2b}\right)\right\}\right],
$$

where the additional pieces are attributed to the Goldstone modes for the reduced global symmetry. In detail,

$$
Z_{\mathrm{G}} = \mathrm{PE}\Bigg[\frac{1}{(1-p)(1-q)}\left(\frac{Q}{1-Q}+\frac{1}{2}\right)\Bigg\{ (1+pq)+\sqrt{pq}\sum_{l=1}^{k}\left(\frac{1}{\sqrt{pq}}e^{m_{k+1}-m_l}+\sqrt{pq}\,e^{m_l-m_{k+1}}\right)
$$
$$
+\sqrt{pq}\sum_{l=1}^{k}\left(\frac{1}{\sqrt{pq}}e^{m_{k+1}-n_l-2b}+\sqrt{pq}\,e^{n_l-m_{k+1}+2b}\right)\Bigg\}\Bigg],
$$
$$
= \mathrm{PE}\Bigg[\frac{\sqrt{pq}}{(1-p)(1-q)}\left(\frac{Q}{1-Q}+\frac{1}{2}\right)\Bigg\{ \left(\frac{1}{\sqrt{pq}}+\sqrt{pq}\right)+\sum_{l=1}^{k}\left(\frac{1}{\sqrt{pq}}e^{m_{k+1}-m_l}+\sqrt{pq}\,e^{m_l-m_{k+1}}\right)
$$
$$
+\sum_{l=1}^{k}\left(\frac{1}{\sqrt{pq}}e^{m_{k+1}-n_l-2b}+\sqrt{pq}\,e^{n_l-m_{k+1}+2b}\right)\Bigg\}\Bigg]. \qquad (A.5)
$$

A.1.2 Higgsing: position dependent VEV

Inspecting the different contributions to (A.1) yields for the position dependent Higgsing (2.36b) the following:

$$
-(1+pq)\sum_{i=1}^{k}\left(e^{a_i-a_{k+1}}+e^{a_{k+1}-a_i}\right)
$$
$$
=-(1+pq)\sum_{i=1}^{k}\left(\frac{1}{\sqrt{pq}}\frac{e^{a_i}}{X}+\sqrt{pq}\frac{X}{e^{a_i}}\right) \qquad (A.6a)
$$

$$
\sqrt{pq}\sum_{l=1}^{k}\left(e^{a_{k+1}}(e^{-m_l+b}+e^{-n_l-b})+e^{a_{k+1}}(e^{m_l-b}+e^{n_l+b})\right)
$$
$$
=\sqrt{pq}\sum_{l=1}^{k}\left(\sqrt{pq}X(e^{-m_l}B+e^{-n_l}B^{-1})+\frac{1}{\sqrt{pq}X}(e^{m_l}B^{-1}+e^{n_l}B)\right)
$$
$$
=\sqrt{pq}\sum_{l=1}^{k}\left(\sqrt{pq}XBe^{-m_l}+\frac{1}{\sqrt{pq}XB}e^{m_l}+\sqrt{pq}Xe^{-n_l}+\frac{1}{\sqrt{pq}X}e^{n_l}\right) \qquad (A.6b)
$$

$$
\sqrt{pq}\sum_{i=1}^{k}\left(e^{a_i}(e^{-m_{k+1}+b}+e^{-n_{k+1}-b})+e^{-a_i}(e^{m_{k+1}-b}+e^{n_{k+1}+b})\right)
$$
$$
=\sqrt{pq}\sum_{i=1}^{k}\left(e^{a_i}X^{-1}\left(\frac{1}{pq}+p^rq^s\right)+e^{-a_i}X\left(pq+\frac{1}{p^rq^s}\right)\right)
$$
$$
=\sqrt{pq}\sum_{i=1}^{k}\left(e^{a_i}X^{-1}\left(1+\frac{1}{pq}-1+p^rq^s\right)+e^{-a_i}X\left(1+pq-1+\frac{1}{p^rq^s}\right)\right)
$$
$$
=(1+pq)\sum_{i=1}^{k}\left(\frac{e^{a_i}}{\sqrt{pq}X}+\frac{\sqrt{pq}X}{e^{a_i}}\right)+(1-p^rq^s)\sqrt{pq}\sum_{i=1}^{k}\left(-\frac{e^{a_i}}{X}+\frac{1}{p^rq^s}\frac{X}{e^{a_i}}\right) \qquad (A.6c)
$$

$$
\sqrt{pq}\left(e^{a_{k+1}}(e^{-m_{k+1}+b}+e^{-n_{k+1}-b})+e^{-a_{k+1}}(e^{m_{k+1}-b}+e^{n_{k+1}+b})\right)
$$
$$
=\sqrt{pq}\left(\sqrt{pq}\left(\frac{1}{pq}+p^rq^s\right)+\frac{1}{\sqrt{pq}}\left(pq+\frac{1}{p^rq^s}\right)\right)
$$
$$
=\frac{1}{p^rq^s}(1+p^rq^s)(1+p^{r+1}q^{s+1}) \qquad (A.6d)
$$

and collecting all the pieces leads to

$$
\begin{aligned}
Z_{\text{pert}}^{k+1} = Z_{\text{pert}}^{k} \cdot \text{PE}\Big[& \frac{1}{(1-p)(1-q)}\left(\frac{Q}{1-Q}+\frac{1}{2}\right) \\
& \times \Big\{ -(1+pq)\sum_{i=1}^{k}\left(\frac{1}{\sqrt{pq}}\frac{e^{a_i}}{X}+\sqrt{pq}\frac{X}{e^{a_i}}\right) - (1+pq) \\
& + \sqrt{pq}\sum_{l=1}^{k}\left(\sqrt{pq}XBe^{-m_l}+\frac{1}{\sqrt{pq}XB}e^{m_l}+\sqrt{pq}\frac{X}{B}e^{-n_l}+\frac{B}{\sqrt{pq}X}e^{n_l}\right) \\
& + (1+pq)\sum_{i=1}^{k}\left(\frac{e^{a_i}}{\sqrt{pq}X}+\frac{\sqrt{pq}X}{e^{a_i}}\right) + (1-p^r q^s)\sqrt{pq}\sum_{i=1}^{k}\left(-\frac{e^{a_i}}{X}+\frac{1}{p^r q^s}\frac{X}{e^{a_i}}\right) \\
& + \frac{1}{p^r q^s}(1+p^r q^s)(1+p^{r+1}q^{s+1})\Big\}\Big] \\
= Z_{\text{pert}}^{k} \cdot \text{PE}\Big[& \frac{1}{(1-p)(1-q)}\left(\frac{Q}{1-Q}+\frac{1}{2}\right) \\
& \times \Big\{ -(1+pq)+\sqrt{pq}\sum_{l=1}^{k}\left(\sqrt{pq}XBe^{-m_l}+\frac{1}{\sqrt{pq}XB}e^{m_l}+\sqrt{pq}\frac{X}{B}e^{-n_l}+\frac{B}{\sqrt{pq}X}e^{n_l}\right) \\
& + (1-p^r q^s)\sqrt{pq}\sum_{i=1}^{k}\left(-\frac{e^{a_i}}{X}+\frac{1}{p^r q^s}\frac{X}{e^{a_i}}\right) + \frac{1}{p^r q^s}(1+p^r q^s)(1+p^{r+1}q^{s+1})\Big\}\Big]. \quad \text{(A.7)}
\end{aligned}
$$

Recalling the contribution (2.33) from the Goldstone bosons, one formally arrives at

$$
\begin{aligned}
Z_{\text{pert}}^{k+1} = Z_{\text{pert}}^{k} \cdot Z_G \cdot \text{PE}\Big[& \frac{1}{(1-p)(1-q)}\left(\frac{Q}{1-Q}+\frac{1}{2}\right)\Big\{(1-p^r q^s)\sqrt{pq}\sum_{i=1}^{k}\left(-\frac{e^{a_i}}{X}+\frac{1}{p^r q^s}\frac{X}{e^{a_i}}\right) \\
& + \frac{1}{p^r q^s}(1+p^r q^s)(1+p^{r+1}q^{s+1}) - 2(1+pq)\Big\}\Big] \quad \text{(A.8)}
\end{aligned}
$$

$$
\begin{aligned}
Z_{\text{pert}}^{(r,s)\text{def}} = \text{PE}\Big[& \frac{1}{(1-p)(1-q)}\left(\frac{Q}{1-Q}+\frac{1}{2}\right)\Big\{ -2(1+pq)+\frac{1}{p^r q^s}(1+p^r q^s)(1+p^{r+1}q^{s+1}) \\
& + (1-p^r q^s)\sqrt{pq}\sum_{i=1}^{k}\left(-\frac{e^{a_i}}{X}+\frac{1}{p^r q^s}\frac{X}{e^{a_i}}\right)\Big\}\Big] \\
= \text{PE}\Big[& \frac{1}{(1-p)(1-q)}\left(\frac{Q}{1-Q}+\frac{1}{2}\right)\Big\{ \frac{1}{p^r q^s}(1-p^r q^s)(1-p^{r+1}q^{s+1}) \quad \text{(A.9)} \\
& + (1-p^r q^s)\sqrt{pq}\sum_{i=1}^{k}\left(-\frac{e^{a_i}}{X}+\frac{1}{p^r q^s}\frac{X}{e^{a_i}}\right)\Big\}\Big] \\
= \text{PE}\Big[& \frac{(1-p^r q^s)}{(1-p)(1-q)}\left(\frac{Q}{1-Q}+\frac{1}{2}\right)\Big\{ \frac{(1-p^{r+1}q^{s+1})}{p^r q^s}+\sqrt{pq}\sum_{i=1}^{k}\left(-\frac{e^{a_i}}{X}+\frac{1}{p^r q^s}\frac{X}{e^{a_i}}\right)\Big\}\Big]
\end{aligned}
$$

and (2.45) contains the additional contributions from the codimension 2 defect, i.e.

$$
Z_{\text{pert}}^{k+(r,s)\text{def}} = Z_{\text{pert}}^{k} \cdot Z_{\text{pert}}^{(r,s)\text{def}} \quad \text{(A.10)}
$$

where the Goldstone mode contribution have been removed. Note that $Z_{\text{pert}}^{(r,s)\text{def}} = 1$ for $(r,s)=(0,0)$.

A.2 Elliptic functions

The non-perturbative contributions of the 6d partition function on $\mathbb{T}^2 \times \mathbb{R}^4_{\epsilon_1,\epsilon_2}$ equals the infinite sum of 2d elliptic genera. These elliptic genera are naturally composed of elliptic modular forms, whose definitions and properties are summarised in this appendix.

A.2.1 Theta functions

There are various different definitions; here, the relevant definitions are recalled. Use the conventions $Q = e^{2\pi i \tau}$, $x = e^{2\pi i z}$ and the Dedekind eta function [44, eq. (A.1)]

$$\eta(\tau) = Q^{\frac{1}{24}} \prod_{n=1}^{\infty} (1 - Q^n).$$

(A.11)

Then, the different definitions are as follows:

[44, eq. (A.3)]
[73, eq. (D.6)]
[74, eq. (A.10)]
$$\theta_1(\tau|z) = -i Q^{\frac{1}{8}} x^{\frac{1}{2}} \prod_{k=1}^{\infty} (1 - Q^k)(1 - xQ^k)(1 - x^{-1}Q^{k-1}),$$

(A.12a)

[18, eq. (3.7)]
$$\overline{\theta}_1(\tau|z) = i Q^{\frac{1}{8}} x^{\frac{1}{2}} \prod_{k=1}^{\infty} (1 - Q^k)(1 - xQ^k)(1 - x^{-1}Q^{k-1}),$$

(A.12b)

[73, eq. (D.9)]
[18, eq. (3.43)]
$$\widehat{\theta}_1(\tau|z) = \prod_{n=0}^{\infty} (1 - xQ^n)(1 - Q^{n+1})(1 - x^{-1}Q^{n+1}),$$

(A.12c)

[18, eq. (A.4)]
$$\widetilde{\theta}_1(\tau|z) = \prod_{j=0}^{\infty} (1 - x^{-1}Q^{j+1})(1 - xQ^j).$$

(A.12d)

Notice that $\widehat{\theta}_1(\tau|z)$ has been called *basic pseudo-elliptic θ-function* in [73, appendix D]. For this note, the following definition is useful

$$\vartheta_1(\tau|z) := \frac{\theta_1(\tau|z)}{Q^{\frac{1}{12}}\eta(\tau)} \quad \text{such that} \quad \begin{cases} \vartheta_1(\tau|-z) = -\vartheta_1(\tau|z) \\ \partial_z^k \vartheta_1(\tau|z) = \partial_z^k \theta_1(\tau|z) \\ \lim_{\tau \to i\infty} \vartheta_1(\tau|z) = i\left(\frac{1}{\sqrt{x}} - \sqrt{x}\right) \end{cases}, \quad x \equiv e^{2\pi i z}.$$

(A.13)

Comparison. The differently defined functions are related as follows:

$$\overline{\theta}_1(\tau|z) = -\theta_1(\tau|z),$$

(A.14a)

$$\widehat{\theta}_1(\tau|z) = \frac{x^{\frac{1}{2}}}{i Q^{\frac{1}{8}}} \theta_1(\tau|z),$$

(A.14b)

$$\widetilde{\theta}_1(\tau|z) = \frac{x^{\frac{1}{2}} Q^{-\frac{1}{12}}}{i \eta(\tau)} \theta_1(\tau|z).$$

(A.14c)

Reflection property. Consider the shift property following [44, eq. (A.5)]:

$$\theta_1(\tau|-z) = -\theta_1(\tau|z),$$

(A.15a)

$$\overline{\theta}_1(\tau|-z) = -\overline{\theta}_1(\tau|z),$$

(A.15b)

$$\widehat{\theta}_1(\tau|-z) = -x^{-1}\widehat{\theta}_1(\tau|z),$$

(A.15c)

$$\widetilde{\theta}_1(\tau|-z) = -x^{-1}\widetilde{\theta}_1(\tau|z).$$

(A.15d)

Note that the transformation rule for $\widehat{\theta}_1$ agrees with [73, eq. (D.10)].

Shift properties. Next, compute the shift properties following [44, eq. (A.4)] for $a, b \in \mathbb{Z}$:

$$\theta_1(\tau | z + a + b\tau) = (-1)^{a+b} x^{-b} Q^{-\frac{b^2}{2}} \theta_1(\tau | z) , \tag{A.16a}$$

$$\overline{\theta}_1(\tau | z + a + b\tau) = (-1)^{a+b} x^{-b} Q^{-\frac{b^2}{2}} \overline{\theta}_1(\tau | z) , \tag{A.16b}$$

$$\widehat{\theta}_1(\tau | z + a + b\tau) = (-1)^b x^{-b} Q^{-\frac{b(b-1)}{2}} \widehat{\theta}_1(\tau | z) , \tag{A.16c}$$

$$\widetilde{\theta}_1(\tau | z + a + b\tau) = \eta(\tau)(-1)^b x^{-b} Q^{-\frac{b(b-1)}{2}} \widetilde{\theta}_1(\tau | z) . \tag{A.16d}$$

Residue. According to [43, eq. (B.7)] or [44, eq. (A.7)], the residue at the pole $a + b\tau$ is

$$\frac{1}{2\pi i} \oint_{u=a+b\tau} \frac{du}{\theta_1(u)} = \frac{(-1)^{a+b} e^{i\pi b^2 \tau}}{2\pi \eta^3} \quad \Rightarrow \quad \oint_{u=0} \frac{du}{\theta_1(u)} = \frac{i}{\eta^3} , \tag{A.17a}$$

which implies

$$\oint_{u=0} \frac{du}{\vartheta_1(u)} = \frac{iQ^{\frac{1}{12}} \eta}{\eta^3} = \frac{iQ^{\frac{1}{12}}}{\eta^2} \tag{A.17b}$$

for the modified function (A.13).

A.2.2 Hierarchy of multiple elliptic gamma functions

Following for instance [18, appendix A], the definition of the multiple elliptic gamma function $G_r(z | \underline{\tau})$ includes

$$G_0(z | \tau) = \widetilde{\theta}_1(z, \tau) \qquad \text{and} \qquad G_1(z | \tau, \sigma) = \Gamma(z, \tau, \sigma) , \tag{A.18}$$

see (A.14) for the definition of $\widetilde{\theta}_1$. These functions satisfy the following useful identity

$$G_r(z + \tau_j | \underline{\tau}) = G_{r-1}(z | \underline{\tau}^-(j)) G_r(z | \underline{\tau}) , \tag{A.19}$$

such that one finds

$$\log \widetilde{\theta}_1(z, \tau) = \log \Gamma(z + \epsilon_1, \tau, \epsilon_1) - \log \Gamma(z, \tau, \epsilon_1) . \tag{A.20}$$

A.3 Conventions for NS-limit

The NS-limit $\epsilon_2 \to 0$ only yields a finite defect partition function if a suitable normalisation is chosen. In addition, the expansion coefficients in the ϵ_2 expansion need to be defined.

A.3.1 Normalised defect partition function

For the 6d theory with and without a defect, one has the following q_ϕ expansions:

$$Z^{6d} = Z^{6d}_{\text{pert}} \left(1 + \sum_{l=1}^{\infty} Z_l^{6d} \right) , \qquad Z^{6d+\text{def}} = Z^{6d}_{\text{pert+def}} \left(1 + \sum_{l=1}^{\infty} Z_l^{6d+\text{def}} \right) , \tag{A.21}$$

such that the *normalised defect partition function* is defined as

$$\widetilde{Z}^{\text{6d+def}} := \frac{Z^{\text{6d+def}}}{Z^{\text{6d}}} \equiv \widetilde{Z}^{\text{6d+def}}_{\text{pert}} \left(1 + \sum_{l=1}^{\infty} \widetilde{Z}^{\text{6d+def}}_l q_\phi^l \right) . \tag{A.22}$$

The q_ϕ expansion of the normalisation factor reads

$$\frac{1}{Z^{6d}} = \frac{1}{Z^{\text{6d}}_{\text{pert}}} \left[1 - Z^{\text{6d}}_1 \phi - \left(Z^{\text{6d}}_2 - \left(Z^{\text{6d}}_1 \right)^2 \right) \phi^2 - \left(Z^{\text{6d}}_3 - 2 Z^{\text{6d}}_1 Z^{\text{6d}}_2 + \left(Z^{\text{6d}}_1 \right)^3 \right) \phi^3 + \mathcal{O}(\phi^4) \right] \tag{A.23}$$

and the standard expansion coefficients of the normalised defect partition function $\widetilde{Z}^{6d+\text{def}}$ are

$$\widetilde{Z}^{\text{6d+def}}_{\text{pert}} = \frac{Z^{\text{6d+def}}_{\text{pert}}}{Z^{\text{6d}}_{\text{pert}}} , \tag{A.24a}$$

$$\widetilde{Z}^{\text{6d+def}}_1 = Z^{\text{6d+def}}_1 - Z^{\text{6d}}_1 , \tag{A.24b}$$

$$\widetilde{Z}^{\text{6d+def}}_2 = Z^{\text{6d+def}}_2 - Z^{\text{6d}}_2 - Z^{\text{6d}}_1 \left(Z^{\text{6d+def}}_1 - Z^{\text{6d}}_1 \right) , \tag{A.24c}$$

and similarly for higher orders in q_ϕ.

A.3.2   Notation and expansion coefficients

Some frequently appearing combinations of Theta functions have the following expansions:

$$\frac{\vartheta_1(2\epsilon_+)}{\vartheta_1(\epsilon_1)\vartheta_1(\epsilon_2)} = \frac{1}{\vartheta'_1(0)} \frac{1}{\epsilon_2} + B^{(0)} + \mathcal{O}(\epsilon_2) \qquad \text{with } B^{(0)} = \frac{1}{\vartheta'_1(0)} L(\epsilon_1) , \tag{A.25a}$$

$$\frac{\vartheta_1(2\epsilon_+)\vartheta_1(s\epsilon_2)}{\vartheta_1(\epsilon_1)\vartheta_1(\epsilon_2)} = s + A^{(1)}\epsilon_2 + \mathcal{O}(\epsilon_2^2) \qquad \text{with } A^{(1)} = sL(\epsilon_1) . \tag{A.25b}$$

The ϵ_2 expansion of functions defined in (2.14), (2.16), (2.50), and (2.65) are given by

$$V_{(0,s)}(u - \epsilon_+) = 1 + V^{(1)}_s(u - \epsilon_+) \cdot \epsilon_2 + V^{(2)}_s(u - \epsilon_+) \cdot \epsilon_2^2 + \mathcal{O}(\epsilon_2^3) \tag{A.25c}$$

$$\text{with} \quad V^{(1)}_s(u - \epsilon_+) = sL\left(u - x - \frac{1}{2}\epsilon_1\right)$$

$$V^{(2)}_s(u - \epsilon_+) = \frac{s}{2}L\left(u - x - \frac{1}{2}\epsilon_1\right)^2 + \frac{s}{2}(s-1)K\left(u - x - \frac{1}{2}\epsilon_1\right) ,$$

$$Q^\vee(a_i - \epsilon_+) = Q^\vee_{(0)}(a_i - \epsilon_+) + Q^\vee_{(1)}(a_i - \epsilon_+) \cdot \epsilon_2 + \mathcal{O}(\epsilon_2^2) \tag{A.25d}$$

$$\text{with} \quad Q^\vee_{(0)}(a_i - \epsilon_+) = Q^\vee(a_i - \epsilon_+)\big|_{\epsilon_2=0}$$

$$Q^\vee_{(1)}(a_i - \epsilon_+) = Q^\vee_{(0)}(a_i - \epsilon_+) \sum_k \left[L(a_j - a_k - \epsilon_1) \right.$$

$$\left. - \frac{1}{2}L\left(a_j - \frac{1}{2}\epsilon_1 - m_k + b\right) - \frac{1}{2}L\left(a_j - \frac{1}{2}\epsilon_1 - n_k - b\right) \right] ,$$

$$W(a_i - \epsilon_+) = 1 + W_{(1)}(a_i - \epsilon_+) \cdot \epsilon_2 + W_{(2)}(a_i - \epsilon_+) \cdot \epsilon_2^2 + \mathcal{O}(\epsilon_2^3) \tag{A.25e}$$

$$\text{with} \quad W_{(1)}(a_i - \epsilon_+) = L(u - z - \epsilon_1) - L(u - z)$$

$$W_{(2)}(a_i - \epsilon_+) = \frac{1}{2} \left(K(u - z) - K(u - z - \epsilon_1) \right)$$
$$+ L(u - z - \epsilon_1) \left(L(u - z - \epsilon_1) - L(u - z) \right) ,$$

with $L(\cdot)$, $K(\cdot)$ as defined in (2.17). In addition, for certain relevant combinations one finds

$$V_{(0,s)}(u_1 - \epsilon_+) V_{(0,s)}(u_2 - \epsilon_+) - 1 = V_{(1)}\left(u_1 - \frac{1}{2}\epsilon_1, u_2 - \frac{1}{2}\epsilon_1 \right) \cdot \epsilon_2 \tag{A.25f}$$
$$+ V_{(2)}\left(u_1 - \frac{1}{2}\epsilon_1, u_2 - \frac{1}{2}\epsilon_1 \right) \cdot \epsilon_2^2 + \mathcal{O}(\epsilon_2^3)$$

with $V_{(1)}\left(u_1 - \frac{1}{2}\epsilon_1, u_2 - \frac{1}{2}\epsilon_1 \right) = s\left(L\left(u_1 - x - \frac{1}{2}\epsilon_1 \right) + L\left(u_2 - x - \frac{1}{2}\epsilon_1 \right) \right)$

$$V_{(2)}\left(u_1 - \frac{1}{2}\epsilon_1, u_2 - \frac{1}{2}\epsilon_1 \right) = \frac{s}{2} \left\{ L\left(u_1 - x - \frac{1}{2}\epsilon_1 \right)^2 + L\left(u_2 - x - \frac{1}{2}\epsilon_1 \right)^2 \right\}$$
$$+ s^2 L\left(u_1 - x - \frac{1}{2}\epsilon_1 \right) L\left(u_2 - x - \frac{1}{2}\epsilon_1 \right)$$
$$+ \frac{s}{2}(s-1)\left\{ K\left(u_1 - x - \frac{1}{2}\epsilon_1 \right) + K\left(u_2 - x - \frac{1}{2}\epsilon_1 \right) \right\}$$

$$W(u_1 - \epsilon_+) W(u_2 - \epsilon_+) - 1 = W_{(1)}\left(u_1 - \frac{1}{2}\epsilon_1, u_2 - \frac{1}{2}\epsilon_1 \right) \cdot \epsilon_2 \tag{A.25g}$$
$$+ W_{(2)}\left(u_1 - \frac{1}{2}\epsilon_1, u_2 - \frac{1}{2}\epsilon_1 \right) \cdot \epsilon_2^2 + \mathcal{O}(\epsilon_2^3)$$

with $W_{(1)}\left(u_1 - \frac{1}{2}\epsilon_1, u_2 - \frac{1}{2}\epsilon_1 \right) = \sum_{J=1}^{2} [L(u_J - z - \epsilon_1) - L(u_J - z)]$

$$W_{(2)}\left(u_1 - \frac{1}{2}\epsilon_1, u_2 - \frac{1}{2}\epsilon_1 \right) = \sum_{J=1}^{2} \left[\frac{1}{2}(K(u_J - z) - K(u_J - z - \epsilon_1)) \right.$$
$$\left. + L(u_J - z - \epsilon_1)(L(u_J - z - \epsilon_1) - L(u_J - z)) \right]$$
$$+ L(u_1 - z) L(u_2 - z) - L(u_1 - z - \epsilon_1) L(u_2 - z)$$
$$- L(u_1 - z) L(u_2 - z - \epsilon_1) + L(u_1 - z - \epsilon_1) L(u_2 - z - \epsilon_1)$$

$$W(a_j - \epsilon_+) W(a_j - \epsilon_+ - \epsilon_\kappa) - 1 = W_{(1)}(a_j - \epsilon_+, a_j - \epsilon_+ - \epsilon_\kappa) \cdot \epsilon_2 \tag{A.25h}$$
$$+ W_{(2)}(a_j - \epsilon_+, a_j - \epsilon_+ - \epsilon_\kappa) \cdot \epsilon_2^2 + \mathcal{O}(\epsilon_2^3), \qquad \kappa \in \{1,2\}$$

with $W_{(1)}(a_j - \epsilon_+, a_j - \epsilon_+ - \epsilon_1) = L(a_j - z - 2\epsilon_1) - L(a_j - z)$

$W_{(1)}(a_j - \epsilon_+, a_j - \epsilon_+ - \epsilon_2) = 2[L(a_j - z - \epsilon_1) - L(a_j - z)]$

$W_{(2)}(a_j - \epsilon_+, a_j - \epsilon_+ - \epsilon_1) = \frac{1}{2}[K(a_j - z) - K(a_j - z - 2\epsilon_1)]$
$$+ L(a_j - z - 2\epsilon_1)[L(a_j - z - 2\epsilon_1) - L(a_j - z)]$$

$W_{(2)}(a_j - \epsilon_+, a_j - \epsilon_+ - \epsilon_2) = 2[K(a_j - z) - K(a_j - z - \epsilon_1)]$
$$+ 4L(a_j - z - 2\epsilon_1)[L(a_j - z - \epsilon_1) - L(a_j - z)]$$

$$\widetilde{Q}\left(a_j - \frac{1}{2}\epsilon_1\right) = \frac{1}{\vartheta_1'(0)} Q_{(0)}^{\vee}\left(a_j - \frac{1}{2}\epsilon_1\right) \left\{ \sum_i \left[\frac{3}{2} L\left(a_j - \frac{1}{2}\epsilon_1 - m_i + b\right)\right.\right.$$

$$\left.+ \frac{3}{2} L\left(a_j - \frac{1}{2}\epsilon_1 - n_i - b\right) - 2L(a_j - a_i - \epsilon_1)\right]$$

$$\left. - \sum_{i \neq j} L(a_j - a_i) \right\}$$

A.4 Elliptic genera for theory without defect

For the theory without defects of section 2.1, the non-perturbative contributions can be computed via (2.15). In this section, the details of the 1 and 2-string calculation are presented. As detailed in [43, 44], the JK-residue prescription requires the choice of an auxiliary vector that determines the poles which contribute to the contour integral. While the final result is independent of the choice made, the individual residues do not have an invariant meaning. For this paper, the auxiliary vector is chosen to be $+1$ on 1-string level and $(1, 1)$ on 2-string level.

A.4.1 1-string

For the evaluation of the 1-string contribution, the residues of the following poles are relevant:

$$\epsilon_+ + u - a_i = 0\,. \tag{A.26}$$

Since $Q(u) = \frac{M(u)}{P_0(u)P_0(u+2\epsilon_+)}$, this choice of poles corresponds to the zeros of $P_0(u + 2\epsilon_+)$. Using (A.17), one computes

$$\oint du \frac{f(u)}{P_0(u + 2\epsilon_+)} = \sum_{i=1}^{k} \frac{f(u)}{\prod_{j \neq i} \vartheta_1(u - a_j + \epsilon_+)}\bigg|_{u=a_i-\epsilon_+} \oint_{u=a_i-\epsilon_+} \frac{du}{\vartheta_1(u - a_i + \epsilon_+)}$$

$$= \sum_{i=1}^{k} \frac{f(u)}{\prod_{j \neq i} \vartheta_1(u - a_j + \epsilon_+)}\bigg|_{u=a_i-\epsilon_+} \frac{iQ^{\frac{1}{12}}\eta}{\eta^3}$$

$$= \frac{iQ^{\frac{1}{12}}\eta}{\eta^3} \sum_{i=1}^{k} \frac{f(u)}{P_0^{\vee}(u + 2\epsilon_+)}\bigg|_{u=a_i-\epsilon_+} = \frac{iQ^{\frac{1}{12}}\eta}{\eta^3} \sum_{i=1}^{k} \frac{f(a_i - \epsilon_+)}{P_0^{\vee}(a_i + \epsilon_+)} \tag{A.27}$$

where the definitions (2.16) have been used. With this preparation, the elliptic genus becomes

$$Z_1 = \oint \frac{du}{(2\pi i)} \left(\frac{2\pi \eta^3 \theta_1(2\epsilon_+)}{\theta_1(\epsilon_1)\theta_1(\epsilon_2)}\right) Q(u) = \frac{\vartheta_1(2\epsilon_+)}{\vartheta_1(\epsilon_1)\vartheta_1(\epsilon_2)} \sum_{i=1}^{k} Q^{\vee}(a_i - \epsilon_+)\,, \tag{A.28}$$

using (A.13).

A.4.2 2-string

For $l = 2$, the elliptic genus becomes

$$Z_2 = \frac{1}{2} \oint \frac{du_1 du_2}{(2\pi i)^2} \left(\frac{2\pi \eta^3 \theta_1(2\epsilon_+)}{\theta_1(\epsilon_1)\theta_1(\epsilon_2)}\right)^2 D(u_1 - u_2)D(u_2 - u_1) \prod_{p=1}^{2} Q(u_p) \tag{A.29}$$

and the relevant poles are as follows:

- Both poles originate from $P_0(u_p + \epsilon_1 + \epsilon_2)$ i.e.

$$(u_1, u_2) = (a_i - \epsilon_+, a_j - \epsilon_+) \qquad \text{for } i \neq j. \tag{A.30}$$

- One pole from $P_0(u_p + \epsilon_1 + \epsilon_2)$ and one from $D(\pm(u_1 - u_2))$, i.e.

$$
\begin{aligned}
(u_1, u_2) &= (a_m - \epsilon_+, a_m - \epsilon_+ - \epsilon_{1,2}) \qquad \text{and} \\
(u_1, u_2) &= (a_m - \epsilon_+ - \epsilon_{1,2}, a_m - \epsilon_+).
\end{aligned}
\tag{A.31}
$$

In order to compute the residues, the following intermediate results are useful:

$$
\begin{aligned}
\oint_{u=-\epsilon_2} \mathrm{d}u\, f(u) D(u) &= \oint_{u=-\epsilon_2} \mathrm{d}u\, f(u) \frac{\vartheta_1(u)\vartheta_1(u+\epsilon_1+\epsilon_2)}{\vartheta_1(u+\epsilon_1)\vartheta_1(u+\epsilon_2)} \\
&= \frac{iQ^{\frac{1}{12}}\eta}{\eta^3} f(-\epsilon_2) \frac{\vartheta_1(-\epsilon_2)\vartheta_1(\epsilon_1)}{\vartheta_1(\epsilon_1-\epsilon_2)}
\end{aligned}
\tag{A.32a}
$$

$$
\begin{aligned}
\oint_{u=-\epsilon_1} \mathrm{d}u\, f(u) D(u) &= \oint_{u=-\epsilon_1} \mathrm{d}u\, f(u) \frac{\vartheta_1(u)\vartheta_1(u+\epsilon_1+\epsilon_2)}{\vartheta_1(u+\epsilon_1)\vartheta_1(u+\epsilon_2)} \\
&= \frac{iQ^{\frac{1}{12}}\eta}{\eta^3} f(-\epsilon_1) \frac{\vartheta_1(-\epsilon_1)\vartheta_1(\epsilon_2)}{\vartheta_1(\epsilon_2-\epsilon_1)},
\end{aligned}
\tag{A.32b}
$$

as well as

$$
D(\epsilon_1) = \frac{\vartheta_1(\epsilon_1)\vartheta_1(2\epsilon_1+\epsilon_2)}{\vartheta_1(2\epsilon_1)\vartheta_1(\epsilon_1+\epsilon_2)} = \frac{\vartheta_1(\epsilon_1)\vartheta_1(\epsilon_1+2\epsilon_+)}{\vartheta_1(2\epsilon_1)\vartheta_1(2\epsilon_+)}, \tag{A.32c}
$$

$$
D(\epsilon_2) = \frac{\vartheta_1(\epsilon_2)\vartheta_1(\epsilon_1+2\epsilon_2)}{\vartheta_1(\epsilon_1+\epsilon_2)\vartheta_1(2\epsilon_2)} = \frac{\vartheta_1(\epsilon_2)\vartheta_1(\epsilon_2+2\epsilon_+)}{\vartheta_1(2\epsilon_+)\vartheta_1(2\epsilon_2)}. \tag{A.32d}
$$

Firstly, consider the contributions for $(u_1, u_2) = (a_i - \epsilon_+, a_j - \epsilon_+)$

$$
Z_2 \supset \frac{1}{2}\left(\frac{\vartheta_1(2\epsilon_+)}{\vartheta_1(\epsilon_1)\vartheta_1(\epsilon_2)}\right)^2 D(a_i-a_j)D(a_j-a_i)Q^\vee(a_i-\epsilon_+)Q^\vee(a_j-\epsilon_+). \tag{A.33}
$$

Secondly, both $(u_1, u_2) = (a_m - \epsilon_+, a_m - \epsilon_+ - \epsilon_1)$ and $(u_1, u_2) = (a_m - \epsilon_+ - \epsilon_1, a_m - \epsilon_+)$ yield

$$
Z_2 \supset \frac{1}{2}\frac{\vartheta_1(2\epsilon_+)\vartheta_1(\epsilon_1+2\epsilon_+)}{\vartheta_1(\epsilon_2)\vartheta_1(2\epsilon_-)\vartheta_1(2\epsilon_1)} Q^\vee(a_m-\epsilon_+)Q(a_m-\epsilon_+-\epsilon_1). \tag{A.34}
$$

Thirdly, both $(u_1, u_2) = (a_m - \epsilon_+, a_m - \epsilon_+ - \epsilon_2)$ and $(u_1, u_2) = (a_m - \epsilon_+ - \epsilon_2, a_m - \epsilon_+)$ yield

$$
Z_2 \supset \frac{1}{2}\frac{-1 \cdot \vartheta_1(2\epsilon_+)\vartheta_1(\epsilon_2+2\epsilon_+)}{\vartheta_1(\epsilon_1)\vartheta_1(2\epsilon_-)\vartheta_1(2\epsilon_2)} Q^\vee(a_m-\epsilon_+)Q(a_m-\epsilon_+-\epsilon_2). \tag{A.35}
$$

Summing up all the individual contributions yields

$$
Z_2 = \left(\frac{\vartheta_1(2\epsilon_+)}{\vartheta_1(\epsilon_1)\vartheta_1(\epsilon_2)}\right)^2 \sum_{1\leq i<j\leq k} D(a_i-a_j)D(a_j-a_i)Q^\vee(a_i-\epsilon_+)Q^\vee(a_j-\epsilon_+) \tag{A.36}
$$

$$
+ \frac{\vartheta_1(2\epsilon_+)}{\vartheta_1(2\epsilon_-)} \sum_{j=1}^{k} Q^\vee(a_j-\epsilon_+)\left[\frac{\vartheta_1(\epsilon_1+2\epsilon_+)}{\vartheta_1(\epsilon_2)\vartheta_1(2\epsilon_1)}Q(a_j-\epsilon_+-\epsilon_1) - \frac{\vartheta_1(\epsilon_2+2\epsilon_+)}{\vartheta_1(\epsilon_1)\vartheta_1(2\epsilon_2)}Q(a_j-\epsilon_+-\epsilon_2)\right]
$$

where the notation (2.16) has been used.

A.5 Elliptic genera for theory with codimension 2 defect

In section 2.3, the theory with codiemnsion 2 defect is introduced. The non-perturbative contributions are computed via (2.49), and in this section the 1 and 2-string results are detailed. Again, choice of the auxiliary vector in the JK-residue is $+1$ on 1-string level and $(1,1)$ on 2-string level.

A.5.1 1-string

The 1-string elliptic genus is given by

$$Z_1^{(0,s)\text{def}} = \oint \frac{du}{2\pi\mathrm{i}} \, Z_{1-\text{loop}}^{(0,s)\text{def}}(k,1) \equiv \oint \frac{du}{2\pi\mathrm{i}} \, Z_{1-\text{loop}}(k,1) \cdot V_{(0,s)}(u) \tag{A.37}$$

and the contour integral is evaluated by selecting the residues of the following poles:

- $u = a_i - \epsilon_+$ for $i = 1, \ldots, k$

$$Z_1^{(0,s)\text{def}} \supset \frac{\vartheta_1(2\epsilon_+)}{\vartheta_1(\epsilon_1)\,\vartheta_1(\epsilon_2)} \sum_{i=1}^{k} \left(Q^\vee(a_i - \epsilon_+) \cdot V_{(0,s)}(a_i - \epsilon_+) \right) \tag{A.38}$$

- $u = x$

$$Z_1^{(0,s)\text{def}} \supset \frac{\vartheta_1(2\epsilon_+)\vartheta_1(s\epsilon_2)}{\vartheta_1(\epsilon_1)\vartheta_1(\epsilon_2)} Q(x) \tag{A.39}$$

such that the elliptic genus for $l = 1$ reads

$$Z_1^{(0,s)\text{def}} = \frac{\vartheta_1(2\epsilon_+)}{\vartheta_1(\epsilon_1)\,\vartheta_1(\epsilon_2)} \left[\sum_{i=1}^{k} \left(Q^\vee(a_i - \epsilon_+) \cdot V_{(0,s)}(a_i - \epsilon_+) \right) + \vartheta_1(s\epsilon_2) \cdot Q(x) \right].$$

Following section A.3.1, the normalised 1-string contribution in the NS-limit reads

$$\widetilde{Z}_1^{(0,s)\text{def}} = Z_1^{(0,s)\text{def}} - Z_1$$

$$\lim_{\epsilon_2 \to 0} \widetilde{Z}_1^{(0,s)\text{def}} = \lim_{\epsilon_2 \to 0} \left\{ \frac{\vartheta_1(2\epsilon_+)}{\vartheta_1(\epsilon_1)\,\vartheta_1(\epsilon_2)} \sum_{i=1}^{k} \left(Q^\vee(a_i - \epsilon_+) \cdot \left[V_{(0,s)}(a_i - \epsilon_+) - 1 \right] \right) \right.$$

$$\left. + \frac{\vartheta_1(2\epsilon_+)\vartheta_1(s\epsilon_2)}{\vartheta_1(\epsilon_1)\vartheta_1(\epsilon_2)} Q(x) \right\}. \tag{A.40}$$

To further evaluate the limit, consider

$$\lim_{\epsilon_2 \to 0} \frac{V_{(0,s)}(a_i - \epsilon_+) - 1}{\vartheta_1(\epsilon_2)} = s \frac{1}{\vartheta_1'(0)} L\left(a_i - x - \frac{1}{2}\epsilon_1\right),$$

$$\lim_{\epsilon_2 \to 0} \frac{\vartheta_1(s\epsilon_2)}{\vartheta_1(\epsilon_2)} = s \quad \text{and} \quad \lim_{\epsilon_2 \to 0} \frac{\vartheta_1(2\epsilon_+)}{\vartheta_1(\epsilon_1)} = 1, \tag{A.41a}$$

such that

$$\lim_{\epsilon_2 \to 0} \widetilde{Z}_1^{(0,s)\text{def}} = \frac{s}{\vartheta_1'(0)} \sum_{i=1}^{k} \left(Q_{(0)}^\vee \left(a_i - \frac{1}{2}\epsilon_1\right) \cdot L\left(a_i - x - \frac{1}{2}\epsilon_1\right) \right) + s \cdot Q_{(0)}(x) \tag{A.42}$$

$$= s \cdot \lim_{\epsilon_2 \to 0} \widetilde{Z}_1^{(0,1)\text{def}},$$

using the notation (A.25). Therefore, the $(0, s)$ defect part is the product of s $(0, 1)$ defect contributions.

A.5.2 2-string

Consider the $l = 2$ elliptic genus with defect given by

$$Z_2^{(0,s)\text{def}} = \frac{1}{2} \oint \frac{\mathrm{d}u_1 \mathrm{d}u_2}{(2\pi\mathrm{i})^2} \left(\frac{2\pi\,\eta^3\theta_1(2\epsilon_+)}{\theta_1(\epsilon_1)\theta_1(\epsilon_2)} \right)^2 D(u_1-u_2)D(u_2-u_1) \prod_{p=1}^{2} Q(u_p)V_{(0,s)}(u_p) \quad \text{(A.43)}$$

and the relevant poles can be split into poles that come from the theory without defect such as:

- Both poles originate from $P_0(u_p + \epsilon_1 + \epsilon_2)$ i.e.

$$(u_1, u_2) = (a_i - \epsilon_+, a_j - \epsilon_+) \qquad \text{for } i \neq j. \quad \text{(A.44)}$$

- One pole from $P_0(u_p + \epsilon_1 + \epsilon_2)$ and one from $D(\pm(u_1 - u_2))$, i.e.

$$\begin{aligned}(u_1, u_2) &= (a_m - \epsilon_+, a_m - \epsilon_+ - \epsilon_{1,2}) \qquad \text{and} \\ (u_1, u_2) &= (a_m - \epsilon_+ - \epsilon_{1,2}, a_m - \epsilon_+).\end{aligned} \quad \text{(A.45)}$$

In addition, there are new poles from the defect part. These are

- One pole from $P_0(u_p + \epsilon_1 + \epsilon_2)$ and one from $V_{(0,s)}(u_p)$, i.e.

$$(u_1, u_2) = (a_m - \epsilon_+, x) \qquad \text{and} \qquad (u_1, u_2) = (x, a_m - \epsilon_+). \quad \text{(A.46)}$$

- One pole from $D(\pm(u_1 - u_2))$ and one from $V_{(0,s)}(u_p)$, i.e.

$$(u_1, u_2) = (x, x - \epsilon_{1,2}) \qquad \text{and} \qquad (u_1, u_2) = (x - \epsilon_{1,2}, x). \quad \text{(A.47)}$$

Now, one can work out the residues for the individual poles as before: firstly, consider the contributions for $(u_1, u_2) = (a_i - \epsilon_+, a_j - \epsilon_+)$

$$\begin{aligned} Z_2^{(0,s)\text{def}} \supset \frac{1}{2} &\left(\frac{\vartheta_1(2\epsilon_+)}{\vartheta_1(\epsilon_1)\vartheta_1(\epsilon_2)} \right)^2 D(a_i - a_j)D(a_j - a_i) \\ &\cdot Q^\vee(a_i - \epsilon_+)Q^\vee(a_j - \epsilon_+)V_{(0,s)}(a_i - \epsilon_+)V_{(0,s)}(a_j - \epsilon_+). \end{aligned} \quad \text{(A.48)}$$

Secondly, both $(u_1, u_2) = (a_m - \epsilon_+, a_m - \epsilon_+ - \epsilon_1)$ and $(u_1, u_2) = (a_m - \epsilon_+ - \epsilon_1, a_m - \epsilon_+)$ yield

$$\begin{aligned} Z_2^{(0,s)\text{def}} \supset \frac{1}{2} &\frac{\vartheta_1(2\epsilon_+)\vartheta_1(\epsilon_1 + 2\epsilon_+)}{\vartheta_1(\epsilon_2)\vartheta_1(2\epsilon_-)\vartheta_1(2\epsilon_1)} Q^\vee(a_m - \epsilon_+)Q(a_m - \epsilon_+ - \epsilon_1) \\ &\cdot V_{(0,s)}(a_m - \epsilon_+)V_{(0,s)}(a_m - \epsilon_+ - \epsilon_1) \end{aligned} \quad \text{(A.49)}$$

Thirdly, both $(u_1, u_2) = (a_m - \epsilon_+, a_m - \epsilon_+ - \epsilon_2)$ and $(u_1, u_2) = (a_m - \epsilon_+ - \epsilon_2, a_m - \epsilon_+)$ yield

$$\begin{aligned} Z_2^{(0,s)\text{def}} \supset \frac{1}{2} &\frac{-1 \cdot \vartheta_1(2\epsilon_+)\vartheta_1(\epsilon_2 + 2\epsilon_+)}{\vartheta_1(\epsilon_1)\vartheta_1(2\epsilon_-)\vartheta_1(2\epsilon_2)} Q^\vee(a_m - \epsilon_+)Q(a_m - \epsilon_+ - \epsilon_2) \\ &\cdot V_{(0,s)}(a_m - \epsilon_+)V_{(0,s)}(a_m - \epsilon_+ - \epsilon_2) \end{aligned} \quad \text{(A.50)}$$

Fourthly, both $(u_1, u_2) = (a_m - \epsilon_+, x)$ and $(u_1, u_2) = (x, a_m - \epsilon_+)$ yield

$$Z_2^{(0,s)\text{def}} \supset \frac{1}{2} \left(\frac{\vartheta_1(2\epsilon_+)}{\vartheta_1(\epsilon_1)\vartheta_1(\epsilon_2)} \right)^2 D(a_m - x - \epsilon_+)D(x + \epsilon_+ - a_m)$$
$$\cdot Q^\vee(a_m - \epsilon_+)Q(x)V_{(0,s)}(a_m - \epsilon+)\vartheta_1(s\epsilon_2) \tag{A.51}$$

Fifthly, both $(u_1, u_2) = (x, x - \epsilon_1)$ and $(u_1, u_2) = (x - \epsilon_1, x)$

$$Z_2^{(0,s)\text{def}} \supset \frac{1}{2} \frac{\vartheta_1(2\epsilon_+)\vartheta_1(\epsilon_1 + 2\epsilon_+)}{\vartheta_1(2\epsilon_-)\vartheta_1(\epsilon_2)\vartheta_1(2\epsilon_1)} \cdot Q(x)Q(x - \epsilon_1)V_{(0,s)}(x - \epsilon_1)\vartheta_1(s\epsilon_2) \tag{A.52}$$

Lastly, both $(u_1, u_2) = (x, x - \epsilon_2)$ and $(u_1, u_2) = (x - \epsilon_2, x)$ yield

$$Z_2^{(0,s)\text{def}} \supset -\frac{1}{2} \frac{\vartheta_1(2\epsilon_+)\vartheta_1(\epsilon_2 + 2\epsilon_+)}{\vartheta_1(\epsilon_1)\vartheta_1(2\epsilon_-)\vartheta_1(2\epsilon_2)} \cdot Q(x)Q(x - \epsilon_2)V_{(0,s)}(x - \epsilon_2)\vartheta_1(s\epsilon_2) \tag{A.53}$$

Summing up all the individual contributions leads to

$$Z_2^{(0,s)\text{def}} = \left(\frac{\vartheta_1(2\epsilon_+)}{\vartheta_1(\epsilon_1)\vartheta_1(\epsilon_2)} \right)^2 \sum_{1 \le i < j \le k} D(a_i - a_j)D(a_j - a_i) \tag{A.54}$$
$$\cdot Q^\vee(a_i - \epsilon_+)Q^\vee(a_j - \epsilon_+)V_{(0,s)}(a_i - \epsilon_+)V_{(0,s)}(a_j - \epsilon_+)$$
$$+ \frac{\vartheta_1(2\epsilon_+)}{\vartheta_1(2\epsilon_-)} \sum_{j=1}^{k} Q^\vee(a_j - \epsilon_+)V_{(0,s)}(a_j - \epsilon_+)$$
$$\cdot \left[\frac{\vartheta_1(\epsilon_1 + 2\epsilon_+)}{\vartheta_1(\epsilon_2)\vartheta_1(2\epsilon_1)} Q(a_j - \epsilon_+ - \epsilon_1)V_{(0,s)}(a_j - \epsilon_+ - \epsilon_1) \right.$$
$$\left. - \frac{\vartheta_1(\epsilon_2 + 2\epsilon_+)}{\vartheta_1(\epsilon_1)\vartheta_1(2\epsilon_2)} Q(a_j - \epsilon_+ - \epsilon_2)V_{(0,s)}(a_j - \epsilon_+ - \epsilon_2) \right]$$
$$+ \left(\frac{\vartheta_1(2\epsilon_+)}{\vartheta_1(\epsilon_1)\vartheta_1(\epsilon_2)} \right)^2 \vartheta_1(s\epsilon_2) \sum_{j=1}^{k} D(a_j - x - \epsilon_+)D(x + \epsilon_+ - a_j)$$
$$\cdot Q^\vee(a_j - \epsilon_+)Q(x)V_{(0,s)}(a_j - \epsilon+)$$
$$+ \frac{\vartheta_1(2\epsilon_+)}{\vartheta_1(2\epsilon_-)} \cdot Q(x)\vartheta_1(s\epsilon_2) \cdot \left[\frac{\vartheta_1(\epsilon_1 + 2\epsilon_+)}{\vartheta_1(\epsilon_2)\vartheta_1(2\epsilon_1)} Q(x - \epsilon_1)V_{(0,s)}(x - \epsilon_1) \right.$$
$$\left. - \frac{\vartheta_1(\epsilon_2 + 2\epsilon_+)}{\vartheta_1(\epsilon_1)\vartheta_1(2\epsilon_2)} Q(x - \epsilon_2)V_{(0,s)}(x - \epsilon_2) \right].$$

Next, consider the normalised 2-string elliptic genus, see appendix A.3.1,

$$\widetilde{Z}_2^{(0,s)\text{def}} = Z_2^{(0,s)\text{def}} - Z_2 - Z_1 \left(Z_1^{(0,s)\text{def}} - Z_1 \right) \qquad \text{and} \qquad Z_2^{\text{aux}} = Z_2^{(0,s)\text{def}} - Z_2. \tag{A.55}$$

Firstly, focus on the 1-string contributions

$$Z_1^{(0,s)\text{def}} - Z_1 = \widetilde{Z}_1 \big|_{\text{fin}} + \widetilde{Z}_1 \big|_{\epsilon_2} \cdot \epsilon_2 + \mathcal{O}(\epsilon_2^2) \tag{A.56}$$

with ϵ_2 expansion coefficients

$$\widetilde{Z}_1\big|_{\text{fin}} = \frac{1}{\vartheta_1'(0)}\sum_j Q_{(0)}^\vee\left(a_j - \frac{1}{2}\epsilon_1\right)V_s^{(1)}\left(a_j - \frac{1}{2}\epsilon_1\right) + sQ_{(0)}(x)\,,$$

$$\widetilde{Z}_1\big|_{\epsilon_2} = \frac{1}{\vartheta_1'(0)}\sum_j Q_{(0)}^\vee\left(a_j - \frac{1}{2}\epsilon_1\right)V_s^{(2)}\left(a_j - \frac{1}{2}\epsilon_1\right)$$

$$+ \frac{1}{\vartheta_1'(0)}\sum_j Q_{(1)}^\vee\left(a_j - \frac{1}{2}\epsilon_1\right)V_s^{(1)}\left(a_j - \frac{1}{2}\epsilon_1\right)$$

$$+ B^{(0)}\sum_j Q_{(0)}^\vee\left(a_j - \frac{1}{2}\epsilon_1\right)V_s^{(1)}\left(a_j - \frac{1}{2}\epsilon_1\right) + A^{(1)}Q_{(0)}(x) + sQ_{(1)}(x)\,.$$

Secondly, consider pure 2-string contributions

$$Z_{k=2}^{\text{aux}} = I_1 + I_2 + I_3 + I_4 \tag{A.57}$$

with the following four parts:

$$I_1 = \left(\frac{\vartheta_1(2\epsilon_+)}{\vartheta_1(\epsilon_1)\vartheta_1(\epsilon_2)}\right)^2 \sum_{1\le i < j \le k} D(a_i - a_j)D(a_j - a_i)$$

$$\cdot\, Q^\vee(a_i - \epsilon_+)Q^\vee(a_j - \epsilon_+)\left[V_{(0,s)}(a_i - \epsilon_+)V_{(0,s)}(a_j - \epsilon_+) - 1\right],$$

$$I_2 = \frac{\vartheta_1(2\epsilon_+)}{\vartheta_1(2\epsilon_-)}\sum_{j=1}^k Q^\vee(a_j - \epsilon_+)$$

$$\cdot\left[\frac{\vartheta_1(\epsilon_1 + 2\epsilon_+)}{\vartheta_1(\epsilon_2)\vartheta_1(2\epsilon_1)}Q(a_j - \epsilon_+ - \epsilon_1)\left[V_{(0,s)}(a_j - \epsilon_+)V_{(0,s)}(a_j - \epsilon_+ - \epsilon_1) - 1\right]\right.$$

$$\left. - \frac{\vartheta_1(\epsilon_2 + 2\epsilon_+)}{\vartheta_1(\epsilon_1)\vartheta_1(2\epsilon_2)}Q(a_j - \epsilon_+ - \epsilon_2)\left[V_{(0,s)}(a_j - \epsilon_+)V_{(0,s)}(a_j - \epsilon_+ - \epsilon_2) - 1\right]\right],$$

$$I_3 = \left(\frac{\vartheta_1(2\epsilon_+)}{\vartheta_1(\epsilon_1)\vartheta_1(\epsilon_2)}\right)^2 \vartheta_1(s\epsilon_2)\sum_{j=1}^k D(a_j - x - \epsilon_+)D(x + \epsilon_+ - a_j)$$

$$\cdot\, Q^\vee(a_j - \epsilon_+)Q(x)V_{(0,s)}(a_j - \epsilon_+)\,,$$

$$I_4 = \frac{\vartheta_1(2\epsilon_+)}{\vartheta_1(2\epsilon_-)}\vartheta_1(s\epsilon_2)\cdot Q(x)\cdot\left[\frac{\vartheta_1(\epsilon_1 + 2\epsilon_+)}{\vartheta_1(\epsilon_2)\vartheta_1(2\epsilon_1)}Q(x - \epsilon_1)V_{(0,s)}(x - \epsilon_1)\right.$$

$$\left. - \frac{\vartheta_1(\epsilon_2 + 2\epsilon_+)}{\vartheta_1(\epsilon_1)\vartheta_1(2\epsilon_2)}Q(x - \epsilon_2)V_{(0,s)}(x - \epsilon_2)\right].$$

The ϵ_2 expansion is defined as

$$I_\mu = I_\mu\big|_{(\frac{1}{\epsilon_2})^2}\cdot\frac{1}{\epsilon_2^2} + I_\mu\big|_{\frac{1}{\epsilon_2}}\cdot\frac{1}{\epsilon_2} + I_\mu\big|_{\text{fin}} + I_\mu\big|_{\epsilon_2}\cdot\epsilon_2 + \mathcal{O}(\epsilon_2^2)\,. \tag{A.58}$$

The inspection of the most singular terms reveals

$$I_\mu\big|_{(\frac{1}{\epsilon_2})^2} = 0\,,\ \forall\mu \qquad \Rightarrow \qquad \left(Z_{k=2}^{(0,s)\text{def}} - Z_{k=2}\right)\big|_{(\frac{1}{\epsilon_2})^2} = 0\,, \tag{A.59}$$

which is required to vanish by consistency. The less singular expansion coefficients are given by

$$I_1\big|_{\frac{1}{\epsilon_2}} = \left(\frac{1}{\vartheta_1'(0)}\right)^2 \sum_{i<j} Q_{(0)}^\vee\left(a_i - \frac{1}{2}\epsilon_1\right) Q_{(0)}^\vee\left(a_j - \frac{1}{2}\epsilon_1\right) V^{(1)}\left(a_i - \frac{1}{2}\epsilon_1, a_j - \frac{1}{2}\epsilon_1\right)$$

$$I_1\big|_{\mathrm{fin}} = \left(\frac{1}{\vartheta_1'(0)}\right)^2 \sum_{i<j} Q_{(0)}^\vee\left(a_i - \frac{1}{2}\epsilon_1\right) Q_{(0)}^\vee\left(a_j - \frac{1}{2}\epsilon_1\right) V^{(2)}\left(a_i - \frac{1}{2}\epsilon_1, a_j - \frac{1}{2}\epsilon_1\right)$$

$$+ \left(\frac{1}{\vartheta_1'(0)}\right)^2 \sum_{i<j} \left\{ Q_{(0)}^\vee\left(a_i - \frac{1}{2}\epsilon_1\right) Q_{(1)}^\vee\left(a_j - \frac{1}{2}\epsilon_1\right) + Q_{(1)}^\vee\left(a_i - \frac{1}{2}\epsilon_1\right) Q_{(0)}^\vee\left(a_j - \frac{1}{2}\epsilon_1\right) \right\}$$

$$\times V^{(1)}\left(a_i - \frac{1}{2}\epsilon_1, a_j - \frac{1}{2}\epsilon_1\right)$$

$$+ \left(\frac{1}{\vartheta_1'(0)}\right)^2 \sum_{i<j} \left\{ D^{(1)}(a_i - \epsilon_+) + D^{(1)}(a_j - \epsilon_+) \right\} Q_{(0)}^\vee(a_i - \epsilon_+) Q_{(0)}^\vee(a_j - \epsilon_+)$$

$$\times V^{(1)}\left(a_i - \frac{1}{2}\epsilon_1, a_j - \frac{1}{2}\epsilon_1\right)$$

$$+ \frac{2B^{(0)}}{\vartheta_1'(0)} \sum_{i<j} Q_{(0)}^\vee(a_i - \epsilon_+) Q_{(0)}^\vee(a_j - \epsilon_+) V^{(1)}(a_i - \epsilon_+, a_j - \epsilon_+)$$

$$I_2\big|_{\frac{1}{\epsilon_2}} = \left(\frac{1}{\vartheta_1'(0)}\right)^2 \sum_j \left(Q_{(0)}^\vee\left(a_j - \frac{1}{2}\epsilon_1\right)\right)^2 V_s^{(1)}\left(a_j - \frac{1}{2}\epsilon_1\right)$$

$$I_2\big|_{\mathrm{fin}} = \frac{1}{\vartheta_1'(0)} \sum_j Q_{(0)}^\vee\left(a_j - \frac{1}{2}\epsilon_1\right) Q^{(0)}\left(a_j - \frac{3}{2}\epsilon_1\right) \left(V_s^{(1)}\left(a_j - \frac{1}{2}\epsilon_1\right) + V_s^{(1)}\left(a_j - \frac{3}{2}\epsilon_1\right)\right)$$

$$- \frac{1}{\vartheta_1'(0)} \sum_j Q_{(0)}^\vee\left(a_j - \frac{1}{2}\epsilon_1\right) \widetilde{Q}\left(a_j - \frac{1}{2}\epsilon_1\right) V_s^{(1)}\left(a_j - \frac{1}{2}\epsilon_1\right)$$

$$+ 2\frac{B^{(0)}}{\vartheta_1'(0)} \sum_j \left(Q_{(0)}^\vee\left(a_j - \frac{1}{2}\epsilon_1\right)\right)^2 V_s^{(1)}\left(a_j - \frac{1}{2}\epsilon_1\right)$$

$$+ \left(\frac{1}{\vartheta_1'(0)}\right)^2 \sum_j \left(2\frac{\vartheta_1'(\epsilon_1)}{\vartheta_1(\epsilon_1)} Q_{(0)}^\vee\left(a_j - \frac{1}{2}\epsilon_1\right) + Q_{(1)}^\vee\left(a_j - \frac{1}{2}\epsilon_1\right)\right)$$

$$\times Q_{(0)}^\vee\left(a_j - \frac{1}{2}\epsilon_1\right) V_s^{(1)}\left(a_j - \frac{1}{2}\epsilon_1\right)$$

$$+ \frac{1}{2(\vartheta_1'(0))^2} \sum_j \left(Q_{(0)}^\vee\left(a_j - \frac{1}{2}\epsilon_1\right)\right)^2 V^{(2)}(a_j - \epsilon_+, a_j - \epsilon_+ - \epsilon_2)$$

$$I_3\big|_{\frac{1}{\epsilon_2}} = \frac{1}{\vartheta_1'(0)} s Q_{(0)}(x) \sum_j Q_{(0)}^\vee\left(a_j - \frac{1}{2}\epsilon_1\right)$$

$$I_3\big|_{\mathrm{fin}} = \frac{1}{\vartheta_1'(0)} s Q_{(0)}(x) \sum_j \left[D^{(1)}\left(a_j - x - \frac{1}{2}\epsilon_1\right) + D^{(1)}\left(x + \frac{1}{2}\epsilon_1 - a_j\right) \right] Q_{(0)}^\vee\left(a_j - \frac{1}{2}\epsilon_1\right)$$

$$+\frac{1}{\vartheta_1'(0)}s\sum_j\left[Q_{(1)}^\vee\left(a_j-\frac{1}{2}\epsilon_1\right)Q_{(0)}(x)+Q_{(0)}^\vee\left(a_j-\frac{1}{2}\epsilon_1\right)Q_{(1)}(x)\right.$$

$$\left.+Q_{(0)}^\vee\left(a_j-\frac{1}{2}\epsilon_1\right)Q_{(0)}(x)V_s^{(1)}\left(a_j-\frac{1}{2}\epsilon_1\right)\right]$$

$$+2B^{(0)}s\sum_jQ_{(0)}^\vee\left(a_j-\frac{1}{2}\epsilon_1\right)Q_{(0)}(x)$$

$$I_4\big|_{\frac{1}{\epsilon_2}}=0$$

$$I_4\big|_{\text{fin}}=sQ_{(0)}(x)\left[Q_{(0)}(x-\epsilon_1)+\frac{s-1}{2}Q_{(0)}(x)\right].$$

Another consistency check is given by the vanishing of the $\frac{1}{\epsilon_2}$ terms if one considers the pure 2-string terms together with the product of the 1-string contributions. Explicitly, one finds

$$\left(Z_2^{(0,s)\text{def}}-Z_2\right)\big|_{\frac{1}{\epsilon_2}}-Z_1\big|_{\frac{1}{\epsilon_2}}\cdot\left(Z_1^{(0,s)\text{def}}-Z_1\right)\big|_{\text{fin}}=0\,,\tag{A.60}$$

as expected. Recalling the notation (A.25), the full normalised 2-string elliptic genus for the codimension 2 defect in the NS-limit is given by

$$\widetilde{Z}_{l=2}^{(0,s)\text{def}}=\frac{s(s+1)}{2}\sum_{j=1}^k\left(\frac{Q_{(0)}^\vee\left(a_j-\frac{\epsilon_1}{2}\right)}{\vartheta_1'(0)}L\left(a_j-x-\frac{\epsilon_1}{2}\right)\right)^2$$

$$-\frac{s}{2}\sum_{j=1}^k\left(\frac{Q_{(0)}^\vee\left(a_j-\frac{\epsilon_1}{2}\right)}{\vartheta_1'(0)}\right)^2K\left(a_j-x-\frac{\epsilon_1}{2}\right)$$

$$+\frac{s^2}{2}\sum_{\substack{i,j=1\\i\neq j}}^k\frac{Q_{(0)}^\vee\left(a_i-\frac{\epsilon_1}{2}\right)}{\vartheta_1'(0)}\frac{Q_{(0)}^\vee\left(a_j-\frac{\epsilon_1}{2}\right)}{\vartheta_1'(0)}L\left(a_i-x-\frac{\epsilon_1}{2}\right)L\left(a_j-x-\frac{\epsilon_1}{2}\right)$$

$$+2s\cdot L(\epsilon_1)\sum_{j=1}^k\left(\frac{Q_{(0)}^\vee\left(a_j-\frac{\epsilon_1}{2}\right)}{\vartheta_1'(0)}\right)^2L\left(a_j-x-\frac{\epsilon_1}{2}\right)$$

$$+s\sum_{\substack{i,j=1\\i\neq j}}^k\frac{Q_{(0)}^\vee\left(a_i-\frac{\epsilon_1}{2}\right)}{\vartheta_1'(0)}\frac{Q_{(0)}^\vee\left(a_j-\frac{\epsilon_1}{2}\right)}{\vartheta_1'(0)}L\left(a_i-x-\frac{\epsilon_1}{2}\right)\left[L(a_i-a_j+\epsilon_1)-L(a_i-a_j)\right.$$

$$\left.+L(a_j-a_i+\epsilon_1)-L(a_j-a_i)\right]$$

$$+s\sum_{j=1}^k\frac{Q_{(0)}^\vee\left(a_j-\frac{\epsilon_1}{2}\right)}{\vartheta_1'(0)}Q_{(0)}\left(a_j-\frac{3\epsilon_1}{2}\right)\left[L\left(a_j-x-\frac{\epsilon_1}{2}\right)+L\left(a_j-x-\frac{3\epsilon_1}{2}\right)\right]$$

$$+s\sum_{j=1}^k\left(\frac{Q_{(0)}^\vee\left(a_j-\frac{\epsilon_1}{2}\right)}{\vartheta_1'(0)}\right)^2L\left(a_j-x-\frac{\epsilon_1}{2}\right)\left[\sum_{i=1}^kL(a_j-a_i-\epsilon_1)+\sum_{\substack{i=1\\i\neq j}}^kL(a_j-a_i)\right.$$

$$\left.-\sum_{i=1}^k\left(L\left(a_j-\frac{\epsilon_1}{2}-m_i+b\right)+L\left(a_j-\frac{\epsilon_1}{2}-n_i-b\right)\right)\right]$$

$$+ s \cdot Q_{(0)}(x) \sum_{j=1}^{l} \frac{Q_{(0)}^{\vee} \left(a_j - \frac{\epsilon_1}{2} \right)}{\vartheta_1'(0)} \left[L \left(a_j - x + \frac{\epsilon_1}{2} \right) - L \left(a_j - x - \frac{\epsilon_1}{2} \right) \right.$$

$$\left. + L \left(x - a_j + \frac{3\epsilon_1}{2} \right) - L \left(x - a_j + \frac{\epsilon_1}{2} \right) + s L \left(a_j - x - \frac{\epsilon_1}{2} \right) \right]$$

$$+ s Q_{(0)}(x) \left(Q_{(0)}(x - \epsilon_1) - \frac{1-s}{2} Q_{(0)}(x) \right) \tag{A.61}$$

the computation has been check against the NS-limit performed with `Mathematica` for $k = 2, 3$.

A.6 Shift operator acting on defect partition function

The shift operator Y defined in (3.26) acts on the codimension 2 defect fugacity x. In the appendix, the action on the perturbative and non-perturbative part of the partition function is derived.

A.6.1 Perturbative contribution

The normalised perturbative part (2.48) for an $(0, s)$ defect can be written as

$$\widetilde{Z}_{\text{pert}}^{(0,s)\text{def}} = \text{PE} \left[\frac{s}{2(1-p)} \left(\frac{1+Q}{1-Q} \right) \left\{ (1-p) + \sqrt{p} \sum_{i=1}^{k} \left(\frac{X}{A_i} - \frac{A_i}{X} \right) \right\} \right] \quad \text{with } A_i = e^a$$

$$= \text{PE} \left[\frac{1}{(1-p)} \left(\frac{1+Q}{1-Q} \right) \left\{ (1-p) + \sqrt{p} \sum_{i=1}^{k} \left(\frac{X}{A_i} - \frac{A_i}{X} \right) \right\} \right]^{\frac{s}{2}} \tag{A.62}$$

$$= \prod_{j,h=0}^{\infty} \text{PE} \left[\left(Q^j + Q^{j+1} \right) \left\{ 1 + p^{h+\frac{1}{2}} \sum_{i=1}^{k} \left(L_i^{-1} - L_i \right) \right\} \right]^{\frac{s}{2}} \quad \text{with } L_i = \frac{A_i}{X}.$$

Focusing only on the X-dependent part, one proceeds further

$$f(X) = \prod_{i=1}^{k} \prod_{j,h=0}^{\infty} \text{PE} \left[\left(Q^j + Q^{j+1} \right) p^{h+\frac{1}{2}} \left(L_i^{-1} - L_i \right) \right]^{\frac{1}{2}}$$

$$= \prod_{i=1}^{k} \prod_{j,h=0}^{\infty} \text{PE} \left[Q^j p^{h+\frac{1}{2}} L_i^{-1} - Q^j p^{h+\frac{1}{2}} L_i Q^{j+1} p^{h+\frac{1}{2}} L_i^{-1} - Q^{j+1} p^{h+\frac{1}{2}} L_i \right]^{\frac{1}{2}}$$

$$= \prod_{i=1}^{k} \prod_{j,h=0}^{\infty} \sqrt{ \frac{(1 - Q^j p^{h+\frac{1}{2}} L_i)(1 - Q^{j+1} p^{h+\frac{1}{2}} L_i)}{(1 - Q^j p^{h+\frac{1}{2}} L_i^{-1})(1 - Q^{j+1} p^{h+\frac{1}{2}} L_i^{-1})} },$$

which can be expressed in different forms:
- As elliptic Gamma functions

$$f(X) = \prod_{i=1}^{k} \sqrt{ \left(\prod_{j,h=0}^{\infty} \frac{(1 - Q^{j+1} p^{h+1} \frac{L_i}{\sqrt{p}})}{(1 - Q^j p^h \frac{\sqrt{p}}{L_i})} \right)^2 \cdot \prod_{h=0}^{\infty} \left(1 - p^h \frac{\sqrt{p}}{L_i} \right) \left(1 - p^{h+1} \frac{L_i}{\sqrt{p}} \right) }$$

$$= \prod_{i=1}^{k} \sqrt{ \left(\Gamma \left(x + \frac{1}{2} \epsilon_1 - a_i, \tau, \epsilon_1 \right) \right)^2 \cdot \widetilde{\theta}_1 \left(x + \frac{1}{2} \epsilon_1 - a_i, \epsilon_1 \right) }, \tag{A.63}$$

and the silly looking notation turns out to be useful to resolve a potential sign issues.

- As inverse of Gamma functions

$$f(X) = \prod_{i=1}^{k} \sqrt{\left(\prod_{j,h=0}^{\infty} \frac{(1 - p^h Q^j(\sqrt{p}L_i))}{(1 - p^{h+1}Q^{j+1}\frac{1}{\sqrt{p}L_i})}\right)^2 \cdot \prod_{h=0}^{\infty} \frac{1}{(1 - p^h(\sqrt{p}L_i))(1 - p^{h+1}\frac{1}{\sqrt{p}L_i})}}$$

$$= \prod_{i=1}^{k} \sqrt{\frac{1}{\widetilde{\theta}_1(a_i - x + \frac{1}{2}\epsilon_1, \epsilon_1)} \cdot \left(\frac{1}{\Gamma(a_i - x + \frac{1}{2}\epsilon_1, \tau, \epsilon_1)}\right)^2}, \tag{A.64}$$

and the clumpsy looking notation is kept on purpose.

The perturbative part becomes

$$\widetilde{Z}_{\text{pert}}^{(0,s)\text{def}} = \left(\text{PE}\left[\frac{1+Q}{1-Q}\right] \cdot f(X)\right)^{\frac{s}{2}}. \tag{A.65}$$

Using the shift property in (A.16) and the expression in terms of elliptic Gamma functions (A.63) and (A.64), one can straightforwardly show that

$$Y\sqrt{\widetilde{\theta}_1(y_i, \epsilon_1) \cdot (\Gamma(y_i, \tau, \epsilon_1))^2} = \sqrt{\widetilde{\theta}_1(y_i - \epsilon_1, \epsilon_1) \cdot (\Gamma(y_i - \epsilon_1, \tau, \epsilon_1))^2} \quad \text{with} \quad y_i = x + \frac{1}{2}\epsilon_1 - a_i$$

$$= \sqrt{-e^{(y_i - \epsilon_1)}\widetilde{\theta}_1(y_i, \epsilon_1) \cdot \left(\frac{\Gamma(y_i, \tau, \epsilon_1)}{\widetilde{\theta}_1(y - \epsilon_1, \tau)}\right)^2} \quad \text{using (A.20)}$$

$$= \sqrt{-e^{(y_i - \epsilon_1)}\widetilde{\theta}_1(y_i, \epsilon_1) \cdot \left(\frac{iQ^{\frac{1}{12}}\eta(\tau)}{e^{\frac{1}{2}(y_i - \epsilon_1)}\theta_1(y - \epsilon_1, \tau)}\right)^2 (\Gamma(y_i, \tau, \epsilon_1))^2}$$

$$= \sqrt{\left(\frac{1}{\vartheta_1(y - \epsilon_1, \tau)}\right)^2} \cdot \sqrt{\widetilde{\theta}_1(y_i, \epsilon_1)(\Gamma(y_i, \tau, \epsilon_1))^2} \tag{A.66}$$

and similarly

$$Y\sqrt{\frac{1}{\widetilde{\theta}_1(z_i, \epsilon_1)} \cdot \frac{1}{(\Gamma(z_i, \tau, \epsilon_1))^2}} = \sqrt{\frac{1}{\widetilde{\theta}_1(z_i + \epsilon_1, \epsilon_1)} \cdot \frac{1}{(\Gamma(z_i + \epsilon_1, \tau, \epsilon_1))^2}} \quad \text{with} \quad z_i = a_i - x + \frac{1}{2}\epsilon_1$$

$$= \sqrt{\frac{1}{-e^{-z_i}\widetilde{\theta}_1(z_i, \epsilon_1)} \cdot \frac{1}{\left(\widetilde{\theta}_1(z_i, \tau)\Gamma(z_i, \tau, \epsilon_1)\right)^2}} \quad \text{using (A.20)}$$

$$= \sqrt{\frac{1}{-e^{-z_i}} \cdot \left(\frac{iQ^{\frac{1}{12}}\eta(\tau)}{e^{\frac{z_i}{2}}\theta_1(z_i, \tau)}\right)^2 \frac{1}{\left(\widetilde{\theta}_1(z_i, \epsilon_1)\Gamma(z_i, \tau, \epsilon_1)\right)^2}}$$

$$= \sqrt{\left(\frac{1}{\vartheta_1(z_i, \tau)}\right)^2} \cdot \sqrt{\frac{1}{\widetilde{\theta}_1(z_i, \epsilon_1)(\Gamma(z_i, \tau, \epsilon_1))^2}} \tag{A.67}$$

such that both calculations (A.66) and (A.67) lead to (3.28).

A.6.2 Elliptic genus

The defect part (3.15d) can be written as

$$
\begin{aligned}
\mathcal{V}_1^{(0,s)} &= s \cdot \partial_u \log \vartheta_1(u-x) = s \cdot \partial_u \log \left[\frac{\mathrm{i}\,\eta Q^{\frac{1}{12}}}{e^{\frac{u-x}{2}}} \widetilde{\theta}_1(u-x) \right] \qquad \text{using (A.14)} \\
&= s \cdot \partial_u \left(\log \widetilde{\theta}_1(u-x) + \log\left[\mathrm{i}\,\eta Q^{\frac{1}{12}} \right] - \frac{1}{2}(u-x) \right) \\
&= s \cdot \partial_u \left(\log \Gamma(u-x+\epsilon_1,\tau,\epsilon_1) - \log \Gamma(u-x,\tau,\epsilon_1) + \log\left[\mathrm{i}\,\eta Q^{\frac{1}{12}} \right] - \frac{1}{2}(u-x) \right) \\
&= s \cdot \partial_u \left(\log \Gamma(u-x+\epsilon_1,\tau,\epsilon_1) - \log \Gamma(u-x,\tau,\epsilon_1) - \frac{1}{2}(u-x) \right)
\end{aligned}
\tag{A.68}
$$

where the $\log[\mathrm{i}\,\eta Q^{\frac{1}{12}}]$ term vanishes due to the derivative. Then, the shift has the following effect:

$$
\begin{aligned}
\mathcal{V}_1^{(0,s)}(x \to x-\epsilon_1) &= s \cdot \partial_u \left(\log \Gamma(u-x+2\epsilon_1,\tau,\epsilon_1) - \log \Gamma(u-x+\epsilon_1,\tau,\epsilon_1) - \frac{1}{2}(u-x+\epsilon_1) \right) \\
&= s \cdot \partial_u \Big(\log \Gamma(u-x+\epsilon_1,\tau,\epsilon_1) + \log \widetilde{\theta}_1(u-x+\epsilon_1,\tau) \\
&\qquad\qquad - \log \Gamma(u-x,\tau,\epsilon_1) - \log \widetilde{\theta}_1(u-x,\tau) - \frac{1}{2}(u-x+\epsilon_1) \Big) \quad \text{using (A.20)} \\
&= \mathcal{V}_1^{(0,s)}(x) + s \cdot \partial_u \left(\log \widetilde{\theta}_1(u-x+\epsilon_1,\tau) - \log \widetilde{\theta}_1(u-x,\tau) - \frac{1}{2}\epsilon_1 \right) \\
&= \mathcal{V}_1^{(0,s)}(x) + s \cdot \partial_u \left(\log \left[\frac{\widetilde{\theta}_1(u-x+\epsilon_1,\tau)}{\widetilde{\theta}_1(u-x,\tau)} \cdot e^{-\frac{1}{2}\epsilon_1} \right] \right) \\
&= \mathcal{V}_1^{(0,s)}(x) + s \cdot \partial_u \left(\log \left[\frac{\vartheta_1(u-x+\epsilon_1,\tau)}{\vartheta_1(u-x,\tau)} \right] \right) \qquad \text{using (A.14), (A.13)} \\
&= \mathcal{V}_1^{(0,s)}(x) - s \cdot \partial_x \left(\log \left[\frac{\vartheta_1(u-x+\epsilon_1,\tau)}{\vartheta_1(u-x,\tau)} \right] \right).
\end{aligned}
\tag{A.69}
$$

Therefore, it follows that

$$
\begin{aligned}
\int \mathrm{d}u \; \rho_*(u) \mathcal{V}_1^{(0,s)}(x \to x-\epsilon_1) &= \int \mathrm{d}u \; \rho_*(u) \mathcal{V}_1^{(0,s)}(x) - s \cdot \int \mathrm{d}u \; \rho_*(u) \partial_x \left(\log \left[\frac{\theta_1(u-x+\epsilon_1,\tau)}{\theta_1(u-x-b,\tau)} \right] \right) \\
&= \int \mathrm{d}u \; \rho_*(u) \mathcal{V}_1^{(0,s)}(x) + \left(\log \left[\frac{\mathcal{Y}(x-\epsilon_1)}{\mathcal{Y}(x)} \right] \right)^s
\end{aligned}
\tag{A.70}
$$

and one arrives at (3.31).

A.7 Elliptic genera for theory with codimension 4 defect

In section 2.4, the theory in the presence of a codimension 4 defect has been considered. The elliptic genus can be computed via (2.66), and the 1 and 2-string computations are detailed here. The chosen auxiliary vector in the JK-residue is $+1$ on 1-string level and $(1,1)$ on 2-string level.

A.7.1   1-string

For 1-string contribution, one needs to evaluate the contour integral of (2.66) for $l = 1$, i.e.

$$Z_1^{\text{Wilson}} = \oint \frac{\mathrm{d}u}{2\pi\mathrm{i}} \left(\frac{2\pi\,\eta^3\theta_1(2\epsilon_+)}{\theta_1(\epsilon_1)\,\theta_1(\epsilon_2)} \right) \cdot Q(u) \cdot W(u)\,. \tag{A.71}$$

Similar to the codimension 2 defect computation (2.53), there are two types of poles:

- $u = a_i - \epsilon_+$ for $i = 1, \ldots, k$

$$Z_1^{\text{Wilson}} \supset \frac{\vartheta_1(2\epsilon_+)}{\vartheta_1(\epsilon_1)\,\vartheta_1(\epsilon_2)} \sum_{i=1}^{k} \left(Q^\vee(a_i - \epsilon_+) \cdot W(a_i - \epsilon_+) \right)\,. \tag{A.72}$$

- $u = z + \epsilon_+$

$$Z_1^{\text{Wilson}} \supset Q(z + \epsilon_+)\,. \tag{A.73}$$

In total, the $l = 1$ genus reads

$$Z_1^{\text{Wilson}} = \frac{\vartheta_1(2\epsilon_+)}{\vartheta_1(\epsilon_1)\,\vartheta_1(\epsilon_2)} \sum_{i=1}^{k} Q^\vee(a_i - \epsilon_+) \cdot W(a_i - \epsilon_+) + Q(z + \epsilon_+)\,, \tag{A.74}$$

where the notation (2.16) has been used. The normalised 1-string contribution in the NS-limit is derived as follows:

$$\widetilde{Z}_1^{\text{Wilson}} = Z_1^{\text{Wilson}} - Z_1 \tag{A.75}$$

$$\lim_{\epsilon_2 \to 0} \widetilde{Z}_1^{\text{Wilson}} = \lim_{\epsilon_2 \to 0} \left(\frac{\vartheta_1(2\epsilon_+)}{\vartheta_1(\epsilon_1)\,\vartheta_1(\epsilon_2)} \sum_{i=1}^{k} Q^\vee(a_i - \epsilon_+) \cdot [W(a_i - \epsilon_+) - 1] + Q(z + \epsilon_+) \right)$$

$$= \frac{1}{\vartheta_1'(0)} \sum_{i=1}^{k} Q_{(0)}^\vee \left(a_i - \frac{1}{2}\epsilon_1 \right) \cdot [L(a_i - z - \epsilon_1) - L(a_i - z)] + Q_{(0)} \left(z + \frac{1}{2}\epsilon_1 \right)\,,$$

using (2.17), (A.25), and

$$\lim_{\epsilon_2 \to 0} \frac{W(a_i - \epsilon_+) - 1}{\vartheta_1(\epsilon_2)} = \frac{1}{\vartheta_1'(0)} [L(a_i - z - \epsilon_1) - L(a_i - z)]\,. \tag{A.76}$$

A.7.2   2-string

Consider the following $l = 2$ elliptic genus

$$Z_2^{\text{Wilson}} = \frac{1}{2} \oint \frac{\mathrm{d}u_1\mathrm{d}u_2}{(2\pi\mathrm{i})^2} \left(\frac{2\pi\,\eta^3\theta_1(2\epsilon_+)}{\theta_1(\epsilon_1)\theta_1(\epsilon_2)} \right)^2 D(u_1 - u_2)D(u_2 - u_1) \prod_{p=1}^{2} Q(u_p)W(u_p) \tag{A.77}$$

and the relevant poles can be split into poles that come from the theory without defect such as:

- Both poles originate from $P_0(u_p + \epsilon_1 + \epsilon_2)$ i.e.

$$(u_1, u_2) = (a_i - \epsilon_+, a_j - \epsilon_+) \qquad \text{for } i \neq j\,. \tag{A.78}$$

- One pole from $P_0(u_p + \epsilon_1 + \epsilon_2)$ and one from $D(\pm(u_1 - u_2))$, i.e.

$$(u_1, u_2) = (a_m - \epsilon_+, a_m - \epsilon_+ - \epsilon_{1,2}) \qquad \text{and}$$
$$(u_1, u_2) = (a_m - \epsilon_+ - \epsilon_{1,2}, a_m - \epsilon_+). \tag{A.79}$$

In addition, there are new poles from the codimension 4 defect part. These are

- One pole from $P_0(u_p + \epsilon_1 + \epsilon_2)$ and one from $W(u_p)$, i.e.

$$(u_1, u_2) = (a_m - \epsilon_+, z + \epsilon_+) \qquad \text{and} \qquad (u_1, u_2) = (z + \epsilon_+, a_m - \epsilon_+). \tag{A.80}$$

- One pole from $D(\pm(u_1 - u_2))$ and one from $V_{(0,s)}(u_p)$, i.e.

$$(u_1, u_2) = (z + \epsilon_+, z + \epsilon_+ - \epsilon_{1,2}) \qquad \text{and} \qquad (u_1, u_2) = (z + \epsilon_+ - \epsilon_{1,2}, z + \epsilon_+). \tag{A.81}$$

Now, one can work out the residues for the individual poles as before: firstly, consider the contributions for $(u_1, u_2) = (a_i - \epsilon_+, a_j - \epsilon_+)$

$$Z_2^{\text{Wilson}} \supset \frac{1}{2} \left(\frac{\vartheta_1(2\epsilon_+)}{\vartheta_1(\epsilon_1)\vartheta_1(\epsilon_2)} \right)^2 D(a_i - a_j)D(a_j - a_i)$$
$$\cdot Q^\vee(a_i - \epsilon_+)Q^\vee(a_j - \epsilon_+)W(a_i - \epsilon_+)W(a_j - \epsilon_+). \tag{A.82}$$

Secondly, both $(u_1, u_2) = (a_m - \epsilon_+, a_m - \epsilon_+ - \epsilon_1)$ and $(u_1, u_2) = (a_m - \epsilon_+ - \epsilon_1, a_m - \epsilon_+)$ yield

$$Z_2^{\text{Wilson}} \supset \frac{1}{2} \frac{\vartheta_1(2\epsilon_+)\vartheta_1(\epsilon_1 + 2\epsilon_+)}{\vartheta_1(\epsilon_2)\vartheta_1(2\epsilon_-)\vartheta_1(2\epsilon_1)} Q^\vee(a_m - \epsilon_+)Q(a_m - \epsilon_+ - \epsilon_1)$$
$$\cdot W(a_m - \epsilon_+)W(a_m - \epsilon_+ - \epsilon_1). \tag{A.83}$$

Thirdly, both $(u_1, u_2) = (a_m - \epsilon_+, a_m - \epsilon_+ - \epsilon_2)$ and $(u_1, u_2) = (a_m - \epsilon_+ - \epsilon_2, a_m - \epsilon_+)$ yield

$$Z_2^{\text{Wilson}} \supset -\frac{1}{2} \frac{\vartheta_1(2\epsilon_+)\vartheta_1(\epsilon_2 + 2\epsilon_+)}{\vartheta_1(\epsilon_1)\vartheta_1(2\epsilon_-)\vartheta_1(2\epsilon_2)} Q^\vee(a_m - \epsilon_+)Q(a_m - \epsilon_+ - \epsilon_2)$$
$$\cdot W(a_m - \epsilon_+)W(a_m - \epsilon_+ - \epsilon_2). \tag{A.84}$$

Fourthly, both $(u_1, u_2) = (a_m - \epsilon_+, z + \epsilon_+)$ and $(u_1, u_2) = (z + \epsilon_+, a_m - \epsilon_+)$ yield

$$Z_2^{\text{Wilson}} \supset \frac{1}{2} \frac{\vartheta_1(2\epsilon_+)}{\vartheta_1(\epsilon_1)\vartheta_1(\epsilon_2)} D(a_m - z - 2\epsilon_+)D(z + 2\epsilon_+ - a_m)$$
$$\cdot Q^\vee(a_m - \epsilon_+)Q(z + \epsilon_+)W(a_m - \epsilon_+). \tag{A.85}$$

Fifthly, both $(u_1, u_2) = (z + \epsilon_+, z + \epsilon_+ - \epsilon_1)$ and $(u_1, u_2) = (z + \epsilon_+ - \epsilon_1, z + \epsilon_+)$

$$Z_2^{\text{Wilson}} \supset \frac{1}{2} \frac{\vartheta_1(\epsilon_1)\vartheta_1(\epsilon_1 + 2\epsilon_+)}{\vartheta_1(2\epsilon_-)\vartheta_1(2\epsilon_1)} \cdot Q(z + \epsilon_+ - \epsilon_1)Q(z + \epsilon_+)W(z + \epsilon_+ - \epsilon_1) = 0, \tag{A.86}$$

because $W(z + \epsilon_+ - \epsilon_1) = 0$. Lastly, both $(u_1, u_2) = (z + \epsilon_+, z + \epsilon_+ - \epsilon_2)$ and $(u_1, u_2) = (z + \epsilon_+ - \epsilon_2, z + \epsilon_+)$ yield

$$Z_2^{\text{Wilson}} \supset -\frac{1}{2} \frac{\vartheta_1(\epsilon_2)\vartheta_1(\epsilon_2 + 2\epsilon_+)}{\vartheta_1(2\epsilon_-)\vartheta_1(2\epsilon_2)} \cdot Q(z + \epsilon_+ - \epsilon_2)Q(z + \epsilon_+)W(z + \epsilon_+ - \epsilon_2) = 0, \tag{A.87}$$

because $W(z + \epsilon_+ - \epsilon_2) = 0$. Summing up all the individual contributions leads to

$$Z_2^{\text{Wilson}} = \left(\frac{\vartheta_1(2\epsilon_+)}{\vartheta_1(\epsilon_1)\vartheta_1(\epsilon_2)}\right)^2 \sum_{1 \le i < j \le k} D(a_i - a_j)D(a_j - a_i) \tag{A.88}$$

$$\cdot Q^\vee(a_i - \epsilon_+)Q^\vee(a_j - \epsilon_+)W(a_i - \epsilon_+)W(a_j - \epsilon_+)$$

$$+ \frac{\vartheta_1(2\epsilon_+)}{\vartheta_1(2\epsilon_-)} \sum_{j=1}^k Q^\vee(a_j - \epsilon_+)W(a_j - \epsilon_+)$$

$$\cdot \left[\frac{\vartheta_1(\epsilon_1 + 2\epsilon_+)}{\vartheta_1(\epsilon_2)\vartheta_1(2\epsilon_1)} Q(a_j - \epsilon_+ - \epsilon_1)W(a_j - \epsilon_+ - \epsilon_1) \right.$$

$$\left. - \frac{\vartheta_1(\epsilon_2 + 2\epsilon_+)}{\vartheta_1(\epsilon_1)\vartheta_1(2\epsilon_2)} Q(a_j - \epsilon_+ - \epsilon_2)W(a_j - \epsilon_+ - \epsilon_2) \right]$$

$$+ \frac{\vartheta_1(2\epsilon_+)}{\vartheta_1(\epsilon_1)\vartheta_1(\epsilon_2)} \sum_{j=1}^k D(a_j - z - 2\epsilon_+)D(z + 2\epsilon_+ - a_j)$$

$$\cdot Q^\vee(a_j - \epsilon_+)Q(z + \epsilon_+)W(a_j - \epsilon_+)$$

For the evaluation of the normalised partition function in the NS-limit, the computation is split into several steps as above:

$$Z_2^{\text{Wilson}} - Z_2 = J_1 + J_2 + J_3 \tag{A.89}$$

with the following parts:

$$J_1 = \left(\frac{\vartheta_1(2\epsilon_+)}{\vartheta_1(\epsilon_1)\vartheta_1(\epsilon_2)}\right)^2 \sum_{1 \le i < j \le k} D(a_i - a_j)D(a_j - a_i) \tag{A.90}$$

$$\cdot Q^\vee(a_i - \epsilon_+)Q^\vee(a_j - \epsilon_+)\left[W(a_i - \epsilon_+)W(a_j - \epsilon_+) - 1\right],$$

$$J_2 = \frac{\vartheta_1(2\epsilon_+)}{\vartheta_1(2\epsilon_-)} \sum_{j=1}^k Q^\vee(a_j - \epsilon_+) \tag{A.91}$$

$$\cdot \left[\frac{\vartheta_1(\epsilon_1 + 2\epsilon_+)}{\vartheta_1(\epsilon_2)\vartheta_1(2\epsilon_1)} Q(a_j - \epsilon_+ - \epsilon_1)\left[W(a_j - \epsilon_+)W(a_j - \epsilon_+ - \epsilon_1) - 1\right] \right.$$

$$\left. - \frac{\vartheta_1(\epsilon_2 + 2\epsilon_+)}{\vartheta_1(\epsilon_1)\vartheta_1(2\epsilon_2)} Q(a_j - \epsilon_+ - \epsilon_2)\left[W(a_j - \epsilon_+)W(a_j - \epsilon_+ - \epsilon_2) - 1\right] \right],$$

$$J_3 = \frac{\vartheta_1(2\epsilon_+)}{\vartheta_1(\epsilon_1)\vartheta_1(\epsilon_2)} \sum_{j=1}^k D(a_j - z - 2\epsilon_+)D(z + 2\epsilon_+ - a_j)$$

$$\cdot Q^\vee(a_j - \epsilon_+)Q(z + \epsilon_+)W(a_j - \epsilon_+), \tag{A.92}$$

and the ϵ_2 expansion yields

$$J_1\big|_{\frac{1}{\epsilon_2}} = \left(\frac{1}{\vartheta_1'(0)}\right)^2 \sum_{i<j} Q_{(0)}^\vee\left(a_i - \frac{1}{2}\epsilon_1\right)Q_{(0)}^\vee\left(a_j - \frac{1}{2}\epsilon_1\right)W_{(1)}\left(a_i - \frac{1}{2}\epsilon_1, a_j - \frac{1}{2}\epsilon_1\right)$$

$$J_1\big|_{\text{fin}} = \left(\frac{1}{\vartheta_1'(0)}\right)^2 \sum_{i<j} Q_{(0)}^\vee\left(a_i - \frac{1}{2}\epsilon_1\right)Q_{(0)}^\vee\left(a_j - \frac{1}{2}\epsilon_1\right)W_{(2)}\left(a_i - \frac{1}{2}\epsilon_1, a_j - \frac{1}{2}\epsilon_1\right)$$

$$+ \left(\frac{1}{\vartheta_1'(0)}\right)^2 \sum_{i<j} \left\{ Q_{(0)}^\vee\left(a_i - \frac{1}{2}\epsilon_1\right)Q_{(1)}^\vee\left(a_j - \frac{1}{2}\epsilon_1\right) + Q_{(1)}^\vee\left(a_i - \frac{1}{2}\epsilon_1\right)Q_{(0)}^\vee\left(a_j - \frac{1}{2}\epsilon_1\right) \right\}$$

$$
\cdot W_{(1)}\left(a_i-\frac{1}{2}\epsilon_1,a_j-\frac{1}{2}\epsilon_1\right)
$$

$$
+\left(\frac{1}{\vartheta_1'(0)}\right)^2\sum_{i<j}\left(D^{(1)}(a_i-a_j)+D^{(1)}(a_j-a_i)\right)Q_{(0)}^\vee\left(a_i-\frac{1}{2}\epsilon_1\right)Q_{(0)}^\vee\left(a_j-\frac{1}{2}\epsilon_1\right)
$$

$$
\cdot W_{(1)}\left(a_i-\frac{1}{2}\epsilon_1,a_j-\frac{1}{2}\epsilon_1\right)
$$

$$
+2\frac{B^{(0)}}{\vartheta_1'(0)}\sum_{i<j}Q_{(0)}^\vee\left(a_i-\frac{1}{2}\epsilon_1\right)Q_{(0)}^\vee\left(a_j-\frac{1}{2}\epsilon_1\right)W_{(1)}\left(a_i-\frac{1}{2}\epsilon_1,a_j-\frac{1}{2}\epsilon_1\right),
$$

$$
J_2\big|_{\frac{1}{\epsilon_2}}=\frac{1}{2}\left(\frac{1}{\vartheta_1'(0)}\right)^2\sum_j Q_{(0)}^\vee(a_j-\epsilon_+-\epsilon_2)Q_{(0)}^\vee(a_j-\epsilon_+)W_{(1)}(a_j-\epsilon_+,a_j-\epsilon_+-\epsilon_2)
$$

$$
J_2\big|_{\rm fin}=\frac{1}{\vartheta_1'(0)}\sum_j Q_{(0)}^\vee\left(a_j-\frac{1}{2}\epsilon_1\right)Q^{(0)}\left(a_j-\frac{3}{2}\epsilon_1\right)W_{(1)}(a_j-\epsilon_+,a_j-\epsilon_+-\epsilon_1)
$$

$$
-\frac{1}{2\vartheta_1'(0)}\sum_j Q_{(0)}^\vee\left(a_j-\frac{1}{2}\epsilon_1\right)\tilde{Q}\left(a_j-\frac{1}{2}\epsilon_1\right)W_{(1)}(a_j-\epsilon_+,a_j-\epsilon_+-\epsilon_2)
$$

$$
+\frac{B^{(0)}}{\vartheta_1'(0)}\sum_j\left(Q_{(0)}^\vee\left(a_j-\frac{1}{2}\epsilon_1\right)\right)^2 W_{(1)}(a_j-\epsilon_+,a_j-\epsilon_+-\epsilon_2)
$$

$$
+\frac{1}{2}\left(\frac{1}{\vartheta_1'(0)}\right)^2\sum_j\left[2L(\epsilon_1)Q_{(0)}^\vee\left(a_j-\frac{1}{2}\epsilon_1\right)+Q_{(1)}^\vee\left(a_j-\frac{1}{2}\epsilon_1\right)\right]
$$

$$
\cdot Q_{(0)}^\vee\left(a_j-\frac{1}{2}\epsilon_1\right)W_{(1)}(a_j-\epsilon_+,a_j-\epsilon_+-\epsilon_2)
$$

$$
+\frac{1}{2(\vartheta_1'(0))^2}\sum_j\left(Q_{(0)}^\vee\left(a_j-\frac{1}{2}\epsilon_1\right)\right)^2 W_{(2)}(a_j-\epsilon_+,a_j-\epsilon_+-\epsilon_2),
$$

$$
J_3\big|_{\frac{1}{\epsilon_2}}=\frac{1}{\vartheta_1'(0)}Q_{(0)}(z+\epsilon_+)\sum_j Q_{(0)}^\vee(a_j-\epsilon_+)
$$

$$
J_3\big|_{\rm fin}=\frac{1}{\vartheta_1'(0)}\sum_j\left(D^{(1)}(a_j-z-2\epsilon_+)+D^{(1)}(z+2\epsilon_+-a_j)\right)Q_{(0)}^\vee\left(a_j-\frac{1}{2}\epsilon_1\right)Q_{(0)}(z+\epsilon_+)
$$

$$
+\frac{1}{\vartheta_1'(0)}\sum_j\left[Q_{(1)}^\vee\left(a_j-\frac{1}{2}\epsilon_1\right)Q_{(0)}(z+\epsilon_+)+Q_{(0)}^\vee\left(a_j-\frac{1}{2}\epsilon_1\right)Q_{(1)}(z+\epsilon_+)\right.
$$

$$
\left.+Q_{(0)}^\vee\left(a_j-\frac{1}{2}\epsilon_1\right)Q_{(0)}(z+\epsilon_+)W_{(1)}(a_j-\epsilon_+)\right]
$$

$$
+B^{(0)}Q_{(0)}(z+\epsilon_+)\sum_j Q_{(0)}^\vee(a_j-\epsilon_+).
$$

With the conventions (2.17) and (A.25), the normalised 2-string elliptic genus in presence of a codimension 4 defect reads

$$
\widetilde{Z}_2^{\rm Wilson}=\sum_{\substack{i,j=1\\i\neq j}}^{k}\frac{Q_{(0)}^\vee(a_i-\frac{\epsilon_1}{2})}{\vartheta_1'(0)}\frac{Q_{(0)}^\vee(a_j-\frac{\epsilon_1}{2})}{\vartheta_1'(0)}\left[\frac{1}{2}L(a_i-z)L(a_j-z)\right.
$$

$$
\left.-L(a_i-z)L(a_j-z-\epsilon_1)+\frac{1}{2}L(a_i-z-\epsilon_1)L(a_j-z-\epsilon_1)\right]
$$

$$+ \frac{1}{2} \sum_{j=1}^{k} \left(\frac{Q_{(0)}^{\vee}(a_j - \frac{\epsilon_1}{2})}{\vartheta_1'(0)} \right)^2 [K(a_j - z) - K(a_j - z - \epsilon_1)]$$

$$+ \frac{1}{2} \sum_{j=1}^{k} \left(\frac{Q_{(0)}^{\vee}(a_j - \frac{\epsilon_1}{2})}{\vartheta_1'(0)} \right)^2 L(a_j - z - \epsilon_1) \left[L(a_j - z - \epsilon_1) - L(a_j - z) \right]$$

$$+ 2L(\epsilon_1) \sum_{j=1}^{k} \left(\frac{Q_{(0)}^{\vee}(a_j - \frac{\epsilon_1}{2})}{\vartheta_1'(0)} \right)^2 [L(a_j - z - \epsilon_1) - L(a_j - z)]$$

$$+ \sum_{j=1}^{k} \left(\frac{Q_{(0)}^{\vee}(a_j - \frac{\epsilon_1}{2})}{\vartheta_1'(0)} \right)^2 [L(a_j - z - \epsilon_1) - L(a_j - z)]$$

$$\cdot \left[\sum_{i=1}^{k} L(a_j - a_i - \epsilon_1) + \sum_{\substack{i=1 \\ i \neq j}}^{k} L(a_j - a_i - \epsilon_1) \right.$$

$$\left. - \sum_{i=1}^{k} \left(L\left(a_j - \frac{\epsilon_1}{2} - m_i + b \right) + L\left(a_j - \frac{\epsilon_1}{2} - n_i - b \right) \right) \right]$$

$$+ \sum_{\substack{i,j=1 \\ i \neq j}}^{k} \frac{Q_{(0)}^{\vee}(a_i - \frac{\epsilon_1}{2})}{\vartheta_1'(0)} \frac{Q_{(0)}^{\vee}(a_j - \frac{\epsilon_1}{2})}{\vartheta_1'(0)} [L(a_i - z - \epsilon_1) - L(a_i - z)]$$

$$\cdot \left[L(a_i - a_j + \epsilon_1) - L(a_i - a_j) + L(a_j - a_i + \epsilon_1) - L(a_j - a_i) \right]$$

$$+ \sum_{j=1}^{k} \frac{Q_{(0)}^{\vee}\left(a_j - \frac{\epsilon_1}{2} \right)}{\vartheta_1'(0)} Q_{(0)}\left(a_j - \frac{3\epsilon_1}{2} \right) [L(a_j - z - 2\epsilon_1) - L(a_j - z)]$$

$$+ Q_{(0)}\left(z + \frac{\epsilon_1}{2} \right) \sum_{j=1}^{k} \frac{Q_{(0)}^{\vee}(a_j - \frac{\epsilon_1}{2})}{\vartheta_1'(0)} [L(z - a_j + 2\epsilon_1) - L(z - a_j + \epsilon_1)] , \quad \text{(A.93)}$$

which has been checked against the explicit NS-limit for $k = 2, 3$ via `Mathematica`.

A.8 Computation of $P(x + \epsilon_1)$ coefficients

In section 3, the function $P(x)$ appeared in the derivation of the difference equation (3.34). The main focus of section 3.4 is to argue that P is related to the expectation value of a Wilson surface. Here, the details of the 1 and 2-string comparison are presented.

A.8.1 1-string

Consider the prediction (3.43), then start by computing

$$(Y^{-1} - 1)\widetilde{Z}_1^{(0,1)\text{def}} = \frac{1}{\vartheta_1'(0)} \sum_{i=1}^{k} Q_{(0)}^{\vee}\left(a_i - \frac{1}{2}\epsilon_1 \right) \cdot \left[L\left(a_i - x - \frac{3}{2}\epsilon_1 \right) - L\left(a_i - x - \frac{1}{2}\epsilon_1 \right) \right]$$
$$+ Q_{(0)}(x + \epsilon_1) - Q_{(0)}(x) \tag{A.94}$$

such that the addition of $Q_{(0)}(x)$ results in (3.44).

A.8.2 2-string

Work out the 2-string prediction (3.47) with the results from above. To begin with, set $s = 1$ then detail $(Y^{-1} - 1)Z_2$ with $Y^{-1}f(x) = f(x + \epsilon_1)$

$$
\begin{aligned}
(Y^{-1}-1)\widetilde{Z}_{k=2}^{(0,1)\mathrm{def}} = &\sum_{j=1}^{k} \left(\frac{Q_{(0)}^{\vee}(a_j - \frac{\epsilon_1}{2})}{\vartheta_1'(0)} \right)^2 \left(L\left(a_j - x - \frac{3\epsilon_1}{2}\right) \right)^2 - \left(L\left(a_j - x - \frac{\epsilon_1}{2}\right) \right)^2 \\
&- \frac{1}{2}\sum_{j=1}^{k} \left(\frac{Q_{(0)}^{\vee}(a_j - \frac{\epsilon_1}{2})}{\vartheta_1'(0)} \right)^2 \left[K\left(a_j - x - \frac{3\epsilon_1}{2}\right) - K\left(a_j - x - \frac{\epsilon_1}{2}\right) \right] \\
&+ \frac{1}{2}\sum_{\substack{i,j=1 \\ i \neq j}}^{k} \frac{Q_{(0)}^{\vee}(a_i - \frac{\epsilon_1}{2})}{\vartheta_1'(0)} \frac{Q_{(0)}^{\vee}(a_j - \frac{\epsilon_1}{2})}{\vartheta_1'(0)} \\
&\cdot \left[L\left(a_i - x - \frac{3\epsilon_1}{2}\right) L\left(a_j - x - \frac{3\epsilon_1}{2}\right) - L\left(a_i - x - \frac{\epsilon_1}{2}\right) L\left(a_j - x - \frac{\epsilon_1}{2}\right) \right] \\
&+ 2 \cdot L(\epsilon_1) \sum_{j=1}^{k} \left(\frac{Q_{(0)}^{\vee}(a_j - \frac{\epsilon_1}{2})}{\vartheta_1'(0)} \right)^2 \left[L\left(a_j - x - \frac{3\epsilon_1}{2}\right) - L\left(a_j - x - \frac{\epsilon_1}{2}\right) \right] \\
&+ \sum_{\substack{i,j=1 \\ i \neq j}}^{k} \frac{Q_{(0)}^{\vee}(a_i - \frac{\epsilon_1}{2})}{\vartheta_1'(0)} \frac{Q_{(0)}^{\vee}(a_j - \frac{\epsilon_1}{2})}{\vartheta_1'(0)} L\left(a_i - x - \frac{\epsilon_1}{2}\right) \left[L(a_i - a_j + \epsilon_1) - L(a_i - a_j) \right. \\
&\left. \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad + L(a_j - a_i + \epsilon_1) - L(a_j - a_i) \right] \\
&+ \sum_{j=1}^{k} \frac{Q_{(0)}^{\vee}(a_j - \frac{\epsilon_1}{2})}{\vartheta_1'(0)} Q_{(0)}\left(a_j - \frac{3\epsilon_1}{2}\right) \left[L\left(a_j - x - \frac{3\epsilon_1}{2}\right) + L\left(a_j - x - \frac{\epsilon_1}{2}\right) \right] \\
&+ \sum_{j=1}^{k} \left(\frac{Q_{(0)}^{\vee}(a_j - \frac{\epsilon_1}{2})}{\vartheta_1'(0)} \right)^2 \left[L\left(a_j - x - \frac{3\epsilon_1}{2}\right) - L\left(a_j - x - \frac{\epsilon_1}{2}\right) \right] \\
&\cdot \left[\sum_{i=1}^{k} L(a_j - a_i - \epsilon_1) + \sum_{\substack{i=1 \\ i \neq j}}^{N} L(a_j - a_i) \right. \\
&\left. \qquad - \sum_{i=1}^{k} \left(L\left(a_j - \frac{\epsilon_1}{2} - m_i + b\right) + L\left(a_j - \frac{\epsilon_1}{2} - n_i - b\right) \right) \right] \\
&+ \sum_{j=1}^{k} \frac{Q_{(0)}^{\vee}(a_j - \frac{\epsilon_1}{2})}{\vartheta_1'(0)} \left\{ Q_{(0)}(x + \epsilon_1) \left[L\left(a_j - x - \frac{\epsilon_1}{2}\right) + L\left(x - a_j + \frac{5\epsilon_1}{2}\right) \right.\right. \\
&\left.\left. \qquad\qquad\qquad\qquad\qquad\qquad - L\left(x - a_j + \frac{3\epsilon_1}{2}\right) \right] \right. \\
&\left. - Q_{(0)}(x) \left[+ L\left(a_j - x + \frac{\epsilon_1}{2}\right) + L\left(x - a_j + \frac{3\epsilon_1}{2}\right) - L\left(x - a_j + \frac{\epsilon_1}{2}\right) \right] \right\} \\
&+ Q_{(0)}(x) \left[Q_{(0)}(x + \epsilon_1) - Q_{(0)}(x - \epsilon_1) \right].
\end{aligned}
\tag{A.95}
$$

Next, one needs to work out the following contribution:

$$(Y-1)\widetilde{Z}_1^{(0,1)\text{def}} = \sum_{i=1}^{k} \frac{Q_{(0)}^{\vee}\left(a_i - \frac{1}{2}\epsilon_1\right)}{\vartheta_1'(0)} \cdot \left[L\left(a_i - x + \frac{1}{2}\epsilon_1\right) - L\left(a_i - x - \frac{1}{2}\epsilon_1\right) \right]$$
$$+ Q_{(0)}(x-\epsilon_1) - Q_{(0)}(x) \,, \tag{A.96}$$

such that

$$Q_{(0)}(x)(Y-1)\widetilde{Z}_1^{(0,1)\text{def}} = Q_{(0)}(x)\sum_{i=1}^{k} \frac{Q_{(0)}^{\vee}(a_i - \frac{1}{2}\epsilon_1)}{\vartheta_1'(0)} \cdot \left[L\left(a_i - x + \frac{1}{2}\epsilon_1\right) - L\left(a_i - x - \frac{1}{2}\epsilon_1\right) \right]$$
$$+ Q_{(0)}(x)\left(Q_{(0)}(x-\epsilon_1) - Q_{(0)}(x)\right) . \tag{A.97}$$

In addition, one needs the following contribution:

$$\widetilde{Z}_1^{(0,1)\text{def}} \cdot (Y^{-1}-1)\widetilde{Z}_1^{(0,1)\text{def}} = \left(\sum_{j=1}^{k} \frac{Q_{(0)}^{\vee}\left(a_j - \frac{1}{2}\epsilon_1\right)}{\vartheta_1'(0)} \cdot L\left(a_j - x - \frac{1}{2}\epsilon_1\right) + Q_{(0)}(x) \right)$$
$$\cdot \left(\sum_{i=1}^{k} \frac{Q_{(0)}^{\vee}\left(a_i - \frac{1}{2}\epsilon_1\right)}{\vartheta_1'(0)} \cdot \left[L\left(a_i - x - \frac{3}{2}\epsilon_1\right) - L\left(a_i - x - \frac{1}{2}\epsilon_1\right) \right] \right.$$
$$\left. + Q_{(0)}(x+\epsilon_1) - Q_{(0)}(x) \right)$$
$$= \sum_{i,j=1}^{k} \frac{Q_{(0)}^{\vee}(a_j - \frac{1}{2}\epsilon_1)}{\vartheta_1'(0)} \frac{Q_{(0)}^{\vee}(a_i - \frac{1}{2}\epsilon_1)}{\vartheta_1'(0)} L\left(a_j - x - \frac{1}{2}\epsilon_1\right)$$
$$\cdot \left[L\left(a_i - x - \frac{3}{2}\epsilon_1\right) - L\left(a_i - x - \frac{1}{2}\epsilon_1\right) \right]$$
$$+ Q_{(0)}(x) \cdot \sum_{i=1}^{k} \frac{Q_{(0)}^{\vee}(a_i - \frac{1}{2}\epsilon_1)}{\vartheta_1'(0)}$$
$$\cdot \left[L\left(a_i - x - \frac{3}{2}\epsilon_1\right) - 2L\left(a_i - x - \frac{1}{2}\epsilon_1\right) \right]$$
$$+ Q_{(0)}(x+\epsilon_1) \cdot \sum_{j=1}^{k} \frac{Q_{(0)}^{\vee}\left(a_j - \frac{1}{2}\epsilon_1\right)}{\vartheta_1'(0)} \cdot L\left(a_j - x - \frac{1}{2}\epsilon_1\right)$$
$$+ Q_{(0)}(x)\left(Q_{(0)}(x+\epsilon_1) - Q_{(0)}(x)\right) . \tag{A.98}$$

Combining the individual terms, one finds (3.48)

References

[1] J.J. Heckman, D.R. Morrison and C. Vafa, *On the classification of* 6*D SCFTs and generalized ADE orbifolds*, *JHEP* **05** (2014) 028 [*Erratum ibid.* **06** (2015) 017] [arXiv:1312.5746] [INSPIRE].

[2] J.J. Heckman, D.R. Morrison, T. Rudelius and C. Vafa, *Atomic classification of* 6*D SCFTs*, *Fortsch. Phys.* **63** (2015) 468 [arXiv:1502.05405] [INSPIRE].

[3] P. Jefferson, H.-C. Kim, C. Vafa and G. Zafrir, *Towards classification of* 5*d SCFTs: single gauge node*, arXiv:1705.05836 [INSPIRE].

[4] P. Jefferson, S. Katz, H.-C. Kim and C. Vafa, *On geometric classification of* 5*d SCFTs*, *JHEP* **04** (2018) 103 [arXiv:1801.04036] [INSPIRE].

[5] L. Bhardwaj and P. Jefferson, *Classifying* 5*d SCFTs via* 6*d SCFTs: rank one*, *JHEP* **07** (2019) 178 [*Addendum ibid.* **01** (2020) 153] [arXiv:1809.01650] [INSPIRE].

[6] L. Bhardwaj and P. Jefferson, *Classifying* 5*d SCFTs via* 6*d SCFTs: arbitrary rank*, *JHEP* **10** (2019) 282 [arXiv:1811.10616] [INSPIRE].

[7] L. Bhardwaj, P. Jefferson, H.-C. Kim, H.-C. Tarazi and C. Vafa, *Twisted circle compactifications of* 6*d SCFTs*, *JHEP* **12** (2020) 151 [arXiv:1909.11666] [INSPIRE].

[8] D. Gaiotto and S.S. Razamat, $N = 1$ *theories of class* $\mathcal{S}_k$, *JHEP* **07** (2015) 073 [arXiv:1503.05159] [INSPIRE].

[9] H.-C. Kim, S.S. Razamat, C. Vafa and G. Zafrir, *E-string theory on Riemann surfaces*, *Fortsch. Phys.* **66** (2018) 1700074 [arXiv:1709.02496] [INSPIRE].

[10] S.S. Razamat, C. Vafa and G. Zafrir, 4*d* $N = 1$ *from* 6*d* $(1, 0)$, *JHEP* **04** (2017) 064 [arXiv:1610.09178] [INSPIRE].

[11] H.-C. Kim, S.S. Razamat, C. Vafa and G. Zafrir, *Compactifications of ADE conformal matter on a torus*, *JHEP* **09** (2018) 110 [arXiv:1806.07620] [INSPIRE].

[12] S.S. Razamat and G. Zafrir, *Compactification of* 6*d minimal SCFTs on Riemann surfaces*, *Phys. Rev. D* **98** (2018) 066006 [arXiv:1806.09196] [INSPIRE].

[13] H.-C. Kim, S.S. Razamat, C. Vafa and G. Zafrir, *D-type conformal matter and SU/USp quivers*, *JHEP* **06** (2018) 058 [arXiv:1802.00620] [INSPIRE].

[14] J. Chen, B. Haghighat, S. Liu and M. Sperling, 4*d* $N = 1$ *from* 6*d D-type* $N = (1, 0)$, *JHEP* **01** (2020) 152 [arXiv:1907.00536] [INSPIRE].

[15] B. Haghighat, A. Iqbal, C. Kozçaz, G. Lockhart and C. Vafa, *M-strings*, *Commun. Math. Phys.* **334** (2015) 779 [arXiv:1305.6322] [INSPIRE].

[16] H.-C. Kim, J. Kim and S. Kim, *Instantons on the* 5*-sphere and M5-branes*, arXiv:1211.0144 [INSPIRE].

[17] B. Haghighat, C. Kozcaz, G. Lockhart and C. Vafa, *Orbifolds of M-strings*, *Phys. Rev. D* **89** (2014) 046003 [arXiv:1310.1185] [INSPIRE].

[18] B. Haghighat, W. Yan and S.-T. Yau, *ADE string chains and mirror symmetry*, *JHEP* **01** (2018) 043 [arXiv:1705.05199] [INSPIRE].

[19] D. Gaiotto and H.-C. Kim, *Surface defects and instanton partition functions*, *JHEP* **10** (2016) 012 [arXiv:1412.2781] [INSPIRE].

[20] N. Nekrasov and E. Witten, *The omega deformation, branes, integrability, and Liouville theory*, *JHEP* **09** (2010) 092 [arXiv:1002.0888] [INSPIRE].

[21] N.A. Nekrasov and S.L. Shatashvili, *Quantization of integrable systems and four dimensional gauge theories*, in 16$^{\text{th}}$ *international congress on mathematical physics*, World Scientific, Singapore (2009), pg. 265 [arXiv:0908.4052] [INSPIRE].

[22] B. Nazzal and S.S. Razamat, *Surface defects in E-string compactifications and the van Diejen model*, *SIGMA* **14** (2018) 036 [arXiv:1801.00960] [INSPIRE].

[23] T. Dimofte, D. Gaiotto and S. Gukov, *Gauge theories labelled by three-manifolds*, *Commun. Math. Phys.* **325** (2014) 367 [arXiv:1108.4389] [INSPIRE].

[24] S. Gukov, *Three-dimensional quantum gravity, Chern-Simons theory, and the A polynomial*, *Commun. Math. Phys.* **255** (2005) 577 [hep-th/0306165] [INSPIRE].

[25] S. Garoufalidis, *On the characteristic and deformation varieties of a knot*, *Geom. Topol. Monogr.* **7** (2004) 291 [math.GT/0306230].

[26] S. Garoufalidis and T.T.Q. Le, *The colored Jones function is q-holonomic*, *Geom. Topol.* **9** (2005) 1253 [math.GT/0309214].

[27] S. Gukov and P. Sulkowski, *A-polynomial, B-model, and quantization*, *JHEP* **02** (2012) 070 [arXiv:1108.0002] [INSPIRE].

[28] M. Aganagic and C. Vafa, *Large N duality, mirror symmetry, and a Q-deformed A-polynomial for knots*, arXiv:1204.4709 [INSPIRE].

[29] M. Aganagic, T. Ekholm, L. Ng and C. Vafa, *Topological strings, D-model, and knot contact homology*, *Adv. Theor. Math. Phys.* **18** (2014) 827 [arXiv:1304.5778] [INSPIRE].

[30] R. Kashaev and M. Mariño, *Operators from mirror curves and the quantum dilogarithm*, *Commun. Math. Phys.* **346** (2016) 967 [arXiv:1501.01014] [INSPIRE].

[31] E. Witten, *Some comments on string dynamics*, in *Strings '95: future perspectives in string theory*, (1995), pg. 501 [hep-th/9507121] [INSPIRE].

[32] A. Strominger, *Open p-branes*, *Phys. Lett. B* **383** (1996) 44 [hep-th/9512059] [INSPIRE].

[33] D. Gaiotto, $N = 2$ *dualities*, *JHEP* **08** (2012) 034 [arXiv:0904.2715] [INSPIRE].

[34] A. Hanany and A. Zaffaroni, *Branes and six-dimensional supersymmetric theories*, *Nucl. Phys. B* **529** (1998) 180 [hep-th/9712145] [INSPIRE].

[35] I. Brunner and A. Karch, *Branes at orbifolds versus Hanany Witten in six-dimensions*, *JHEP* **03** (1998) 003 [hep-th/9712143] [INSPIRE].

[36] D. Gaiotto and A. Tomasiello, *Holography for* $(1,0)$ *theories in six dimensions*, *JHEP* **12** (2014) 003 [arXiv:1404.0711] [INSPIRE].

[37] M. Del Zotto, J.J. Heckman, A. Tomasiello and C. Vafa, 6d *conformal matter*, *JHEP* **02** (2015) 054 [arXiv:1407.6359] [INSPIRE].

[38] K. Ohmori, H. Shimizu, Y. Tachikawa and K. Yonekura, 6d $N = (1,0)$ *theories on* S^1/T^2 *and class S theories: part II*, *JHEP* **12** (2015) 131 [arXiv:1508.00915] [INSPIRE].

[39] M. Del Zotto and G. Lockhart, *Universal features of BPS strings in six-dimensional SCFTs*, *JHEP* **08** (2018) 173 [arXiv:1804.09694] [INSPIRE].

[40] A. Hanany and G. Zafrir, *Discrete gauging in six dimensions*, *JHEP* **07** (2018) 168 [arXiv:1804.08857] [INSPIRE].

[41] H. Hayashi, S.-S. Kim, K. Lee and F. Yagi, *Equivalence of several descriptions for 6d SCFT*, *JHEP* **01** (2017) 093 [arXiv:1607.07786] [INSPIRE].

[42] J. Kim, S. Kim and K. Lee, *Higgsing towards E-strings*, *JHEP* **01** (2021) 110 [arXiv:1510.03128] [INSPIRE].

[43] F. Benini, R. Eager, K. Hori and Y. Tachikawa, *Elliptic genera of two-dimensional N = 2 gauge theories with rank-one gauge groups*, *Lett. Math. Phys.* **104** (2014) 465 [arXiv:1305.0533] [INSPIRE].

[44] F. Benini, R. Eager, K. Hori and Y. Tachikawa, *Elliptic genera of 2d N = 2 gauge theories*, *Commun. Math. Phys.* **333** (2015) 1241 [arXiv:1308.4896] [INSPIRE].

[45] N.A. Nekrasov, *Seiberg-Witten prepotential from instanton counting*, *Adv. Theor. Math. Phys.* **7** (2003) 831 [hep-th/0206161] [INSPIRE].

[46] F. Fucito, J.F. Morales, D.R. Pacifici and R. Poghossian, *Gauge theories on Ω-backgrounds from non commutative Seiberg-Witten curves*, *JHEP* **05** (2011) 098 [arXiv:1103.4495] [INSPIRE].

[47] L.C. Jeffrey and F.C. Kirwan, *Localization for non-Abelian group actions*, *Topology* **34** (1995) 291.

[48] D. Gaiotto, L. Rastelli and S.S. Razamat, *Bootstrapping the superconformal index with surface defects*, *JHEP* **01** (2013) 022 [arXiv:1207.3577] [INSPIRE].

[49] A. Bourget et al., *The Higgs mechanism — Hasse diagrams for symplectic singularities*, *JHEP* **01** (2020) 157 [arXiv:1908.04245] [INSPIRE].

[50] Y. Ito and Y. Yoshida, *Superconformal index with surface defects for class $\mathcal{S}_k$*, *Nucl. Phys. B* **962** (2021) 115277 [arXiv:1606.01653] [INSPIRE].

[51] L.F. Alday, D. Gaiotto, S. Gukov, Y. Tachikawa and H. Verlinde, *Loop and surface operators in N = 2 gauge theory and Liouville modular geometry*, *JHEP* **01** (2010) 113 [arXiv:0909.0945] [INSPIRE].

[52] A. Gadde and S. Gukov, *2d index and surface operators*, *JHEP* **03** (2014) 080 [arXiv:1305.0266] [INSPIRE].

[53] H. Mori and Y. Sugimoto, *Surface operators from M-strings*, *Phys. Rev. D* **95** (2017) 026001 [arXiv:1608.02849] [INSPIRE].

[54] P. Koroteev and S. Shakirov, *The quantum DELL system*, *Lett. Math. Phys.* **110** (2020) 969 [arXiv:1906.10354] [INSPIRE].

[55] O.J. Ganor, *Six-dimensional tensionless strings in the large N limit*, *Nucl. Phys. B* **489** (1997) 95 [hep-th/9605201] [INSPIRE].

[56] B. Chen, W. He, J.-B. Wu and L. Zhang, *M5-branes and Wilson surfaces*, *JHEP* **08** (2007) 067 [arXiv:0707.3978] [INSPIRE].

[57] N. Nekrasov, *BPS/CFT correspondence: non-perturbative Dyson-Schwinger equations and qq-characters*, *JHEP* **03** (2016) 181 [arXiv:1512.05388] [INSPIRE].

[58] H.-C. Kim, *Line defects and 5d instanton partition functions*, *JHEP* **03** (2016) 199 [arXiv:1601.06841] [INSPIRE].

[59] M. Bullimore and H.-C. Kim, *The superconformal index of the (2, 0) theory with defects*, *JHEP* **05** (2015) 048 [arXiv:1412.3872] [INSPIRE].

[60] P. Agarwal, J. Kim, S. Kim and A. Sciarappa, *Wilson surfaces in M5-branes*, *JHEP* **08** (2018) 119 [arXiv:1804.09932] [INSPIRE].

[61] D. Tong, *The holographic dual of $AdS_3 \times S^3 \times S^3 \times S^1$*, *JHEP* **04** (2014) 193 [arXiv:1402.5135] [INSPIRE].

[62] K. Maruyoshi and J. Yagi, *Surface defects as transfer matrices*, *PTEP* **2016** (2016) 113B01 [arXiv:1606.01041] [INSPIRE].

[63] J. Yagi, *Surface defects and elliptic quantum groups*, *JHEP* **06** (2017) 013 [arXiv:1701.05562] [INSPIRE].

[64] M. Bullimore, H.-C. Kim and P. Koroteev, *Defects and quantum Seiberg-Witten geometry*, *JHEP* **05** (2015) 095 [arXiv:1412.6081] [INSPIRE].

[65] F. Ferrari and M. Piatek, *On a path integral representation of the Nekrasov instanton partition function and its Nekrasov-Shatashvili limit*, arXiv:1212.6787 [INSPIRE].

[66] R. Poghossian, *Deforming SW curve*, *JHEP* **04** (2011) 033 [arXiv:1006.4822] [INSPIRE].

[67] N. Lambert and C. Papageorgakis, *Non-Abelian $(2,0)$ tensor multiplets and 3-algebras*, *JHEP* **08** (2010) 083 [arXiv:1007.2982] [INSPIRE].

[68] M.R. Douglas, *On $D = 5$ super Yang-Mills theory and $(2,0)$ theory*, *JHEP* **02** (2011) 011 [arXiv:1012.2880] [INSPIRE].

[69] N. Lambert, C. Papageorgakis and M. Schmidt-Sommerfeld, *M5-branes, D4-branes and quantum 5D super-Yang-Mills*, *JHEP* **01** (2011) 083 [arXiv:1012.2882] [INSPIRE].

[70] C. Hwang, J. Kim, S. Kim and J. Park, *General instanton counting and 5d SCFT*, *JHEP* **07** (2015) 063 [*Addendum ibid.* **04** (2016) 094] [arXiv:1406.6793] [INSPIRE].

[71] S.S. Razamat, E. Sabag and G. Zafrir, *From 6d flows to 4d flows*, *JHEP* **12** (2019) 108 [arXiv:1907.04870] [INSPIRE].

[72] B. Haghighat, J. Kim, W. Yan and S.-T. Yau, *D-type fiber-base duality*, *JHEP* **09** (2018) 060 [arXiv:1806.10335] [INSPIRE].

[73] N. Nekrasov and V. Pestun, *Seiberg-Witten geometry of four dimensional $N = 2$ quiver gauge theories*, arXiv:1211.2240 [INSPIRE].

[74] J. Kim, S. Kim, K. Lee, J. Park and C. Vafa, *Elliptic genus of E-strings*, *JHEP* **09** (2017) 098 [arXiv:1411.2324] [INSPIRE].