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Elliptic structures on weighted three-dimensional Fano hypersurfaces

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Abstract. We classify birational transformations into elliptic fibrations of a general quasi-smooth hypersurface in $\mathbb{P}(1,a_1,a_2,a_3,a_4)$ of degree $\sum_{i=1}^4 a_i$ that has terminal singularities.

§ 1. Introduction

Let X be a quasi-smooth hypersurface in $\mathbb{P}(1,a_1,a_2,a_3,a_4)$ of degree $d=\sum_{i=1}^4 a_i$ that has terminal singularities, where $a_1 \leqslant a_2 \leqslant a_3 \leqslant a_4$. Then X is a Fano threefold, and there are exactly 95 possibilities for the four-tuple (a_1,a_2,a_3,a_4). We shall use the symbol n to denote the place of such a family in the list in [1].

Suppose that the hypersurface X is general. The following result is proved in [2].

Theorem 1.1. *The hypersurface X is birationally rigid* [1] *and*, *in particular*, *non-rational.*

There are finitely many involutions $\tau_1,\dots,\tau_{k_n} \in \mathrm{Bir}(X)$ that generate the group $\mathrm{Bir}(X)$ up to biregular automorphisms (see [2]). In the case when $n \notin \{7,20,60\}$ and $k_n > 0$, the hypersurface X can be birationally transformed into an elliptic fibration that is invariant under the induced action of the group $\mathrm{Bir}(X)$. This fact is used in [4] to find the relations between the involutions $\tau_1,\dots,\tau_{k_n}$.

It is natural to try to classify all birational transformations of the hypersurface X into elliptic fibrations. This is equivalent to the following problem: find all rational maps $X \dashrightarrow \mathbb{P}^2$ whose generic fibre is birational to an elliptic curve. Here are a few examples.

Example 1.2. Let $n=1$. Then X is a quartic threefold. Let $\xi\colon X \dashrightarrow \mathbb{P}^2$ be projection from a line contained in X. Then a generic fibre of the map ξ is an elliptic curve.

Example 1.3. Let $n=2$. Then X is a hypersurface in $\mathbb{P}(1,1,1,1,2)$ of degree 5 which has one singular point of type $\frac{1}{2}(1,1,1)$. There is a commutative diagram

$$\begin{tikzcd} Y \arrow[r, "\gamma"] \arrow[d, "\pi"'] & Z \arrow[d, "\eta"] \\ X \arrow[r, dashed, "\psi"'] & \mathbb{P}^3, \end{tikzcd}$$

[1]Let V be a Fano variety of Picard rank 1 having terminal $\mathbb{Q}$-factorial singularities. Then V is said to be *birationally rigid* (see [3]) if it is not birational to any other Mori fibre space, and birationally superrigid if it is birationally rigid and $\mathrm{Bir}(V)=\mathrm{Aut}(V)$.

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where ψ is the natural projection, π is a weighted blow-up of the singular point of the hypersurface X with weights $(1,1,1)$, γ is a birational morphism that contracts 15 irreducible smooth rational curves $C_1,\dots,C_{15}$, and η is a double cover. Put $\xi=\chi\circ\psi$, where $\chi\colon\mathbb{P}^3\dashrightarrow\mathbb{P}^2$ is the projection from the point $\eta\circ\gamma(C_i)$. Then a generic fibre of the map ξ is an elliptic curve.

Example 1.4. Let $n=17$. Then X is a hypersurface in $\mathbb{P}(1,1,3,4,4)$ of degree 12 whose singularities consist of three singular points of type $\frac{1}{4}(1,1,3)$. There is a commutative diagram

$$\begin{tikzcd} U \arrow[d, "\alpha"'] & & W \arrow[ll, "\beta"'] \arrow[d, "\omega"] \\ X \arrow[rr, dashed, "\xi"'] & & \mathbb{P}(1,1,4), \end{tikzcd}$$

where ξ is a projection, α is a weighted blow-up of a singular point of type $\frac{1}{4}(1,1,3)$ with weights $(1,1,3)$, β is a weighted blow-up with weights $(1,1,2)$ of the singular point that is contained in the exceptional divisor of the morphism α, and ω is an elliptic fibration.

Example 1.5. Let $n=26$. Then X is a hypersurface in $\mathbb{P}(1,1,3,5,6)$ of degree 15 that has two singular points of type $\frac{1}{3}(1,1,2)$. There is a commutative diagram

$$\begin{tikzcd} & U \arrow[dl, "\sigma"'] \arrow[dr, "\omega"] & \\ X \arrow[rr, dashed, "\xi"'] & & \mathbb{P}(1,1,6), \end{tikzcd}$$

where ξ is a projection, σ is a weighted blow-up of a singular point of type $\frac{1}{3}(1,1,2)$ with weights $(1,1,2)$, and ω is the morphism given by the linear system $|-6K_X|$. Then the normalization of a generic fibre of the rational map ξ is an elliptic curve.

Example 1.6. Let $n=31$. Then X is a hypersurface in $\mathbb{P}(1,1,4,5,6)$ of degree 16 that has a singular point of type $\frac{1}{5}(1,1,4)$. There is a commutative diagram

$$\begin{tikzcd} U \arrow[d, "\alpha"'] & & W \arrow[ll, "\beta"'] \arrow[d, "\omega"] \\ X \arrow[rr, dashed, "\xi"'] & & \mathbb{P}(1,1,6), \end{tikzcd}$$

where ξ is a projection, α is a weighted blow-up of the singular point of type $\frac{1}{5}(1,1,4)$ with weights $(1,1,4)$, β is a weighted blow-up with weights $(1,1,3)$ of the singular point that is contained in the exceptional divisor of the morphism α, and ω is an elliptic fibration.

Example 1.7. Let $n\in\{7,11,19\}$. Then $a_2=a_3$, and the hypersurface X has $\frac{d}{a_2}$ singular points of type $\frac{1}{a_2}(1,1,a_2-1)$. Let $\xi\colon X\dashrightarrow\mathbb{P}(1,a_1,a_2)$ be the rational map induced by a linear subsystem in the linear system $|-a_2K_X|$ consisting of surfaces that pass through a given singular point of type $\frac{1}{a_2}(1,1,a_2-1)$. Then the normalization of a generic fibre of the map ξ is an elliptic curve.

Example 1.8. Let $n \in \{7,9,20,30,36,44,49,51,64\}$. Then X can be given by the equation

$$w^2t + wg(x,y,z,t) + f(x,y,z,t) = 0$$

or by the equation

$$tz^k + \sum_{i=0}^{k-1} g_i(x,y,t,w)z^i = 0 \subset \mathrm{Proj}\big(\mathbb{C}[x,y,z,t,w]\big),$$

where $\mathrm{wt}(x) = 1$, $\mathrm{wt}(y) = a_1$, $\mathrm{wt}(z) = a_2$, $\mathrm{wt}(t) = a_3$, $\mathrm{wt}(w) = a_4$, and g_i is a quasi-homogeneous polynomial. Let $\xi\colon X \dashrightarrow \mathbb{P}(1,a_1,a_3)$ be the rational map given by a linear system consisting of surfaces that are cut out by $f(x,y,t) = 0$, where $f(x,y,t)$ is a quasi-homogeneous polynomial of degree a_1a_3. Then the normalization of a generic fibre of the map ξ is an elliptic curve.

Example 1.9. Let $n \notin \{1,2,3,7,11,19,60,75,84,87,93\}$ and $\xi\colon X \dashrightarrow \mathbb{P}(1,a_1,a_2)$ be the natural projection. Then the normalization of a generic fibre of the map ξ is an elliptic curve.

The purpose of this paper is to prove the following result.[2]

Theorem 1.10. *Let $\rho\colon X \dashrightarrow \mathbb{P}^2$ be a rational map whose generic fibre is birational to an elliptic curve. Then there is a commutative diagram*

$$\begin{tikzcd} X \arrow[r, dashed, "\sigma"] \arrow[d, dashed, "\xi"'] & X \arrow[d, dashed, "\rho"] \\ \mathbb{P}(1,a_1,a_i) \arrow[r, dashed, "\varphi"'] & \mathbb{P}^2, \end{tikzcd}$$

where φ is a birational map, $\sigma \in \mathrm{Bir}(X)$, and ξ is one of the dominant rational maps constructed in Examples 1.2–1.9.

Corollary 1.11. *Let $\rho\colon X \dashrightarrow \mathbb{P}^2$ be a rational map whose generic fibre is birational to an elliptic curve. Suppose that $n \notin \{1,2,7,9,11,17,19,20,26,30,31,36,44,49,51,64\}$. Then there is a commutative diagram*

$$\begin{tikzcd} & X \arrow[dl, dashed, "\psi"'] \arrow[dr, dashed, "\rho"] & \\ \mathbb{P}(1,a_1,a_2) \arrow[rr, dashed, "\varphi"'] & & \mathbb{P}^2, \end{tikzcd}$$

where ψ is the natural projection and φ is a birational map.

Corollary 1.12. *The hypersurface X can be birationally transformed into an elliptic fibration if and only if $n \notin \{3,60,75,84,87,93\}$.*

[2]In the case when $n \in \{1,3,14,22,28,34,37,39,52,53,57,59,60,66,70,72,73,75,78,81,84,86,87,88,89,90,92,93,94,95\}$, the assertion of Theorem 1.10 is proved in [4]–[6].

We illustrate our technique by proving the following result.

Proposition 1.13. *The assertion of Theorem 1.10 holds for* $n = 14$.

Proof. Let $n = 14$. Then X is a hypersurface in $\mathbb{P}(1,1,1,4,6)$ of degree 12 with just one singular point, which is of type $\frac{1}{2}(1,1,1)$. Let $\xi\colon X \dashrightarrow \mathbb{P}^2$ be the natural projection and $\pi\colon U \to X$ a weighted blow-up with weights $(1,1,1)$ of the singular point of X. Then $\xi \circ \pi$ is a morphism.

Let $\rho\colon X \dashrightarrow \mathbb{P}^2$ be a rational map such that the normalization of its generic fibre is an irreducible elliptic curve. Consider commutative diagram

$$\begin{tikzcd} & V \arrow[dl, "\alpha"'] \arrow[dr, "\beta"] & \\ X \arrow[rr, dashed, "\rho"'] & & \mathbb{P}^2, \end{tikzcd}$$

where V is smooth, α is a birational morphism and β is a morphism. Let $\mathcal{M}$ be the proper transform of $|\beta^*(\mathcal{O}_{\mathbb{P}^2}(1))|$ on the hypersurface X. To complete the proof, we have to show that the proper transform of the linear system $\mathcal{M}$ on the variety U lies in the fibres of the fibration $\xi \circ \pi$.

There is a natural number $k > 0$ such that $\mathcal{M} \sim -kK_X$. Consider the mobile linear system $\left(X, \frac{1}{k}\mathcal{M}\right)$. Then

$$K_V + \frac{1}{k}\mathcal{B} \sim_{\mathbb{Q}} \alpha^*\left(K_X + \frac{1}{k}\mathcal{M}\right) + \sum_{i=1}^{\delta} a_i E_i \sim_{\mathbb{Q}} \sum_{i=1}^{\delta} a_i E_i,$$

where E_i is an α-exceptional divisor, a_i is a rational number and δ is the number of exceptional divisors of α. It follows from [2] that $a_i \geqslant 0$ for every i, but that there is a j such that $a_j \leqslant 0$ by Lemma 2.1. Put $Z_j = \alpha(E_j)$.

Suppose that Z_j is a smooth point of X. Let S_1 and S_2 be general surfaces in the linear system $\mathcal{M}$. Then $\operatorname{mult}_{Z_j}(S_1 \cdot S_2) \geqslant 4k^2$ by [7], Lemma 1.10. But the linear system $|-4K_X|$ induces a double cover $X \to \mathbb{P}(1,1,1,4)$. Thus, we have

$$2k^2 = H \cdot S_1 \cdot S_2 \geqslant \operatorname{mult}_{Z_j}(S_1 \cdot S_2) \geqslant 4k^2,$$

where H is a sufficiently general divisor in the linear system $|-4K_X|$ that passes through the point Z_j, a contradiction.

It follows from Corollary 2.8 that Z_j is not a curve, which implies that Z_j is the unique singular point of the hypersurface X. Let $\mathcal{D}$ be the proper transform of the linear system $\mathcal{M}$ on the variety U. Then $\mathcal{D} \sim_{\mathbb{Q}} -kK_U$ by Theorem 2.2. Hence, the linear system $\mathcal{D}$ lies in the fibres of the elliptic fibration $\xi \circ \pi$, which completes the proof.

Let us describe the structure of the paper. We give some auxiliary results in § 2. The first steps of the proof of Theorem 1.10 are done in § 3, where we prove Theorem 1.10 for $n \in \{1, 3, 5, 11, 14, 22, 28, 34, 37, 39, 52, 53, 57, 59, 60, 66, 70, 72, 73, 75, 78, 81, 84, 86, 87, 88, 89, 90, 92, 93, 94, 95\}$. Then we prove Theorem 1.10 in the remaining cases.

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§ 2. Preliminaries

Let X be a threefold having terminal $\mathbb{Q}$-factorial singularities, $\mathcal{M}$ a linear system on X such that $\mathcal{M}$ does not have fixed components, and λ an arbitrary non-negative rational number. In this section we consider technical results describing properties of the mobile log pair $(X, \lambda\mathcal{M})$ which are used in the proof of Theorem 1.10. Elementary properties of mobile log pairs can be found in [3]. As usual, the set of centres of canonical singularities of $(X, \lambda\mathcal{M})$ is denoted by $\mathbb{CS}(X, \lambda\mathcal{M})$.

The main idea in the proof of Theorem 1.10 is to use iteratively the following result, which is a generalization of the classical Noether–Fano inequality.

Lemma 2.1. *Let $\rho\colon X \dashrightarrow \mathbb{P}^2$ be a rational map whose generic fibre is birational to an elliptic curve and $\pi\colon V \to X$ a resolution of the indeterminacies of ρ. Suppose that $\mathcal{M}$ is a proper transform of the linear system $|\rho\circ\pi^*(\mathcal{O}_{\mathbb{P}^2}(1))|$, the divisor $-K_X$ is nef and big and the equivalence $K_X + \lambda\mathcal{M} \sim_{\mathbb{Q}} 0$ holds. Then the singularities of the log pair $(X, \lambda\mathcal{M})$ are not terminal.*

Proof. See [3], proof of Theorem 1.4.4.

In the course of the proof of Theorem 1.10, applications of Lemma 2.1 are usually followed by applications of the following well-known result.

Theorem 2.2. *Let O be a singular point of X of type $\frac{1}{r}(1, a, r-a)$, where a and r are coprime natural numbers such that $r > a$, and let $\operatorname{mult}_O(\mathcal{M})$ be a rational number such that*

$$\mathcal{D} \sim_{\mathbb{Q}} \pi^*(\mathcal{M}) - \operatorname{mult}_O(\mathcal{M})G,$$

where $\pi\colon U \to X$ is a weighted blow-up of O with weights $(1, a, r-a)$, G is the exceptional divisor of π and $\mathcal{D}$ is a proper transform of $\mathcal{M}$ on the variety U. Suppose that $\mathbb{CS}(X, \lambda\mathcal{M})$ contains either the point O or a curve passing through O. Then $\operatorname{mult}_O(\mathcal{M}) \geqslant 1/(r\lambda)$.

Proof. This is proved in [8].

In the course of the proof of Theorem 1.10, applications of Theorem 2.2 are usually followed by applications of the following result.

Lemma 2.3. *With the assumptions and notation of Theorem 2.2, suppose that the singularities of the log pair $(X, \lambda\mathcal{M})$ are canonical, $\mathbb{CS}(X, \lambda\mathcal{M}) = \{O\}$ and $\mathbb{CS}(U, \lambda\mathcal{D}) \neq \varnothing$. Then the following assertions hold:*

the set $\mathbb{CS}(U, \lambda\mathcal{D})$ does not contain smooth points of $G \cong \mathbb{P}(1, a, r-a)$;

if the set $\mathbb{CS}(U, \lambda\mathcal{D})$ contains a curve L, then $L \in \left|\mathcal{O}_{\mathbb{P}(1,\,a,\,r-a)}(1)\right|$, and every singular point of the surface G is contained in the set $\mathbb{CS}(U, \lambda\mathcal{D})$.

Proof. We consider only the case when $r = 5$ and $a = 2$ because the proof is similar in the general case. Thus, we have $G \cong \mathbb{P}(1, 2, 3)$.

Let P and Q be singular points of G and L the curve in $\left|\mathcal{O}_{\mathbb{P}(1,2,3)}(1)\right|$. Then L passes through P and Q, but $\mathrm{mult}_O(\mathcal{M}) = 1/(5\lambda)$ by Theorem 2.2, which implies that $\mathcal{D}|_G \sim_{\mathbb{Q}} \lambda L$.

Suppose that the set $\mathbb{CS}(U, \lambda\mathcal{D})$ contains a subvariety Z of the subvariety U that is different from the curve L and the points P and Q. We will show that this assumption leads to a contradiction, which is enough to complete the proof by Theorem 2.2. We have $Z \subset G$.

Suppose that Z is a point. Then Z is smooth on the variety U, which implies the inequality $\mathrm{mult}_Z(\mathcal{D}) > 1/\lambda$. Let C be a general curve in the linear system $|\mathcal{O}_{\mathbb{P}(1,2,3)}(6)|$ that passes through Z. Then C is not contained in the base locus of the linear system $\mathcal{D}$. Hence, we have $1/\lambda = C \cdot \mathcal{D} \geqslant \mathrm{mult}_Z(C)\,\mathrm{mult}_Z(\mathcal{D}) > 1/\lambda$, which is a contradiction.

Therefore, the subvariety Z is a curve. Then $\mathrm{mult}_Z(\mathcal{D}) \geqslant 1/\lambda$. Let C be a sufficiently general curve in the linear system $|\mathcal{O}_{\mathbb{P}(1,2,3)}(6)|$. Then

$$\frac{1}{\lambda} = C \cdot \mathcal{D} \geqslant \mathrm{mult}_Z(\mathcal{D}) C \cdot Z \geqslant \frac{C \cdot Z}{\lambda},$$

which implies that $C \cdot Z = 1$. Hence, the curve Z is contained in $|\mathcal{O}_{\mathbb{P}(1,2,3)}(1)|$. Thus, the curves L and Z coincide, which is a contradiction. The lemma is proved.

In the course of the proof of Theorem 1.10, applications of Lemma 2.3 are sometimes followed by applications of the following result.

Lemma 2.4. *Let C be a curve on X such that $C \in \mathbb{CS}(X, \lambda\mathcal{M})$. Suppose that the complete linear system $|-mK_X|$ is base-point-free for some natural number $m > 0$. Then $-K_X \cdot C \leqslant -K_X^3$.*

Proof. Let M_1 and M_2 be general surfaces in $\mathcal{M}$. Then

$$\mathrm{mult}_C(M_1 \cdot M_2) \geqslant \mathrm{mult}_C(M_1)\,\mathrm{mult}_C(M_2) \geqslant \frac{1}{\lambda^2}.$$

Let H be a general surface in $|-mK_X|$. Then

$$\frac{-mK_X^3}{\lambda^2} = H \cdot M_1 \cdot M_2 \geqslant (-mK_X \cdot C)\,\mathrm{mult}_C(M_1 \cdot M_2) \geqslant \frac{-mK_X \cdot C}{\lambda^2},$$

which implies that $-K_X \cdot C \leqslant -K_X^3$.

We now consider a simple result that will substantially simplify the proof of Theorem 1.10.

Lemma 2.5. *Suppose that the linear system $\mathcal{M}$ is not composed of a pencil. Then there is no proper Zariski-closed subset $\Sigma \subsetneq X$ such that*

$$\mathrm{Supp}(S_1) \cap \mathrm{Supp}(S_2) \subset \Sigma \subsetneq X,$$

where S_1 and S_2 are general divisors in the linear system $\mathcal{M}$.

Proof. Suppose that there is a proper Zariski-closed subset $\Sigma \subset X$ such that the set-theoretic intersection of the sufficiently general divisors S_1 and S_2 of the linear system $\mathcal{M}$ is contained in the set Σ. We claim that this assumption leads to a contradiction.

Let $\rho\colon X \dashrightarrow \mathbb{P}^n$ be a rational map induced by the linear system $\mathcal{M}$, where n is the dimension of $\mathcal{M}$. Then there is a commutative diagram

$$\begin{tikzcd} & W \arrow[dl, "\alpha"'] \arrow[dr, "\beta"] & \\ X \arrow[rr, dashed, "\rho"'] & & \mathbb{P}^n, \end{tikzcd}$$

where W is a smooth variety, α is a birational morphism and β is a morphism. Let Y be the image of β. Then $\dim(Y) \geqslant 2$ because $\mathcal{M}$ is not composed of a pencil.

Let Λ be a Zariski-closed subset of W such that the morphism

$$\alpha|_{W\setminus\Lambda}\colon W \setminus \Lambda \longrightarrow X \setminus \alpha(\Lambda)$$

is an isomorphism, and let Δ be the union of the subset $\Lambda \subset W$ and the closure of the proper transform of the set $\Sigma \setminus \alpha(\Lambda)$ on W. Then Δ is a Zariski-closed proper subset of W.

Let B_1 and B_2 be general hyperplane sections of the variety Y and let D_1 and D_2 be proper transforms of the divisors B_1 and B_2 on the variety W, respectively. Then $\alpha(D_1)$ and $\alpha(D_2)$ are general divisors of the linear system $\mathcal{M}$. Hence, in the set-theoretic sense we have

$$\varnothing \neq \beta^{-1}\bigl(\mathrm{Supp}(B_1) \cap \mathrm{Supp}(B_2)\bigr) = \mathrm{Supp}(D_1) \cap \mathrm{Supp}(D_2) \subset \Delta \subsetneq W \tag{2.1}$$

because $\dim(Y) \geqslant 2$. However, the set-theoretic identity (2.1) is an absurdity.

The following result is implied by [9], Lemma 0.3.3, and Lemma 2.5.

Corollary 2.6. *Suppose that $\mathcal{M}$ is not composed of a pencil. Let D be a divisor on X that is big and nef. Then $D \cdot S_1 \cdot S_2 > 0$, where S_1 and S_2 are sufficiently general surfaces in the linear system $\mathcal{M}$.*

The proof of Lemma 2.5 implies the following result.

Lemma 2.7. *Suppose that $\mathcal{M}$ is not composed of a pencil. Let $\mathcal{D}$ be a linear system on X that does not have fixed components. Then there is no Zariski-closed subset $\Sigma \subsetneq X$ such that*

$$\mathrm{Supp}(S) \cap \mathrm{Supp}(D) \subset \Sigma \subsetneq X,$$

where S and D are sufficiently general divisors of the linear system $\mathcal{M}$ and $\mathcal{D}$, respectively.

Lemma 2.5 and the proof of Lemma 2.4 imply the following result.

Corollary 2.8. *With the assumptions and notation of Lemma 2.4, suppose that the linear system $\mathcal{M}$ is not composed of a pencil and the divisor $-K_X$ is nef and big. Then $-K_X \cdot C < -K_X^3$.*

Many applications of Lemma 2.7 use the following simple result.

Lemma 2.9. *Let S be a surface and D an effective divisor on S such that $D \equiv \sum_{i=1}^{r} a_i C_i$, where $a_i \in \mathbb{Q}$ and $C_1, \dots, C_r$ are irreducible curves on S whose intersection form is negative definite. Then $D = \sum_{i=1}^{r} a_i C_i$.*

Proof. Let $D = \sum_{i=1}^{k} c_i B_i$, where B_i is an irreducible curve on S and c_i is a non-negative rational number. Suppose that

$$\sum_{i=1}^{k} c_i B_i \neq \sum_{i=1}^{r} a_i C_i$$

and none of the B_i are among the curves $C_1, \dots, C_r$. We may assume that not all of $c_1, \dots, c_k$ are zero. We claim that these assumptions lead to a contradiction, which implies the desired result.

The intersection form of the curves $C_1, \dots, C_r$ is negative definite. Thus, we have

$$\begin{aligned} 0 &\geqslant \biggl(\sum_{a_i>0} a_i C_i\biggr)\biggl(\sum_{a_i>0} a_i C_i\biggr) \\ &= \biggl(\sum_{i=1}^{k} c_i B_i\biggr)\biggl(\sum_{a_i>0} a_i C_i\biggr) - \biggl(\sum_{a_i\leqslant 0} a_i C_i\biggr)\biggl(\sum_{a_i>0} a_i C_i\biggr) \geqslant 0, \end{aligned}$$

which implies that

$$\sum_{c_i\geqslant 0} c_i B_i \equiv \sum_{a_i\leqslant 0} a_i C_i.$$

Hence, we have $c_i = 0$ and $a_i = 0$ for every i, which is a contradiction.

§ 3. Beginning of the classification

We shall use the notation and assumptions of § 1. In this section we begin the proof of Theorem 1.10. Suppose that there is a birational map $\rho\colon X \dashrightarrow V$ and an elliptic fibration $\nu\colon V \to \mathbb{P}^2$ such that V is smooth and the fibres of ν are connected. We must show that there is a commutative diagram

$$\begin{tikzcd} X \arrow[r, dashed, "\sigma"] \arrow[d, dashed, "\xi"'] & X \arrow[r, dashed, "\rho"] & V \arrow[d, "\nu"] \\ \mathbb{P}(1, a_1, a_i) \arrow[rr, dashed, "\zeta"'] & & \mathbb{P}^2, \end{tikzcd} \tag{3.1}$$

where ζ and σ are birational maps and ξ is one of the rational maps constructed in Examples 1.2–1.9.

The commutative diagram (3.1) implies the commutative diagram

$$\begin{tikzcd} X \arrow[r, dashed, "\rho"] \arrow[d, dashed, "\xi"'] & V \arrow[d, "\nu"] \\ \mathbb{P}(1, a_1, a_i) \arrow[r, dashed, "\zeta"'] & \mathbb{P}^2 \end{tikzcd} \tag{3.2}$$

in the case when $\xi \circ \sigma = \chi \circ \xi$ for every $\sigma \in \mathrm{Bir}(X)$, where $\chi \in \mathrm{Bir}(\mathbb{P}(1, a_1, a_i))$.

Example 3.1. Let $\psi\colon X \dashrightarrow \mathbb{P}(1,a_1,a_2)$ be a projection and σ any birational automorphism of the threefold X. Suppose that $n \notin \{1,2,3,7,11,19,20,36,60,75,84,87,93\}$. Then it follows from [2] that there is a birational automorphism χ of the surface $\mathbb{P}(1,a_1,a_2)$ such that $\psi\circ\sigma = \chi\circ\psi$.

Let $\mathcal{M}$ be a proper transform of the linear system $|\nu^*(\mathcal{O}_{\mathbb{P}^2}(1))|$ on the hypersurface X. Then $\mathcal{M} \sim_{\mathbb{Q}} -kK_X$ for some natural number k, but the singularities of the log pair $\left(X, \frac{1}{k}\mathcal{M}\right)$ are not terminal by Lemma 2.1.

Remark 3.2. It follows from [2] that there is a birational automorphism $\sigma \in \mathrm{Bir}(X)$ such that the singularities of the log pair $(X, \frac{1}{k'}\sigma(\mathcal{M}))$ are canonical, where $k' \in \mathbb{N}$ is such that $\mathcal{M} \sim_{\mathbb{Q}} -k'K_X$.

We may assume that the singularities of the log pair $\left(X, \frac{1}{k}\mathcal{M}\right)$ are canonical.

Theorem 3.3. *The set $\mathbb{CS}\left(X, \frac{1}{k}\mathcal{M}\right)$ does not contain smooth points of X when $n \neq 1$ or 2.*

Proof. This follows from the proof of Theorem 5.1.2 in [2].

The following corollary is implied by Lemma 2.4.

Corollary 3.4. *The set $\mathbb{CS}\left(X, \frac{1}{k}\mathcal{M}\right)$ of centres of canonical singularities does not contain curves that do not contain singular points of X when $n \geqslant 6$.*

In particular, the set $\mathbb{CS}\left(X, \frac{1}{k}\mathcal{M}\right)$ contains a singular point of X when $n \geqslant 6$ by Theorem 2.2.

Proposition 3.5. *Suppose that the set $\mathbb{CS}\left(X, \frac{1}{k}\mathcal{M}\right)$ contains a singular point O of the hypersurface X that is a singularity of type $\frac{1}{r}(1, r-a, a)$, where a and r are coprime natural numbers and $r > a$. Let $\pi\colon Y \to X$ be a weighted blow-up of the point O with weights $(1, a, r-a)$. Then $-K_Y^3 \geqslant 0$.*

Proof. Suppose that $-K_Y^3 = -K_X^3 - 1/(ra(r-a)) < 0$. Let E be the π-exceptional divisor and $\mathcal{B}$ a proper transform of $\mathcal{M}$ on the variety Y. Then $\mathcal{B} \sim_{\mathbb{Q}} -kK_Y$ by Theorem 2.2.

Let $\overline{\mathbb{NE}}(Y)$ be the closure in $\mathbb{R}^2$ of the cone generated by effective one-dimensional cycles of the variety Y. Then $-E\cdot E$ generates an extremal ray of $\overline{\mathbb{NE}}(Y)$, but it follows from [2], Corollary 5.4.6, that there are integers $b > 0$ and $c \geqslant 0$ such that the cycle $-K_Y\cdot(-bK_Y + cE)$ is numerically equivalent to an effective, irreducible and reduced curve Γ on the variety Y that generates the extremal ray of the cone $\overline{\mathbb{NE}}(Y)$ different from the ray generated by $-E\cdot E$.

Let S_1 and S_2 be general surfaces in $\mathcal{B}$. Then $S_1\cdot S_2 \in \overline{\mathbb{NE}}(Y)$, but $S_1\cdot S_2 \equiv k^2K_Y^2$, which implies that the cycle $S_1\cdot S_2$ generates an extremal ray of the cone $\overline{\mathbb{NE}}(Y)$ that contains the curve Γ. Moreover, for every effective cycle $C \in \mathbb{R}^+\Gamma$ we have

$$\mathrm{Supp}(C) = \mathrm{Supp}(S_1\cdot S_2)$$

because $S_1\cdot\Gamma < 0$ and $S_2\cdot\Gamma < 0$, which is impossible by Lemma 2.5 because the linear system $\mathcal{B}$ is not composed of a pencil.

The following result is implied by Proposition 3.5.

Proposition 3.6. *The assertion of Theorem 1.10 holds for* $n \in \{14, 22, 28, 34, 37, 39, 52, 53, 57, 59, 66, 70, 72, 73, 78, 81, 86, 88, 89, 90, 92, 94, 95\}$.

Proof. We must show the existence of a commutative diagram

$$\begin{tikzcd} X \arrow[r, dashed, "\rho"] \arrow[d, dashed, "\psi"'] & V \arrow[d, "\nu"] \\ \mathbb{P}(1, a_1, a_2) \arrow[r, dashed, "\varphi"'] & \mathbb{P}^2, \end{tikzcd} \tag{3.3}$$

where ψ is the natural projection and φ is a birational map.

It follows from Theorems 3.3, Lemma 2.4 and Theorem 2.2 that $\mathbb{CS}\big(X, \frac{1}{k}\mathcal{M}\big)$ contains a singular point P of the hypersurface X of type $\frac{1}{r}(1, a, r-a)$, where a and r are coprime natural numbers and $r > a$.

Let $\pi\colon Y \to X$ be a weighted blow-up of the point P with weights $(1, a, r-a)$ and $\mathcal{B}$ the proper transform of the linear system $\mathcal{M}$ on the variety Y. Then the equivalence $\mathcal{B} \sim_{\mathbb{Q}} -kK_Y$ holds by Theorem 2.2. Moreover, for every value of n, either the inequality $-K_Y^3 < 0$ holds or the inequality $-K_Y^3 = 0$ holds.

Then $-K_Y^3 = 0$ by Proposition 3.5. Then the linear system $|-rK_Y|$ does not have base points for $r \gg 0$ and induces a morphism $\eta\colon Y \to \mathbb{P}(1, a_1, a_2)$ such that the diagram

$$\begin{tikzcd} & Y \arrow[dl, "\pi"'] \arrow[dr, "\eta"] & \\ X \arrow[rr, dashed, "\psi"'] & & \mathbb{P}(1, a_1, a_2) \end{tikzcd}$$

is commutative. Let C be a generic fibre of the morphism η and S a general surface in the linear system $\mathcal{B}$. Then $S \sim_{\mathbb{Q}} -kK_Y$ by Theorem 2.2. Hence, the equality $S \cdot C = 0$ holds, which implies that the surface S lies in the fibres of the elliptic fibration η. The latter implies the existence of the commutative diagram (3.3).

The following result implies Corollary 1.12.

Lemma 3.7. *Our assumptions imply that* $n \notin \{3, 60, 75, 84, 87, 93\}$.

Proof. It follows from Proposition 3.5 that $n \notin \{75, 84, 87, 93\}$.

Suppose that $n = 3$. Then the hypersurface X is smooth and the set $\mathbb{CS}\big(X, \frac{1}{k}\mathcal{M}\big)$ contains an irreducible curve Γ such that $-K_X \cdot \Gamma = 1$ by Lemma 2.4. Let $\gamma\colon \bar{X} \to X$ be a blow-up of the curve Γ and G the exceptional divisor of γ. Then the divisor $\gamma^*(-3K_X) - G$ is nef and big. But

$$\big(\eta^*(-3K_X) - G\big) \cdot \bar{S}_1 \cdot \bar{S}_1 = 0,$$

where $\bar{S}_i$ is the proper transform of a general surface in $\mathcal{M}$ on $\bar{X}$, which is impossible by Corollary 2.6.

We have $n = 60$. Then X is a hypersurface in $\mathbb{P}(1, 4, 5, 6, 9)$ of degree 24.

It is easy to check that $\mathbb{CS}\big(X, \frac{1}{k}\mathcal{M}\big)$ does not contain curves by Corollary 2.8, which implies that the set $\mathbb{CS}\big(X, \frac{1}{k}\mathcal{M}\big)$ consists of the singular point O of the hypersurface X that is a quotient singularity of type $\frac{1}{9}(1, 4, 5)$ by Proposition 3.5.

Let $\pi\colon Y \to X$ be a weighted blow-up of the singular point O with weights $(1,4,5)$ and $\mathcal{D}$ a proper transform of the linear system $\mathcal{M}$ on the threefold Y. Then $\mathcal{D} \sim_{\mathbb{Q}} -kK_Y$ by Theorem 2.2. Let P and Q be the points of Y contained in the π-exceptional divisor that are singularities of types $\frac{1}{4}(1,1,3)$ and $\frac{1}{5}(1,1,4)$, respectively. Then $\mathbb{CS}\big(Y, \frac{1}{k}\mathcal{D}\big) \subseteq \{P, Q\}$ by Lemmas 2.3 and 2.4.

Suppose that $\mathbb{CS}\big(Y, \frac{1}{k}\mathcal{D}\big)$ contains the point Q. Let $\alpha\colon U \to Y$ be a weighted blow-up of Q with weights $(1,1,4)$ and $\mathcal{B}$ a proper transform of $\mathcal{M}$ on the variety U. Then $\mathcal{B} \sim_{\mathbb{Q}} -kK_U$ by Theorem 2.2, the linear system $|-4K_U|$ is a proper transform of the pencil $|-4K_X|$ and the base locus of the pencil $|-4K_U|$ consists of an irreducible reduced curve Z on U such that the curve $\pi\circ\alpha(Z)$ is a base curve of the pencil $|-4K_X|$. Let H be a general surface in $|-4K_U|$. Then $Z^2 = -1/30$ on H. But the equivalence $\mathcal{B}|_H \sim_{\mathbb{Q}} kZ$ holds, which is impossible by Lemmas 2.7 and 2.9.

We see that $\mathbb{CS}\big(Y, \frac{1}{k}\mathcal{D}\big) = \{P\}$. Let $\beta\colon W \to Y$ be a weighted blow-up of the point P with weights $(1,1,3)$ and D the proper transform of the general surface in $|-5K_X|$ on the variety W. Then D is nef and big, but the equality $D\cdot H_1\cdot H_2 = 0$ holds, where H_1 and H_2 are general surfaces in the proper transform of $\mathcal{M}$ on W, which is impossible by Corollary 2.6.

We now consider a very simple case when there are many ways of birationally transforming the hypersurface X into elliptic fibrations.

Proposition 3.8. *Suppose that $n = 11$. Then the diagram (3.2) exists, where ξ is one of the five rational maps constructed in Example 1.7.*

Proof. The threefold X is a hypersurface in $\mathbb{P}(1,1,2,2,5)$ of degree 10 whose singularities consist of points P_1, P_2, P_3, P_4 and P_5 that are singularities of types $\frac{1}{2}(1,1,1)$. The hypersurface X is birationally superrigid. It follows from the construction in Example 1.7 that there is a commutative diagram

$$\begin{tikzcd} & U_i \arrow[dl, "\pi_i"'] \arrow[dr, "\eta_i"] & \\ X \arrow[rr, dashed, "\xi_i"'] & & \mathbb{P}(1,1,2), \end{tikzcd}$$

where ξ_i is a projection, π_i is the weighted blow-up of P_i with weights $(1,1,1)$ and η_i is an elliptic fibration. It follows from Theorem 2.2, Lemma 2.4 and Theorem 3.3 that the set $\mathbb{CS}\big(X, \frac{1}{k}\mathcal{M}\big)$ contains the singular point P_i for some $i \in \{1,2,3,4,5\}$. Let $\mathcal{D}_i$ be the proper transform of $\mathcal{M}$ on U_i. Then $\mathcal{D}_i \sim_{\mathbb{Q}} -kK_{U_i}$ by Theorem 2.2, which implies the existence of the commutative diagram (3.2) for $\xi = \xi_i$.

The following result is obtained in [10].

Theorem 3.9. *Suppose that $\mathbb{CS}\big(X, \frac{1}{k}\mathcal{M}\big)$ contains an irreducible curve C on the hypersurface X and $n \geqslant 3$. Then there are different surfaces S_1 and S_2 in the linear system $|-K_X|$ such that the curve C is a component of the curve $S_1 \cap S_2$.*

The following result is obtained in the thesis of D. Ryder.

Proposition 3.10. *The assertion of Theorem 1.10 holds for $n = 5$.*

Proof. Suppose that $n = 5$. Then X is a general hypersurface in $\mathbb{P}(1,1,1,2,3)$ of degree 7, the equality $-K_X^3 = 7/6$ holds, and the singularities of the hypersurface X consist of two isolated singular points P and Q of types $\frac{1}{2}(1,1,1)$ and $\frac{1}{3}(1,1,2)$, respectively.

The hypersurface X can be given by the equation

$$w^2 f_1(x,y,z) + f_4(x,y,z,t)w + f_7(x,y,z,t) = 0 \subset \mathbb{P}(1,1,1,2,3) \cong \operatorname{Proj}\big(\mathbb{C}[x,y,z,t,w]\big),$$

where $\mathrm{wt}(x) = \mathrm{wt}(y) = \mathrm{wt}(z) = 1$, $\mathrm{wt}(t) = 2$ and $\mathrm{wt}(w) = 3$, f_i is a general quasi-homogeneous polynomial of degree i, and Q is given by the equations $x = y = z = t = 0$. There is a commutative diagram

$$\begin{tikzcd}
 & & W \arrow[dll, "\gamma"'] \arrow[d, "\alpha"] & & \\
Y \arrow[dr, "\eta"'] & & X \arrow[dl, dashed, "\xi"'] \arrow[dr, dashed, "\psi"] & & U \arrow[ull, "\beta"'] \arrow[dl, "\omega"] \\
 & \mathbb{P}(1,1,1,2) \arrow[rr, dashed, "\chi"'] & & \mathbb{P}^2, &
\end{tikzcd}$$

where χ and ξ are the natural projections, the morphism ω is an elliptic fibration, the morphism α is a weighted blow-up of the point Q with weights $(1,1,2)$, the morphism γ is the birational morphism that contracts 14 smooth irreducible rational curves $C_1,\dots,C_{14}$ into 14 isolated ordinary double points $P_1,\dots,P_{14}$, respectively, of the variety Y, the morphism η is a double cover branched over the surface $R \subset \mathbb{P}(1,1,1,2)$ that is given by the equation

$$f_4(x,y,z,t)^2 - 4f_1(x,y,z)f_7(x,y,z,t) = 0 \subset \mathbb{P}(1,1,1,2) \cong \operatorname{Proj}\big(\mathbb{C}[x,y,z,t]\big)$$

and has 14 isolated ordinary double points $\eta(P_1),\dots,\eta(P_{14})$, and β is the composite of the weighted blow-ups with the weights $(1,1,1)$ of two singular points of the variety W that are singularities of types $\frac{1}{2}(1,1,1)$.

It follows from Theorem 3.9 that the set $\mathbb{CS}\big(X, \frac{1}{k}\mathcal{M}\big)$ does not contain curves (see the proof of Lemma 8.3).

Suppose that $\mathbb{CS}\big(X, \frac{1}{k}\mathcal{M}\big) = \{Q\}$. Let O be the singular point of the threefold W such that $\alpha(O) = Q$ and $\mathcal{D}$ the proper transform of $\mathcal{M}$ on the threefold W. Then $O \in \mathbb{CS}\big(W, \frac{1}{k}\mathcal{D}\big)$ by Theorem 2.2 and Lemmas 2.1 and 2.3. Let $\mathcal{B}$ be the proper transform of $\mathcal{D}$ on the variety Y. Then it follows from Theorem 2.2 and Lemmas 2.3 and 2.4 that $\mathbb{CS}\big(Y, \frac{1}{k}\mathcal{B}\big)$ contains a curve C such that $-K_Y \cdot C = 1/2$ and $\chi \circ \eta(C)$ is a point.

There is an irreducible curve Z on Y such that $\eta(Z) = \eta(C)$ and $Z \neq C$. Let S be a general surface in the linear system $|-K_Y|$ that contains the curve C. Then $Z^2 < 0$ on S. But $\mathcal{B}|_S \sim_{\mathbb{Q}} kC + kZ$ on S, which is impossible by Lemma 2.7 because $\mathcal{B}$ is not composed of a pencil.

It follows from Theorem 2.2 and Lemmas 2.3 and 2.4 that $\mathbb{CS}\big(X, \frac{1}{k}\mathcal{M}\big) = \big\{P, Q\big\}$. Hence, we have $O \in \mathbb{CS}\big(W, \frac{1}{k}\mathcal{D}\big)$ by Corollary 2.8 and Lemmas 2.1 and 2.3. Thus, the proper transform of $\mathcal{M}$ on U is contained in the fibres of the elliptic fibration ω by Theorem 2.2.

The assertion of Theorem 3.9 implies the following result.

Lemma 3.11. *Suppose that $a_2 \neq 1$. Then the set $\mathbb{CS}\big(X, \frac{1}{k}\mathcal{M}\big)$ does not contain curves.*

Proof. Suppose that $\mathbb{CS}\big(X, \frac{1}{k}\mathcal{M}\big)$ contains a curve C. Then the assertion of Theorem 3.9 implies that there are surfaces S_1 and S_2 in the pencil $|-K_X|$ such that C is contained in the intersection $S_1 \cap S_2$. Now the inequality $a_2 \neq 1$ implies that the cycle $S_1 \cdot S_2$ is a reduced and irreducible curve. Hence, we have $-K_X \cdot C = -K_X^3$, which is impossible by Lemma 2.4.

We illustrate the application of Lemma 3.11 by proving the following result.

Proposition 3.12. *The assertion of Theorem 1.10 holds for $n = 18$.*

Proof. Let $n = 18$. Then X is a hypersurface in $\mathbb{P}(1, 2, 2, 3, 5)$ of degree 12 whose singularities consist of points $O_1, \ldots, O_6$ that are quotient singularities of type $\frac{1}{2}(1, 1, 1)$ and a point P that is a quotient singularity of type $\frac{1}{5}(1, 2, 3)$. The equality $-K_X^3 = 1/5$ holds and there is a commutative diagram

$$\begin{tikzcd} U \arrow[d, "\alpha"'] & & W \arrow[ll, "\beta"'] \arrow[d, "\eta"] \\ X \arrow[rr, dashed, "\psi"'] & & \mathbb{P}(1,2,2), \end{tikzcd}$$

where ψ is the natural projection, the morphism α is the weighted blow-up of the point P with weights $(1, 2, 3)$, β is the weighted blow-up with weights $(1, 1, 2)$ of the singular point of the variety U that is a quotient singularity of type $\frac{1}{3}(1, 1, 2)$, and η is an elliptic fibration.

It follows from Theorem 3.3, Lemma 3.11 and Proposition 3.5 that $\mathbb{CS}\big(X, \frac{1}{k}\mathcal{M}\big) = \{P\}$.

Let $\mathcal{D}$ be the proper transform of $\mathcal{M}$ on U and let Q and O be the singular points of U contained in the exceptional divisor of the birational morphism α that are singularities of types $\frac{1}{3}(1, 1, 2)$ and $\frac{1}{2}(1, 1, 1)$, respectively. Then it follows from Theorem 2.2 and Lemmas 2.1, 2.3 and 2.4 that $\mathcal{D} \sim_{\mathbb{Q}} -kK_U$, the set $\mathbb{CS}\big(U, \frac{1}{k}\mathcal{D}\big)$ is non-empty and the set $\mathbb{CS}\big(U, \frac{1}{k}\mathcal{D}\big)$ contains either Q or O.

Suppose that $\mathbb{CS}\big(U, \frac{1}{k}\mathcal{D}\big)$ contains O. Let $\pi\colon Y \to U$ be the weighted blow-up of O with weights $(1, 1, 1)$, F the π-exceptional divisor and $\mathcal{H}$ and $\mathcal{P}$ the proper transforms of $\mathcal{M}$ and $|-3K_U|$ on Y, respectively. Then $\mathcal{H} \sim_{\mathbb{Q}} -kK_Y$ by Theorem 2.2, but

$$\mathcal{P} \sim_{\mathbb{Q}} \pi^*(-3K_U) - \frac{1}{2}F,$$

and the base locus of the linear system $\mathcal{P}$ consists of the irreducible curve Z such that $\alpha \circ \pi(Z)$ is the base curve of the linear system $|-3K_X|$. Moreover, for a general surface S in $\mathcal{P}$, the inequality $S \cdot Z > 0$ holds, which implies that the divisor $\pi^*(-6K_U) - F$ is nef and big. Let D_1 and D_2 be general surfaces in the linear system $\mathcal{H}$. Then

$$\big(\pi^*(-6K_U) - F\big) \cdot D_1 \cdot D_2 = \big(\pi^*(-6K_U) - F\big) \cdot \left(\pi^*(-kK_U) - \frac{k}{2}F\right)^2 = 0,$$

which is impossible by Corollary 2.6.

Therefore, the set $\mathbb{CS}\big(U, \frac{1}{k}\mathcal{D}\big)$ contains the point Q. Let $\mathcal{B}$ be the proper transform of $\mathcal{M}$ on W. Then the equivalence $\mathcal{B} \sim_{\mathbb{Q}} -kK_W$ holds by Theorem 2.2, which easily implies the desired assertion.

We conclude this section by proving the following result, which is obtained in [5] and [6].

Proposition 3.13. *The assertion of Theorem 1.10 holds for $n = 1$.*

Proof. Let X be a general hypersurface in $\mathbb{P}^4$ of degree 4. Then we must show that there is a line $L \subset X$ such that there is a commutative diagram

$$\begin{tikzcd} X \arrow[r, dashed, "\rho"] \arrow[d, dashed, "\psi"'] & V \arrow[d, "\nu"] \\ \mathbb{P}^2 \arrow[r, dashed, "\sigma"'] & \mathbb{P}^2, \end{tikzcd} \tag{3.4}$$

where ψ is a projection from L and σ is a birational map.

Suppose that $\mathbb{CS}\big(X, \frac{1}{k}\mathcal{M}\big)$ contains a point P of the quartic X. Let H be a general hyperplane section of X passing through P. Then it follows from [7], Lemma 1.10, that

$$4k^2 \leqslant \mathrm{mult}_P(D_1 \cdot D_2) \leqslant D_1 \cdot D_2 \cdot H = 4k^2,$$

where D_1 and D_2 are general surfaces in $\mathcal{M}$. Therefore, the support of the effective one-dimensional cycle $D_1 \cdot D_2$ is contained in the union of a finite number of lines on the quartic X that pass through the point P. This is impossible by Lemma 2.5.

Therefore, $\mathbb{CS}\big(X, \frac{1}{k}\mathcal{M}\big)$ contains a curve C. Thus, the inequality $\mathrm{mult}_C(\mathcal{M}) \geqslant k$ holds, but it follows from Lemma 2.4 that $\deg(C) \leqslant 3$.

Suppose that C is not contained in any plane in $\mathbb{P}^4$. Then C is either a smooth curve of degree 3 or 4, or a rational curve of degree 4 that has one double point.

Suppose that C is smooth. Let $\alpha \colon U \to X$ be the blow-up of C, F the exceptional divisor of α and $\mathcal{D}$ the proper transform of $\mathcal{M}$ on the variety U. Then the base locus of the linear system $|\alpha^*(-\deg(C)K_X) - F|$ does not contain curves. We have

$$\big(\alpha^*(-\deg(C)K_X) - F\big) \cdot D_1 \cdot D_2 < 0,$$

where D_1 and D_2 are general surfaces in the linear system $\mathcal{D}$, which is a contradiction.

Thus, the curve C is a quartic curve with a double point P. Let $\beta \colon W \to X$ be the composite of the blow-up of P and the blow-up of the proper transform of C. Let G and E be the exceptional divisors of β such that $\beta(E) = C$ and $\beta(G) = P$. Then the base locus of the linear system $|\beta^*(-4K_X) - E - 2G|$ does not contain curves. We have

$$\big(\beta^*\big(-4K_X\big) - E - 2G\big) \cdot D_1 \cdot D_2 < 0,$$

where D_1 and D_2 are general surfaces in the linear system $\mathcal{D}$, which is a contradiction. Hence, we see that C is contained in some two-dimensional linear subspace of $\mathbb{P}^4$.

Suppose that $\deg(C) \neq 1$. Then we have the following possibilities:

(i) C is a smooth conic;
(ii) C is a smooth plane cubic;
(iii) C is a plane singular cubic.

Suppose that C is smooth. Let $\alpha\colon U \to X$ be a blow-up of C, F the exceptional divisor of the birational morphism α and $\mathcal{D}$ the proper transform of $\mathcal{M}$ on U. Then one can easily check that the base locus of the linear system $|\alpha^*(-\deg(C)K_X)-F|$ does not contain curves. Therefore, the divisor $\alpha^*(-\deg(C)K_X)-F$ is nef and big. On the other hand, we have

$$\bigl(\alpha^*\bigl(-\deg(C)K_X\bigr)-F\bigr)\cdot D_1\cdot D_2=0,$$

where D_1 and D_2 are general surfaces in the linear system $\mathcal{D}$, which is impossible by Corollary 2.6.

Hence, C is a plane cubic with a double point P. Let $\beta\colon W \to X$ be the composite of the blow-up of P and the blow-up of the proper transform of C. Let G and E be the exceptional divisors of the morphism β such that $\beta(E)=C$ and $\beta(G)=P$. Then the base locus of the linear system $|\beta^*(-3K_X)-E-2G|$ does not contain curves, which implies that the divisor $\beta^*(-3K_X)-E-2G$ is nef and big. On the other hand, the inequality

$$\bigl(\beta^*(-3K_X)-E-2G\bigr)\cdot D_1\cdot D_2\leqslant 0$$

holds, where D_1 and D_2 are general surfaces in $\mathcal{D}$, which is impossible by Corollary 2.6.

Thus, we see that C is a line. The equality $\operatorname{mult}_C(D)=k$ implies the existence of the commutative diagram (3.4) for $L=C$.

§ 4. The case $n=2$: a hypersurface of degree 5 in $\mathbb{P}(1,1,1,1,2)$

We use the notation and assumptions of § 3. Let $n=2$. Then X is a sufficiently general hypersurface in $\mathbb{P}(1,1,1,1,2)$ of degree 5, the equality $-K_X^3=5/2$ holds and the singularities of the hypersurface X consist of a point O that is a quotient singularity of type $\frac{1}{2}(1,1,1)$.

The hypersurface X can be given by the equation

$$w^2f_1(x,y,z,t)+f_3(x,y,z,t)w+f_5(x,y,z,t)=0\subset\mathbb{P}(1,1,1,1,2)\cong\operatorname{Proj}\bigl(\mathbb{C}[x,y,z,t,w]\bigr),$$

where $\mathrm{wt}(x)=\mathrm{wt}(y)=\mathrm{wt}(z)=\mathrm{wt}(t)=1$ and $\mathrm{wt}(w)=2$, $f_i(x,y,z,t)$ is a homogeneous polynomial of degree i, and the point O is given by the equations $x=y=z=t=0$. Let $\psi\colon X \dashrightarrow \mathbb{P}^3$ be the natural projection. Then there is a commutative diagram

$$\begin{tikzcd}
& Y \arrow[dl, "\pi"'] \arrow[dr, "\gamma"] & & W_i \arrow[ll, "\alpha_i"'] \arrow[dr, "\omega_i"] & \\
X \arrow[dr, dashed, "\psi"'] & & Z \arrow[dl, "\eta"] & & U_i, \arrow[ll, "\beta_i"'] \arrow[dl, "\xi_i"] \\
& \mathbb{P}^3 \arrow[rr, dashed, "\chi_i"'] & & \mathbb{P}^2 &
\end{tikzcd}$$

where π is the weighted blow-up of the point O with weights $(1,1,1)$, the morphism γ is the birational morphism that contracts 15 smooth rational curves $C_1,\ldots,C_{15}$ to 15 isolated ordinary double points $P_1,\ldots,P_{15}$, respectively, of the variety Z, η is a double cover branched over the surface $R\subset\mathbb{P}^3$ of degree 6 given by the equation

$$f_3(x,y,z,t)^2-4f_1(x,y,z,t)f_5(x,y,z,t)=0\subset\mathbb{P}^3\cong\operatorname{Proj}\bigl(\mathbb{C}[x,y,z,t]\bigr)$$

and has 15 isolated ordinary double points $\eta(P_1),\ldots,\eta(P_{15})$, α_i is a blow-up of C_i, β_i is a blow-up of the point P_i, w_i is a birational morphism, χ_i is a projection from $\eta(P_i)$, and ξ_i is an elliptic fibration. Moreover, the points $\eta(P_1),\ldots,\eta(P_{15})$ are given by the equations $f_3=f_1=f_5=0$.

Remark 4.1. Let τ be the birational involution of X induced by the double covering η. According to [2], the group $\operatorname{Bir}(X)$ is generated by τ and biregular automorphisms of X. Moreover, it follows from [2] that generic fibres of $\chi_i\circ\psi$ are $\operatorname{Bir}(X)$-invariant.

In the rest of this section we prove the following result.

Proposition 4.2. *There is a commutative diagram*

$$\begin{tikzcd} U_i \arrow[r, dashed, "\rho"] \arrow[d, "\xi_i"'] & V \arrow[d, "\nu"] \\ \mathbb{P}^2 \arrow[r, dashed, "\varphi"'] & \mathbb{P}^2 \end{tikzcd} \tag{4.1}$$

for some $i\in\{1,\ldots,15\}$, *where* φ *is a birational map.*

Proof. Let $\mathcal{B}_i$ be the proper transform of the linear system $\mathcal{M}$ on the variety W_i. To prove the existence of the commutative diagram (4.1), it is enough to show that $\mathcal{B}_i$ lies in the fibres of the elliptic fibration $\xi_i\circ\omega_i$. The latter follows easily from the equivalence $\mathcal{B}_i\sim_{\mathbb{Q}}-kK_{W_i}$.

Lemma 4.3. *Suppose that the set* $\mathbb{CS}\bigl(X,\frac{1}{k}\mathcal{M}\bigr)$ *contains a smooth point of* X. *Then the commutative diagram* (4.1) *exists for some* $i\in\{1,\ldots,15\}$.

Proof. Suppose that $\mathbb{CS}\bigl(X,\frac{1}{k}\mathcal{M}\bigr)$ contains a smooth point P of X. Let S be a sufficiently general surface in $|-K_X|$ that passes through P. In the case when $P\notin\bigcup_{i=1}^{15}\pi(C_i)$, the surface S does not contain irreducible components of the effective cycle $D_1\cdot D_2$, where D_1 and D_2 are general surfaces in the linear system $\mathcal{M}$. Therefore, in this case we have

$$\operatorname{mult}_P(D_1\cdot D_2)\leqslant D_1\cdot D_2\cdot S=-k^2K_X^3=\frac{5}{2}\,k^2,$$

which is impossible by [7], Lemma 1.10. Thus, the point P is contained in $\pi(C_i)$ for some i. Following the arguments of [2], put $C=\pi(C_i)$ and

$$\mathcal{M}|_S=\mathcal{L}+\operatorname{mult}_C(\mathcal{M})C,$$

where $\mathcal{L}$ is a linear system on the surface S without fixed components. Then the log pair

$$\left(S,\ \frac{1}{k}\mathcal{L}+\frac{\mathrm{mult}_C\left(\mathcal{M}\right)}{k}C\right)$$

is not log terminal at P by [11], Theorem 7.5. Let L_1 and L_2 be general curves in $\mathcal{L}$. Then

$$\mathrm{mult}_P(L_1\cdot L_2)\geqslant 4\left(1-\frac{\mathrm{mult}_C(\mathcal{M})}{k}\right)k^2$$

by [7], Theorem 3.1. On the other hand, the equality

$$L_1\cdot L_2=\frac{5}{2}k^2-\mathrm{mult}_C(\mathcal{M})k-\frac{3}{2}\mathrm{mult}_C^2(\mathcal{M})$$

holds on the surface S because $C^2=-3/2$. Hence, we have

$$\frac{5}{2}k^2-\mathrm{mult}_C(\mathcal{M})k-\frac{3}{2}\mathrm{mult}_C^2(\mathcal{M})\geqslant 4\left(1-\frac{\mathrm{mult}_C(\mathcal{M})}{k}\right)k^2,$$

which implies the equality $\mathrm{mult}_C(\mathcal{M})=k$. Thus, the curve $\pi(C_i)$ is also contained in the set $\mathbb{CS}\big(X,\frac{1}{k}\mathcal{M}\big)$. It now follows from Theorem 2.2 that the equivalence $\mathcal{B}_i\sim_{\mathbb{Q}}-kK_{W_i}$ holds, which implies the desired assertion.

Hence, we may assume that $\mathbb{CS}\big(X,\frac{1}{k}\mathcal{M}\big)$ does not contain smooth points of X.

Lemma 4.4. *Let C be a curve on X such that $C\cap\mathrm{Sing}(X)=\varnothing$. Then $C\notin\mathbb{CS}\big(X,\frac{1}{k}\mathcal{M}\big)$.*

Proof. Suppose that $\mathbb{CS}\big(X,\frac{1}{k}\mathcal{M}\big)$ contains C. Then $\mathrm{mult}_C(\mathcal{M})=k$. Let H be a very ample divisor on X. Then $H\sim_{\mathbb{Q}}-\lambda K_X$ holds for some natural number λ. Thus, we have

$$\frac{5\lambda k^2}{2}=-\lambda k^2K_X^3=H\cdot S_1\cdot S_2\geqslant\mathrm{mult}_C^2(\mathcal{M})H\cdot C\geqslant-\lambda k^2K_X\cdot C,$$

where S_1 and S_2 are general surfaces in $\mathcal{M}$. Therefore, we have the following possibilities:

(i) the equality $-K_X\cdot C=1$ holds and C is smooth and rational;
(ii) the equality $-K_X\cdot C=2$ holds and C is smooth and rational;
(iii) the equality $-K_X\cdot C=2$ holds and the arithmetic genus of C is 1.

Let $\sigma\colon\check{X}\to X$ be the blow-up of the ideal sheaf of C and G the exceptional divisor of the birational morphism σ. Then the variety $\check{X}$ is smooth in the neighbourhood of G whenever C is smooth. Moreover, in the case when C has an ordinary double point, the singularities of $\check{X}$ in the neighbourhood of G consist of a single isolated ordinary double point. In the case when C has a cuspidal singularity, the singularities of the variety $\check{X}$ in the neighbourhood of G consist of an isolated double point such that, in the neighbourhood of this point, $\check{X}$ is locally isomorphic to the hypersurface

$$x_1^2+x_2^2+x_3^2+x_4^3=0\subset\mathbb{C}\cong\mathrm{Spec}\big(\mathbb{C}[x_0,x_1,x_2,x_3]\big).$$

Let $\check{S}_1$ and $\check{S}_2$ be the proper transforms of S_1 and S_2 on $\check{X}$, respectively.

Suppose that $-K_X \cdot C = 1$. Then C is cut out in the set-theoretic sense by the surfaces in the linear system $|-2K_X|$ that pass through C. Moreover, the scheme-theoretic intersection of two general surfaces in $|-2K_X|$ passing through C is reduced at a general point of C. Thus, the divisor $\sigma^*(-2K_X) - G$ is nef and big (see [2], Lemma 5.2.5). But the equality

$$\big(\sigma^*(-2K_X) - G\big) \cdot \check{S}_1 \cdot \check{S}_2 = 0$$

holds, which is impossible by Corollary 2.6.

Suppose that $-K_X \cdot C = 2$ and C is smooth and rational. Then $\sigma^*(-2K_X) - G$ is nef because C is cut out in the set-theoretic sense by the surfaces in the linear system $|-2K_X|$ that pass through it, but the scheme-theoretic intersection of two general surfaces in $|-2K_X|$ passing through C is reduced at a general point of C. We have

$$0 > -3k^2 = \big(\sigma^*(-2K_X) - G\big) \cdot \check{S}_1 \cdot \check{S}_2 \geqslant 0,$$

which is a contradiction.

Hence, the arithmetic genus of C is 1 and $-K_X \cdot C = 2$. C is the set-theoretic intersection of the surfaces in $|-4K_X|$ that pass through C. Moreover, the scheme-theoretic intersection of two general surfaces in $|-4K_X|$ passing through C is reduced at a general point of C. Hence, the divisor $\sigma^*(-4K_X) - G$ is nef and big. On the other hand, the equality

$$\big(\sigma^*(-4K_X) - G\big) \cdot \check{S}_1 \cdot \check{S}_2 = 0$$

holds, which is impossible by Corollary 2.6.

It follows from Theorem 2.2 that $O \in \big(X, \frac{1}{k}\mathcal{M}\big)$. Let $\mathcal{D}$ be the proper transform of $\mathcal{M}$ on Y. Then it follows from Theorem 2.2 that $\mathcal{D} \sim_{\mathbb{Q}} -kK_Y$. Thus, the commutative diagram (4.1) exists in the case when the set $\mathbb{CS}\big(Y, \frac{1}{k}\mathcal{D}\big)$ contains the curve C_i. Therefore, we may assume that

$$\mathbb{CS}\bigg(Y, \frac{1}{k}\mathcal{D}\bigg) \cap \{C_1, \dots, C_{15}\} = \varnothing.$$

Lemma 4.5. *$\mathbb{CS}\big(Y, \frac{1}{k}\mathcal{D}\big)$ does not contain smooth points of Y.*

Proof. $\mathbb{CS}\big(X, \frac{1}{k}\mathcal{M}\big)$ does not contain smooth points of X. Therefore, to complete the proof it is enough to show that $\mathbb{CS}\big(Y, \frac{1}{k}\mathcal{D}\big)$ does not contain points of the exceptional divisor of the morphism π, and this follows from Lemma 2.3.

Put $\mathcal{H} = \gamma(\mathcal{D})$. Then $\mathcal{H} \sim_{\mathbb{Q}} -kK_Z$ and the singularities of the log pair $\big(Z, \frac{1}{k}\mathcal{H}\big)$ are canonical. Thus, it follows from Lemma 2.1 that $\mathbb{CS}\big(Z, \frac{1}{k}\mathcal{H}\big)$ is non-empty.

Lemma 4.6. *$\mathbb{CS}\big(Z, \frac{1}{k}\mathcal{H}\big)$ does not contain points of Z.*

Proof. It follows from Lemma 4.5 that smooth points of Z are not contained in $\mathbb{CS}\big(Z, \frac{1}{k}\mathcal{H}\big)$. On the other hand, if $P_i \in \mathbb{CS}\big(Z, \frac{1}{k}\mathcal{H}\big)$, then $\mathbb{CS}\big(Y, \frac{1}{k}\mathcal{D}\big)$ contains either the curve C_i or one of its points, which is impossible.

Thus, there is a curve Γ on Z that is contained in $\mathbb{CS}\big(Z, \frac{1}{k}\mathcal{H}\big)$. In particular, the inequality $\mathrm{mult}_{\Gamma}(\mathcal{H}) = k$ holds.

Lemma 4.7. *The equality $-K_Z \cdot \Gamma = 1$ holds.*

Proof. Let H be a general divisor of the linear system $|-K_Z|$. Then

$$2k^2 = H \cdot D_1 \cdot D_2 \geqslant \mathrm{mult}_\Gamma(D_1 \cdot D_2) H \cdot \Gamma \geqslant -k^2 K_Z \cdot \Gamma,$$

where D_1 and D_2 are sufficiently general surfaces in the linear system $\mathcal{H}$. Therefore, the inequality $-K_Z \cdot \Gamma \leqslant 2$ holds. Moreover, if the equality $-K_Z \cdot \Gamma = 2$ holds, then the support of the effective cycle $D_1 \cdot D_2$ coincides with the curve Γ. The latter is impossible by Lemma 2.5.

The curve $\eta(\Gamma)$ is a line in $\mathbb{P}^3$ and $\eta|_\Gamma \colon \Gamma \to \eta(\Gamma)$ is an isomorphism. However, the arguments used in the proof of Lemma 4.7 easily imply that the set $\mathbb{CS}\big(Z, \frac{1}{k}\mathcal{H}\big)$ does not contain subvarieties of Z besides the curve Γ, and the inequality $\mathrm{mult}_\Gamma(D_1 \cdot D_2) < 2k^2$ holds, where D_1 and D_2 are general surfaces in $\mathcal{H}$.

Lemma 4.8. *The line $\eta(\Gamma)$ is contained in the ramification surface R of the double cover η.*

Proof. Suppose that $\eta(\Gamma)$ is not contained in R. Let S be a general surface in $|-K_Z|$ that passes through Γ. Then

$$\mathcal{H}|_S = \mathrm{mult}_\Gamma(\mathcal{H})\Gamma + \mathrm{mult}_\Omega(\mathcal{H})\Omega + \mathcal{L},$$

where $\mathcal{L}$ is a linear system on the surface S that does not have fixed components and Ω is a smooth rational curve on Z such that $\eta(\Omega) = \eta(\Gamma)$ but $\Omega \neq \Gamma$. We have

$$\mathrm{Sing}(Z) \cap \Gamma = \{P_{i_1}, \ldots, P_{i_r}\} \subsetneq \{P_1, \ldots, P_{15}\} = \mathrm{Sing}(Z),$$

but P_{i_j} is an ordinary double point of S. The equalities $\Gamma^2 = \Omega^2 = -2 + r/2$ hold on S but $r \leqslant 3$. Hence, the inequality $\Omega^2 < 0$ holds on S. We have

$$(k - \mathrm{mult}_\Omega(\mathcal{H}))\Omega^2 = (\mathrm{mult}_\Gamma(\mathcal{H}) - k)\Gamma \cdot \Omega + L \cdot \Omega = L \cdot \Omega \geqslant 0,$$

where L is a general curve in $\mathcal{L}$. Therefore, the inequality $\mathrm{mult}_\Omega(\mathcal{H}) \geqslant k$ holds. Then $\Omega \in \mathbb{CS}\big(Z, \frac{1}{k}\mathcal{H}\big)$, which is impossible.

Let H be a general hyperplane in $\mathbb{P}^3$ passing through the line $\eta(\Gamma)$. Then the curve

$$\Delta = H \cap R = \eta(\Gamma) \cup \Upsilon$$

is reduced and $\eta(\Gamma) \not\subset \mathrm{Supp}(\Upsilon)$, where Υ is a plane curve of degree 5. Moreover, the reducible curve Δ is singular at every singular point $\eta(P_i)$ of R that lies on $\eta(\Gamma)$, but the set $\eta(\Gamma) \cap \Upsilon$ contains at most 5 points. On the other hand, we have

$$\mathrm{Sing}(\Delta) \cap \eta(\Gamma) = \Upsilon \cap \eta(\Gamma),$$

which implies that $|\mathrm{Sing}(Z) \cap \Gamma| \leqslant 5$. Moreover, R is given by the equation

$$f_3(x,y,z,t)^2 = 4 f_1(x,y,z,t) f_5(x,y,z,t) \subset \mathbb{P}^3 \cong \mathrm{Proj}\big(\mathbb{C}[x,y,z,t]\big)$$

and singular points of R are given by the equations $f_1 = f_3 = f_5 = 0$. We may assume that the equations $f_1 = f_3 = 0$ and $f_1 = f_5 = 0$ define irreducible curves in $\mathbb{P}^3$, which implies that at most 3 points of the subset $\mathrm{Sing}(R) \subset \mathbb{P}^3$ lie on a single line. Therefore, Bertini's theorem implies that the intersection $\eta(\Gamma) \cap \Upsilon$ contains at least two different points O_1 and O_2 that are not contained in the set $\mathrm{Sing}(R)$.

Remark 4.9. The hyperplane H is tangent to R at the points O_1 and O_2.

Let L_j be a general line on the plane H that passes through the point O_j, $\widetilde{O}_j$ the proper transform of O_j on Z and $\tilde{L}_j$ the proper transform of L_j on Z. Then L_j is tangent to R at O_j and the curve $\tilde{L}_j$ is irreducible and singular at the point $\widetilde{O}_j$, but $-K_Z \cdot \tilde{L}_j = 2$. Let $\widetilde{H}$ be the proper transform of H on Z and D a general surface in the linear system $\mathcal{H}$. Then

$$D|_{\widetilde{H}} = \mathrm{mult}_{\Gamma}(\mathcal{H})\Gamma + \Psi,$$

where Ψ is an effective divisor on $\widetilde{H}$ such that $\Gamma \not\subset \mathrm{Supp}(\Psi)$. Let $\Lambda_j = (D \setminus \Gamma) \cap \tilde{L}_j$. Then

$$\begin{aligned} 2k = D \cdot \tilde{L}_j &\geqslant \mathrm{mult}_{\widetilde{O}_j}(\tilde{L}_j)\,\mathrm{mult}_{\Gamma}(D) + \sum_{P \in \Lambda_j} \mathrm{mult}_P(D)\,\mathrm{mult}_P(\tilde{L}_j) \\ &\geqslant 2k + \sum_{P \in \Lambda_j} \mathrm{mult}_P(D)\,\mathrm{mult}_P(\tilde{L}_j), \end{aligned}$$

which implies that $D \cap \tilde{L}_j \subset \Gamma$. On the other hand, when we vary the lines L_1 and L_2 on the plane H, the curves $\tilde{L}_1$ and $\tilde{L}_2$ span two different pencils on the surface $\widetilde{H}$, whose base loci consist of the points $\widetilde{O}_1$ and $\widetilde{O}_2$, respectively. Hence, we have $\mathrm{Supp}(D) \cap \mathrm{Supp}(\widetilde{H}) = \mathrm{Supp}(\Gamma)$, where $\widetilde{H}$ is a general divisor in $|-K_Z|$ that passes through Γ and D is a general divisor in $\mathcal{H}$. But this is impossible by Lemma 2.7. The assertion of Proposition 4.2 is proved.

§ 5. The case $n = 4$: a hypersurface of degree 6 in $\mathbb{P}(1,1,1,2,2)$

We use the notation and assumptions of § 3. Let $n = 4$. Then X is a general hypersurface in $\mathbb{P}(1,1,1,2,2)$ of degree 6. The equality $-K_X^3 = 3/2$ holds. The singularities of X consist of points P_1, P_2 and P_3 that are quotient singularities of type $\frac{1}{2}(1,1,1)$.

Let $\psi\colon X \dashrightarrow \mathbb{P}^2$ be the natural projection. Then a generic fibre of the rational map ψ is an elliptic curve and the composite $\psi \circ \eta$ is a morphism, where $\eta\colon Y \to X$ is the composite of the weighted blow-ups of the singular points P_1, P_2 and P_3 with weights $(1,1,1)$.

Proposition 5.1. *The assertion of Theorem 1.10 holds for $n = 4$.*

Proof. To prove this, we must show the existence of a commutative diagram

$$\begin{tikzcd} X \arrow[r, dashed, "\rho"] \arrow[d, dashed, "\psi"'] & V \arrow[d, "\nu"] \\ \mathbb{P}^2 \arrow[r, dashed, "\varphi"'] & \mathbb{P}^2, \end{tikzcd} \tag{5.1}$$

where $\varphi \in \mathrm{Bir}(\mathbb{P}^2)$. Let Z be an element of $\mathbb{CS}\bigl(X, \frac{1}{k}\mathcal{M}\bigr)$. We have the following possibilities:

(i) Z is a curve contained in $X \setminus \mathrm{Sing}(X)$;
(ii) Z is a curve containing a singular point of X;
(iii) Z is a singular point of X.

Suppose that Z is an irreducible curve that does not contain singular points of X. Then the equality $-K_X \cdot Z = 1$ holds by Lemma 2.4, which implies that Z is smooth. Let $\gamma\colon W \to X$ be the blow-up of Z and G the exceptional divisor of the morphism γ. Then the divisor $\gamma^*(-4K_X) - G$ is nef, which implies that

$$\big(\gamma^*(-4K_X) - G\big) \cdot \overline{S}_1 \cdot \overline{S}_2 \geqslant 0,$$

where the $\overline{S}_i$ are the proper transforms on W of sufficiently general surfaces in $\mathcal{M}$. We have

$$0 \leqslant \big(\gamma^*(-4K_X) - G\big) \cdot \overline{S}_1 \cdot \overline{S}_2 = -k^2,$$

which is a contradiction.

Suppose that Z is an irreducible curve that passes through some singular point of X. Then the inequality $-K_X \cdot Z \leqslant 1$ holds by Lemma 2.4. The curve Z is contracted by the rational map ψ to a point, and either $-K_X \cdot Z = 1/2$ or $-K_X \cdot Z = 1$.

Let F be a sufficiently general surface in $|-K_X|$ that passes through Z. Then F is smooth outside P_1, P_2 and P_3, which are isolated ordinary double points of F. Let $\widetilde{Z}$ be a fibre of ψ over the point $\psi(Z)$. Then the generality of X implies that Z is an irreducible component of $\widetilde{Z}$.

Suppose that $\widetilde{Z}$ consists of two irreducible components, Z and $\overline{Z}$. Then the inequality $\overline{Z}^2 < 0$ holds on F. But the equivalence $\mathcal{M}|_F \sim_{\mathbb{Q}} kZ + k\overline{Z}$ holds. On the other hand, we have

$$\mathcal{M}|_F = \operatorname{mult}_Z(\mathcal{M})Z + \operatorname{mult}_{\overline{Z}}(\mathcal{M})\overline{Z} + \mathcal{F},$$

where $\mathcal{F}$ is a linear system on F that does not have fixed components. We have $(k - \operatorname{mult}_{\overline{Z}}(\mathcal{M}))\overline{Z} \sim_{\mathbb{Q}} \mathcal{F}$, which implies that $\operatorname{mult}_{\overline{Z}}(\mathcal{M}) = k$, and the support of the effective cycle $S_1 \cdot S_2$ is contained in $Z \cup \overline{Z}$. But this is impossible by Lemma 2.5.

Suppose that $\widetilde{Z}$ consists of three irreducible components, Z, $\widehat{Z}$ and $\check{Z}$. Then

$$-K_X \cdot \widehat{Z} = -K_X \cdot \check{Z} = -K_X \cdot Z = \frac{1}{2},$$

and the intersection form of the curves $\widehat{Z}$ and $\check{Z}$ on F is negative definite. Thus, the support of the cycle $S_1 \cdot S_2$ is contained in the union $Z \cup \widehat{Z} \cup \check{Z}$, where S_1 and S_2 are general surfaces in $\mathcal{M}$. However, this is impossible by Lemma 2.5. Hence, we see that $\mathbb{CS}\big(X, \frac{1}{k}\mathcal{M}\big) \subseteq \{P_1, P_2, P_3\}$.

Let $\pi\colon U \to X$ be the composite of the weighted blow-ups with weights $(1,1,1)$ of the singular points of X that are contained in the set $\mathbb{CS}\big(X, \frac{1}{k}\mathcal{M}\big)$ and $\mathcal{D}$ the proper transform of $\mathcal{M}$ on U. Then it follows from Theorem 2.2 that the equivalence $\mathcal{D} \sim_{\mathbb{Q}} -kK_U$ holds, but the divisor $-K_U$ is nef.

Suppose that $\mathbb{CS}\big(U, \frac{1}{k}\mathcal{D}\big)$ contains a subvariety Δ of U. Then Δ is contained in an exceptional divisor, G say, of π. Then Δ is a line on the surface $G \cong \mathbb{P}^2$ by Lemma 2.3. On the other hand, the linear system $|\pi^*(-2K_X) - G|$ does not have base points and the divisor $\pi^*(-2K_X) - G$ is nef and big. It follows from [9], Lemma 0.3.3, that there is a proper Zariski-closed subset $\Lambda \subset U$ that contains all curves on U having trivial intersection with the divisor $\pi^*(-2K_X) - G$. We have

$$2k^2 = \big(\pi^*(-2K_X) - G\big) \cdot \widetilde{S}_1 \cdot \widetilde{S}_2 \geqslant \operatorname{mult}^2_{\Delta}(\mathcal{D})\big(\pi^*(-2K_X) - G\big) \cdot \Delta = 2k^2,$$

where $\widetilde{S}_1$ and $\widetilde{S}_2$ are the proper transforms of S_1 and S_2 on U, respectively. Thus, the support of the effective cycle $\widetilde{S}_1 \cdot \widetilde{S}_2$ is contained in the subset $\Lambda \cup \Delta$, which is impossible by Lemma 2.5. Hence, the set $\mathbb{CS}\big(U, \frac{1}{k}\mathcal{D}\big)$ is empty.

The equality $-K_U^3 = 0$ holds by Lemma 2.1. Then $\pi = \eta$ and $-K_U \sim (\psi \circ \pi)^*(\mathcal{O}_{\mathbb{P}^2}(1))$. Hence, the linear system $\mathcal{D}$ lies in the fibres of the elliptic fibration $\psi \circ \pi$. Therefore, the commutative diagram (5.1) exists.

§ 6. The case $n = 6$: a hypersurface of degree 8 in $\mathbb{P}(1, 1, 1, 2, 4)$

We use the notation and assumptions of § 3. Let $n = 6$. Then X is a sufficiently general hypersurface in $\mathbb{P}(1, 1, 1, 2, 4)$ of degree 8, the equality $-K_X^3 = 1$ holds and the singularities of X consist of points P_1 and P_2 that are singularities of type $\frac{1}{2}(1, 1, 1)$. There is a commutative diagram

$$\begin{tikzcd} & U \arrow[dl, "\pi"'] \arrow[dr, "\eta"] & \\ X \arrow[rr, dashed, "\psi"'] & & \mathbb{P}^2, \end{tikzcd} \tag{6.1}$$

where ψ is the natural projection, π is the composite of the weighted blow-ups of the singular points P_1 and P_2 with weights $(1, 1, 1)$ and η is an elliptic fibration.

Proposition 6.1. *The assertion of Theorem 1.10 holds for $n = 6$.*

Proof. It follows from Theorem 3.3 that $\mathbb{CS}\big(X, \frac{1}{k}\mathcal{M}\big)$ does not contain smooth points of X. Therefore, $\mathbb{CS}\big(X, \frac{1}{k}\mathcal{M}\big)$ contains a singular point of X by Corollary 3.4 and Theorem 2.2.

Remark 6.2. Suppose that $\mathbb{CS}\big(X, \frac{1}{k}\mathcal{M}\big)$ contains both of the points P_1 and P_2. Then it follows easily from Theorem 2.2 that the assertion of the Theorem 1.10 holds for X. We may thus assume that $P_1 \in \mathbb{CS}\big(X, \frac{1}{k}\mathcal{M}\big) \not\ni P_2$.

Lemma 6.3. *The set $\mathbb{CS}\big(X, \frac{1}{k}\mathcal{M}\big)$ does not contain curves.*

Proof. Suppose that $\mathbb{CS}\big(X, \frac{1}{k}\mathcal{M}\big)$ contains an irreducible curve C. Then $-K_X \cdot C = 1/2$ by Lemma 2.4.

Let $\check{C}$ be the proper transform of C on U. Then $-K_U \cdot \check{C} = 0$, which implies that $\check{C}$ is a component of a fibre of η. Therefore, C is contracted by the rational map $\psi\colon X \dashrightarrow \mathbb{P}^2$ to a point. In particular, C is smooth and rational.

Let S be a general surface in $|-K_X|$ that contains C. Then S is smooth outside the points P_1 and P_2, which are isolated ordinary double points on S. Let F be a fibre of the rational map ψ over the point $\psi(C)$. Then F consists of two irreducible components and C is one of them. Let Z be the other component of F. Then $Z^2 < 0$ on S but

$$\mathcal{M}|_S = \operatorname{mult}_C(\mathcal{M})C + \operatorname{mult}_Z(\mathcal{M})Z + \mathcal{L},$$

where $\mathcal{L}$ is a linear system with no fixed components. We have $(k - \operatorname{mult}_Z(\mathcal{M}))Z \sim_{\mathbb{Q}} \mathcal{L}$, which implies that $\operatorname{mult}_Z(\mathcal{M}) = k$. It follows from Theorem 2.2 that $\mathbb{CS}\big(X, \frac{1}{k}\mathcal{M}\big)$ contains P_2, which is a contradiction.

Thus, the set $\mathbb{CS}\big(X, \frac{1}{k}\mathcal{M}\big)$ consists of the point P_1. Let $\zeta\colon Y \to X$ be the weighted blow-up of P_1 with weights $(1,1,1)$ and $\mathcal{D}$ the proper transform of $\mathcal{M}$ on Y. Then $\mathcal{D} \sim_{\mathbb{Q}} -kK_Y$ by Theorem 2.2 and $\mathbb{CS}\big(Y, \frac{1}{k}\mathcal{D}\big) \neq \varnothing$ by Lemma 2.1.

Let T be a subvariety of Y contained in $\mathbb{CS}\big(Y, \frac{1}{k}\mathcal{D}\big)$. Then $\zeta(T) \in \mathbb{CS}\big(X, \frac{1}{k}\mathcal{M}\big)$, which implies that $T \subset G$, where G is the exceptional divisor of ζ. It follows from Lemma 2.3 that T is a line in $G \cong \mathbb{P}^2$. But this is impossible by Lemma 2.4.

§ 7. The case $n = 7$: a hypersurface of degree 8 in $\mathbb{P}(1,1,2,2,3)$

We use the notation and assumptions of § 3. Let $n = 7$. Then X is a sufficiently general hypersurface in $\mathbb{P}(1,1,2,2,3)$ of degree 8 whose singularities consist of points P_1, P_2, P_3 and P_4 that are quotient singularities of type $\frac{1}{2}(1,1,1)$ and a point Q that is a quotient singularity of type $\frac{1}{3}(1,1,2)$.

Let $\xi_i\colon X \dashrightarrow \mathbb{P}(1,1,2)$ be a projection such that there is a commutative diagram

$$\begin{tikzcd} & & Y_i \arrow[dll, "\beta_i"'] \arrow[ddr, "\eta_i"] & \\ U_i \arrow[dr, "\alpha_i"'] & & & \\ & X \arrow[rr, dashed, "\xi_i"'] & & \mathbb{P}(1,1,2), \end{tikzcd}$$

where α_i is the weighted blow-up of P_i with weights $(1,1,1)$, β_i is the weighted blow-up of the proper transform of Q on the variety U_i with weights $(1,1,2)$, and η_i is an elliptic fibration.

Let $\xi_0\colon X \dashrightarrow \mathbb{P}(1,1,2)$ be a projection such that there is a commutative diagram

$$\begin{tikzcd} U_0 \arrow[d, "\alpha_0"'] & Y_0 \arrow[l, "\beta_0"'] \arrow[dr, "\eta_0"] & \\ X \arrow[rr, dashed, "\xi_0"'] & & \mathbb{P}(1,1,2), \end{tikzcd}$$

where α_0 is the weighted blow-up of Q with weights $(1,1,2)$, β_0 is the blow-up with weights $(1,1,1)$ of the singular point of U_0 that dominates Q, and η_0 is an elliptic fibration.

Proposition 7.1. *There is a commutative diagram*

$$\begin{tikzcd} X \arrow[r, dashed, "\sigma"] \arrow[d, dashed, "\xi_i"'] & X \arrow[r, dashed, "\rho"] & V \arrow[d, "\nu"] \\ \mathbb{P}(1,1,2) \arrow[rr, dashed, "\varphi"'] & & \mathbb{P}^2 \end{tikzcd} \tag{7.1}$$

for some $i \in \{0,1,2,3,4\}$, where σ and φ are birational maps.

Proof. The set $\mathbb{CS}\big(X, \frac{1}{k}\mathcal{M}\big)$ is non-empty and does not contain smooth points of X. It follows from Lemma 3.11 that $\mathbb{CS}\big(X, \frac{1}{k}\mathcal{M}\big) \subseteq \{P_1, P_2, P_3, P_4, Q\}$.

Lemma 7.2. *The set* $\mathbb{CS}\big(X, \frac{1}{k}\mathcal{M}\big)$ *does not contain two points of the set* $\{P_1, P_2, P_3, P_4\}$.

Proof. Suppose that $\mathbb{CS}\big(X, \frac{1}{k}\mathcal{M}\big)$ contains the points P_1 and P_2. Let $\pi\colon W \to X$ be the composite of the weighted blow-ups of P_1 and P_2 with weights $(1,1,1)$ and $\mathcal{B}$ the proper transform of $\mathcal{M}$ on W. Then $\mathcal{B} \sim_{\mathbb{Q}} -kK_W$ by Theorem 2.2. But the base locus of $|-K_W|$ consists of a curve C such that $C^2 = -1/3$ on a general surface in the pencil $|-K_W|$. But this is impossible by Lemma 2.7.

Let $\mathcal{D}_i$ be the proper transform of $\mathcal{M}$ on U_i.

Remark 7.3. If the set $\mathbb{CS}\big(X, \frac{1}{k}\mathcal{M}\big)$ contains the point Q, then $\mathbb{CS}\big(U_0, \frac{1}{k}\mathcal{D}_0\big)$ is non-empty by Lemma 2.1. Similarly, $\mathbb{CS}\big(U_i, \frac{1}{k}\mathcal{D}_i\big)$ is non-empty if $\mathbb{CS}\big(X, \frac{1}{k}\mathcal{M}\big)$ contains P_i.

Lemma 7.4. *The set* $\mathbb{CS}\big(X, \frac{1}{k}\mathcal{M}\big)$ *does not consist of the point* P_i.

Proof. Suppose that $\mathbb{CS}\big(X, \frac{1}{k}\mathcal{M}\big) = \{P_i\}$. Then the set $\mathbb{CS}\big(U_i, \frac{1}{k}\mathcal{D}_i\big)$ contains an irreducible subvariety $Z \subset U_i$ by Lemma 2.1. Let E_i be the exceptional divisor of α_i. Then Z is a line on the surface $E_i \cong \mathbb{P}^2$ by Lemma 2.3, which is impossible by Lemma 2.4.

Therefore, if $P_i \in \mathbb{CS}\big(X, \frac{1}{k}\mathcal{M}\big)$, then $\mathbb{CS}\big(X, \frac{1}{k}\mathcal{M}\big) = \{P_i, Q\}$. This implies that the proper transform of $\mathcal{M}$ on Y_i lies in the fibres of the elliptic fibration η_i. Thus, if $P_i \in \mathbb{CS}\big(X, \frac{1}{k}\mathcal{M}\big)$, then the commutative diagram (7.1) exists. Thus, we may assume that the set $\mathbb{CS}\big(X, \frac{1}{k}\mathcal{M}\big)$ consists of the point Q.

Let O be a singular point of the variety U_0 contained in the exceptional divisor of the morphism α_0. Then O is contained in $\mathbb{CS}\big(U_0, \frac{1}{k}\mathcal{D}_0\big)$ by Lemma 2.3, which implies the existence of the commutative diagram (7.1) by Theorem 2.2. The assertion is proved.

§ 8. The case $n = 8$: a hypersurface of degree 9 in $\mathbb{P}(1,1,1,3,4)$

We use the notation and assumptions of § 3. Let $n = 8$. Then X is a sufficiently general hypersurface in $\mathbb{P}(1,1,1,3,4)$ of degree 9, the equality $-K_X^3 = 3/4$ holds and the singularities of X consist of one point O that is a quotient singularity of type $\frac{1}{4}(1,1,3)$.

Proposition 8.1. *The assertion of Theorem* 1.10 *holds for* $n = 8$.

Proof. The hypersurface X can be given by the equation

$$w^2z + f_5(x,y,z,t)w + f_9(x,y,z,t) = 0 \subset \mathbb{P}(1,1,1,3,5) \cong \operatorname{Proj}\big(\mathbb{C}[x,y,z,t,w]\big), \quad (8.1)$$

where $\operatorname{wt}(x) = \operatorname{wt}(y) = \operatorname{wt}(z) = 1$, $\operatorname{wt}(t) = 3$, $\operatorname{wt}(w) = 4$, and f_i is a quasi-homogeneous polynomial of degree i. The point O is given by $x = y = z = t = 0$.

There is a commutative diagram

$$\begin{tikzcd}
 & & W \arrow[dll, "\sigma"'] \arrow[d, "\alpha"] & U \arrow[l, "\beta"'] & \\
Y \arrow[dr, "\omega"'] & & X \arrow[dl, dashed, "\xi"'] \arrow[dr, dashed, "\psi"] & & Z \arrow[ul, "\gamma"'] \arrow[dl, "\eta"] \\
 & \mathbb{P}(1,1,1,3) \arrow[rr, dashed, "\chi"'] & & \mathbb{P}^2, &
\end{tikzcd}$$

where ξ, ψ and χ are natural projections, α is the weighted blow-up of the singular point O with weights $(1,1,3)$, β is the weighted blow-up with weights $(1,1,2)$ of the singular point of W that is a quotient singularity of type $\frac{1}{3}(1,1,2)$, γ is the weighted blow-up with weights $(1,1,1)$ of the singular point of U that is a quotient singularity of type $\frac{1}{2}(1,1,1)$, η is an elliptic fibration, σ is a birational morphism that contracts 15 smooth rational curves to 15 isolated ordinary double points $P_1,\ldots,P_{15}$ of Y, respectively, and ω is a double cover branched over the surface $R\subset\mathbb{P}(1,1,1,3)$ given by the equation

$$f_5(x,y,z,t)^2-4zf_9(x,y,z,t)=0\subset\mathbb{P}(1,1,1,3)\cong\operatorname{Proj}\big(\mathbb{C}[x,y,z,t]\big)$$

and has 15 isolated ordinary double points $\omega(P_1),\ldots,\omega(P_{15})$ given by $z=f_5=f_9=0$.

Let G be the exceptional divisor of the morphism α, F the exceptional divisor of the morphism β, P the singular point of W, Q the singular point of U, $\mathcal{D}$ the proper transform of $\mathcal{M}$ on W, $\mathcal{H}$ the proper transform of $\mathcal{M}$ on U, $\mathcal{P}$ the proper transform of $\mathcal{M}$ on Z, and $\mathcal{B}$ the proper transform of $\mathcal{M}$ on Y.

Remark 8.2. The divisors $-K_W$ and $-K_U$ are nef and big, and $\omega\circ\sigma(G)$ is given by $z=0$.

It follows from Theorem 3.3 that the set $\mathbb{CS}\big(X,\frac{1}{k}\mathcal{M}\big)$ does not contain smooth points of X. It follows from Corollary 3.4 and Theorem 2.2 that $O\in\mathbb{CS}\big(X,\frac{1}{k}\mathcal{M}\big)$.

Lemma 8.3. *The set* $\mathbb{CS}\big(X,\frac{1}{k}\mathcal{M}\big)$ *does not contain curves.*

Proof. Let L be a curve in $\mathbb{CS}\big(X,\frac{1}{k}\mathcal{M}\big)$. It follows from Theorem 3.9 that there are two different surfaces D and D' in the linear system $|-K_X|$ such that the irreducible curve L is a component of the cycle $D\cdot D'$, which is reduced and contains at most two components.

Let $\mathcal{P}$ be the pencil in $|-K_X|$ generated by D and D'. Then we may assume that D is a sufficiently general surface in $\mathcal{P}$. Applying Lemma 2.7 together with the proof of Lemma 2.4 to $\mathcal{M}$ and $\mathcal{P}$, we immediately obtain a contradiction in the case when the equality $-K_X\cdot L=3/4$ holds. Therefore, we may assume that either $-K_X\cdot L=1/4$ or $-K_X\cdot L=1/2$. We consider only the case $-K_X\cdot L=1/4$ because the case $-K_X\cdot L=1/2$ is simpler and very similar.

Let S_W be the proper transform on W of the surface given by $z=0$ and L_W the proper transform on W of L. Then S_W must contain L_W because

$$S_W\sim_{\mathbb{Q}}\alpha^*(-K_X)-\frac{5}{4}G,$$

but $G \cdot L_W \geqslant 1/3$. Thus, either the curve L_W is contracted by σ or the curve $\omega(L_Y)$ is a ruling of the cone $\mathbb{P}(1,1,1,3)$ contained in the surface $\omega \circ \sigma(G)$, where $L_Y = \sigma(L_W)$.

Suppose that L_W is not contracted by σ. Then $\omega(L_Y)$ is not contained in the surface R, which implies that $\omega(L_Y)$ contains at most one singular point of R different from the point $\omega\circ\sigma(P)$. Moreover, the curve $\omega(L_Y)$ must contain a singular point of R different from $\omega \circ \sigma(P)$ because otherwise $-K_X \cdot L = 3/4$. Thus, we may assume that $\omega(L_Y)$ contains the point $\omega(P_1)$.

Let D_Y and D'_Y be the proper transforms on Y of D and D', respectively. Then P_1 is an isolated ordinary double point of D_Y. Thus, we see that the proper transform of the curve L' on the threefold W is contracted to the point P_1 by σ and

$$D_Y \cdot D'_Y = L_Y + \bar{L}_Y,$$

where $\bar{L}_Y$ is a ruling of the cone $G \cong \mathbb{P}(1,1,3)$. In particular, we have $-K_X \cdot L' = 1/4$, which is impossible by the equality $-K_X \cdot L = 1/4$. Hence, L_W is contracted by σ.

Let L'_W be the proper transform on W of L' and $L'_Y = \sigma(L'_W)$. Then $\omega(L'_Y)$ is a ruling of the cone $\mathbb{P}(1,1,1,3)$ contained in the surface $\omega \circ \sigma(G)$. $\omega(L'_Y)$ is not contained in R, which implies that $\omega(L'_Y)$ contains at most one singular point of R different from $\omega \circ \sigma(P)$. $\omega(L'_Y)$ must contain a singular point of R different from $\omega \circ \sigma(P)$ because $-K_X \cdot L' \neq 3/4$. Thus, we may assume that $\omega(L'_Y)$ contains $\omega(P_1)$.

Since P_1 is an isolated ordinary double point of D_Y and $\sigma(L_W) = P_1$, we have

$$D_Y \cdot D'_Y = L'_Y + \bar{L}'_Y,$$

where $\bar{L}'_Y$ is a ruling of $G \cong \mathbb{P}(1,1,3)$. The intersection $L_W \cap L'_W$ consists of a point $O' \notin G$. Hence, the intersection $L \cap L'$ contains the point $\alpha(O')$, which is different from O.

The surface D is smooth at the point $\alpha(O')$ but $(L + L') \cdot L' = 1/2$ on D, which implies that $L' \cdot L' < 0$ on D. Therefore, we have

$$\mathcal{M}|_D = m_1 L + m_2 L' + \mathcal{L} \equiv kL + kL',$$

where $\mathcal{L}$ is a linear system on D that does not have fixed components and m_1 and m_2 are natural numbers such that $m_1 \geqslant k$. In particular, we have

$$0 \leqslant (m_1 - k)L' \cdot L + \mathcal{L} \cdot L' = (k - m_2)L' \cdot L',$$

which implies that $m_2 = m_1 = k$ and $\mathcal{M}|_D = kL + kL'$, which is impossible by Lemma 2.7.

It follows from Theorem 3.3 that $\mathbb{CS}\big(X, \frac{1}{k}\mathcal{M}\big) = \{O\}$. Hence, the assertion of Theorem 2.2 implies that $\mathcal{D} \sim_{\mathbb{Q}} -kK_W$. But it follows from Corollary 2.8 and Lemmas 2.1 and 2.3 that $\mathbb{CS}\big(W, \frac{1}{k}\mathcal{D}\big) = \{P\}$.

The equivalence $\mathcal{H} \sim_{\mathbb{Q}} -kK_U$ holds by Theorem 2.2. Hence, the set $\mathbb{CS}\big(U, \frac{1}{k}\mathcal{H}\big)$ contains the point Q by Lemmas 2.1 and 2.3. We see that $\mathcal{P}$ lies in the fibres of η by Theorem 2.2.

§ 9. The case $n = 9$: a hypersurface of degree 9 in $\mathbb{P}(1,1,2,3,3)$

We use the notation and assumptions of § 3. Let $n = 9$. Then X is a sufficiently general hypersurface in $\mathbb{P}(1,1,2,3,3)$ of degree 9, the equality $-K_X^3 = 1/2$ holds and the singularities of X consist of points P_1, P_2 and P_3 that are quotient singularities of type $\frac{1}{3}(1,1,2)$ and a point O that is a quotient singularity of type $\frac{1}{2}(1,1,1)$.

Let $\xi\colon X \dashrightarrow \mathbb{P}(1,1,2)$ be a projection. There is a commutative diagram

$$\begin{tikzcd}
 & U_{ij} \arrow[dl, "\beta_{ij}"'] & & Y \arrow[ll, "\gamma_{ij}"'] \arrow[dd, "\eta"] \\
U_i \arrow[dr, "\alpha_i"'] & & & \\
 & X \arrow[rr, dashed, "\xi"'] & & \mathbb{P}(1,1,2),
\end{tikzcd}$$

where α_i is the blow-up of P_i with weights $(1,1,2)$, β_{ij} is the weighted blow-up with weights $(1,1,2)$ of the proper transform of P_j on the variety U_i, γ_{ij} is the weighted blow-up with weights $(1,1,2)$ of the proper transform of P_k on U_{ij}, and η is an elliptic fibration, where $i \neq j$ and $k \notin \{i,j\}$.

Remark 9.1. The divisors $-K_{U_i}$ and $-K_{U_{ij}}$ are nef and big.

There is a commutative diagram

$$\begin{tikzcd}
 & Z \arrow[dl, "\pi"'] \arrow[dr, "\sigma"] & \\
X \arrow[rr, dashed, "\chi"'] & & \mathbb{P}(1,1,3),
\end{tikzcd}$$

where χ is a projection, π is the blow-up of Q with weights $(1,1,1)$ and σ is an elliptic fibration.

Proposition 9.2. *Either there is a commutative diagram*

$$\begin{tikzcd}
X \arrow[r, dashed, "\rho"] \arrow[d, dashed, "\xi"'] & V \arrow[d, "\nu"] \\
\mathbb{P}(1,1,2) \arrow[r, dashed, "\varphi"'] & \mathbb{P}^2
\end{tikzcd} \tag{9.1}$$

or there is a commutative diagram

$$\begin{tikzcd}
X \arrow[r, dashed, "\omega"] \arrow[d, dashed, "\chi"'] & X \arrow[r, dashed, "\rho"] & V \arrow[d, "\nu"] \\
\mathbb{P}(1,1,3) \arrow[rr, dashed, "\upsilon"'] & & \mathbb{P}^2,
\end{tikzcd} \tag{9.2}$$

where φ, ω and υ are birational maps.

Proof. Note that this proposition implies the assertion of Theorem 1.10 for $n = 9$.

Lemma 9.3. *Suppose that $Q \in \mathbb{CS}\big(X, \frac{1}{k}\mathcal{M}\big)$. Then there is a commutative diagram* (9.2).

Proof. Let $\mathcal{B}$ be the proper transform of $\mathcal{M}$ on Z. Then $\mathcal{B} \sim_{\mathbb{Q}} -kK_X$ by Theorem 2.2, which implies the equality $S \cdot C = 0$, where S is a general surface in $\mathcal{M}$ and C is a generic fibre of the morphism σ. Thus, the commutative diagram (9.2) exists.

We may assume that $\mathbb{CS}\big(X, \frac{1}{k}\mathcal{M}\big) \subseteq \{P_1, P_2, P_3\}$ by Theorem 3.3 and Lemma 3.11.

Lemma 9.4. *The commutative diagram* (9.1) *exists whenever $\mathbb{CS}\big(X, \frac{1}{k}\mathcal{M}\big) = \{P_1, P_2, P_3\}$.*

Proof. See the proof of Lemma 9.3.

Therefore, we may assume that the set $\mathbb{CS}\big(X, \frac{1}{k}\mathcal{M}\big)$ does not contain the point P_3 but does contain the point P_1.

Lemma 9.5. *The set $\mathbb{CS}\big(X, \frac{1}{k}\mathcal{M}\big)$ contains the point P_2.*

Proof. Suppose that $\mathbb{CS}\big(X, \frac{1}{k}\mathcal{M}\big)$ consists of the point P_1. Let $\mathcal{B}$ be the proper transform of $\mathcal{M}$ on U_1. Then $\mathcal{B} \sim_{\mathbb{Q}} -kK_{U_1}$ by Theorem 2.2, and Lemma 2.1 implies that the set $\mathbb{CS}\big(U_1, \frac{1}{k}\mathcal{B}\big)$ is non-empty.

Let Z be an element of the set $\mathbb{CS}\big(U_1, \frac{1}{k}\mathcal{B}\big)$ and G the exceptional divisor of the birational morphism α_1. Then it follows from Lemmas 2.3 and 2.4 that Z is a singular point of G which is a quotient singularity of type $\frac{1}{2}(1,1,1)$ on U_1.

Let $\delta \colon W \to U_1$ be the weighted blow-up of the point Z with weights $(1,1,1)$, $\mathcal{D}$ the proper transform of $\mathcal{M}$ on W, and F a general surface in the linear system $|-K_W|$. Then the inequality $\Delta^2 = -1/2$ holds on F, but $\mathcal{D}|_F \sim_{\mathbb{Q}} k\Delta$ by Theorem 2.2, which is impossible by Lemmas 2.9 and 2.7.

We can now apply the arguments of the proof of Lemma 9.5 to get a contradiction.

§ 10. The case $n = 10$: a hypersurface of degree 10 in $\mathbb{P}(1,1,1,3,5)$

We use the notation and assumptions of § 3. Let $n = 10$. Then X is a hypersurface in $\mathbb{P}(1,1,1,3,5)$ of degree 10, the equality $-K_X^3 = 2/3$ holds and the singularities of X consist of a point O that is a quotient singularity of type $\frac{1}{3}(1,1,2)$. There is a commutative diagram

$$\begin{tikzcd}
 & & W \arrow[dl, "\alpha"'] & \\
 & X \arrow[dl, "\xi"'] \arrow[drr, dashed, "\psi"] & & Y \arrow[ul, "\beta"'] \arrow[d, "\eta"] \\
\mathbb{P}(1,1,1,3) \arrow[rrr, dashed, "\chi"'] & & & \mathbb{P}^2,
\end{tikzcd}$$

where ξ, ψ and χ are natural projections, α is the weighted blow-up of O with weights $(1,1,1)$, β is the weighted blow-up with weights $(1,1,1)$ of the singular point of W, and η is an elliptic fibration.

Proposition 10.1. *The assertion of Theorem 1.10 holds for* $n=10$.

Proof. Let Q be the unique singular point of the variety W, $\mathcal{D}$ the proper transform of $\mathcal{M}$ on W and $\mathcal{H}$ the proper transform of $\mathcal{M}$ on Y.

Lemma 10.2. *The set* $\mathbb{CS}\big(X, \frac{1}{k}\mathcal{M}\big)$ *does not contains curves.*

Proof. Suppose that $\mathbb{CS}\big(X, \frac{1}{k}\mathcal{M}\big)$ contains a curve C. Then $-K_X\cdot C = 1/3$ by Lemma 2.4, which implies that C is contracted by the rational map ψ to a point.

The variety $\mathbb{P}(1,1,1,3)$ is cone whose vertex is the point $\xi(O)$. The curve $\xi(C)$ is a ruling of the cone $\mathbb{P}(1,1,1,3)$. The generality of the hypersurface X implies that $\xi(C)$ is not contained in the ramification surface of the morphism ξ. Thus, there is an irreducible smooth rational curve Z on X such that $Z\neq C$ and $\xi(Z)=\xi(C)$.

Let S be a general surface in the linear system $|-K_X|$ that contains the curves C and Z. Then $Z^2<0$ on S. But $\mathcal{M}|_S \sim_{\mathbb{Q}} kC+kZ$, which easily leads to a contradiction using Lemmas 2.7 and 2.9 because $\mathrm{mult}_C(\mathcal{M})=k$.

The set $\mathbb{CS}\big(X, \frac{1}{k}\mathcal{M}\big)$ consists of the point O by Theorem 3.3. The set $\mathbb{CS}\big(W, \frac{1}{k}\mathcal{D}\big)$ contains the point Q by Lemmas 2.1 and 2.3. But $\mathcal{H}\sim_{\mathbb{Q}} -kK_Y$ by Theorem 2.2, which implies that $\mathcal{H}$ is contained in the fibres of η. The assertion is proved.

§ 11. The case $n=12$: a hypersurface of degree 10 in $\mathbb{P}(1,1,2,3,4)$

We use the notation and assumptions of § 3. Let $n=12$. Then X is a general hypersurface in $\mathbb{P}(1,1,2,3,4)$ of degree 10 whose singularities consist of points P_1 and P_2 that are singularities of type $\frac{1}{2}(1,1,1)$, a point P_3 that is a singularity of type $\frac{1}{3}(1,1,2)$ and a point P_4 that is a singularity of type $\frac{1}{4}(1,1,3)$.

Proposition 11.1. *The assertion of Theorem 1.10 holds for* $n=12$.

There is a commutative diagram

$$\begin{tikzcd}
 & & Y \arrow[dl, "\gamma_5"'] \arrow[dr, "\gamma_3"] \arrow[drrr, "\eta"] & & & \\
 & U_{34} \arrow[dl, "\beta_4"'] \arrow[dr, "\beta_3"] & & U_{45} \arrow[dl, "\beta_5"] & & \mathbb{P}(1,1,2), \\
U_3 \arrow[dr, "\alpha_3"'] & & U_4 \arrow[dl, "\alpha_4"'] & & & \\
 & X \arrow[uurrrr, dashed, "\psi"'] & & & &
\end{tikzcd}$$

where ψ is a projection, α_3 is the weighted blow-up of P_3 with weights $(1,1,2)$, α_4 is the weighted blow-up of P_4 with weights $(1,1,3)$, β_4 is the weighted blow-up with weights $(1,1,3)$ of the proper transform of the point P_4 on U_3, β_3 is the weighted blow-up with weights $(1,1,2)$ of the proper transform of the point P_3 on U_4, β_5 is the weighted blow-up with weights $(1,1,2)$ of the singular point of U_4 that is contained in the exceptional divisor of α_4, γ_3 is the weighted blow-up with weights $(1,1,2)$ of the proper transform of the point P_3 on the variety U_{45}, γ_5 is the weighted blow-up with weights $(1,1,2)$ of the singular point of U_{34} that is contained in the exceptional divisor of the morphism β_4, and η is an elliptic fibration.

Remark 11.2. The divisors $-K_{U_3}$, $-K_{U_4}$, $-K_{U_{34}}$ and $-K_{U_{45}}$ are nef and big.

Proof of Proposition 11.1. It follows from Theorem 3.3 and Lemma 3.11 that

$$\mathbb{CS}\left(X, \frac{1}{k}\mathcal{M}\right) \subseteq \{P_1, P_2, P_3, P_4\}.$$

Lemma 11.3. *The set $\mathbb{CS}\big(X, \frac{1}{k}\mathcal{M}\big)$ does not contain P_1 or P_2.*

Proof. We assume that $P_1 \in \mathbb{CS}\big(X, \frac{1}{k}\mathcal{M}\big)$ and seek a contradiction.

Let $\pi\colon W \to X$ be the weighted blow-up of the singular point P_1 with weights $(1,1,1)$, $\mathcal{B}$ the proper transform of $\mathcal{M}$ on W, S a general surface in the pencil $|-K_W|$, and Z the base curve of $|-K_W|$. Then $Z^2 = -1/24$ on S. But $\mathcal{B}|_S \sim_{\mathbb{Q}} kZ$ by Theorem 2.2, which is impossible by Lemmas 2.9 and 2.7.

Let $\mathcal{D}_3$, $\mathcal{D}_4$, $\mathcal{D}_{34}$ and $\mathcal{D}_{45}$ be the proper transforms of $\mathcal{M}$ on U_3, U_4, U_{34} and U_{45}, respectively. Then it follows from Lemma 2.1 that $\mathbb{CS}\big(U_\mu, \frac{1}{k}\mathcal{D}_\mu\big) \neq \varnothing$ in the case when $\mathcal{D}_\mu \sim_{\mathbb{Q}} -kK_{U_\mu}$.

Lemma 11.4. *Suppose that the set $\mathbb{CS}\big(X, \frac{1}{k}\mathcal{M}\big)$ contains the point P_4. Let $\bar{P}_3$ be the proper transform of P_3 on U_4 and $\bar{P}_5$ the singular point of U_4 such that $\alpha_4(\bar{P}_5) = P_4$. Then $\mathcal{D}_4 \sim_{\mathbb{Q}} -kK_{U_4}$ and $\mathbb{CS}\big(U_4, \frac{1}{k}\mathcal{D}_4\big) \subseteq \{\bar{P}_3, \bar{P}_5\}$.*

Proof. The equivalence $\mathcal{D}_4 \sim_{\mathbb{Q}} -kK_{U_4}$ follows from Theorem 2.2. We must show that $\mathbb{CS}\big(U_4, \frac{1}{k}\mathcal{D}_4\big) \subseteq \{\bar{P}_3, \bar{P}_5\}$. Suppose that the set $\mathbb{CS}\big(U_4, \frac{1}{k}\mathcal{D}_4\big)$ is not contained in $\{\bar{P}_3, \bar{P}_5\}$. Let C be an element of $\mathbb{CS}\big(U_4, \frac{1}{k}\mathcal{D}_4\big)$ that is different from the points $\bar{P}_3$ and $\bar{P}_5$. Then $\alpha_4(C)$ is contained in the set $\mathbb{CS}\big(X, \frac{1}{k}\mathcal{M}\big)$, which implies that $\alpha_4(C) = P_4$.

Let G be an exceptional divisor of the blow-up α_4. Then $G \cong \mathbb{P}(1,1,3)$, and we can identify G with a cone over a smooth rational curve in $\mathbb{P}^3$ of degree 3. It follows from Lemma 2.3 that C is a ruling of the cone G. But this is impossible by Corollary 2.8.

Lemma 11.5. *Suppose that $P_3 \in \mathbb{CS}\big(X, \frac{1}{k}\mathcal{M}\big)$. Let $\bar{P}_4$ be the proper transform of P_4 on U_3. Then $\mathcal{D}_3 \sim_{\mathbb{Q}} -kK_{U_3}$ and $\mathbb{CS}\big(U_3, \frac{1}{k}\mathcal{D}_3\big) = \{\bar{P}_4\}$.*

Proof. See the proof of Lemma 9.5.

It follows from Theorem 2.2 that either $\mathcal{D}_{34} \sim_{\mathbb{Q}} -kK_{U_{34}}$ or $\mathcal{D}_{45} \sim_{\mathbb{Q}} -kK_{U_{45}}$. Let $\mathcal{D}$ be the proper transform of $\mathcal{M}$ on Y.

Lemma 11.6. *Suppose that $\mathcal{D}_{34} \sim_{\mathbb{Q}} -kK_{U_{34}}$. Then $\mathcal{D} \sim_{\mathbb{Q}} -kK_Y$.*

Proof. Let F be the exceptional divisor of the morphism β_3, G the exceptional divisor of the morphism β_4, $\check{P}_5$ the singular point of the surface G and $\check{P}_6$ the singular point of the surface F. Then $\mathcal{D} \sim_{\mathbb{Q}} -kK_Y$ by Theorem 2.2 if the set $\mathbb{CS}\big(U_{34}, \frac{1}{k}\mathcal{D}_{34}\big)$ contains the point $\check{P}_5$.

We may assume that $\check{P}_5 \notin \mathbb{CS}\big(U_{34}, \frac{1}{k}\mathcal{D}_{34}\big)$. Then $\check{P}_6 \in \mathbb{CS}\big(U_{34}, \frac{1}{k}\mathcal{D}_{34}\big)$ by Lemmas 2.1 and 2.3. Let $\pi\colon W \to U_{34}$ be the weighted blow-up of the point $\check{P}_6$ with weights $(1,1,1)$, $\mathcal{B}$ the proper transform of $\mathcal{M}$ on W and S a general surface in the linear system $|-K_W|$. Then the base locus of $|-K_W|$ consists of the irreducible curve Δ such that $\mathcal{B}|_S \sim_{\mathbb{Q}} k\Delta$. But $\Delta^2 < 0$ on S, which is impossible by Lemmas 2.9 and 2.7.

Arguing as in the proof of Lemma 11.6, we see that $\mathcal{D} \sim_{\mathbb{Q}} -kK_Y$ in the case when $\mathcal{D}_{45} \sim_{\mathbb{Q}} -kK_{U_{45}}$. But it follows from the equivalence $\mathcal{D} \sim_{\mathbb{Q}} -kK_Y$ that the linear system $\mathcal{D}$ lies in the fibres of the elliptic fibration η. Hence, the assertion is proved.

§ 12. The case $n = 13$: a hypersurface of degree 11 in $\mathbb{P}(1,1,2,3,5)$

We use the notation and assumptions of § 3. Let $n = 13$. Then X is a sufficiently general hypersurface in $\mathbb{P}(1,1,2,3,5)$ of degree 11, the singularities of X consist of a point P_1 that is a quotient singularity of type $\frac{1}{2}(1,1,1)$, a point P_2 that is a singularity of type $\frac{1}{3}(1,1,2)$ and a point P_3 that is a singularity of type $\frac{1}{5}(1,2,3)$. The equality $-K_X^3 = 11/30$ holds.

There is a commutative diagram

$$\begin{tikzcd}
 & & Y \arrow[dl, "\gamma_4"'] \arrow[dr, "\gamma_2"] \arrow[rrrd, "\eta"] & & & \\
 & U_{23} \arrow[dl, "\beta_3"'] \arrow[dr, "\beta_2"] & & U_{34} \arrow[dl, "\beta_4"'] & & \mathbb{P}(1,1,2), \\
U_2 \arrow[dr, "\alpha_2"'] & & U_3 \arrow[dl, "\alpha_3"'] & & & \\
 & X \arrow[rrrruu, dashed, "\psi"'] & & & &
\end{tikzcd}$$

where ψ is a projection, α_2 is the weighted blow-up of P_2 with weights $(1,1,2)$, α_3 is the weighted blow-up of P_3 with weights $(1,2,3)$, β_3 is the weighted blow-up with weights $(1,2,3)$ of the proper transform of the point P_3 on U_2, β_2 is the weighted blow-up with weights $(1,1,2)$ of the proper transform of P_2 on U_3, β_4 is the weighted blow-up with weights $(1,1,2)$ of the singular point of U_3 that is the quotient singularity of type $\frac{1}{3}(1,1,2)$ contained in the exceptional divisor of α_3, γ_2 is the weighted blow-up with weights $(1,1,2)$ of the proper transform of P_2 on U_{34}, γ_4 is the weighted blow-up with weights $(1,1,2)$ of the singular point of U_{23} that is the quotient singularity of type $\frac{1}{3}(1,1,2)$ contained in the exceptional divisor of the morphism β_3, and η is an elliptic fibration.

Proposition 12.1. *The assertion of Theorem 1.10 holds for $n = 13$.*

Proof. It follows from Theorem 3.3, Lemma 3.11 and Proposition 3.5 that $\mathbb{CS}\big(X, \frac{1}{k}\mathcal{M}\big) \subseteq \{P_2, P_3\}$.

Lemma 12.2. *The set $\mathbb{CS}\big(X, \frac{1}{k}\mathcal{M}\big)$ contains the point P_3.*

Proof. Suppose that $P_3 \notin \mathbb{CS}\big(X, \frac{1}{k}\mathcal{M}\big)$. Then $\mathbb{CS}\big(X, \frac{1}{k}\mathcal{M}\big) = \{P_2\}$. Let $\mathcal{D}_2$ be the proper transform of $\mathcal{M}$ on U_2. Then it follows from Theorem 2.2 that $\mathcal{D}_2 \sim_{\mathbb{Q}} -kK_{U_2}$, but the set $\mathbb{CS}\big(U_2, \frac{1}{k}\mathcal{D}_2\big)$ is non-empty by Lemma 2.1.

Let E be the exceptional divisor of the morphism α_2. Then E can be identified with a cone over the smooth rational curve in $\mathbb{P}^3$ of degree 3. Let Z be a subvariety of U_2 that is contained in the set $\mathbb{CS}\big(U_2, \frac{1}{k}\mathcal{D}_2\big)$. Then it follows from Lemmas 2.3 and 2.4 that Z is the vertex of the cone E.

The point Z is a quotient singularity of type $\frac{1}{2}(1,1,1)$ of U_2. Let $\pi\colon W \to U_2$ be the blow-up of Z with weights $(1,1,1)$ and S a sufficiently general surface in the pencil $|-K_W|$. Then the base locus of $|-K_W|$ consists of an irreducible curve Δ such that $\Delta^2 = -3/10$ on the surface S, but $\mathcal{B}|_S \sim_{\mathbb{Q}} k\Delta$, where $\mathcal{B}$ is the proper transform of $\mathcal{M}$ on W. We have $\mathcal{B}|_S = k\Delta$, which is impossible by Lemma 2.7.

Lemma 12.3. *The set $\mathbb{CS}\big(X, \frac{1}{k}\mathcal{M}\big)$ contains the point P_2.*

Proof. Suppose that $\mathbb{CS}\big(X, \frac{1}{k}\mathcal{M}\big)$ does not contain P_2. Then $\mathbb{CS}\big(X, \frac{1}{k}\mathcal{M}\big)$ consists of the point P_3. Let $\mathcal{D}_3$ be the proper transform of $\mathcal{M}$ on U_3. Then $\mathbb{CS}\big(U_3, \frac{1}{k}\mathcal{D}_3\big) \neq \varnothing$ by Lemma 2.1.

Let G be the exceptional divisor of the morphism α_3, P_4 the singular point of G that is a quotient singularity of type $\frac{1}{3}(1,1,2)$ on U_3, and P_5 the singular point of the surface G that is a quotient singularity of type $\frac{1}{2}(1,1,1)$ on U_3. Then $G \cong \mathbb{P}(1,2,3)$, and it follows from Lemma 2.3 that either $\mathbb{CS}\big(U_3, \frac{1}{k}\mathcal{D}_3\big) = \{P_4\}$, or $P_5 \in \mathbb{CS}\big(U_3, \frac{1}{k}\mathcal{D}_3\big)$.

Suppose that the set $\mathbb{CS}\big(U_3, \frac{1}{k}\mathcal{D}_3\big)$ contains P_5. Let $\pi\colon W \to U_3$ be the weighted blow-up of P_5 with weights $(1,1,1)$, $\mathcal{B}$ the proper transform of $\mathcal{M}$ on W, L the curve on the surface G that is contained in $\big|\mathcal{O}_{\mathbb{P}(1,2,3)}(1)\big|$, $\bar{L}$ the proper transform of L on W and S a general surface in $|-K_W|$. Then $\mathcal{B}|_S \sim_{\mathbb{Q}} k\Delta + k\bar{L}$ by Theorem 2.2 and the base locus of $|-K_W|$ consists of $\bar{L}$ and the irreducible curve Δ such that $\alpha \circ \pi(\Delta)$ is the base curve of the pencil $|-K_X|$. The equalities

$$\Delta^2 = -\frac{5}{6}, \qquad \bar{L}^2 = -\frac{4}{3}, \qquad \Delta \cdot \bar{L} = 1,$$

hold on the surface S, and this implies that the intersection form of Δ and $\bar{L}$ on S is negative definite. But this is impossible by Lemmas 2.7 and 2.9.

Hence, the set $\mathbb{CS}\big((U_3, \lambda\mathcal{D}_3\big)$ consists of the point P_4. Let D_{34} be the proper transform of $\mathcal{M}$ on U_{34}. Then $\mathbb{CS}\big(U_{34}, \frac{1}{k}\mathcal{D}_{34}\big) \neq \varnothing$ by Lemma 2.1 because the equivalence $D_{34} \sim_{\mathbb{Q}} -kK_{U_{34}}$ holds by Theorem 2.2.

Let E be the exceptional divisor of the morphism β_4 and P_6 the singular point of the surface E. Then $E \cong \mathbb{P}(1,1,3)$, P_6 is a quotient singularity of type $\frac{1}{2}(1,1,1)$ on U_{34}, and $P_6 \in \mathbb{CS}\big(U_{34}, \frac{1}{k}\mathcal{D}_{34}\big)$ by Lemma 2.3.

Let $\zeta\colon Z \to U_{34}$ be the weighted blow-up of P_6 with weights $(1,1,1)$, $\mathcal{H}$ the proper transform of $\mathcal{M}$ on Z and F a general surface in the pencil $|-K_Z|$. Then the base locus of $|-K_Z|$ consists of irreducible curves $\check{L}$ and $\check{\Delta}$ such that the equalities

$$\check{\Delta}^2 = -\frac{5}{6}, \qquad \check{L}^2 = -\frac{3}{2}, \qquad \check{\Delta} \cdot \check{L} = 1$$

hold on F. Thus, the intersection form of $\check{\Delta}$ and $\check{L}$ on F is negative definite, but $\mathcal{H}|_F \sim_{\mathbb{Q}} k\check{\Delta} + k\check{L}$ by Theorem 2.2, which is impossible by Lemmas 2.7 and 2.9.

We see that $\mathbb{CS}\big(X, \frac{1}{k}\mathcal{M}\big) = \{P_2, P_3\}$. Let $\mathcal{D}_{23}$ be the proper transform of $\mathcal{M}$ on U_{23}. Then $\mathcal{D}_{23} \sim_{\mathbb{Q}} -kK_{U_{23}}$ by Theorem 2.2, and $\mathbb{CS}\big(U_{23}, \frac{1}{k}\mathcal{D}_{23}\big) \neq \varnothing$ by Lemma 2.1.

Remark 12.4. The proof of Lemma 12.2 implies that the set $\mathbb{CS}\big(U_{23}, \frac{1}{k}\mathcal{D}_{23}\big)$ does not contain the singular point of the variety U_{23} that is contained in the exceptional divisor of β_2. But the proof of Lemma 12.3 implies that $\mathbb{CS}\big(U_{23}, \frac{1}{k}\mathcal{D}_{23}\big)$ does not contain the singular point of U_{23} that is a singularity of type $\frac{1}{2}(1,1,1)$ contained in the exceptional divisor of β_3.

Therefore, the set $\mathbb{CS}\big(U_{23}, \frac{1}{k}\mathcal{D}_{23}\big)$ contains the singular point of U_{23} that is a singularity of type $\frac{1}{3}(1,1,2)$ contained in the β_3-exceptional divisor. Let $\mathcal{P}$ be the proper transform of $\mathcal{M}$ on Y. Then $\mathcal{P} \sim_{\mathbb{Q}} -kK_Y$ by Theorem 2.2, which implies the assertion of the proposition.

§ 13. The case $n = 15$: a hypersurface of degree 12 in $\mathbb{P}(1,1,2,3,6)$

We use the notation and assumptions of § 3. Let $n = 15$. Then X is a general hypersurface in $\mathbb{P}(1,1,2,3,6)$ of degree 12, the equality $-K_X^3 = 1/3$ holds and the singularities of X consist of points P_1 and P_2 that are quotient singularities of type $\frac{1}{2}(1,1,1)$ and points P_3 and P_4 that are quotient singularities of type $\frac{1}{3}(1,1,2)$.

Proposition 13.1. *The assertion of Theorem 1.10 holds for* $n = 15$.

Proof. There is a commutative diagram

$$\begin{tikzcd}
 & Y \arrow[dl, "\gamma"'] \arrow[dr, "\delta"] \arrow[rrr, "\eta"] & & & \mathbb{P}(1,1,2), \\
U \arrow[dr, "\alpha"'] & & W \arrow[dl, "\beta"] & & \\
 & X \arrow[rrruu, dashed, "\psi"'] & & &
\end{tikzcd}$$

where ψ is the natural projection, α is the weighted blow-up of P_3 with weights $(1,1,2)$, β is the weighted blow-up of P_4 with weights $(1,1,2)$, γ is the weighted blow-up with weights $(1,1,2)$ of the proper transform of P_4 on U, δ is the weighted blow-up with weights $(1,1,2)$ of the proper transform of P_3 on W, and η is an elliptic fibration.

We have $\mathbb{CS}\big(X, \frac{1}{k}\mathcal{M}\big) \subseteq \{P_3, P_4\}$ by Theorem 3.3, Proposition 3.5 and Lemma 3.11.

Let $\mathcal{H}$ be the proper transform of $\mathcal{M}$ on Y. To complete the proof we must show that $\mathcal{H}$ lies in the fibres of the morphism η, which follows easily from the condition $\mathbb{CS}\big(X, \frac{1}{k}\mathcal{M}\big) = \{P_3, P_4\}$ by Theorem 2.2.

We may assume that $P_4 \notin \mathbb{CS}\big(X, \frac{1}{k}\mathcal{M}\big) \ni P_3$.

Let $\mathcal{B}$ be the proper transform of $\mathcal{M}$ on U and Q the singular point of U that is contained in the exceptional divisor of the morphism α. Then $\mathcal{B} \sim_{\mathbb{Q}} -kK_U$ by Theorem 2.2, and it follows from Lemmas 2.1 and 2.3 that the set $\mathbb{CS}\big(U, \frac{1}{k}\mathcal{B}\big)$ contains Q.

Let $\zeta\colon Z \to U$ be the weighted blow-up of Q with weights $(1,1,1)$, $\mathcal{D}$ the proper transform of $\mathcal{M}$ on the threefold Z and S a general surface in $|-K_Z|$. Then the base locus of the pencil $|-K_Z|$ consists of an irreducible curve Δ such that $\Delta^2<0$ on S, which is impossible by Lemmas 2.9 and 2.7 because $\mathcal{D}|_S \sim_{\mathbb{Q}} -k\Delta$ by Theorem 2.2.

§ 14. The case $n=16$: a hypersurface of degree 12 in $\mathbb{P}(1,1,2,4,5)$

We use the notation and assumptions of § 3. Let $n=16$. Then X is a general hypersurface in $\mathbb{P}(1,1,2,4,5)$ of degree 12, the equality $-K_X^3=3/10$ holds and the singularities of X consist of points P_1, P_2 and P_3 that are quotient singularities of type $\frac{1}{2}(1,1,1)$ and a point P_4 that is a quotient singularity of type $\frac{1}{5}(1,1,4)$.

There is a commutative diagram

$$\begin{tikzcd} U \arrow[d, "\alpha"'] & W \arrow[l, "\beta"'] & Y \arrow[l, "\gamma"'] \arrow[d, "\eta"] \\ X \arrow[rr, dashed, "\psi"'] & & \mathbb{P}(1,1,2), \end{tikzcd}$$

where ψ is the natural projection, α is the weighted blow-up of P_4 with weights $(1,1,4)$, β is the weighted blow-up with weights $(1,1,3)$ of the singular point of U that is contained in the exceptional divisor of α, γ is the weighted blow-up with weights $(1,1,2)$ of the singular point of W that is contained in the exceptional divisor of β, and η is an elliptic fibration.

Proposition 14.1. *The assertion of Theorem 1.10 holds for $n=16$.*

Proof. The divisors $-K_U$ and $-K_W$ are nef and big. The morphism η coincides with the map given by the linear system $|-2K_Y|$.

It follows from Theorem 3.3, Lemma 3.11 and Proposition 3.5 that $\mathbb{CS}\big(X,\frac{1}{k}\mathcal{M}\big)$ consists of the point P_4. Let G be the exceptional divisor of the morphism α, $\overline{P}_5$ the singular point of the surface G and $\mathcal{D}$ the proper transform of $\mathcal{M}$ on U. Then G is a cone over a smooth rational curve of degree 4 and $\overline{P}_5$ is a singularity of type $\frac{1}{4}(1,1,3)$ on U.

We see that $\mathcal{D}\sim_{\mathbb{Q}} -kK_U$ by Theorem 2.2 and $\mathbb{CS}\big(U,\frac{1}{k}\mathcal{D}\big)\neq\varnothing$ by Lemma 2.1.

Lemma 14.2. *The set $\mathbb{CS}\big(U,\frac{1}{k}\mathcal{D}\big)$ consists of the point $\overline{P}_5$.*

Proof. Suppose that $\mathbb{CS}\big(U,\frac{1}{k}\mathcal{D}\big)$ contains a subvariety C of U that is different from the point $\overline{P}_5$. Then it follows from Lemma 2.3 that C is a ruling of the cone G, which is impossible by Corollary 2.8.

Let $\mathcal{H}$ be the proper transform of $\mathcal{M}$ on W. Then it follows from Theorem 2.2 and Lemmas 2.1 and 2.3 that the set $\mathbb{CS}\big(W,\frac{1}{k}\mathcal{H}\big)$ contains the singular point of W that is contained in the exceptional divisor of β. Therefore, the proper transform of $\mathcal{M}$ on Y lies in the fibres of the morphism η by Theorem 2.2, which completes the proof.

§ 15. The case $n = 17$: a hypersurface of degree 12 in $\mathbb{P}(1,1,3,4,4)$

We use the notation and assumptions of § 3. Let $n = 17$. Then X is a general hypersurface in $\mathbb{P}(1,1,3,4,4)$ of degree 12 whose singularities consist of points P_1, P_2 and P_3 that are singularities of type $\frac{1}{4}(1,1,3)$. There is a commutative diagram

$$
\begin{tikzcd}
 & Z \arrow[dl, "\pi"'] \arrow[dr, "\omega"] & \\
X \arrow[rr, dashed, "\psi"'] & & \mathbb{P}(1,1,3),
\end{tikzcd}
$$

where ψ is a projection, π is the composite of the weighted blow-ups of P_1, P_2 and P_3 with weights $(1,1,3)$, and ω is an elliptic fibration.

It follows from Example 1.4 that there is a commutative diagram

$$
\begin{tikzcd}
 & Y_i \arrow[dl, "\beta_i"'] \arrow[ddrr, "\eta_i"] & & \\
U_i \arrow[dr, "\alpha_i"'] & & & \\
 & X \arrow[rr, dashed, "\xi_i"'] & & \mathbb{P}(1,1,4),
\end{tikzcd}
$$

where ξ_i is a projection, α_i is the blow-up of P_i with weights $(1,1,3)$, β_i is the weighted blow-up with weights $(1,1,2)$ of the singular point of U_i that is contained in the exceptional divisor of the morphism α_i, and η_i is an elliptic fibration.

The assertion of Theorem 1.10 for $n = 17$ follows from the next result.

Proposition 15.1. *Either there is a commutative diagram*

$$
\begin{tikzcd}
X \arrow[r, dashed, "\rho"] \arrow[d, dashed, "\psi"'] & V \arrow[d, "\nu"] \\
\mathbb{P}(1,1,3) \arrow[r, dashed, "\varphi"'] & \mathbb{P}^2
\end{tikzcd}
\tag{15.1}
$$

or there is a commutative diagram

$$
\begin{tikzcd}
X \arrow[r, dashed, "\sigma"] \arrow[d, dashed, "\xi_i"'] & X \arrow[r, dashed, "\rho"] & V \arrow[d, "\nu"] \\
\mathbb{P}(1,1,4) \arrow[rr, dashed, "\upsilon"'] & & \mathbb{P}^2
\end{tikzcd}
\tag{15.2}
$$

for some $i \in \{1,2,3\}$, where φ, σ and υ are birational maps.

Proof. It follows from Theorem 3.3 and Lemma 3.11 that $\mathbb{CS}\big(X, \frac{1}{k}\mathcal{M}\big) \subseteq \{P_1, P_2, P_3\}$.

Remark 15.2. It follows from Theorem 2.2 that the commutative diagram (15.1) exists in the case when the set $\mathbb{CS}\big(X, \frac{1}{k}\mathcal{M}\big)$ contains the points P_1, P_2 and P_3.

Hence, we may assume that $P_1 \in \mathbb{CS}\big(X, \frac{1}{k}\mathcal{M}\big) \not\ni P_3$.

Remark 15.3. The divisor $-K_{U_1}$ is nef and big.

Let $\mathcal{D}$ be the proper transform of $\mathcal{M}$ on U_1, $\bar{P}_2$ the proper transform of P_2 on U_1 and $\bar{P}_4$ the singular point of U_1 that is contained in the exceptional divisor of α_1. Then $\mathcal{D} \sim_{\mathbb{Q}} -kK_{U_1}$ by Theorem 2.2, and it follows from Lemma 2.1 that the set $\mathbb{CS}\big(U_1, \frac{1}{k}\mathcal{D}\big)$ is non-empty.

Remark 15.4. It follows easily from Theorem 2.2 that the commutative diagram (15.2) exists in the case when $\mathbb{CS}\big(U_1, \frac{1}{k}\mathcal{D}\big)$ contains $\bar{P}_4$.

Therefore, we may assume that $\mathbb{CS}\big(U_1, \frac{1}{k}\mathcal{D}\big)$ does not contain $\bar{P}_4$. Hence, it follows from the proof of Lemma 9.5 that $\mathbb{CS}\big(U_1, \frac{1}{k}\mathcal{D}\big)$ does not contain a subvariety of U_1 that is contained in the exceptional divisor of α_1. Thus, the set $\mathbb{CS}\big(U_1, \frac{1}{k}\mathcal{D}\big)$ contains $\bar{P}_2$.

Let $\gamma\colon W \to U_1$ be the weighted blow-up of $\bar{P}_2$ with weights $(1,1,3)$, $\mathcal{B}$ the proper transform of $\mathcal{M}$ on W, S a general surface in $|-K_W|$, and C the base curve of the pencil $|-K_W|$. Then $\mathcal{B} \sim_{\mathbb{Q}} -kK_W$ by Theorem 2.2, the curve C is irreducible, the inequality $C^2 = -1/24$ holds on the normal surface S, and the equivalence $\mathcal{B}|_S \sim_{\mathbb{Q}} kC$ holds, which is impossible by Lemmas 2.9 and 2.7. The assertion is proved.

§ 16. The case $n = 19$: a hypersurface of degree 12 in $\mathbb{P}(1,2,3,3,4)$

We use the notation and assumptions of § 3. Let $n = 19$. Then X is a general hypersurface in $\mathbb{P}(1,2,3,3,4)$ of degree 12, the singularities of X consist of points O_1, O_2, O_3 that are quotient singularities of type $\frac{1}{2}(1,1,1)$, and points P_1, P_2 P_3, P_4 that are quotient singularities of type $\frac{1}{3}(1,1,2)$. The equality $-K_X^3 = 1/6$ holds.

It follows from Example 1.7 that for every $i \in \{1,2,3,4\}$ there is a commutative diagram

$$\begin{tikzcd} & U_i \arrow[dl, "\pi_i"'] \arrow[dr, "\eta_i"] & \\ X \arrow[rr, dashed, "\xi_i"'] & & \mathbb{P}(1,2,3), \end{tikzcd}$$

where ξ_i is a projection, π_i is the blow-up of P_i with weights $(1,1,2)$, and η_i is an elliptic fibration.

Proposition 16.1. *There is a commutative diagram*

$$\begin{tikzcd} X \arrow[r, dashed, "\rho"] \arrow[d, dashed, "\xi_i"'] & V \arrow[d, "\nu"] \\ \mathbb{P}(1,2,3) \arrow[r, dashed, "\sigma"'] & \mathbb{P}^2 \end{tikzcd} \tag{16.1}$$

for some $i \in \{1,2,3,4\}$, where σ is a birational map.

Proof. It follows from Theorem 3.3, Corollary 3.4, Theorem 2.2 and Proposition 3.5 that the set $\mathbb{CS}\bigl(X, \frac{1}{k}\mathcal{M}\bigr)$ contains the point P_i for some $i \in \{1,2,3,4\}$.

Let $\mathcal{D}$ be the proper transform of $\mathcal{M}$ on U_i. Then $\mathcal{D} \sim_{\mathbb{Q}} -kK_{U_i}$ by Theorem 2.2, which implies the existence of the commutative diagram (16.1).

The assertion of Proposition 16.1 implies the assertion of Theorem 1.10 for $n=19$.

§ 17. The case $n = 20$: a hypersurface of degree 13 in $\mathbb{P}(1,1,3,4,5)$

We use the notation and assumptions of § 3. Let $n = 20$. Then X is a general hypersurface in $\mathbb{P}(1,1,3,4,5)$ of degree 13, the equality $-K_X^3 = 13/60$ holds, and the singularities of X consist of a point P_1 that is a singularity of type $\frac{1}{3}(1,1,2)$, a point P_2 that is a singularity of type $\frac{1}{4}(1,1,3)$, and a point P_3 that is a singularity of type $\frac{1}{5}(1,1,4)$. There is a commutative diagram

$$
\begin{tikzcd}
 & & Y \arrow[dl, "\gamma_4"'] \arrow[dr, "\gamma_2"] \arrow[drrr, "\eta"] & & & \\
 & U_{23} \arrow[dl, "\beta_3"'] \arrow[dr, "\beta_2"] & & U_{34} \arrow[dl, "\beta_4"'] & & \mathbb{P}(1,1,3), \\
U_2 \arrow[dr, "\alpha_2"'] & & U_3 \arrow[dl, "\alpha_3"'] & & & \\
 & X \arrow[uurrrr, dashed, "\psi"'] & & & &
\end{tikzcd}
$$

where ψ is the natural projection, α_2 is the weighted blow-up of P_2 with weights $(1,1,3)$, α_3 is the weighted blow-up of P_3 with weights $(1,1,4)$, β_3 is the weighted blow-up with weights $(1,1,4)$ of the proper transform of P_3 on U_2, β_2 is the weighted blow-up with weights $(1,1,3)$ of the proper transform of P_2 on U_3, β_4 is the weighted blow-up with weights $(1,1,3)$ of the singular point of U_3 that is contained in the exceptional divisor of α_3, γ_2 is the weighted blow-up with weights $(1,1,3)$ of the proper transform of P_2 on U_{34}, γ_4 is the weighted blow-up with weights $(1,1,3)$ of the singular point of U_{23} that is contained in the exceptional divisor of the morphism β_3, and η is an elliptic fibration.

It follows from Example 1.7 that there is a commutative diagram

$$
\begin{tikzcd}
 & Y \arrow[dl, "\gamma_3"'] \arrow[dr, "\gamma_1"] \arrow[drrrr, "\omega"] & & & & \\
U_1 \arrow[dr, "\alpha_1"'] & & U_3 \arrow[dl, "\alpha_3"] & & & \mathbb{P}(1,1,4), \\
 & X \arrow[urrrr, dashed, "\xi"'] & & & &
\end{tikzcd}
$$

where ξ is a projection, α_1 is the blow-up of P_1 with weights $(1,1,2)$, α_3 is the weighted blow-up of P_3 with weights $(1,1,4)$, γ_3 is the weighted blow-up with weights $(1,1,4)$ of the proper transform of P_3 on U_1, γ_1 is the weighted blow-up with weights $(1,1,2)$ of the proper transform of P_1 on U_3, and ω is an elliptic fibration.

Proposition 17.1. *Either there is a commutative diagram*

$$\begin{tikzcd} X \arrow[r, dashed, "\sigma"] \arrow[d, dashed, "\psi"'] & X \arrow[r, dashed, "\rho"] & V \arrow[d, "\nu"] \\ \mathbb{P}(1,1,3) \arrow[rr, dashed, "\upsilon"'] & & \mathbb{P}^2 \end{tikzcd} \tag{17.1}$$

or there is a commutative diagram

$$\begin{tikzcd} X \arrow[r, dashed, "\sigma"] \arrow[d, dashed, "\xi"'] & X \arrow[r, dashed, "\rho"] & V \arrow[d, "\nu"] \\ \mathbb{P}(1,1,4) \arrow[rr, dashed, "\zeta"'] & & \mathbb{P}^2 \end{tikzcd} \tag{17.2}$$

where σ, υ and ζ are birational maps.

Proof. It follows from Theorem 3.3 and Lemma 3.11 that $\mathbb{CS}\big(X, \frac{1}{k}\mathcal{M}\big) \subseteq \{P_1, P_2, P_3\}$.

Lemma 17.2. *Suppose that $\{P_1, P_3\} \subseteq \mathbb{CS}\big(X, \frac{1}{k}\mathcal{M}\big)$. Then the commutative diagram (17.2) exists.*

Proof. Let $\mathcal{B}$ be the proper transform of $\mathcal{M}$ on Y and S a general surface in $\mathcal{B}$. Then $S \sim_{\mathbb{Q}} -kK_Y$ by Theorem 2.2. We see that $S \cdot C = 0$, where C is a generic fibre of ω, which implies the existence of the commutative diagram (17.2).

Lemma 17.3. *The set $\mathbb{CS}\big(X, \frac{1}{k}\mathcal{M}\big)$ does not contain the set $\{P_1, P_2\}$.*

Proof. Suppose that $\{P_1, P_3\} \subseteq \mathbb{CS}\big(X, \frac{1}{k}\mathcal{M}\big)$. Let $\pi\colon W \to U_1$ be the weighted blow-up with weights $(1,1,3)$ of the proper transform of P_2 on U_1 and $\mathcal{B}$ the proper transform of $\mathcal{M}$ on W. Then $\mathcal{B} \sim_{\mathbb{Q}} -kK_W$ by Theorem 2.2.

The linear system $|-K_W|$ is a pencil whose base locus is the irreducible curve Δ such that $\alpha_1 \circ \pi(C)$ is the curve cut out on X by the equations $z = y = 0$.

Let S be a sufficiently general surface in the linear system $|-K_W|$, $\bar{P}_3$ the proper transform of the singular point P_3 on W and $\bar{P}_5$ and $\bar{P}_6$ other singular points of W such that $\alpha_1 \circ \pi(\bar{P}_5) = P_1$ and $\alpha_1 \circ \pi(\bar{P}_6) = P_2$. Then $\bar{P}_5$ is a quotient singularity of type $\frac{1}{2}(1,1,1)$ on W, $\bar{P}_6$ is a quotient singularity of type $\frac{1}{3}(1,1,2)$ on W, the surface S is smooth outside the points $\bar{P}_3$, $\bar{P}_5$ and $\bar{P}_6$ and the singularities of S at the points $\bar{P}_3$, $\bar{P}_5$ and $\bar{P}_6$ are Du Val singularities of types $\mathbb{A}_4$, $\mathbb{A}_1$ and $\mathbb{A}_2$, respectively.

The equality $\Delta^2 = -1/30$ holds on S. But the equivalence $\mathcal{B}|_S \sim_{\mathbb{Q}} k\Delta$ holds, which implies that $\mathcal{B}|_S = k\Delta$. We can now easily obtain a contradiction using Lemmas 2.7 and 2.9.

Lemma 17.4. *The set $\mathbb{CS}\big(X, \frac{1}{k}\mathcal{M}\big)$ does not consist only of the point P_i.*

Proof. Suppose that $\mathbb{CS}\big(X, \frac{1}{k}\mathcal{M}\big) = \{P_i\}$. Let $\mathcal{D}$ be the proper transform of $\mathcal{M}$ on U_i. Then the equivalence $\mathcal{D} \sim_{\mathbb{Q}} -kK_{U_i}$ holds by Theorem 2.2. Moreover, it follows from Lemma 2.1 and the proof of Lemma 9.5 that the set $\mathbb{CS}\big(U_i, \frac{1}{k}\mathcal{D}\big)$ contains the singular point of U_i that is the singular point of the exceptional divisor of the birational morphism α_i.

Let $\pi\colon W \to U_i$ be the weighted blow-up with weights $(1,1,i)$ of the singular point of the variety U_i that is contained in the exceptional divisor of the morphism α_i, and let $\mathcal{B}$ be the proper transform of $\mathcal{M}$ on W. Then $\mathcal{B} \sim_{\mathbb{Q}} -kK_W$ by Theorem 2.2.

Let S be a sufficiently general surface in the pencil $|-K_W|$ and Δ the unique base curve of $|-K_W|$. Then S is normal but the curve Δ is irreducible, rational and smooth. Moreover, simple computations imply that

$$\Delta^2 = \begin{cases} -9/20 & \text{if } i = 1, \\ -1/30 & \text{if } i = 2, \\ 0 & \text{if } i = 3, \end{cases}$$

on S. However, we have the equivalence $\mathcal{B}|_S \sim_{\mathbb{Q}} k\Delta$, which implies (see the proof of Lemma 17.3) that the curve $\alpha_i \circ \pi(\Delta)$ is contained in the set $\mathbb{CS}\big(X, \frac{1}{k}\mathcal{M}\big)$ if $i \neq 3$. It follows that $i = 3$.

Let G be the exceptional divisor of α_3 and $\bar{P}_4$ the singular point of U_3 that is contained in G. Then $\bar{P}_4$ is a quotient singularity of type $\frac{1}{4}(1,1,3)$ on U_3.

It follows from Lemmas 2.3 and 2.4 that the set $\mathbb{CS}\big(U_3, \frac{1}{k}\mathcal{D}\big)$ consists of the singular point $\bar{P}_4$.

The variety W is the variety U_{34} and the birational morphism π is the morphism β_4. Thus, the divisor $-K_W$ is nef and big. Therefore, it follows from Lemmas 2.1 and 2.3 that the set $\mathbb{CS}\big(W, \frac{1}{k}\mathcal{B}\big)$ contains the singular point of W that is contained in the exceptional divisor of π.

Let $\mu\colon Z \to W$ be the weighted blow-up with weights $(1,1,2)$ of the singular point of W that is contained in the exceptional divisor of π and let $\mathcal{P}$ be the proper transform of $\mathcal{M}$ on Z. Then the equivalence $\mathcal{P} \sim_{\mathbb{Q}} -kK_Z$ holds by Theorem 2.2. Let F be a general surface and Γ the unique base curve of the pencil $|-K_Z|$. Then F is irreducible and normal. But Γ is irreducible, rational and smooth. The equality $\Gamma^2 = -1/24$ holds on F. But $\mathcal{P}|_S \sim_{\mathbb{Q}} k\Gamma$, which easily leads to a contradiction using Lemmas 2.7 and 2.9.

It follows from Theorem 3.3 and Lemma 3.11 that we may assume that $\mathbb{CS}\big(X, \frac{1}{k}\mathcal{M}\big) = \{P_2, P_3\}$.

Let $\mathcal{D}$ be the proper transform of $\mathcal{M}$ on U_{23}. Then $\mathcal{D} \sim_{\mathbb{Q}} -kK_{U_{23}}$ by Theorem 2.2. It now follows from Lemmas 2.1 and 2.3 that the set $\mathbb{CS}\big(U_{23}, \frac{1}{k}\mathcal{D}\big)$ contains either the singular point of U_{23} that is contained in the exceptional divisor of the morphism β_3 or the singular point of U_{23} that is contained in the exceptional divisor of β_2.

Lemma 17.5. *The set $\mathbb{CS}\big(U_{23}, \frac{1}{k}\mathcal{D}\big)$ does not contain the singular point of U_{23} that is contained in the exceptional divisor of β_2.*

Proof. Let E be the exceptional divisor of β_2 and $\bar{P}_6$ the singular point of the surface E. Then $\bar{P}_6$ is a quotient singularity of type $\frac{1}{3}(1,1,2)$ of U_{23}. Suppose that the set $\mathbb{CS}\big(U_{23}, \frac{1}{k}\mathcal{D}\big)$ contains $\bar{P}_6$.

Let $\pi\colon W \to U_{23}$ be the weighted blow-up of $\bar{P}_6$ with weights $(1,1,2)$, $\mathcal{B}$ the proper transform of $\mathcal{M}$ on W, S a general surface in $|-K_W|$ and Δ the base curve of $|-K_W|$. Then S is normal, Δ is irreducible, and $\mathcal{B}|_S \sim_{\mathbb{Q}} k\Delta$ by Theorem 2.2. But the equality $\Delta^2 = -1/24$ holds on S, which is impossible by Lemmas 2.9 and 2.7.

Hence, the set $\mathbb{CS}\big(U_{23}, \frac{1}{k}\mathcal{D}\big)$ contains the singular point of U_{23} that is contained in the exceptional divisor of β_3. Thus, the existence of the commutative diagram (17.2) follows easily from Theorem 2.2. The proposition is proved.

§ 18. The case $n = 23$: a hypersurface of degree 14 in $\mathbb{P}(1,2,3,4,5)$

We use the notation and assumptions of § 3. Let $n = 23$. Then X is a general hypersurface in $\mathbb{P}(1,2,3,4,5)$ of degree 14, the singularities of X consist of points P_1, P_2 and P_3 that are quotient singularities of type $\frac{1}{2}(1,1,1)$, a point P_4 that is a quotient singularity of type $\frac{1}{3}(1,1,2)$, a point P_5 that is a quotient singularity of type $\frac{1}{4}(1,1,3)$ and a point P_6 that is quotient singularity of type $\frac{1}{5}(1,2,3)$. The equality $-K_X^3 = 7/60$ holds.

Proposition 18.1. *The assertion of Theorem 1.10 holds for $n = 23$.*

Proof. It follows from Theorem 3.3, Lemma 3.11 and Proposition 3.5 that $\mathbb{CS}\big(X, \frac{1}{k}\mathcal{M}\big) \subseteq \{P_5, P_6\}$. There is a commutative diagram

$$\begin{tikzcd}
 & Y \arrow[dl, "\beta_6"'] \arrow[dr, "\beta_5"] \arrow[drrr, "\eta"] & & & \\
U_5 \arrow[dr, "\alpha_5"'] & & U_6 \arrow[dl, "\alpha_6"] & & \mathbb{P}(1,2,3), \\
 & X \arrow[urrr, dashed, "\psi"'] & & &
\end{tikzcd}$$

where ψ is the natural projection, α_5 is the weighted blow-up of the singular point P_5 with weights $(1,1,3)$, α_6 is the weighted blow-up of P_6 with weights $(1,2,3)$, β_5 is the weighted blow-up with weights $(1,1,3)$ of the proper transform of P_5 on U_6, β_6 is the weighted blow-up with weights $(1,2,3)$ of the proper transform of P_6 on U_5, and η is an elliptic fibration.

Lemma 18.2. *The set $\mathbb{CS}\big(X, \frac{1}{k}\mathcal{M}\big)$ contains the point P_6.*

Proof. Suppose that $\mathbb{CS}\big(X, \frac{1}{k}\mathcal{M}\big)$ does not contain P_6. Then the set $\mathbb{CS}\big(X, \frac{1}{k}\mathcal{M}\big)$ consists of the point P_5. Let $\mathcal{D}_5$ be the proper transform of $\mathcal{M}$ on U_5. Then $\mathcal{D}_5 \sim_{\mathbb{Q}} -kK_{U_5}$ by Theorem 2.2, but the set $\mathbb{CS}\big(U_5, \frac{1}{k}\mathcal{D}_5\big)$ is non-empty by Lemma 2.1.

Let $\bar{P}_7$ be the singular point of U_5 that is contained in the exceptional divisor of the morphism α_5. Then $\bar{P}_7$ is a quotient singularity of type $\frac{1}{3}(1,1,2)$ on U_5, and it follows from Lemma 2.3 that $\mathbb{CS}\big(U_5, \frac{1}{k}\mathcal{D}_5\big)$ contains $\bar{P}_7$. Let $\pi\colon U \to U_5$ be the weighted blow-up of $\bar{P}_7$ with weights $(1,1,3)$. Then the linear system $|-2K_U|$ is a proper transform of the pencil $|-2K_X|$, and the base locus of $|-2K_U|$ consists of a single irreducible curve Z such that $\alpha_5 \circ \pi(Z)$ is the unique base curve of $|-2K_X|$.

Let S be a sufficiently general surface in $|-2K_U|$. Then S is normal and contains the curve Z, and the inequality $Z^2 < 0$ holds on S because $-K_U^3 < 0$. However, the equivalence $\mathcal{B}|_S \sim_{\mathbb{Q}} kZ$ holds by Theorem 2.2, where $\mathcal{B}$ is the proper transform of $\mathcal{M}$ on W. It follows from Lemma 2.9 that

$$\operatorname{Supp}(S) \cap \operatorname{Supp}(D) = \operatorname{Supp}(Z),$$

where D is a sufficiently general surface in the linear system $\mathcal{B}$. But this is impossible by Lemma 2.7 because $\mathcal{B}$ is not composed of a pencil.

It follows easily from Theorem 2.2 that the assertion of Theorem 1.10 holds for X whenever $\mathbb{CS}\big(X, \frac{1}{k}\mathcal{M}\big)$ contains P_5 and P_6. Thus, we may assume that $P_5 \notin \mathbb{CS}\big(X, \frac{1}{k}\mathcal{M}\big)$. We claim that this assumption leads to a contradiction.

Let $\mathcal{D}_6$ be the proper transform of $\mathcal{M}$ on U_6. Then $\mathcal{D}_6 \sim_{\mathbb{Q}} -kK_{U_6}$ by Theorem 2.2, which implies that the set $\mathbb{CS}\big(U_6, \frac{1}{k}\mathcal{D}_6\big)$ is non-empty by Lemma 2.1. Let $\bar{P}_7$ and $\bar{P}_8$ be the singular points of U_6 that are quotient singularities of types $\frac{1}{3}(1,1,2)$ and $\frac{1}{2}(1,1,1)$, respectively, contained in the exceptional divisor of α_6.

Lemma 18.3. *The set $\mathbb{CS}\big(U_6, \frac{1}{k}\mathcal{D}_6\big)$ does not contain the point $\bar{P}_7$.*

Proof. Suppose that $\bar{P}_7 \in \mathbb{CS}\big(U_6, \frac{1}{k}\mathcal{D}_6\big)$. Let $\gamma\colon W \to U_6$ be the weighted blow-up of $\bar{P}_7$ with weights $(1,1,2)$ and let S be a sufficiently general surface in $|-2K_W|$. Then S is irreducible and normal, the linear system $|-2K_W|$ is the proper transform of the pencil $|-2K_X|$, and the base locus of $|-2K_W|$ consists of the irreducible curve Δ such that the equality

$$\Delta^2 = -2K_W^3 = -\frac{1}{6}$$

holds on S. Moreover, the equivalence $\mathcal{B}|_S \sim_{\mathbb{Q}} k\Delta$ holds, where $\mathcal{B}$ is the proper transform of $\mathcal{M}$ on W. But this is impossible by Lemmas 2.9 and 2.7.

Therefore, the set $\mathbb{CS}\big(U_6, \frac{1}{k}\mathcal{D}_6\big)$ contains the point $\bar{P}_8$ by Lemma 2.3.

Remark 18.4. The linear system $|-3K_{U_6}|$ is the proper transform of the linear system $|-3K_X|$ and the base locus of $|-3K_{U_6}|$ consists of the irreducible fibre of $\psi \circ \alpha_6$ that passes through the singular point $\bar{P}_8$.

Let $\pi\colon U \to U_6$ be the weighted blow-up of the point $\bar{P}_8$ with weights $(1,1,1)$, F the exceptional divisor of the morphism π, $\mathcal{D}$ the proper transform of $\mathcal{M}$ on U and $\mathcal{H}$ the proper transform of $|-3K_{U_6}|$ on U. Then $\mathcal{D} \sim_{\mathbb{Q}} -kK_U$ by Theorem 2.2. Simple computations imply that

$$\mathcal{H} \sim_{\mathbb{Q}} \pi^*(-3K_{U_6}) - \frac{1}{2}F,$$

and the base locus of $\mathcal{H}$ consists of the irreducible curve Z such that $\alpha_6 \circ \pi(Z)$ is the base curve of $|-3K_X|$. Moreover, the equality $S \cdot Z = 1/12$ holds, where S is a general surface in $\mathcal{H}$.

Let D_1 and D_2 be general surfaces in the linear system $\mathcal{D}$. Then $S \cdot D_1 \cdot D_2 = -k^2/4$, which is a contradiction. The assertion is proved.

§ 19. The case $n = 25$: a hypersurface of degree 15 in $\mathbb{P}(1,1,3,4,7)$

We use the notation and assumptions of § 3. Let $n = 25$. Then X is a general hypersurface in $\mathbb{P}(1,1,3,4,7)$ of degree 15, the equality $-K_X^3 = 5/28$ holds, and the singularities of X consist of a point P_1 that is a quotient singularity of type $\frac{1}{4}(1,1,3)$ and a point P_2 that is a quotient singularity of type $\frac{1}{7}(1,3,4)$.

Proposition 19.1. *The assertion of Theorem 1.10 holds for $n = 25$.*

There is a commutative diagram

$$\begin{tikzcd}
 & & Y \arrow[dl, "\gamma_3"'] \arrow[dr, "\gamma_1"] \arrow[drr, "\eta"] & & \\
 & U_{12} \arrow[dl, "\beta_2"'] \arrow[dr, "\beta_1"] & & U_{23} \arrow[dl, "\beta_3"] & \mathbb{P}(1,1,3), \\
U_1 \arrow[dr, "\alpha_1"'] & & U_2 \arrow[dl, "\alpha_2"'] & & \\
 & X \arrow[uurrr, dashed, "\psi"'] & & &
\end{tikzcd}$$

where ψ is the natural projection, α_1 is the weighted blow-up of P_1 with weights $(1,1,3)$, α_2 is the weighted blow-up of P_2 with weights $(1,3,4)$, β_2 is the weighted blow-up with weights $(1,3,4)$ of the proper transform of P_2 on U_1, β_1 is the weighted blow-up with weights $(1,1,3)$ of the proper transform of P_1 on U_2, β_3 is the weighted blow-up with weights $(1,1,3)$ of the singular point of U_2 that is a quotient singularity of type $\frac{1}{4}(1,1,3)$ contained in the exceptional divisor of the morphism α_2, γ_1 is the weighted blow-up with weights $(1,1,3)$ of the proper transform of P_1 on U_{23}, γ_3 is the weighted blow-up with weights $(1,1,3)$ of the point of U_{12} that is a quotient singularity of type $\frac{1}{4}(1,1,3)$ contained in the exceptional divisor of β_2, and η is an elliptic fibration.

Remark 19.2. The divisors $-K_{U_1}$, $-K_{U_2}$, $-K_{U_{12}}$ and $-K_{U_{23}}$ are nef and big.

Proof of Proposition 19.1. It follows from Theorem 3.3, Lemma 3.11 and Proposition 3.5 that $\mathbb{CS}\big(X, \frac{1}{k}\mathcal{M}\big) \subseteq \{P_1, P_2\}$.

Lemma 19.3. *The set* $\mathbb{CS}\big(X, \frac{1}{k}\mathcal{M}\big)$ *contains the point* P_2.

Proof. Suppose that $\mathbb{CS}\big(X, \frac{1}{k}\mathcal{M}\big)$ does not contain P_2. Let $\mathcal{D}_1$ be the proper transform of $\mathcal{M}$ on U_1. Then $\mathbb{CS}\big(X, \frac{1}{k}\mathcal{M}\big) = \{P_1\}$, and $\mathbb{CS}\big(U_1, \frac{1}{k}\mathcal{D}_1\big) \neq \varnothing$ by Lemma 2.1 because the equivalence $\mathcal{D}_1 \sim_{\mathbb{Q}} -kK_{U_1}$ holds by Theorem 2.2.

Let P_5 be the singular point of U_1 that is contained in the exceptional divisor of α_1. Then P_5 is a quotient singularity of type $\frac{1}{3}(1,1,2)$ on U_1, and it follows from Lemma 2.3 that $\mathbb{CS}\big(U_1, \frac{1}{k}\mathcal{D}_1\big)$ contains it.

Let $\pi\colon W \to U_1$ be the blow-up of P_5 with weights $(1,1,2)$ and let S be a sufficiently general surface in the pencil $|-K_W|$. Then S is irreducible and normal and the base locus of the pencil $|-K_W|$ consists of the irreducible curve Δ such that $\Delta^2 = -1/14$ on S. But $\mathcal{B}|_S \sim_{\mathbb{Q}} k\Delta$, where $\mathcal{B}$ is the proper transform of $\mathcal{M}$ on W. Therefore, we have $\mathcal{B}|_S = k\Delta$, which implies that

$$\mathrm{Supp}(S) \cap \mathrm{Supp}(D) = \mathrm{Supp}(\Delta),$$

where D is a general surface in $\mathcal{B}$. But this contradicts Lemma 2.7.

Let G be the exceptional divisor of α_2, $\mathcal{D}_2$ the proper transform of $\mathcal{M}$ on U_2, $\overline{P}_1$ the proper transform of P_1 on U_2, and $\overline{P}_3$ and $\overline{P}_4$ the singular points of U_2 that are quotient singularities of types $\frac{1}{4}(1,1,3)$ and $\frac{1}{3}(1,1,2)$, respectively, contained in the exceptional divisor G. Then $G \cong \mathbb{P}(1,3,4)$, $\bar{P}_3$ and $\bar{P}_4$ are singular points of G, and $\mathcal{D}_2 \sim_{\mathbb{Q}} -kK_{U_2}$ by Theorem 2.2. Hence, the set $\mathbb{CS}\big(U_2, \frac{1}{k}\mathcal{D}_2\big)$ is non-empty by Lemma 2.1. Moreover, the proof of Lemma 19.3 implies that $\mathbb{CS}\big(U_2, \frac{1}{k}\mathcal{D}_2\big) \neq \{\bar{P}_1\}$.

Lemma 19.4. *The set $\mathbb{CS}\big(U_2, \frac{1}{k}\mathcal{D}_2\big)$ does not contain both $\bar{P}_3$ and $\bar{P}_4$.*

Proof. Suppose that $\{\bar{P}_3, \bar{P}_4\} \subseteq \mathbb{CS}\big(U_2, \frac{1}{k}\mathcal{D}_2\big)$. Let $\pi\colon W \to U_2$ be the composite of the weighted blow-ups of $\bar{P}_3$ and $\bar{P}_4$ with weights $(1,1,3)$ and $(1,1,2)$, respectively, and let $\mathcal{B}$ be the proper transform of $\mathcal{M}$ on W. Then $\mathcal{B} \sim_{\mathbb{Q}} -kK_W$ by Theorem 2.2.

Let S be a general surface in the pencil $|-K_W|$. Then S is irreducible and normal, but the base locus of $|-K_W|$ consists of the irreducible curves C and L such that the curve $\alpha_2 \circ \pi(C)$ is the unique base curve of $|-K_X|$, the curve $\pi(L)$ is contained in G, and $\pi(L)$ is the unique curve in $\big|\mathcal{O}_{\mathbb{P}(1,\,3,\,4)}(1)\big|$. We have

$$\mathcal{B}|_S \sim_{\mathbb{Q}} -kK_W|_S \sim_{\mathbb{Q}} kS|_S = kC + kL,$$

but the intersection form of L and C on S is negative definite. Now applying Lemma 2.9, we obtain a contradiction to Lemma 2.7.

It follows from Lemma 2.3 that $\mathbb{CS}\big(U_2, \frac{1}{k}\mathcal{D}_2\big) \subsetneq \{\bar{P}_1, \bar{P}_3, \bar{P}_4\}$.

Lemma 19.5. *The set $\mathbb{CS}\big(U_2, \frac{1}{k}\mathcal{D}_2\big)$ contains either $\bar{P}_1$ or $\bar{P}_4$.*

Proof. Suppose that the set $\mathbb{CS}\big(U_2, \frac{1}{k}\mathcal{D}_2\big)$ does not contain either of the singular points $\bar{P}_1$, $\bar{P}_4$. Then $\mathbb{CS}\big(U_2, \frac{1}{k}\mathcal{D}_2\big)$ consists of the point $\bar{P}_3$.

The linear system $|-K_{U_2}|$ is the proper transform of $|-K_X|$ and the base locus of $|-K_{U_2}|$ consists of the irreducible curves L and Δ such that $\alpha_2(\Delta)$ is the unique base curve of $|-K_X|$, L is contained in the divisor G and L is the unique curve of the linear system $\big|\mathcal{O}_{\mathbb{P}(1,\,3,\,4)}(1)\big|$.

Let $\widetilde{P}_6$ be the singular point of the variety U_{23} that is contained in the exceptional divisor of β_3 and let $\mathcal{D}_{23}$ be the proper transform of $\mathcal{M}$ on U_{23}. Then $\mathcal{D}_{23} \sim_{\mathbb{Q}} -kK_{U_{23}}$ by Theorem 2.2, and it follows from Lemmas 2.1 and 2.3 that the set $\mathbb{CS}\big(U_{23}, \frac{1}{k}\mathcal{D}_{23}\big)$ contains the point $\widetilde{P}_6$, which is a quotient singularity of type $\frac{1}{3}(1,1,2)$ on U_{23}.

Let $\pi\colon W \to U_{23}$ be the weighted blow-up of $\widetilde{P}_6$ with weights $(1,1,2)$, $\mathcal{D}$ the proper transform of $\mathcal{M}$ on W, and $\bar{L}$ and $\bar{\Delta}$ be the proper transforms of L and Δ on W, respectively. Then $\mathcal{D} \sim_{\mathbb{Q}} -kK_W$ by Theorem 2.2, $|-K_W|$ is the proper transform of $|-K_{U_2}|$, and the base locus of $|-K_W|$ consists of the curves $\bar{L}$ and $\bar{\Delta}$.

Let S be a general surface in $|-K_W|$. Then S is irreducible and normal and the equivalence $\mathcal{D}|_S \sim_{\mathbb{Q}} k\bar{\Delta} + k\bar{L}$ holds. But the equalities $\bar{\Delta}^2 = -7/12$, $\bar{L}^2 = -5/6$ and $\bar{\Delta}\cdot\bar{L} = 2/3$ hold on S. Therefore, the intersection form of the curves $\bar{\Delta}$ and $\bar{L}$ on S is negative definite, which is impossible by Lemmas 2.9 and 2.7.

The hypersurface X can be given by the equation

$$\begin{aligned} w^2y + wt^2 + wtf_4(x,y,z) + wf_8(x,y,z)& \\ + tf_{11}(x,y,z) + f_{15}(x,y,z) &= 0 \subset \operatorname{Proj}\big(\mathbb{C}[x,y,z,t,w]\big), \end{aligned}$$

where $\mathrm{wt}(x) = 1$, $\mathrm{wt}(y) = 1$, $\mathrm{wt}(z) = 3$, $\mathrm{wt}(t) = 4$, $\mathrm{wt}(w) = 7$ and $f_i(x,y,t)$ is a sufficiently general quasi-homogeneous polynomial of degree i.

Remark 19.6. Suppose that the set $\mathbb{CS}\big(U_2, \frac{1}{k}\mathcal{D}_2\big)$ contains both $\bar{P}_1$ and $\bar{P}_3$. Then the assertion of Theorem 2.2 easily implies the existence of a commutative diagram

$$\begin{tikzcd} X \arrow[rr, dashed, "\rho"] \arrow[d, dashed, "\psi"'] & & V \arrow[d, "\nu"] \\ \mathbb{P}(1,1,3) \arrow[rr, dashed, "\zeta"'] & & \mathbb{P}^2, \end{tikzcd}$$

where ζ is a birational map.

Therefore, we may assume that $\mathbb{CS}\big(U_2, \frac{1}{k}\mathcal{D}_2\big)$ does not contain both $\bar{P}_1$ and $\bar{P}_3$.

Lemma 19.7. *The set $\mathbb{CS}\big(U_2, \frac{1}{k}\mathcal{D}_2\big)$ contains the point $\bar{P}_1$.*

Proof. Suppose that $\bar{P}_1 \notin \mathbb{CS}\big(U_2, \frac{1}{k}\mathcal{D}_2\big)$. Then $\mathbb{CS}\big(U_2, \frac{1}{k}\mathcal{D}_2\big) = \{\bar{P}_4\}$.

Let $\pi\colon W \to U_2$ be the weighted blow-up of $\bar{P}_4$ with weights $(1,1,2)$, E the exceptional divisor of π, and $\bar{G}$ and $\mathcal{B}$ the proper transforms of G and $\mathcal{M}$ on W, respectively. Then it follows from Theorem 2.2 that the equivalence $\mathcal{B} \sim_{\mathbb{Q}} -kK_W$ holds. But the proof of Lemma 19.5 implies that the set $\mathbb{CS}\big(W, \frac{1}{k}\mathcal{B}\big)$ does not contain the singular point of the variety W that is contained in the exceptional divisor of π. Therefore, the singularities of the log pair $\big(W, \frac{1}{k}\mathcal{B}\big)$ are terminal by Lemma 2.3.

Let S_x, S_y, S_z, S_t and S_w be the proper transforms on W of the surfaces that are cut out on X by the equations $x = 0$, $y = 0$, $z = 0$, $t = 0$ and $w = 0$, respectively. Then

$$\begin{gathered} S_x \sim_{\mathbb{Q}} (\alpha_2 \circ \pi)^*(-K_X) - \frac{3}{7}E - \frac{1}{7}\bar{G}, \\ S_y \sim_{\mathbb{Q}} (\alpha_2 \circ \pi)^*(-K_X) - \frac{10}{7}E - \frac{8}{7}\bar{G}, \\ S_z \sim_{\mathbb{Q}} (\alpha_2 \circ \pi)^*(-3K_X) - \frac{2}{7}E - \frac{3}{7}\bar{G}, \\ S_t \sim_{\mathbb{Q}} (\alpha_2 \circ \pi)^*(-4K_X) - \frac{5}{7}E - \frac{4}{7}\bar{G}, \\ S_w \sim_{\mathbb{Q}} (\alpha_2 \circ \pi)^*(-7K_X). \end{gathered} \tag{19.1}$$

The base locus of the pencil $|-K_W|$ consists of the irreducible curves C and L such that the curve $\alpha_2 \circ \pi(C)$ is cut out by the equations $x = y = 0$ on the hypersurface X, the curve $\pi(L)$ is contained in G, and the curve $\pi(L)$ is contained in the linear system $\big|\mathcal{O}_{\mathbb{P}(1,3,4)}(1)\big|$.

The equivalences (19.1) imply that the rational functions y/x, zy/x^4, ty/x^5 and wy^3/x^{10} are contained in the linear system $|aS_x|$, where $a = 1$, 4, 5 and 10, respectively. Therefore, the linear system $|-20K_W|$ induces a birational map $\chi\colon W \dashrightarrow X'$, where X' is a hypersurface with canonical singularities in $\mathbb{P}(1,1,4,5,10)$ of degree 20. In particular, the divisor $-K_W$ is big.

It follows from [12] that there is a birational map $\zeta\colon W \dashrightarrow Z$ such that ζ is regular outside C and L, the map ζ is an isomorphism in codimension 1, and the divisor $-K_Z$ is nef and big. Let $\mathcal{P}$ be the proper transform of $\mathcal{M}$ on Z. Then the singularities of the log pair $\big(Z, \frac{1}{k}\mathcal{P}\big)$ are terminal because ζ is a log-flop with respect to the log pair $\big(W, \frac{1}{k}\mathcal{B}\big)$, which has terminal singularities. But it follows from Lemma 2.1 that the singularities of $\big(Z, \frac{1}{k}\mathcal{P}\big)$ are not terminal.

Hence, the set $\mathbb{CS}\big(U_2, \frac{1}{k}\mathcal{D}_2\big)$ consists of the points $\bar{P}_1$ and $\bar{P}_4$. Let $\pi\colon W \to U_2$ be the composite of the weighted blow-ups of the points $\bar{P}_1$ and $\bar{P}_4$ with weights $(1,1,3)$ and $(1,1,2)$, respectively, let $\bar{G}$ and $\mathcal{B}$ be the proper transforms of G and $\mathcal{M}$ on W respectively, and let F and E be exceptional divisors of π that dominate the points $\bar{P}_1$ and $\bar{P}_4$, respectively. Then the equivalence $\mathcal{B} \sim_{\mathbb{Q}} -kK_W$ holds by Theorem 2.2. Arguing as in the proof of Lemma 19.5, we see that the singularities of the log pair $\big(W, \frac{1}{k}\mathcal{B}\big)$ are terminal.

Let S_x, S_y, S_z, S_t and S_w be the proper transforms on W of the surfaces that are cut out on X by the equations $x=0$, $y=0$, $z=0$, $t=0$ and $w=0$, respectively. Then

$$S_x \sim_{\mathbb{Q}} (\alpha_2\circ\pi)^*(-K_X) - \frac{3}{7}E - \frac{1}{7}\bar{G} - \frac{1}{4}F,$$

$$S_y \sim_{\mathbb{Q}} (\alpha_2\circ\pi)^*(-K_X) - \frac{10}{7}E - \frac{8}{7}\bar{G} - \frac{1}{4}F,$$

$$S_z \sim_{\mathbb{Q}} (\alpha_2\circ\pi)^*(-3K_X) - \frac{2}{7}E - \frac{3}{7}\bar{G} - \frac{3}{4}F,$$

$$S_t \sim_{\mathbb{Q}} (\alpha_2\circ\pi)^*(-4K_X) - \frac{5}{7}E - \frac{4}{7}\bar{G},$$

$$S_w \sim_{\mathbb{Q}} (\alpha_2\circ\pi)^*(-7K_X) - \frac{11}{4}F,$$

and it follows that the rational functions y/x, zy/x^4, twy^4/x^{15} and wy^3/x^{10} are contained in the linear system $|aS_x|$, where $a=1$, 4, 15 and 10, respectively. The linear system $|-60K_W|$ induces a birational map $\chi\colon W \dashrightarrow X'$ such that the variety X' is a hypersurface in $\mathbb{P}(1,1,4,10,15)$ of degree 30. In particular, the divisor $-K_W$ is big. We can now obtain a contradiction in the same way as in the proof of Lemma 19.7. The proposition is proved.

§ 20. The case $n=26$: a hypersurface of degree 15 in $\mathbb{P}(1,1,3,5,6)$

We use the notation and assumptions of § 3. Let $n=26$. Then X is a sufficiently general hypersurface in $\mathbb{P}(1,1,3,5,6)$ of degree 15, the equality $-K_X^3 = 1/6$ holds, and the singularities of X consist of points P_1 and P_2 that are quotient singularities of type $\frac{1}{3}(1,1,2)$ and a point P_3 that is a quotient singularity of type $\frac{1}{6}(1,1,5)$. There is a commutative diagram

$$\begin{tikzcd} U \arrow[d, "\alpha"'] & W \arrow[l, "\beta"'] & Y \arrow[l, "\gamma"'] \arrow[d, "\eta"] \\ X \arrow[rr, dashed, "\psi"'] & & \mathbb{P}(1,1,3), \end{tikzcd}$$

where ψ is the natural projection, α is the weighted blow-up of P_3 with weights $(1,1,5)$, β is the weighted blow-up with weights $(1,1,4)$ of the singular point of U

that is contained in the exceptional divisor of α, γ is the weighted blow-up with weights $(1,1,3)$ of the singular point of W that is contained in the exceptional divisor of β, and η is an elliptic fibration.

It follows from Example 1.5 that there is a commutative diagram

$$\begin{tikzcd} & U_i \arrow[dl, "\sigma_i"'] \arrow[dr, "\omega_i"] & \\ X \arrow[rr, dashed, "\xi_i"'] & & \mathbb{P}(1,1,6), \end{tikzcd}$$

where ξ_i is a projection, σ_i is the weighted blow-up of P_i with weights $(1,1,2)$ and ω_i is an elliptic fibration induced by the linear system $|-6K_{U_i}|$.

It follows from [2] that the group $\mathrm{Bir}(X)$ is generated by biregular automorphisms of X and a birational involution $\tau \in \mathrm{Bir}(X)$ such that $\psi\circ\tau=\psi$ and $\xi_1\circ\tau=\xi_2$.

The rest of the section is devoted to proving the following result.

Proposition 20.1. *Either there is a commutative diagram*

$$\begin{tikzcd} X \arrow[r, dashed, "\rho"] \arrow[d, dashed, "\psi"'] & V \arrow[d, "\nu"] \\ \mathbb{P}(1,1,3) \arrow[r, dashed, "\varphi"'] & \mathbb{P}^2 \end{tikzcd} \tag{20.1}$$

or there is a commutative diagram

$$\begin{tikzcd} X \arrow[r, dashed, "\rho"] \arrow[d, dashed, "\xi_i"'] & V \arrow[d, "\nu"] \\ \mathbb{P}(1,1,6) \arrow[r, dashed, "\sigma"'] & \mathbb{P}^2, \end{tikzcd} \tag{20.2}$$

where φ and σ are birational maps and $i=1$ or 2.

Proof. It follows from Theorem 3.3 and Lemma 3.11 that $\mathbb{CS}\big(X,\frac{1}{k}\mathcal{M}\big)\subseteq\{P_1,P_2,P_3\}$.

Lemma 20.2. *Suppose that $P_i\in\mathbb{CS}\big(X,\frac{1}{k}\mathcal{M}\big)$, where $i\in\{1,2\}$. Then the commutative diagram (20.2) exists.*

Proof. Let $\mathcal{B}$ be the proper transform of $\mathcal{M}$ on U_i. Then $\mathcal{B}\sim_{\mathbb{Q}}-kK_{U_i}$ by Theorem 2.2, which implies the existence of the commutative diagram (20.2).

Therefore, we may assume that the set $\mathbb{CS}\big(X,\frac{1}{k}\mathcal{M}\big)$ consists of the point P_3.

Let $\mathcal{D}$ be the proper transform of $\mathcal{M}$ on U and $\overline{P}_4$ the singular point of U that is contained in the exceptional divisor of α. Then $\mathcal{D}\sim_{\mathbb{Q}}-kK_U$ by Theorem 2.2.

Lemma 20.3. *The set $\mathbb{CS}\big(U,\frac{1}{k}\mathcal{D}\big)$ consists of the point $\overline{P}_4$.*

Proof. Let G be an exceptional divisor of the blow-up α. Then G is a cone over a smooth rational curve of degree 5 and $\overline{P}_4$ is the vertex of G. Suppose that $\mathbb{CS}\big(U,\frac{1}{k}\mathcal{D}\big)$ contains a subvariety C of U that is different from the point $\overline{P}_4$. Then C is a ruling of G by Lemma 2.3, which is impossible by Corollary 2.8.

Let $\mathcal{H}$ be the proper transform of $\mathcal{M}$ on W. Then $\mathcal{H}\sim_{\mathbb{Q}}-kK_W$ by Theorem 2.2. But it follows from Lemmas 2.1 and 2.3 that $\mathbb{CS}\big(W,\frac{1}{k}\mathcal{H}\big)$ contains the singular point of W that is contained in the exceptional divisor of β. The existence of the diagram (20.1) now follows from Theorem 2.2.

§ 21. The case $n=27$: a hypersurface of degree 15 in $\mathbb{P}(1,2,3,5,5)$

We use the notation and assumptions of § 3. Let $n=27$. Then X is a general hypersurface in $\mathbb{P}(1,2,3,5,5)$ of degree 15, whose singularities consist of a point O that is a singularity of type $\frac{1}{2}(1,1,1)$, and points P_1, P_2 and P_3 that are singularities of type $\frac{1}{5}(1,2,3)$.

Let $\psi\colon X\dashrightarrow\mathbb{P}(1,2,3)$ be the natural projection. Then ψ is undefined only at the points P_1, P_2 and P_3. The generic fibre of ψ is a smooth elliptic curve. There is a commutative diagram

$$\begin{tikzcd}
 & U_{ij} \arrow[dl, "\beta_{ij}"'] & & Y \arrow[ll, "\gamma_{ij}"'] \arrow[ddr, "\eta"] & \\
U_i \arrow[dr, "\alpha_i"'] & & & & \\
 & X \arrow[rrr, dashed, "\psi"'] & & & \mathbb{P}(1,2,3),
\end{tikzcd}$$

where α_i is the blow-up of P_i with weights $(1,2,3)$, β_{ij} is the weighted blow-up with weights $(1,2,3)$ of the proper transform of P_j on U_i, γ_{ij} is the weighted blow-up with weights $(1,2,3)$ of the proper transform of P_k on U_{ij}, and η is an elliptic fibration, where $i\neq j$ and $k\notin\{i,j\}$.

Proposition 21.1. *The assertion of Theorem 1.10 holds for $n=27$.*

Proof. It follows from Lemma 3.11 and Proposition 3.5 that $\mathbb{CS}\big(X,\frac{1}{k}\mathcal{M}\big)\subseteq\{P_1,P_2,P_3\}$.

Remark 21.2. The assertion of Theorem 1.10 holds for the hypersurface X by Theorem 2.2 in the case when $\mathbb{CS}\big(X,\frac{1}{k}\mathcal{M}\big)=\{P_1,P_2,P_3\}$.

We may assume that $\mathbb{CS}\big(X,\frac{1}{k}\mathcal{M}\big)$ contains P_1 but not P_3. We shall show that this assumption leads to a contradiction.

Lemma 21.3. *The set $\mathbb{CS}\big(X,\frac{1}{k}\mathcal{M}\big)$ contains the point P_2.*

Proof. Suppose that $\mathbb{CS}\big(X,\frac{1}{k}\mathcal{M}\big)$ does not contain P_2. Then $\mathbb{CS}\big(X,\frac{1}{k}\mathcal{M}\big)$ consists of the point P_1. Let $\mathcal{D}_1$ be the proper transform of $\mathcal{M}$ on U_1. Then Theorem 2.2 implies that the equivalence $\mathcal{D}_1\sim_{\mathbb{Q}}-kK_{U_1}$ holds. The set $\mathbb{CS}\big(U_1,\frac{1}{k}\mathcal{D}_1\big)$ is non-empty by Lemma 2.1.

Let G be the exceptional divisor of α_1 and O, Q the singular points of G that are quotient singularities of types $\frac{1}{3}(1,1,2)$, $\frac{1}{2}(1,1,1)$ on U_1, respectively. Then it follows from the assertion of Lemma 2.3 that $\mathbb{CS}\big(U_1,\frac{1}{k}\mathcal{D}_1\big)$ contains either O or Q.

The linear system $|-2K_{U_1}|$ is a pencil and its base locus consists of the irreducible curve C which passes through O and is contracted by the rational map $\psi\circ\alpha_1$ to a singular point of the surface $\mathbb{P}(1,2,3)$.

Suppose that the set $\mathbb{CS}\big(U_1, \frac{1}{k}\mathcal{D}_1\big)$ contains the point O. Let $\pi\colon W \to U_1$ be the weighted blow-up of O with weights $(1,1,2)$, $\mathcal{B}$ the proper transform of $\mathcal{M}$ on W and $\bar{C}$ the proper transform of C on W. Then $\mathcal{B} \sim_{\mathbb{Q}} -kK_W$ by Theorem 2.2, the linear system $|-2K_W|$ is the proper transform of the pencil $|-2K_{U_1}|$, and the base locus of $|-2K_W|$ consists of the curve $\bar{C}$. Let S be a general surface in $|-2K_W|$. Then $\mathcal{B}|_S \sim_{\mathbb{Q}} k\bar{C}$. But $\bar{C}^2 < 0$ on S. But this is impossible by Lemmas 2.7 and 2.9.

Hence, $\mathbb{CS}\big(U_1, \frac{1}{k}\mathcal{D}_1\big)$ contains Q. Let $\zeta\colon U \to U_1$ be the weighted blow-up of Q with weights $(1,1,1)$, F the exceptional divisor of ζ, $\mathcal{H}$ the proper transform of $\mathcal{M}$ on U and $\mathcal{P}$ the proper transform of $|-3K_{U_1}|$ on U. Then $\mathcal{H} \sim_{\mathbb{Q}} -kK_U$ by Theorem 2.2. But

$$\mathcal{P} \sim_{\mathbb{Q}} \zeta^*(-3K_{U_1}) - \frac{1}{2}\,F,$$

and the base locus of $\mathcal{P}$ consists of the irreducible curve Z such that the curve $\alpha_1 \circ \zeta(Z)$ is the unique base curve of $|-3K_X|$. Therefore, we have

$$\left(\zeta^*(-3K_{U_1}) - \frac{1}{2}\,F\right)\cdot Z = \frac{1}{10}\,,$$

which implies that

$$\begin{aligned}
-\frac{3k^2}{10} &= \left(\zeta^*(-3K_{U_1}) - \frac{1}{2}\,F\right)\cdot\left(\zeta^*(-kK_{U_1}) - \frac{k}{2}\,F\right)^2 \\
&= \left(\zeta^*(-3K_{U_1}) - \frac{1}{2}\,F\right)\cdot H_1\cdot H_2 \geqslant 0,
\end{aligned}$$

where H_1 and H_2 are general surfaces in $\mathcal{H}$. The resulting contradiction completes the proof of the lemma.

We have $\mathbb{CS}\big(X, \frac{1}{k}\mathcal{M}\big) = \{P_1, P_2\}$. We can now apply the proof of Lemma 21.3 to the proper transform of $\mathcal{M}$ on U_{12} to get a contradiction. The proposition is proved.

§ 22. The case $n = 29$: a hypersurface of degree 16 in $\mathbb{P}(1,1,2,5,8)$

We use the notation and assumptions of § 3. Let $n = 29$. Then X is a sufficiently general hypersurface in $\mathbb{P}(1,1,2,5,8)$ of degree 16, the equality $-K_X^3 = 1/5$ holds, and the singularities of X consist of points O_1, O_2 that are quotient singularities of type $\frac{1}{2}(1,1,1)$ and a point P that is a quotient singularity of type $\frac{1}{5}(1,2,3)$. There is a commutative diagram

$$\begin{tikzcd}
U \arrow[d, "\alpha"'] & & W \arrow[ll, "\beta"'] \arrow[d, "\eta"] \\
X \arrow[rr, dashed, "\psi"'] & & \mathbb{P}(1,1,2),
\end{tikzcd}$$

where ψ is the natural projection, α is the weighted blow-up of P with weights $(1,2,3)$, β is the weighted blow-up with weights $(1,1,2)$ of the singular point of U that is a quotient singularity of type $\frac{1}{3}(1,1,2)$, and η is an elliptic fibration.

Proposition 22.1. *The assertion of Theorem 1.10 holds for* $n = 29$.

Proof. The hypersurface X is birationally superrigid. It follows from Theorem 3.3, Lemma 3.11 and Proposition 3.5 that $\mathbb{CS}\big(X, \frac{1}{k}\mathcal{M}\big) = \{P\}$.

Let $\mathcal{D}$ be the proper transform of $\mathcal{M}$ on U. Then $\mathcal{D} \sim_{\mathbb{Q}} -kK_U$ by Theorem 2.2. Let G be the α-exceptional divisor and let Q, O be the singular points of G that are singularities of types $\frac{1}{3}(1,1,2)$, $\frac{1}{2}(1,1,1)$, respectively. Then it follows from Lemmas 2.1 and 2.3 that the set $\mathbb{CS}\big(U, \frac{1}{k}\mathcal{D}\big)$ either consists of the point O or contains Q.

Suppose that $\mathbb{CS}\big(U, \frac{1}{k}\mathcal{D}\big)$ contains O. Let $\pi\colon Y \to U$ be the weighted blow-up of O with weights $(1,1,1)$, $\mathcal{H}$ the proper transform of $\mathcal{M}$ on Y, L the curve on G that is contained in $\big|\mathcal{O}_{\mathbb{P}(1,2,3)}(1)\big|$, $\bar{L}$ the proper transform of L on Y, and S a general surface in $|-K_Y|$. Then $\mathcal{H} \sim_{\mathbb{Q}} -kK_Y$ by Theorem 2.2, and the base locus of the pencil $|-K_Y|$ consists of the curve $\bar{L}$ and the irreducible curve Δ such that $\alpha\circ\pi(\Delta)$ is the base locus of $|-K_X|$. The equalities $\Delta^2 = -1$, $\bar{L}^2 = -4/3$ and $\Delta\cdot\bar{L} = 1$ hold on S. Thus, the intersection form of the curves Δ and $\bar{L}$ on S is negative definite. We have $\mathcal{H}|_S \sim_{\mathbb{Q}} k\Delta + k\bar{L}$, which is impossible by Lemmas 2.7 and 2.9.

Therefore, $\mathbb{CS}\big(U, \frac{1}{k}\mathcal{D}\big)$ contains the point Q. Let $\mathcal{B}$ be the proper transform of $\mathcal{M}$ on W. Then $\mathcal{B} \sim_{\mathbb{Q}} -kK_W$ by Theorem 2.2, which implies that $\mathcal{B}$ is contained in the fibres of η. The proposition is proved.

§ 23. The case $n = 30$: a hypersurface of degree 16 in $\mathbb{P}(1,1,3,4,8)$

We use the notation and assumptions of § 3. Let $n = 30$. Then X is a general hypersurface in $\mathbb{P}(1,1,3,4,8)$ of degree 16, the equality $-K_X^3 = 1/6$ holds, and the singularities of X consist of a point O that is a quotient singularity of type $\frac{1}{3}(1,1,2)$ and points P_1, P_2 that are singularities of type $\frac{1}{4}(1,1,3)$.

There is a commutative diagram

$$\begin{tikzcd}
& Y \arrow[dl, "\gamma"'] \arrow[dr, "\delta"] \arrow[drrr, "\eta"] & & & \\
U \arrow[dr, "\alpha"'] & & W \arrow[dl, "\beta"] & & \mathbb{P}(1,1,3), \\
& X \arrow[urrr, dashed, "\psi"'] & & &
\end{tikzcd}$$

where ψ is the natural projection, α is the weighted blow-up of P_1 with weights $(1,1,3)$, β is the weighted blow-up of P_2 with weights $(1,1,3)$, γ is the weighted blow-up with weights $(1,1,3)$ of the proper transform of P_2 on U, δ is the weighted blow-up with weights $(1,1,3)$ of the proper transform of P_1 on W, and η is and elliptic fibration.

There is a commutative diagram

$$\begin{tikzcd}
& Z \arrow[dl, "\zeta"'] \arrow[dr, "\omega"] & \\
X \arrow[rr, dashed, "\xi"'] & & \mathbb{P}(1,1,4),
\end{tikzcd}$$

where ξ is a projection, ζ is the blow-up of O with weights $(1,1,2)$, and ω is an elliptic fibration.

Proposition 23.1. *Either there is a commutative diagram*

$$
\begin{tikzcd}
X \arrow[r, dashed, "\rho"] \arrow[d, dashed, "\psi"'] & V \arrow[d, "\nu"] \\
\mathbb{P}(1,1,3) \arrow[r, dashed, "\varphi"'] & \mathbb{P}^2
\end{tikzcd}
\tag{23.1}
$$

or there is a commutative diagram

$$
\begin{tikzcd}
X \arrow[r, dashed, "\theta"] \arrow[d, dashed, "\xi"'] & X \arrow[r, dashed, "\rho"] & V \arrow[d, "\nu"] \\
\mathbb{P}(1,1,4) \arrow[rr, dashed, "\sigma"'] & & \mathbb{P}^2,
\end{tikzcd}
\tag{23.2}
$$

where φ, θ and σ are birational maps.

Proof. Suppose that $O \in \mathbb{CS}\big(X, \frac{1}{k}\mathcal{M}\big)$. Then the existence of the commutative diagram (23.2) follows from Theorem 2.2. Similarly, the existence of the commutative diagram (23.1) follows from Theorem 2.2 in the case when $\mathbb{CS}\big(X, \frac{1}{k}\mathcal{M}\big) = \{P_1, P_2\}$. Therefore, we may assume that $\mathbb{CS}\big(X, \frac{1}{k}\mathcal{M}\big)$ consists of the singular point P_1 by Theorem 3.3, Proposition 3.5 and Lemma 3.11.

Let Q be the singular point of U such that $\alpha(Q) = P_1$ and $\mathcal{B}$ the proper transform of $\mathcal{M}$ on U. Then $Q \in \mathbb{CS}\big(U, \frac{1}{k}\mathcal{B}\big)$ by Theorem 2.2 and Lemmas 2.1 and 2.3.

Let $\upsilon\colon \bar{U} \to U$ be the weighted blow-up of Q with weights $(1,1,2)$, $\mathcal{D}$ the proper transform of $\mathcal{M}$ on $\bar{U}$ and S a sufficiently general surface in the pencil $\big|-K_{\bar{U}}\big|$. Then $\mathcal{D} \sim_{\mathbb{Q}} -kK_{\bar{U}}$ by Theorem 2.2, S is normal and the base locus of $\big|-K_{\bar{U}}\big|$ consists of the irreducible curve Δ such that $\alpha \circ \upsilon(\Delta)$ is the unique base curve of $|-K_X|$. Moreover, the inequality $\Delta^2 < 0$ holds on S. But the equivalence $\mathcal{D}|_S \sim_{\mathbb{Q}} k\Delta$ holds, which is impossible by Lemmas 2.9 and 2.7.

§ 24. The case $n = 31$: a hypersurface of degree 16 in $\mathbb{P}(1,1,4,5,6)$

We use the notation and assumptions of § 3. Let $n = 31$. Then X is a general hypersurface in $\mathbb{P}(1,1,2,3,4)$ of degree 10 and $-K_X^3 = 2/15$. The singularities of X consist of points P_1, P_2 and P_3 that are singularities of type $\frac{1}{2}(1,1,1)$, $\frac{1}{5}(1,1,4)$ and $\frac{1}{6}(1,1,5)$, respectively. There is a commutative diagram

$$
\begin{tikzcd}
& & Y \arrow[dl, "\gamma_4"'] \arrow[dr, "\gamma_2"] \arrow[rrrrd, "\eta"] & & & & \\
& U_{23} \arrow[dl, "\beta_3"'] \arrow[dr, "\beta_2"] & & U_{34} \arrow[dl, "\beta_4"'] & & & \mathbb{P}(1,1,4), \\
U_2 \arrow[dr, "\alpha_2"'] & & U_3 \arrow[dl, "\alpha_3"] & & & & \\
& X \arrow[rrrrruu, dashed, "\psi"'] & & & & &
\end{tikzcd}
$$

where ψ is a projection, α_2 is the weighted blow-up of P_2 with weights $(1,1,4)$, α_3 is the weighted blow-up of P_3 with weights $(1,1,5)$, β_3 is the weighted blow-up with weights $(1,1,5)$ of the proper transform of P_3 on U_2, β_2 is the weighted blow-up with weights $(1,1,4)$ of the proper transform of P_2 on U_3, β_4 is the weighted blow-up with weights $(1,1,4)$ of the singular point of U_3 that is contained in the exceptional divisor of α_3, γ_2 is the weighted blow-up with weights $(1,1,4)$ of the proper transform of P_2 on U_{34}, γ_4 is the weighted blow-up with weights $(1,1,4)$ of the singular point of U_{23} that is contained in the exceptional divisor of β_3, and η is an elliptic fibration. There is a commutative diagram

$$\begin{tikzcd} U_2 \arrow[d, "\alpha_2"'] & W \arrow[l, "\beta"'] \arrow[d, "\omega"] \\ X \arrow[r, dashed, "\xi"'] & \mathbb{P}(1,1,6), \end{tikzcd}$$

where ξ is a projection, β is the weighted blow-up with weights $(1,1,3)$ of the singular point of U_2 that is a quotient singularity of type $\frac{1}{4}(1,1,3)$, and ω is an elliptic fibration.

Proposition 24.1. *Either there is a commutative diagram*

$$\begin{tikzcd} X \arrow[r, dashed, "\rho"] \arrow[d, dashed, "\psi"'] & V \arrow[d, "\nu"] \\ \mathbb{P}(1,1,4) \arrow[r, dashed, "\varphi"'] & \mathbb{P}^2 \end{tikzcd} \tag{24.1}$$

or there is a commutative diagram

$$\begin{tikzcd} X \arrow[r, dashed, "\theta"] \arrow[d, dashed, "\xi"'] & X \arrow[r, dashed, "\rho"] & V \arrow[d, "\nu"] \\ \mathbb{P}(1,1,5) \arrow[rr, dashed, "\sigma"'] & & \mathbb{P}^2, \end{tikzcd} \tag{24.2}$$

where φ, θ and σ are birational maps.

Proof. It follows from Theorem 3.3, Lemma 3.11 and Proposition 3.5 that $\mathbb{CS}\big(X, \frac{1}{k}\mathcal{M}\big) \subseteq \{P_2, P_3\}$. Let $\mathcal{D}_2$, $\mathcal{D}_3$, $\mathcal{D}_{23}$ and $\mathcal{D}_{34}$ be the proper transforms of $\mathcal{M}$ on U_2, U_3, U_{23} and U_{34}, respectively. Then it follows from Lemma 2.1 that the set $\mathbb{CS}\big(U_\mu, \frac{1}{k}\mathcal{D}_\mu\big)$ is non-empty if $\mathcal{D}_\mu \sim_{\mathbb{Q}} -kK_{U_\mu}$.

Lemma 24.2. *Suppose that the set $\mathbb{CS}\big(X, \frac{1}{k}\mathcal{M}\big)$ contains the point P_3. Let $\bar{P}_2$ be the proper transform of P_2 on U_3 and $\bar{P}_4$ the singular point of U_3 that is contained in the exceptional divisor of α_3. Then $\mathcal{D}_3 \sim_{\mathbb{Q}} -kK_{U_3}$ and $\mathbb{CS}\big(U_3, \frac{1}{k}\mathcal{D}_3\big) \subseteq \{\bar{P}_2, \bar{P}_4\}$.*

Proof. The equivalence $\mathcal{D}_3 \sim_{\mathbb{Q}} -kK_{U_3}$ follows from Theorem 2.2. Suppose that

$$\mathbb{CS}\left(U_3, \frac{1}{k}\mathcal{D}_3\right) \not\subseteq \{\bar{P}_2, \bar{P}_4\}.$$

We shall show that this assumption leads to a contradiction.

Let G be the exceptional divisor of α_3. Then $G \cong \mathbb{P}(1,1,5)$, and it follows from Lemma 2.3 that there is a curve $C \subset G$ of the linear system $\left|\mathcal{O}_{\mathbb{P}(1,1,5)}(1)\right|$ that is contained in $\mathbb{CS}\bigl(U_3, \frac{1}{k}\mathcal{D}_3\bigr)$, which is impossible by Lemma 2.4.

Lemma 24.3. *Let $\overline{P}_3$ be the proper transform of P_3 on U_2. Suppose that the set $\mathbb{CS}\bigl(X, \frac{1}{k}\mathcal{M}\bigr)$ contains the point P_2. Then either $\mathbb{CS}\bigl(U_2, \frac{1}{k}\mathcal{D}_2\bigr) = \{\overline{P}_3\}$ or the commutative diagram (24.2) exists.*

Proof. The equivalence $\mathcal{D}_2 \sim_{\mathbb{Q}} -kK_{U_2}$ follows from Theorem 2.2. Suppose that the set $\mathbb{CS}\bigl(U_2, \frac{1}{k}\mathcal{D}_2\bigr)$ does not consist of the point $\overline{P}_3$. Let $\overline{P}_5$ be the singular point of U_2 that is contained in the exceptional divisor of α_2. Then $\overline{P}_5$ is a quotient singularity of type $\frac{1}{4}(1,1,3)$ on U_2, and $\mathbb{CS}\bigl(U_2, \frac{1}{k}\mathcal{D}_2\bigr)$ contains $\overline{P}_5$ by Lemma 2.3.

Let $\mathcal{B}$ be the proper transform of $\mathcal{M}$ on W. Then $\mathcal{B} \sim_{\mathbb{Q}} -kK_W$ by Theorem 2.2, which implies the existence of the commutative diagram (24.2).

By Theorem 2.2, we may assume that either $\mathcal{D}_{23} \sim_{\mathbb{Q}} -kK_{U_{23}}$ or $\mathcal{D}_{34} \sim_{\mathbb{Q}} -kK_{U_{34}}$. Let $\mathcal{D}$ be the proper transform of $\mathcal{M}$ on Y.

Lemma 24.4. *Suppose that $\mathcal{D}_{23} \sim_{\mathbb{Q}} -kK_{U_{23}}$. Then $\mathcal{D} \sim_{\mathbb{Q}} -kK_Y$.*

Proof. Let F be the exceptional divisor of β_2, G the exceptional divisor of β_3, $\check{P}_4$ the singular point of G and $\check{P}_5$ the singular point of the surface F. Arguing as in the proof of Lemma 24.3, we see that $\mathbb{CS}\bigl(U_{23}, \frac{1}{k}\mathcal{D}_{23}\bigr)$ does not contain the point $\check{P}_5$. Hence, it follows from Lemmas 2.3 and 2.1 that the set $\mathbb{CS}\bigl(U_{23}, \frac{1}{k}\mathcal{D}_{23}\bigr)$ contains the point $\check{P}_4$, which implies that $\mathcal{D} \sim_{\mathbb{Q}} -kK_Y$ by Theorem 2.2.

Lemma 24.5. *Suppose that $\mathcal{D}_{34} \sim_{\mathbb{Q}} -kK_{U_{34}}$. Then $\mathcal{D} \sim_{\mathbb{Q}} -kK_Y$.*

Proof. Let G be the exceptional divisor of β_4, $\check{P}_2$ the proper transform of P_2 on U_{34} and $\check{P}_6$ the singular point of the surface G. Then G is a cone over a smooth rational cubic curve and $\check{P}_6$ is a quotient singularity of type $\frac{1}{3}(1,1,2)$ on U_{34}.

The set $\mathbb{CS}\bigl(U_{23}, \frac{1}{k}\mathcal{D}_{23}\bigr)$ is non-empty by Lemma 2.1. But the equivalence $\mathcal{D} \sim_{\mathbb{Q}} -kK_Y$ follows from Theorem 2.2 if $\mathbb{CS}\bigl(U_{23}, \frac{1}{k}\mathcal{D}_{23}\bigr)$ contains $\check{P}_2$. Therefore, we may assume that $\mathbb{CS}\bigl(U_{23}, \frac{1}{k}\mathcal{D}_{23}\bigr)$ contains $\check{P}_6$ by Lemma 2.3.

Let $\pi\colon W \to U_{34}$ be the weighted blow-up of $\check{P}_6$ with weights $(1,1,3)$, $\mathcal{B}$ the proper transform of $\mathcal{M}$ on W and S a sufficiently general surface in the pencil $|-K_W|$. Then the base locus of $|-K_W|$ consists of the irreducible curve Δ such that the equivalence $\mathcal{B}|_S \sim_{\mathbb{Q}} k\Delta$ holds, and the inequality $\Delta^2 < 0$ holds on S. It follows from Lemma 2.9 that $\mathcal{B}|_S = k\Delta$, which is impossible by Lemma 2.7.

Hence, the equivalence $\mathcal{D} \sim_{\mathbb{Q}} -kK_Y$ holds, which implies that $\mathcal{D}$ is contained in fibres of η. But this implies the existence of the commutative diagram (24.1). The proposition is proved.

§ 25. The case $n = 32$: a hypersurface of degree 16 in $\mathbb{P}(1,2,3,4,7)$

We use the notation and assumptions of § 3. Let $n = 32$. Then X is a sufficiently general hypersurface in $\mathbb{P}(1,2,3,4,6)$ of degree 16, and the equality $-K_X^3 = 2/21$ holds. The singularities of X consist of points P_1, P_2, P_3 and P_4 that are quotient singularities of type $\frac{1}{2}(1,1,1)$, a point P_5 that is a quotient singularity of type $\frac{1}{3}(1,1,2)$ and a point P_6 that is a singularity of type $\frac{1}{7}(1,3,4)$.

Proposition 25.1. *The assertion of Theorem 1.10 holds for $n = 32$.*

Proof. It follows from Theorem 3.3, Lemma 3.11 and Proposition 3.5 that $\mathbb{CS}\big(X, \frac{1}{k}\mathcal{M}\big) = \{P_6\}$.

There is a commutative diagram

$$\begin{tikzcd} U \arrow[d, "\alpha"'] & Y \arrow[l, "\beta"'] \arrow[d, "\eta"] \\ X \arrow[r, dashed, "\psi"'] & \mathbb{P}(1,2,3), \end{tikzcd}$$

where ψ is the natural projection, α is the weighted blow-up of P_6 with weights $(1,3,4)$, β is the weighted blow-up with weights $(1,1,3)$ of the singular point of U that is a quotient singularity of type $\frac{1}{4}(1,1,3)$ contained in the exceptional divisor of α, and η is an elliptic fibration.

Let E be the exceptional divisor of α and $\mathcal{D}$ the proper transform of $\mathcal{M}$ on U. Then $E \cong \mathbb{P}(1,3,4)$, and it follows from Theorem 2.2 that $\mathcal{D} \sim_{\mathbb{Q}} -kK_U$. But the set $\mathbb{CS}\big(U, \frac{1}{k}\mathcal{D}\big)$ is non-empty by Lemma 2.1.

Let P_7 and P_8 be the singular points of U contained in the divisor E that are singularities of types $\frac{1}{3}(1,1,2)$ and $\frac{1}{4}(1,1,3)$, respectively.

Lemma 25.2. *Suppose that $P_8 \in \mathbb{CS}\big(X, \frac{1}{k}\mathcal{M}\big)$. Then there is a commutative diagram*

$$\begin{tikzcd} X \arrow[r, dashed, "\rho"] \arrow[d, dashed, "\psi"'] & V \arrow[d, "\nu"] \\ \mathbb{P}(1,2,3) \arrow[r, dashed, "\zeta"'] & \mathbb{P}^2, \end{tikzcd} \tag{25.1}$$

where ζ is a birational map.

Proof. Let $\mathcal{H}$ be the proper transform of $\mathcal{M}$ on Y. Then it follows from Theorem 2.2 that $\mathcal{H} \sim_{\mathbb{Q}} -kK_Y$. Hence, $\mathcal{H}$ lies in the fibres of η, which implies the existence of the commutative diagram (25.1).

We may assume that $\mathbb{CS}\big(U, \frac{1}{k}\mathcal{D}\big) = \{P_7\}$ by Lemma 2.3.

Let $\gamma\colon W \to U$ be the weighted blow-up of P_7 with weights $(1,1,2)$, F the exceptional divisor of γ, $\bar{E}$ the proper transform of the surface E on W and $\mathcal{B}$ the proper transform of $\mathcal{M}$ on W. Then $F \cong \mathbb{P}(1,1,2)$, and it follows from Theorem 2.2 that $\mathcal{B} \sim_{\mathbb{Q}} -kK_W$.

The hypersurface X can be given by the quasi-homogeneous equation

$$w^2y + wf_9(x,y,z,t) + f_{16}(x,y,z,t) = 0 \subset \mathbb{P}(1,2,3,4,7) \cong \operatorname{Proj}\big(\mathbb{C}[x,y,z,t,w]\big),$$

where $\mathrm{wt}(x) = 1$, $\mathrm{wt}(y) = 2$, $\mathrm{wt}(z) = 3$, $\mathrm{wt}(t) = 4$, $\mathrm{wt}(w) = 7$, and f_9 and f_{16} are quasi-homogeneous polynomials of degree 9 and 16, respectively. Let S be the unique surface in the linear system $|-K_X|$ and D a general surface in the pencil $|-2K_X|$. Then S is cut out by $x = 0$ and D is cut out by

$$\lambda x^2 + \mu y = 0,$$

where $(\lambda,\mu)\in\mathbb{P}^1$. The surface D is normal and the base locus of $|-2K_X|$ consists of the curve C such that $C=D\cdot S$.

In the neighbourhood of P_6, the monomials x, z and t can be regarded as weighted local coordinates on X such that $\mathrm{wt}(x)=1$, $\mathrm{wt}(z)=3$ and $\mathrm{wt}(z)=4$. Then in the neighbourhood of the singular point P_6, the surface D can be given by the equation

$$\lambda x^2+\mu(\varepsilon_1x^9+\varepsilon_2zx^6+\varepsilon_3z^2x^3+\varepsilon_4z^3+\varepsilon_5t^2x+\varepsilon_6tx^5+\varepsilon_7tzx^2+\cdots)=0,$$

where $\varepsilon_i\in\mathbb{C}$. In the neighbourhood of P_7, α can be given by the equations

$$x=\tilde{x}\tilde{z}^{\frac{1}{7}},\qquad z=\tilde{z}^{\frac{3}{7}},\qquad t=\tilde{t}\tilde{z}^{\frac{4}{7}},$$

where $\tilde{x}$, $\tilde{y}$ and $\tilde{z}$ are weighted local coordinates on U in the neighbourhood of the singular point P_7 such that $\mathrm{wt}(\tilde{x})=1$, $\mathrm{wt}(\tilde{z})=2$ and $\mathrm{wt}(\tilde{t})=1$. Let $\widetilde{D}$, $\widetilde{S}$ and $\widetilde{C}$ be the proper transforms on U of D, S and C, respectively, and let E be the exceptional divisor of α. Then in the neighbourhood of P_7, E is given by the equation $\tilde{z}=0$, $\widetilde{D}$ is given by the vanishing of the analytic function

$$\lambda\tilde{x}^2+\mu(\varepsilon_1\tilde{x}^9\tilde{z}+\varepsilon_2\tilde{z}\tilde{x}^6+\varepsilon_3\tilde{z}\tilde{x}^3+\varepsilon_4\tilde{z}+\varepsilon_5\tilde{t}^2\tilde{x}\tilde{z}+\varepsilon_6\tilde{t}\tilde{x}^5\tilde{z}+\varepsilon_7\tilde{t}\tilde{z}\tilde{x}^2+\cdots),$$

and S is given by the equation $\tilde{x}=0$.

In the neighbourhood of the singular point of F, β can be given by the equations

$$\tilde{x}=\bar{x}\bar{z}^{\frac{1}{3}},\qquad \tilde{z}=\bar{z}^{\frac{2}{3}},\qquad \tilde{t}=\bar{t}\bar{z}^{\frac{1}{3}},$$

where $\bar{x}$, $\bar{z}$ and $\bar{t}$ are weighted local coordinates on W in the neighbourhood of the singular point of F such that $\mathrm{wt}(\bar{x})=\mathrm{wt}(\bar{z})=\mathrm{wt}(\bar{t})=1$. The surface F is given by the equation $\bar{z}=0$, the proper transform of D on W is given by the vanishing of the analytic function

$$\lambda\bar{x}^2+\mu(\varepsilon_1\bar{x}^9\bar{z}^3+\varepsilon_2\bar{z}^2\bar{x}^6+\varepsilon_3\bar{z}\bar{x}^3+\varepsilon_4+\varepsilon_5\bar{t}^2\bar{x}\bar{z}+\varepsilon_6\bar{t}\bar{x}^5\bar{z}^2+\varepsilon_7\bar{t}\bar{z}\bar{x}^2+\cdots)=0,$$

the proper transform of S on W is given by the equation $\bar{x}=0$, and the proper transform of E on W is given by the equation $\bar{z}=0$.

Let $\mathcal{P}$, $\overline{D}$, $\overline{S}$ and $\overline{C}$ be the proper transforms on W of the pencil $|-2K_X|$, the surface D, the surface S and the curve C, respectively, and let $\overline{H}$ be the proper transform on W of the surface that is cut out on X by the equation $y=0$. Then $\overline{D}$ is a general surface in the pencil $\mathcal{P}$ and

$$\begin{gathered}\overline{E}\sim_{\mathbb{Q}}\gamma^*(E)-\frac{2}{3}F,\\ \overline{D}\sim_{\mathbb{Q}}(\alpha\circ\gamma)^*(-2K_X)-\frac{2}{7}\gamma^*(E)-\frac{2}{3}F\sim_{\mathbb{Q}}(\alpha\circ\gamma)^*(-2K_X)-\frac{2}{7}\overline{E}-\frac{6}{7}F,\\ \overline{S}\sim_{\mathbb{Q}}(\alpha\circ\gamma)^*(-K_X)-\frac{1}{7}\gamma^*(E)-\frac{1}{3}F\sim_{\mathbb{Q}}(\alpha\circ\gamma)^*(-K_X)-\frac{1}{7}\overline{E}-\frac{3}{7}F,\\ \overline{H}\sim_{\mathbb{Q}}\gamma^*\left(\alpha^*(-2K_X)-\frac{9}{7}E\right)\sim_{\mathbb{Q}}(\alpha\circ\gamma)^*(-2K_X)-\frac{9}{7}\overline{E}-\frac{6}{7}F\sim_{\mathbb{Q}}2\overline{S}-\overline{E}.\end{gathered}\tag{25.2}$$

The curve $\overline{C}$ is contained in the base locus of $\mathcal{P}$ but is not the only such curve. To see this, let L be the curve on E that is contained in $|\mathcal{O}_{\mathbb{P}(1,3,4)}(1)|$, which means that

L is given locally by the equations $\tilde{x} = \tilde{z} = 0$, and let $\bar{L}$ be the proper transform of L on W. Then $\bar{L}$ is also contained in the base locus of $\mathcal{P}$. Moreover, it follows from local computations that the base locus of $\mathcal{P}$ does not contain curves outside the union $\bar{C} \cup \bar{L}$. The curve $\bar{C}$ is the intersection of the divisors $\bar{S}$ and $\bar{H}$ and the curve $\bar{L}$ is the intersection of the divisors $\bar{S}$ and $\bar{E}$. Moreover, we have $2\bar{C} = \bar{D} \cdot \bar{H}$ $\bar{C} + \bar{L} = \bar{S} \cdot \bar{D}$ and $2\bar{L} = \bar{D} \cdot \bar{E}$.

The curves $\bar{C}$ and $\bar{L}$ can be regarded as divisors on the normal surface $\bar{D}$. Then it follows from the equivalences (25.2) that

$$\begin{gathered} \bar{L} \cdot \bar{L} = \frac{\bar{E} \cdot \bar{E} \cdot \bar{D}}{4} = -\frac{5}{8}, \\ \bar{C} \cdot \bar{C} = \frac{\bar{H} \cdot \bar{H} \cdot \bar{D}}{4} = \bar{S} \cdot \bar{S} \cdot \bar{D} - \bar{S} \cdot \bar{E} \cdot \bar{D} - \frac{3}{2} \bar{E} \cdot \bar{E} \cdot \bar{D} = -\frac{7}{24}, \\ \bar{C} \cdot \bar{L} = \frac{\bar{H} \cdot \bar{E} \cdot \bar{D}}{4} = \frac{\bar{S} \cdot \bar{E} \cdot \bar{D}}{2} - \frac{\bar{E} \cdot \bar{E} \cdot \bar{D}}{4} = \frac{3}{8}, \end{gathered} \tag{25.3}$$

which implies that the intersection form of $\bar{C}$ and $\bar{L}$ on $\bar{D}$ is negative definite. On the other hand, the equivalence $\mathcal{B}|_{\bar{D}} \sim_{\mathbb{Q}} k\bar{C} + k\bar{L}$ holds, which is impossible by Lemmas 2.9 and 2.7. The proposition is proved.

Remark 25.3. Let $\overline{\mathbb{NE}}(W)$ be the closure in $\mathbb{R}^3$ of the cone that is generated by the effective one-dimensional cycles on W. Then the negative definiteness of the intersection form of $\bar{C}$ and $\bar{L}$ on $\bar{D}$ implies that $\bar{C}$ and $\bar{L}$ generate the two-dimensional extremal face of the cone $\overline{\mathbb{NE}}(W)$ that does not contain irreducible curves on W other than $\bar{C}$ and $\bar{L}$. This can be shown using the equivalences (25.2) without studying the geometry of $\bar{C}$ and $\bar{L}$ on $\bar{D}$.

§ 26. The case $n = 36$: a hypersurface of degree 18 in $\mathbb{P}(1,1,4,6,7)$

We use the notation and assumptions of § 3. Let $n = 36$. Then X is a sufficiently general hypersurface in $\mathbb{P}(1,1,4,6,7)$ of degree 18, the equality $-K_X^3 = 3/28$ holds, and the singularities of X consist of a point P_1 that is a quotient singularity of type $\frac{1}{2}(1,1,1)$, a point P_2 that is a quotient singularity of type $\frac{1}{4}(1,1,3)$ and a point P_3 that is a quotient singularity of type $\frac{1}{7}(1,1,6)$.

There is a commutative diagram

$$\begin{tikzcd} U_3 \arrow[d, "\alpha_3"'] & U_{34} \arrow[l, "\beta_4"'] & Y \arrow[l, "\gamma_5"'] \arrow[d, "\eta"] \\ X \arrow[rr, dashed, "\psi"'] & & \mathbb{P}(1,1,4), \end{tikzcd}$$

where ψ is the natural projection, α_3 is the weighted blow-up of P_3 with weights $(1,1,7)$, β_4 is the weighted blow-up with weights $(1,1,6)$ of the singular point of U_3 that is contained in the exceptional divisor of α_3, γ_5 is the weighted blow-up with weights $(1,1,4)$ of the singular point of U_{34} that is contained in the exceptional divisor of the birational morphism β_4, and η is an elliptic fibration.

Remark 26.1. The divisors $-K_{U_3}$ and $-K_{U_{34}}$ are nef and big.

It follows from Example 1.8 that there is a commutative diagram

$$\begin{tikzcd}
& W \arrow[dl, "\beta_3"'] \arrow[dr, "\beta_2"] \arrow[drrr, "\omega"] & & & \\
U_2 \arrow[dr, "\alpha_2"'] & & U_3 \arrow[dl, "\alpha_3"] & & \mathbb{P}(1,1,6), \\
& X \arrow[urrr, dashed, "\xi"'] & & &
\end{tikzcd}$$

where ξ is a projection, α_2 is the blow-up of P_2 with weights $(1,1,3)$, α_3 is the weighted blow-up of P_3 with weights $(1,1,6)$, β_2 is the weighted blow-up with weights $(1,1,3)$ of the proper transform of P_2 on U_3, β_3 is the weighted blow-up with weights $(1,1,6)$ of the proper transform of P_3 on U_2, and ω is an elliptic fibration.

Remark 26.2. The divisor $-K_{U_2}$ is nef and big.

In this section we prove the following result.

Proposition 26.3. *Either there is a commutative diagram*

$$\begin{tikzcd}
X \arrow[r, dashed, "\zeta"] \arrow[d, dashed, "\psi"'] & X \arrow[r, dashed, "\rho"] & V \arrow[d, "\nu"] \\
\mathbb{P}(1,1,4) \arrow[rr, dashed, "\varphi"'] & & \mathbb{P}^2
\end{tikzcd} \tag{26.1}$$

or there is a commutative diagram

$$\begin{tikzcd}
X \arrow[r, dashed, "\rho"] \arrow[d, dashed, "\xi"'] & V \arrow[d, "\nu"] \\
\mathbb{P}(1,1,6) \arrow[r, dashed, "\sigma"'] & \mathbb{P}^2,
\end{tikzcd} \tag{26.2}$$

where ζ, φ and σ are birational maps.

Proof. It follows from Theorem 3.3, Proposition 3.5 and Lemma 3.11 that $\varnothing \neq \mathbb{CS}\big(X, \frac{1}{k}\mathcal{M}\big) \subseteq \{P_2, P_3\}$ and the existence of the commutative diagram (26.2) is obvious if $\mathbb{CS}\big(X, \frac{1}{k}\mathcal{M}\big) \supset \{P_2, P_3\}$.

Lemma 26.4. *The set $\mathbb{CS}\big(X, \frac{1}{k}\mathcal{M}\big)$ contains the point P_3.*

Proof. Suppose that $P_3 \notin \mathbb{CS}\big(X, \frac{1}{k}\mathcal{M}\big)$. Then $\mathbb{CS}\big(X, \frac{1}{k}\mathcal{M}\big) = \{P_2\}$. Let $\mathcal{D}_2$ be the proper transform of $\mathcal{M}$ on U_2 and P_6 the singular point of U_2 that is contained in the exceptional divisor of α_2. Then $\mathcal{D}_2 \sim_{\mathbb{Q}} -kK_{U_2}$ by Theorem 2.2, P_6 is a quotient singularity of type $\frac{1}{3}(1,1,2)$ on U_2 and it follows from Lemmas 2.1 and 2.3 that $P_6 \in \mathbb{CS}\big(U_2, \frac{1}{k}\mathcal{D}_2\big)$.

Let $\pi\colon Z \to U_2$ be the weighted blow-up of P_6 with weights $(1,1,3)$, $\mathcal{B}$ the proper transform of $\mathcal{M}$ on Z and S a general surface in $|-K_Z|$. Then S is normal and the base locus of $|-K_Z|$ consists of the irreducible curve Δ such that $\mathcal{B}|_S \sim_{\mathbb{Q}} k\Delta$.

The equality $\Delta^2 = 1/7$ holds on S, but this is impossible by Lemmas 2.7 and 2.9.

For the rest of the proof we may assume that $\mathbb{CS}\big(X, \frac{1}{k}\mathcal{M}\big) = \{P_3\}$. Let $\mathcal{D}_3$ be the proper transform of $\mathcal{M}$ on U_3 and P_4 the singular point of U_3 contained in the exceptional divisor of α_3. Then $\mathcal{D}_3 \sim_{\mathbb{Q}} -kK_{U_3}$ and P_4 is a quotient singularity of type $\frac{1}{6}(1,1,5)$ of U_3. It follows from Lemmas 2.1 and 2.3 that $P_4 \in \mathbb{CS}\big(U_3, \frac{1}{k}\mathcal{D}_3\big)$.

Lemma 26.5. *The set* $\mathbb{CS}\big(U_3, \frac{1}{k}\mathcal{D}_3\big)$ *consists of the point* P_4.

Proof. Suppose that $\mathbb{CS}\big(U_3, \frac{1}{k}\mathcal{D}_3\big)$ contains a subvariety C of U_3 that is different from the point P_4. Let G be the exceptional divisor of β_4. Then G is a cone over a rational normal curve in $\mathbb{P}^6$ of degree 6. The curve C must be a ruling of G by Lemma 2.3, but this is impossible by Lemma 2.4.

Hence, $\mathbb{CS}\big(U_3, \frac{1}{k}\mathcal{D}_3\big)$ consists of the point P_4. Let $\mathcal{D}_{34}$ be the proper transform of $\mathcal{M}$ on U_{34} and P_5 the singular point of U_{34} contained in the exceptional divisor of β_4. Then $\mathcal{D}_{34} \sim_{\mathbb{Q}} -kK_{U_{34}}$ by Theorem 2.2, and P_5 is a quotient singularity of type $\frac{1}{5}(1,1,4)$ of U_{34}. But $\mathbb{CS}\big(U_{34}, \frac{1}{k}\mathcal{D}_{34}\big)$ contains P_5 by Lemmas 2.1 and 2.3. It follows from Theorem 2.2 that the proper transform of $\mathcal{M}$ on Y is contained in the fibres of η, which implies the existence of the commutative diagram (26.1). The proposition is proved.

§ 27. The case $n = 38$: a hypersurface of degree 18 in $\mathbb{P}(1,2,3,5,8)$

We use the notation and assumptions of § 3. Let $n = 38$. Then X is a general hypersurface of degree 18 in $\mathbb{P}(1,2,3,5,8)$. The singularities of X consist of points P_1 and P_2 that are quotient singularities of type $\frac{1}{2}(1,1,1)$, a point P_3 that is a quotient singularity of type $\frac{1}{5}(1,2,3)$ and a point P_4 that is a quotient singularity of type $\frac{1}{8}(1,3,5)$. The equality $-K_X^3 = 3/40$ holds.

Proposition 27.1. *The assertion of Theorem* 1.10 *holds for* $n = 38$.

Proof. The singularities of the log pair $\big(X, \frac{1}{k}\mathcal{M}\big)$ are canonical (see Remark 3.2). It follows from Theorem 3.3, Lemma 3.11 and Proposition 3.5 that $\mathbb{CS}\big(X, \frac{1}{k}\mathcal{M}\big) \subseteq \{P_3, P_4\}$. Arguing as in the proof of Proposition 21.1, we see that $P_4 \in \mathbb{CS}\big(X, \frac{1}{k}\mathcal{M}\big)$.

There is a commutative diagram

$$\begin{tikzcd}
& & Y \arrow[dl, "\gamma_5"'] \arrow[dr, "\gamma_3"] \arrow[drrr, "\eta"] & & & \\
& U_{34} \arrow[dl, "\beta_4"'] \arrow[dr, "\beta_3"] & & U_{45} \arrow[dl, "\beta_5"'] & & \mathbb{P}(1,2,3), \\
U_3 \arrow[dr, "\alpha_3"'] & & U_4 \arrow[dl, "\alpha_4"'] & & & \\
& X \arrow[uurrrr, dashed, "\psi"'] & & & &
\end{tikzcd}$$

where ψ is the natural projection, α_3 is the weighted blow-up of P_3 with weights $(1,2,3)$, α_4 is the weighted blow-up of P_4 with weights $(1,3,5)$, β_4 is the weighted blow-up with weights $(1,3,5)$ of the proper transform of P_4 on U_3, β_3 is the weighted blow-up with weights $(1,2,3)$ of the proper transform of P_3 on U_4, β_5 is the weighted blow-up with weights $(1,2,3)$ of the singular point of U_4 that is a quotient singularity of type $\frac{1}{5}(1,2,3)$ contained in the exceptional divisor of α_4, γ_3 is the weighted blow-up with weights $(1,2,3)$ of the proper transform of P_3 on U_{45}, γ_5 is the weighted blow-up with weights $(1,2,3)$ of the singular point of U_{34} that is a quotient singularity of type $\frac{1}{5}(1,2,3)$ contained in the exceptional divisor of β_4, and η is an elliptic fibration.

Let $\mathcal{D}_4$ be the proper transform of $\mathcal{M}$ on U_4, $\bar{P}_3$ the proper transform of P_3 on U_4, and P_5 and P_6 the singular points of U_4 that are quotient singularities of types $\frac{1}{5}(1,2,3)$ and $\frac{1}{3}(1,1,2)$, respectively, contained in the exceptional divisor of α_4. Arguing as in the proof of Proposition 25.1, we see that $\mathbb{CS}\big(U_4,\frac{1}{k}\mathcal{D}_4\big)$ contains either $\bar{P}_3$ or P_5.

Suppose that $\bar{P}_3\in\mathbb{CS}\big(U_4,\frac{1}{k}\mathcal{D}_5\big)$. Arguing as in the proofs of Propositions 21.1 and 25.1, we see that $\mathbb{CS}\big(U_4,\frac{1}{k}\mathcal{D}_5\big)$ contains P_5. Therefore, the assertion of Theorem 2.2 implies that Theorem 1.10 holds for X. Thus, we may assume that $\mathbb{CS}\big(U_4,\frac{1}{k}\mathcal{D}_5\big)$ contains P_5.

Let $\mathcal{D}_{45}$ be the proper transform of $\mathcal{M}$ on U_{45} and P_7, P_8 the singular points of U_{45} that are quotient singularities of types $\frac{1}{2}(1,1,1)$, $\frac{1}{3}(1,1,2)$, respectively, contained in the exceptional divisor of β_5. Then it follows from Lemma 2.1 that $\mathbb{CS}\big(U_{45},\frac{1}{k}\mathcal{D}_{45}\big)\neq\varnothing$, and Theorem 2.2 implies that Theorem 1.10 holds for X in the case when $\mathbb{CS}\big(U_{45},\frac{1}{k}\mathcal{D}_{45}\big)$ contains the proper transform of P_3 on U_{45}.

By Lemma 2.3, we may assume that $\mathbb{CS}\big(U_{45},\frac{1}{k}\mathcal{D}_{45}\big)\cap\{P_7,P_8\}\neq\varnothing$.

Suppose that $P_7\in\mathbb{CS}\big(U_{45},\frac{1}{k}\mathcal{D}_{45}\big)$. Then by considering the proper transform of the complete linear system $|-3K_X|$ on the weighted blow-up of P_7 with weights $(1,1,1)$, we easily obtain a contradiction as in the proof of Lemma 21.3.

Thus, the set $\mathbb{CS}\big(U_{45},\frac{1}{k}\mathcal{D}_{45}\big)$ contains the point P_8. Let $\pi\colon W\to U_{45}$ be the weighted blow-up of P_8, $\mathcal{B}$ the proper transform of $\mathcal{M}$ on W and D a general surface in $|-2K_W|$. Then $\mathcal{B}\sim_{\mathbb{Q}}-kK_W$ by Theorem 2.2, the surface D is normal and the pencil $|-2K_W|$ is the proper transform of the pencil $|-2K_X|$. The base locus of $|-2K_W|$ consists of curves C and L such that $\alpha_4\circ\beta_5\circ\pi(C)$ is the unique base curve of $|-2K_X|$ and the curve $\beta_5\circ\pi(L)$ is contained in the exceptional divisor of α_4.

The intersection form of C and L on D is negative definite. But the equivalence $\mathcal{B}|_D\sim_{\mathbb{Q}}kC+kL$ holds, which is impossible by Lemmas 2.9 and 2.7.

§ 28. The case $n=40$: a hypersurface of degree 19 in $\mathbb{P}(1,3,4,5,7)$

We use the notation and assumptions of § 3. Let $n=40$. Then X is a general hypersurface in $\mathbb{P}(1,3,4,5,7)$ of degree 19. The singularities of X consist of a point P_1 that is a quotient singularity of type $\frac{1}{3}(1,1,2)$, a point P_2 that is a quotient singularity of type $\frac{1}{4}(1,1,3)$, a point P_3 that is a quotient singularity of type $\frac{1}{5}(1,2,3)$ and a point P_4 that is a quotient singularity of type $\frac{1}{7}(1,3,4)$, and $-K_X^3 = 19/420$. There is a commutative diagram

$$\begin{tikzcd}
 & Y \arrow[dl, "\beta_4"'] \arrow[dr, "\beta_3"] \arrow[drrrr, "\eta"] & & & & \\
U_3 \arrow[dr, "\alpha_3"'] & & U_4 \arrow[dl, "\alpha_4"] & & & \mathbb{P}(1,3,4), \\
 & X \arrow[urrrr, dashed, "\psi"'] & & & &
\end{tikzcd}$$

where ψ is the natural projection, α_3 is the weighted blow-up of P_3 with weights $(1,2,3)$, α_4 is the weighted blow-up of P_4 with weights $(1,3,4)$, β_3 is the weighted blow-up with weights $(1,2,3)$ of the proper transform of P_3 on U_4, β_4 is the weighted blow-up with weights $(1,3,4)$ of the proper transform of P_4 on U_3, and η is an elliptic fibration.

Proposition 28.1. *The assertion of Theorem 1.10 holds for* $n = 40$.

Proof. It follows from Theorem 3.3, Lemma 3.11 and Proposition 3.5 that $\mathbb{CS}\big(X, \frac{1}{k}\mathcal{M}\big) \subseteq \{P_3, P_4\}$. The singularities of the log pair $\big(X, \frac{1}{k}\mathcal{M}\big)$ are canonical. Let $\mathcal{D}_3$ and $\mathcal{D}_4$ be the proper transforms of $\mathcal{M}$ on U_3 and U_4, respectively.

Lemma 28.2. *Suppose that* $\mathbb{CS}\big(X, \frac{1}{k}\mathcal{M}\big) = \{P_3, P_4\}$. *Then there is a commutative diagram*

$$\begin{tikzcd}
X \arrow[r, dashed, "\rho"] \arrow[d, dashed, "\psi"'] & V \arrow[d, "\nu"] \\
\mathbb{P}(1,3,4) \arrow[r, dashed, "\zeta"'] & \mathbb{P}^2,
\end{tikzcd} \tag{28.1}$$

where ζ *is a birational map.*

Proof. Let $\mathcal{H}$ be the proper transform of $\mathcal{M}$ on Y. Then it follows from Theorem 2.2 that the equivalence $\mathcal{H} \sim_{\mathbb{Q}} -kK_Y$ holds, which implies that $\mathcal{H}$ lies in the fibres of η. This implies the existence of the commutative diagram (28.1).

Let P_5 and P_6 be the singular points of U_3 contained in the exceptional divisor of α_3 such that P_5 and P_6 are quotient singularities of types $\frac{1}{2}(1,1,1)$ and $\frac{1}{3}(1,1,2)$, respectively, and let P_7 and P_8 be the singular points of U_4 that are quotient singularities of types $\frac{1}{3}(1,1,2)$ and $\frac{1}{4}(1,1,3)$, respectively, contained in the exceptional divisor of α_4. Then it follows from Theorem 2.2 and Lemmas 2.1 and 2.3 that

$\mathbb{CS}\big(U_3, \frac{1}{k}\mathcal{D}_3\big)$ contains either P_5 or P_6 if $\mathbb{CS}\big(X, \frac{1}{k}\mathcal{M}\big)$ does not contain P_4;

$\mathbb{CS}\big(U_4, \frac{1}{k}\mathcal{D}_4\big)$ contains either P_7 or P_8 if $\mathbb{CS}\big(X, \frac{1}{k}\mathcal{M}\big)$ does not contain P_3.

Lemma 28.3. *Suppose that* $P_4 \notin \mathbb{CS}\big(X, \frac{1}{k}\mathcal{M}\big)$. *Then* $P_5 \notin \mathbb{CS}\big(U_3, \frac{1}{k}\mathcal{D}_3\big)$.

Proof. Suppose that $\mathbb{CS}\big(U_3, \frac{1}{k}\mathcal{D}_3\big)$ contains P_5. Let $\gamma\colon W \to U_3$ be the weighted blow-up of P_5 with weights $(1,1,1)$, F the exceptional divisor of γ, $\mathcal{D}$ the proper transform of $\mathcal{M}$ on W, $\mathcal{H}$ the proper transform of the pencil $|-3K_X|$ on W, $\bar{D}$ a general surface in the linear system $\mathcal{D}$ and $\bar{H}$ a general surface in $\mathcal{H}$. Then

$$\bar{H} \sim_{\mathbb{Q}} (\alpha_3 \circ \gamma)^*(-3K_X) - \frac{3}{5}\gamma^*(E) - \frac{1}{2}F,$$

where E is the exceptional divisor of α_3. The base locus of $\mathcal{H}$ consists of a curve $\bar{C}$ such that $\alpha_3 \circ \gamma(C)$ is the unique base curve of $|-3K_X|$. On the other hand, it follows from Theorem 2.2 that $\bar{D} \sim_{\mathbb{Q}} -kK_W$. The equivalence $\bar{D}|_{\bar{H}} \sim_{\mathbb{Q}} k\bar{C}$ holds. But $\bar{C}^2 < 0$ on $\bar{H}$. It follows from Lemma 2.9 that the support of the cycle $\bar{H} \cdot \bar{D}$ consists of $\bar{C}$, which is impossible by Lemma 2.7.

Lemma 28.4. *Suppose that* $P_4 \notin \mathbb{CS}\big(X, \frac{1}{k}\mathcal{M}\big)$. *Then* $P_6 \notin \mathbb{CS}\big(U_3, \frac{1}{k}\mathcal{D}_3\big)$.

Proof. We suppose that $P_6 \in \mathbb{CS}\big(U_3, \frac{1}{k}\mathcal{D}_3\big)$ and seek a contradiction.

Let $\gamma\colon W \to U_3$ be the weighted blow-up of P_6 with weights $(1,1,2)$, let F and G be the exceptional divisors of α_3 and γ, respectively, let $\mathcal{B}$ and $\mathcal{D}$ be the proper transforms of $\mathcal{M}$ and $|-7K_X|$, respectively, on W and let D be a general surface in $\mathcal{D}$. Then it follows from Theorem 2.2 that $\mathcal{B} \sim_{\mathbb{Q}} -kK_W$. But the base locus of $\mathcal{D}$ does not contain curves. Moreover, we have

$$\mathcal{B} \sim_{\mathbb{Q}} -kK_W \sim_{\mathbb{Q}} (\alpha_3 \circ \gamma)^*(-kK_X) - \frac{k}{5}\gamma^*(F) - \frac{k}{3}G,$$

and the divisor D is nef because the base locus of $\mathcal{D}$ does not contain curves. But $D \cdot B_1 \cdot B_2 = -k^2/12$, where B_1 and B_2 are general surfaces in $\mathcal{B}$. This contradiction completes the proof of the lemma.

Lemma 28.5. *Suppose that* $P_3 \notin \mathbb{CS}\big(X, \frac{1}{k}\mathcal{M}\big)$. *Then* $P_7 \notin \mathbb{CS}\big(U_4, \frac{1}{k}\mathcal{D}_4\big)$.

Proof. We suppose that $\mathbb{CS}\big(U_4, \frac{1}{k}\mathcal{D}_4\big)$ contains P_7 and seek a contradiction.

Let $\gamma\colon W \to U_4$ be the weighted blow-up of P_7 with weights $(1,1,2)$, F the exceptional divisor of γ, $\mathcal{D}$ the proper transform of $\mathcal{M}$ on W, $\mathcal{H}$ the proper transform of $|-4K_X|$ on W, $\bar{D}$ a general surface in $\mathcal{D}$ and $\bar{H}$ a general surface in $\mathcal{H}$. Then

$$\bar{H} \sim_{\mathbb{Q}} (\alpha_3 \circ \gamma)^*(-4K_X) - \frac{4}{7}\gamma^*(E) - \frac{1}{3}F,$$

and the base locus of $\mathcal{H}$ consists of a curve $\bar{C}$ such that $\alpha_3 \circ \gamma(C)$ is the base curve of $|-4K_X|$. It follows from Theorem 2.2 that $\bar{D} \sim_{\mathbb{Q}} -kK_W$. The equality $\bar{C}^2 = -1/30$ holds on $\bar{H}$, which implies that the support of the cycle $\bar{H} \cdot \bar{D}$ consists of $\bar{C}$ because $\bar{D}|_{\bar{H}} \sim_{\mathbb{Q}} k\bar{C}$. But this is impossible by Lemma 2.7.

Lemma 28.6. *Suppose that* $P_3 \notin \mathbb{CS}\big(X, \frac{1}{k}\mathcal{M}\big)$. *Then* $P_8 \notin \mathbb{CS}\big(U_4, \frac{1}{k}\mathcal{D}_4\big)$.

Proof. We suppose that $\mathbb{CS}\big(U_4, \frac{1}{k}\mathcal{D}_4\big)$ contains P_8 and seek a contradiction.

Let $\gamma\colon W \to U_4$ be the weighted blow-up of P_8 with weights $(1,1,3)$, $\mathcal{D}$ the proper transform of $\mathcal{M}$ on W, $\bar{H}$ a general surface in $|-3K_W|$ and $\bar{D}$ a general surface in $\mathcal{D}$. Then $\bar{D} \sim_{\mathbb{Q}} -kK_W$. But the base locus of $|-3K_W|$ consists of an irreducible curve $\bar{C}$ such that $\alpha_4 \circ \gamma(C)$ is the base curve of $|-3K_X|$. The equality $\bar{C}^2 = -1/20$ holds on $\bar{H}$. But $\bar{D}|_{\bar{H}} \sim_{\mathbb{Q}} k\bar{C}$, which implies that the support of the cycle $\bar{H} \cdot \bar{D}$ consists of $\bar{C}$. But this is impossible by Lemma 2.7.

The proposition is proved.

§ 29. The case $n = 43$: a hypersurface of degree 20 in $\mathbb{P}(1, 2, 4, 5, 9)$

We use the notation and assumptions of § 3. Let $n = 43$. Then X is a general hypersurface in $\mathbb{P}(1, 2, 4, 5, 9)$ of degree 20 and the equality $-K_X^3 = 1/18$ holds. The singularities of X consist of points P_1 P_2, P_3, P_4 and P_5 that are quotient singularities of type $\frac{1}{2}(1, 1, 1)$ and a point P_6 that is a quotient singularity of type $\frac{1}{9}(1, 4, 5)$.

Proposition 29.1. *The assertion of Theorem 1.10 holds for* $n = 43$.

Proof. It follows from Theorem 3.3, Lemma 3.11 and Proposition 3.5 that $\mathbb{CS}\bigl(X, \frac{1}{k}\mathcal{M}\bigr) = \{P_6\}$. There is a commutative diagram

$$\begin{tikzcd} U \arrow[d, "\alpha"'] & & Y \arrow[ll, "\beta"'] \arrow[d, "\eta"] \\ X \arrow[rr, dashed, "\psi"'] & & \mathbb{P}(1, 2, 4), \end{tikzcd}$$

where ψ is the natural projection, α is the weighted blow-up of P_6 with weights $(1, 4, 5)$, β is the weighted blow-up with weights $(1, 1, 4)$ of the singular point of U that is a quotient singularity of type $\frac{1}{5}(1, 1, 4)$ contained in the exceptional divisor of α, and η is an elliptic fibration.

Let $\mathcal{D}$ be the proper transform of $\mathcal{M}$ on U and P_7, P_8 the singular points of U that are quotient singularities of types $\frac{1}{4}(1, 1, 3)$, $\frac{1}{5}(1, 1, 4)$, respectively, contained in the exceptional divisor of α. Then $\mathcal{D} \sim_{\mathbb{Q}} -kK_U$ by Theorem 2.2.

We must show that the proper transform of $\mathcal{M}$ on Y is contained in the fibres of η. This is implied by Theorem 2.2 if $P_8 \in \mathbb{CS}\bigl(U, \frac{1}{k}\mathcal{D}\bigr)$. Thus, we may assume that $P_8 \notin \mathbb{CS}\bigl(U, \frac{1}{k}\mathcal{D}\bigr)$.

Remark 29.2. The set $\mathbb{CS}\bigl(U, \frac{1}{k}\mathcal{D}\bigr)$ contains P_7 by Lemma 2.3 because $-K_U$ is nef and big.

Let $\gamma\colon W \to U$ be the weighted blow-up of P_7 with weights $(1, 1, 3)$, $\mathcal{B}$ the proper transform of $\mathcal{M}$ on W and P_9 the singular point of W that is a quotient singularity of type $\frac{1}{3}(1, 1, 2)$ contained in the exceptional divisor of γ. Then $\mathcal{B} \sim_{\mathbb{Q}} -kK_W$ by Theorem 2.2.

Lemma 29.3. *The set* $\mathbb{CS}\bigl(W, \frac{1}{k}\mathcal{B}\bigr)$ *does not contain the point* P_9.

Proof. We suppose that $\mathbb{CS}\bigl(W, \frac{1}{k}\mathcal{B}\bigr)$ contains P_9 and seek a contradiction.

Let $\pi\colon Z \to W$ be the weighted blow-up of P_9 with weights $(1, 1, 2)$, $\mathcal{H}$ the proper transform of $\mathcal{M}$ on Z and $\mathcal{P}$ the proper transform of $|-5K_X|$ on Z. Then $\mathcal{H} \sim_{\mathbb{Q}} -kK_Z$ by Theorem 2.2, but the base locus of $\mathcal{P}$ consists of an irreducible curve Γ such that $\alpha \circ \gamma \circ \pi(\Gamma)$ is the base curve of $|-5K_X|$.

Let H_1 and H_2 be general surfaces in $\mathcal{H}$ and D a general surface in $\mathcal{P}$. Then $D \cdot \Gamma = 1$ and $D^3 = 6$. Therefore, the divisor D is nef and big, but elementary computations imply that $D \cdot H_1 \cdot H_2 = 0$, which is impossible by Corollary 2.6.

Therefore, Lemma 2.3 has the following consequence.

Corollary 29.4. *The singularities of the log pair* $(W, \frac{1}{k}\mathcal{B})$ *are terminal.*

The hypersurface X can be given by the following quasi-homogeneous equation of degree 20:

$$w^2y + wf_{11}(x,y,z,t) + f_{20}(x,y,z,t) = 0 \subset \mathbb{P}(1,2,4,5,9) \cong \mathrm{Proj}\big(\mathbb{C}[x,y,z,t,w]\big),$$

where $\mathrm{wt}(x) = 1$, $\mathrm{wt}(y) = 2$, $\mathrm{wt}(z) = 4$, $\mathrm{wt}(t) = 5$, $\mathrm{wt}(w) = 9$, and $f_i(x,y,z,t)$ is a quasi-homogeneous polynomial of degree i. Let D be a general surface in the pencil $|-2K_X|$ and S the surface cut out on X by the equation $x = 0$. Then D is cut out on X by the quasi-homogeneous equation $\lambda x^2 + \mu y = 0$, where $(\lambda,\mu) \in \mathbb{P}^1$. The base locus of $|-2K_X|$ consists of the irreducible curve C cut out on X by the equations $x = y = 0$.

In the neighbourhood of the point P_6, the monomials x, z and t can be regarded as weighted local coordinates on X such that $\mathrm{wt}(x) = 1$, $\mathrm{wt}(z) = 4$ and $\mathrm{wt}(t) = 5$. Then in the neighbourhood of P_7, the weighted blow-up α is given by the equations

$$x = \tilde{x}\tilde{z}^{\frac{1}{9}}, \qquad z = \tilde{z}^{\frac{4}{9}}, \qquad t = \tilde{t}\tilde{z}^{\frac{5}{9}},$$

where $\tilde{x}$, $\tilde{z}$ and $\tilde{t}$ are weighted local coordinates on U in the neighbourhood of P_7 such that $\mathrm{wt}(\tilde{x}) = 1$, $\mathrm{wt}(\tilde{z}) = 3$ and $\mathrm{wt}(\tilde{t}) = 1$.

Let E be the exceptional divisor of α and $\widetilde{D}$, $\widetilde{S}$, $\widetilde{C}$ the proper transforms on U of D, S, C, respectively. Then E is given by the equation $\tilde{z} = 0$ and $\widetilde{S}$ by the equation $\tilde{x} = 0$. Moreover, it follows from the local equation of $\widetilde{D}$ that $\widetilde{D}\cdot\widetilde{S} = \widetilde{C} + 2\tilde{L}_1$, where $\tilde{L}_1$ is the curve given locally by the equations $\tilde{z} = \tilde{x} = 0$. Moreover, $\widetilde{D}$ is not normal at a general point of $\tilde{L}_1$ and $\widetilde{D} \sim_{\mathbb{Q}} 2\widetilde{S}$.

In the neighbourhood of P_9, γ is given by the equations

$$\tilde{x} = \bar{x}\bar{z}^{\frac{1}{4}}, \qquad \tilde{z} = \bar{z}^{\frac{3}{4}}, \qquad \tilde{t} = \bar{t}\bar{z}^{\frac{1}{4}},$$

where $\bar{x}$, $\bar{z}$ and $\bar{t}$ are weighted local coordinates on W in the neighbourhood of P_9 such that $\mathrm{wt}(\bar{x}) = 1$, $\mathrm{wt}(\bar{z}) = 2$ and $\mathrm{wt}(\bar{t}) = 1$. In particular, the exceptional divisor of γ is given by the equation $\bar{z} = 0$ and the proper transform of S on W is given by the equation $\bar{x} = 0$.

Let F be the exceptional divisor of γ and $\bar{D}$, $\bar{S}$, $\bar{E}$, $\bar{C}$, $\bar{L}_1$ the proper transforms on W of D, S, E, C, $\tilde{L}_1$, respectively. Then

$$\begin{aligned} \bar{S} &\sim_{\mathbb{Q}} -K_W \sim_{\mathbb{Q}} (\alpha\circ\gamma)^*(-K_X) - \frac{1}{9}\gamma^*(E) - \frac{1}{4}F, \\ \bar{D} &\sim_{\mathbb{Q}} -2K_W \sim_{\mathbb{Q}} (\alpha\circ\gamma)^*(-2K_X) - \frac{2}{9}\gamma^*(E) - \frac{1}{2}F, \\ \bar{E} &\sim_{\mathbb{Q}} \gamma^*(E) - \frac{3}{4}F. \end{aligned} \tag{29.1}$$

Let $\bar{L}_2$ be the curve on W given by the equations $\bar{z} = \bar{x} = 0$. Then

$$\bar{D}\cdot\bar{S} = \bar{C} + 2\bar{L}_1 + \bar{L}_2, \qquad \bar{D}\cdot\bar{E} = 2\bar{L}_1, \qquad \bar{D}\cdot F = 2\bar{L}_2,$$

and the base locus of $|-2K_W|$ consists of $\bar{C}$, $\bar{L}_1$ and $\bar{L}_2$. The equivalences (29.1) imply that

$$\bar{D}\cdot\bar{C} = 0, \qquad \bar{D}\cdot\bar{L}_1 = -\frac{2}{5}, \qquad \bar{D}\cdot\bar{L}_2 = \frac{2}{3}.$$

Let $\bar{H}$ and $\bar{T}$ be the proper transforms on W of the surfaces cut out on X by the equations $y=0$ and $t=0$, respectively. Then

$$\begin{aligned}\bar{T}&\sim_{\mathbb{Q}}(\alpha\circ\gamma)^*(-5K_X)-\frac{5}{9}\gamma^*(E)-\frac{1}{4}F\sim_{\mathbb{Q}}(\alpha\circ\gamma)^*(-5K_X)-\frac{5}{9}\bar{E}-\frac{2}{3}F,\\ \bar{H}&\sim_{\mathbb{Q}}(\alpha\circ\gamma)^*(-2K_X)-\frac{11}{9}\gamma^*(E)-\frac{3}{2}F\sim_{\mathbb{Q}}(\alpha\circ\gamma)^*(-2K_X)-\frac{11}{9}\bar{E}-\frac{5}{3}F,\end{aligned}\tag{29.2}$$

which implies that $-14K_W\sim_{\mathbb{Q}}14\bar{D}\sim_{\mathbb{Q}}2\bar{T}+2\bar{H}+2\bar{E}$, and the support of the cycle $\bar{T}\cdot\bar{H}$ does not contain $\bar{L}_2$ or $\bar{C}$. Therefore, the base locus of $|-14K_W|$ does not contain curves other than $\bar{L}_1$. The singularities of the mobile log pair $(W,\lambda|-14K_W|)$ are log-terminal for some rational number $\lambda>1/14$, but the divisor $K_W+\lambda|-14K_W|$ has non-negative intersection with all the curves on W except $\bar{L}_1$. It follows from [12] that the log-flip $\zeta\colon W\dashrightarrow Z$ in $\bar{L}_1$ with respect to the log pair $(W,\lambda|-14K_W|)$ exists.

Let $\mathcal{P}$ be the proper transform of $\mathcal{M}$ on Z. Then the singularities of the log pair $(Z,\frac{1}{k}\mathcal{P})$ are terminal because those of the log pair $(W,\frac{1}{k}\mathcal{B})$ are, but the rational map ζ is a log flop with respect to the log pair $(W,\frac{1}{k}\mathcal{B})$ while $-K_Z$ is numerically effective because the base locus of $|-14K_W|$ does not contain curves other than $\bar{L}_1$ and the inequality $-K_W\cdot\bar{L}_1<0$ holds. We claim that $-K_Z$ is big, which is impossible by Lemma 2.1.

The rational functions y/x^2 and ty/x^7 are contained in $|2S|$ and $|7S|$, respectively, but the equivalences (29.2) imply that y/x^2 and ty/x^7 are contained in $\left|2\bar{S}\right|$ and $\left|7\bar{S}\right|$, respectively.

Let $\bar{Z}$ be the proper transform on W of the irreducible surface cut out on X by the equation $z=0$. Then the equivalences

$$\bar{Z}\sim_{\mathbb{Q}}(\alpha\circ\gamma)^*(-4K_X)-\frac{4}{9}\gamma^*(E)\sim_{\mathbb{Q}}(\alpha\circ\gamma)^*(-4K_X)-\frac{4}{9}\bar{E}-\frac{1}{3}F$$

hold, so that $-6K_W\sim_{\mathbb{Q}}\bar{Z}+\bar{H}+\bar{E}$. Thus, the rational function zy/x^6 is contained in the linear system $\left|6\bar{S}\right|$. Thus, the linear system $|-42K_W|$ maps W dominantly onto some three-dimensional variety, which implies that the divisor $-K_Z$ is big. Thus, the proposition is proved.

Remark 29.5. The function wy^3/x^{15} belongs to $|15\bar{S}|$. Thus, $|-210K_W|$ induces a birational map $W\dashrightarrow X'$ such that X' is a hypersurface in $\mathbb{P}(1,2,6,7,15)$ of degree 30.

§ 30. The case $n=44$: a hypersurface of degree 20 in $\mathbb{P}(1,2,5,6,7)$

We use the notation and assumptions of § 3. Let $n=44$. Then X is a general hypersurface in $\mathbb{P}(1,2,5,6,7)$ of degree 20, and the equality $-K_X^3=1/21$ holds. The singularities of X consist of points P_1, P_2 and P_3 that are quotient singularities of type $\frac{1}{2}(1,1,1)$, a point P_4 that is a quotient singularity of type $\frac{1}{6}(1,1,5)$, and a point P_5 that is a quotient singularity of type $\frac{1}{7}(1,2,5)$.

There is a commutative diagram

$$
\begin{tikzcd}
 & Y \arrow[dl, "\beta_5"'] \arrow[dr, "\beta_4"] \arrow[drr, "\eta"] & & \\
U_4 \arrow[dr, "\alpha_4"'] & & U_5 \arrow[dl, "\alpha_5"] & \mathbb{P}(1,2,5), \\
 & X \arrow[urr, dashed, "\psi"'] & &
\end{tikzcd}
$$

where ψ is a projection, α_4 is the weighted blow-up of P_4 with weights $(1,1,5)$, α_5 is the weighted blow-up of P_5 with weights $(1,2,5)$, β_4 is the weighted blow-up with weights $(1,1,5)$ of the proper transform of P_4 on U_5, β_5 is the weighted blow-up with weights $(1,2,5)$ of the proper transform of P_5 on U_4, and η is an elliptic fibration.

There is a commutative diagram

$$
\begin{tikzcd}
U_5 \arrow[d, "\alpha_5"'] & W \arrow[l, "\beta_6"'] \arrow[d, "\omega"] \\
X \arrow[r, dashed, "\xi"'] & \mathbb{P}(1,1,3),
\end{tikzcd}
$$

where ξ is a projection, β_6 is the weighted blow-up with weights $(1,2,3)$ of the singular point of U_5 that is a singularity of type $\frac{1}{5}(1,2,3)$ contained in the α_5-exceptional divisor, and ω is an elliptic fibration.

Proposition 30.1. *Either there is a commutative diagram*

$$
\begin{tikzcd}
X \arrow[r, dashed, "\rho"] \arrow[d, dashed, "\psi"'] & V \arrow[d, "\nu"] \\
\mathbb{P}(1,2,5) \arrow[r, dashed, "\varphi"'] & \mathbb{P}^2
\end{tikzcd}
\tag{30.1}
$$

or there is a commutative diagram

$$
\begin{tikzcd}
X \arrow[r, dashed, "\zeta"] \arrow[d, dashed, "\xi"'] & X \arrow[r, dashed, "\rho"] & V \arrow[d, "\nu"] \\
\mathbb{P}(1,1,3) \arrow[rr, dashed, "\sigma"'] & & \mathbb{P}^2,
\end{tikzcd}
\tag{30.2}
$$

where φ, ζ and σ are birational maps.

Proof. It follows from Lemma 3.11 and Proposition 3.5 that $\mathbb{CS}\big(X, \frac{1}{k}\mathcal{M}\big) \subseteq \{P_4, P_5\}$. Arguing as in the proof of Lemma 18.2, we see that $\mathbb{CS}\big(X, \frac{1}{k}\mathcal{M}\big)$ contains P_5. The existence of the commutative diagram (30.1) follows easily from Theorem 2.2 in the case when $\mathbb{CS}\big(X, \frac{1}{k}\mathcal{M}\big) = \{P_4, P_5\}$. Thus, we may assume that $\mathbb{CS}\big(X, \frac{1}{k}\mathcal{M}\big)$ consists of the point P_5.

Let $\mathcal{D}_5$ be the proper transform of $\mathcal{M}$ on U_5. Then $\mathcal{D}_5 \sim_{\mathbb{Q}} -kK_{U_5}$ by Theorem 2.2, which implies that $\mathbb{CS}\big(U_5, \frac{1}{k}\mathcal{D}_5\big)$ is non-empty by Lemma 2.1. Let G be the exceptional divisor of α_5 and $\bar{P}_6$, $\bar{P}_7$ the singular points of the surface G that are quotient singularities of types $\frac{1}{5}(1,2,3)$, $\frac{1}{2}(1,1,1)$ on U_5, respectively. In the case when $\mathbb{CS}\big(U_5, \frac{1}{k}\mathcal{D}_5\big)$ contains $\bar{P}_6$, the existence of the commutative diagram (30.2) follows from Theorem 2.2. Therefore, we may assume by Lemma 2.3 that $\mathbb{CS}\big(U_5, \frac{1}{k}\mathcal{D}_5\big)$ contains the point $\bar{P}_7$.

Remark 30.2. The linear system $|-5K_{U_5}|$ is a proper transform of $|-5K_X|$ and its base locus consists of the irreducible curve that is the fibre of the rational map $\psi\circ\alpha_5$ passing through $\bar{P}_7$.

Let $\pi\colon U\to U_5$ be the weighted blow-up of $\bar{P}_7$ with weights $(1,1,1)$, let F be the exceptional divisor of the blow-up π, let $\mathcal{D}$ be the proper transform of $\mathcal{M}$ on U and let $\mathcal{H}$ be the proper transform of $|-5K_{U_5}|$ on U. Then $\mathcal{D}\sim_{\mathbb{Q}} -kK_U$ by Theorem 2.2. But

$$\mathcal{H}\sim_{\mathbb{Q}} \pi^*\big(-5K_{U_5}\big)-\frac{1}{2}\,F,$$

and the base locus of $\mathcal{H}$ consists of an irreducible curve Z such that $\alpha_5\circ\pi(Z)$ is the unique curve in the base locus of $|-5K_X|$.

Let S be a general surface in $\mathcal{H}$. Then the equality $S\cdot Z=1/3$ holds, which implies that the divisor $\pi^*(-10K_{U_5})-F$ is nef. Let D_1 and D_2 be general surfaces in $\mathcal{D}$. Then

$$-\frac{2k^2}{3}=\big(\pi^*(-10K_{U_5})-F\big)\cdot\bigg(\pi^*(-kK_{U_5})-\frac{k}{2}\,F\bigg)^2=\big(\pi^*(-10K_{U_5})-F\big)\cdot D_1\cdot D_2\geqslant 0,$$

which is a contradiction. The proposition is proved.

§ 31. The case $n=47$: a hypersurface of degree 21 in $\mathbb{P}(1,1,5,7,8)$

We use the notation and assumptions of § 3. Let $n=47$. Then X is a general hypersurface in $\mathbb{P}(1,1,5,7,8)$ of degree 21 whose singularities consist of a point P_1 that is a singularity of type $\frac{1}{5}(1,2,3)$ and a point P_2 that is a singularity of type $\frac{1}{8}(1,1,7)$.

Proposition 31.1. *The assertion of Theorem 1.10 holds for $n=47$.*

Proof. It follows from Theorem 3.3, Lemma 3.11 and Proposition 3.5 that $\mathbb{CS}\big(X, \frac{1}{k}\mathcal{M}\big)\subseteq\{P_1,P_2\}$.

The hypersurface X can be given by the equation

$$w^2z+\sum_{i=0}^{2}wz^ig_{13-5i}(x,y,t)+\sum_{i=0}^{3}z^ig_{21-5i}(x,y,t)=0\subset \operatorname{Proj}\big(\mathbb{C}[x,y,z,t,w]\big),$$

where $\mathrm{wt}(x)=1$, $\mathrm{wt}(y)=1$, $\mathrm{wt}(z)=5$, $\mathrm{wt}(t)=7$, $\mathrm{wt}(w)=8$ and $g_i(x,y,t)$ is a quasi-homogeneous polynomial of degree i. There is a commutative diagram

$$\begin{tikzcd} U \arrow[d, "\alpha"'] & W \arrow[l, "\beta"'] & Y \arrow[l, "\gamma"'] \arrow[d, "\eta"] \\ X \arrow[rr, dashed, "\psi"'] & & \mathbb{P}(1,1,5), \end{tikzcd}$$

where α is the weighted blow-up of P_2 with weights $(1,1,7)$, β is the weighted blow-up with weights $(1,1,6)$ of the singular point of U that is a quotient singularity of type $\frac{1}{7}(1,1,6)$, γ is the weighted blow-up with weights $(1,1,5)$ of the singular point of W that is a quotient singularity of type $\frac{1}{6}(1,1,5)$, and η is an elliptic fibration.

Lemma 31.2. *The set $\mathbb{CS}\big(X,\frac{1}{k}\mathcal{M}\big)$ contains the point P_2.*

Proof. Suppose that $\mathbb{CS}\big(X,\frac{1}{k}\mathcal{M}\big)$ does not contain P_2. Then $\mathbb{CS}\big(X,\frac{1}{k}\mathcal{M}\big)=\{P_1\}$.

Let $\pi\colon Z\to X$ be the weighted blow-up of P_1 with weights $(1,2,3)$, E the exceptional divisor of π and $\mathcal{B}$ the proper transform of $\mathcal{M}$ on Z. Then $E\cong\mathbb{P}(1,2,35)$ and $\mathcal{B}\sim_{\mathbb{Q}}-kK_Z$ by Theorem 2.2.

Let $\overline{P}_3$ and $\overline{P}_4$ be the singular points of Z contained in E that are singularities of types $\frac{1}{2}(1,1,1)$ and $\frac{1}{3}(1,1,2)$, respectively. Arguing as in the proof of Proposition 22.1, we see that the singularities of the log pair $\big(Z,\frac{1}{k}\mathcal{B}\big)$ are terminal.

The base locus of the pencil $|-K_Z|$ consists of irreducible curves C and L such that the curve $\pi(C)$ is cut out on X by the equations $x=y=0$ and L is contained in the divisor E and in the linear system $\big|\mathcal{O}_{\mathbb{P}(1,2,3)}(1)\big|$. The inequalities $-K_Z\cdot C<0$ and $-K_Z\cdot L>0$ hold. It follows from [12] that there is an antiflip $\zeta\colon Z\dashrightarrow\overline{Z}$ in C. Then the divisor $-K_{\overline{Z}}$ is nef.

Let $\mathcal{P}$ be the proper transform of $\mathcal{M}$ on $\overline{Z}$. Then the singularities of the log pair $\big(\overline{Z},\frac{1}{k}\mathcal{P}\big)$ are terminal because those of the log pair $\big(Z,\frac{1}{k}\mathcal{B}\big)$ are, and the antiflip ζ is a log flop with respect to the log pair $\big(Z,\frac{1}{k}\mathcal{B}\big)$. The rational functions y/x, zy/x^6, ty/x^8, yw/x^9 are contained in the linear system $|-aK_Z|$, where $a=1,6,8,9$, respectively. Therefore, the complete linear system $|-72K_Z|$ induces a birational map $\chi\colon Z\dashrightarrow\overline{X}$ such that $\overline{X}$ is a hypersurface of degree 24 in $\mathbb{P}(1,1,6,8,9)$. Hence, the divisor $-K_{\overline{Z}}$ is big, which is impossible by Lemma 2.1.

Let G be the exceptional divisor of α, $\mathcal{D}$ the proper transform of $\mathcal{M}$ on U, $\overline{P}_1$ the proper transform of P_1 on U and $\overline{P}_5$ the singular point of U that is contained in G. Then $\mathcal{D}\sim_{\mathbb{Q}}-kK_U$ by Theorem 2.2.

Lemma 31.3. *The set $\mathbb{CS}\big(U,\frac{1}{k}\mathcal{D}\big)$ contains the point $\overline{P}_5$.*

Proof. Suppose that $\overline{P}_5\notin\mathbb{CS}\big(U,\frac{1}{k}\mathcal{D}\big)$. Then $\mathbb{CS}\big(U,\frac{1}{k}\mathcal{D}\big)=\{\overline{P}_1\}$ by Lemmas 2.1 and 2.3.

Let $\pi\colon Z\to U$ be the weighted blow-up of $\overline{P}_1$ with weights $(1,2,3)$, E the exceptional divisor of π and $\overline{G}$, $\mathcal{B}$ the proper transforms of G, $\mathcal{M}$ on Z, respectively. Then $\mathcal{B}\sim_{\mathbb{Q}}-kK_Z$ by Theorem 2.2. Arguing as in the proof of Lemma 31.2, we see that the singularities of the log pair $\big(Z,\frac{1}{k}\mathcal{B}\big)$ are terminal.

Let S_x, S_y, S_z, S_t and S_w be the proper transforms on Z of the surfaces cut out on X by the equations $x=0$, $y=0$, $z=0$, $t=0$ and $w=0$, respectively. Then

$$
\begin{aligned}
S_x &\sim_{\mathbb{Q}} (\alpha\circ\pi)^*(-K_X)-\frac{1}{5}\,E-\frac{1}{8}\,\bar{G},\\
S_y &\sim_{\mathbb{Q}} (\alpha\circ\pi)^*(-K_X)-\frac{6}{5}\,E-\frac{1}{8}\,\bar{G},\\
S_z &\sim_{\mathbb{Q}} (\alpha\circ\pi)^*(-5K_X)-\frac{1}{5}\,E-\frac{13}{8}\,\bar{G},\\
S_t &\sim_{\mathbb{Q}} (\alpha\circ\pi)^*(-7K_X)-\frac{2}{5}\,E-\frac{7}{8}\,\bar{G},\\
S_w &\sim_{\mathbb{Q}} (\alpha\circ\pi)^*(-8K_X)-\frac{3}{5}\,E.
\end{aligned}
\tag{31.1}
$$

The base locus of $|-K_Z|$ consists of irreducible curves C and L such that the curve $\alpha\circ\pi(C)$ is cut out on X by the equations $x=y=0$, the curve L is contained in the divisor E, and L is the unique curve of $\big|\mathcal{O}_{\mathbb{P}(1,2,3)}(1)\big|$.

It follows from [12] that there is an antiflip $\zeta\colon Z \dashrightarrow \bar{Z}$ in C such that the divisor $-K_{\bar{Z}}$ is nef.

Let $\mathcal{P}$ be the proper transform of $\mathcal{M}$ on $\bar{Z}$. Then the singularities of the log pair $\big(\bar{Z},\frac{1}{k}\mathcal{P}\big)$ are terminal. The equivalences (31.1) imply that the functions y/x, zy/x^6, ty/x^8 and wzy^2/x^{15} are contained in the complete linear system $|aS_x|$, where $a=1,6,8$ and 15, respectively. Hence, the linear system $|-120K_Z|$ induces a birational map $\chi\colon Z \dashrightarrow \bar{X}$ such that $\bar{X}$ is a hypersurface of degree 30 in $\mathbb{P}(1,1,6,8,15)$. Hence, the divisor $-K_{\bar{Z}}$ is big, which is impossible by Lemma 2.1.

Remark 31.4. It follows from Lemma 2.4 that $\mathbb{CS}\big(U,\frac{1}{k}\mathcal{D}\big)=\{\bar{P}_5\}$.

Let $\mathcal{H}$ be the proper transform of $\mathcal{M}$ on W, F the exceptional divisor of β, $\widetilde{P}_1$ the proper transform of P_1 on W and $\widetilde{P}_6$ the singular point of W contained in F. Then $\mathcal{H}\sim_{\mathbb{Q}} -kK_W$ by Theorem 2.2, which implies by Lemma 2.1 that $\mathbb{CS}\big(W,\frac{1}{k}\mathcal{H}\big)\neq\varnothing$.

Lemma 31.5. *Suppose that $\widetilde{P}_6\in\mathbb{CS}\big(W,\frac{1}{k}\mathcal{H}\big)$. Then there is a commutative diagram*

$$
\begin{tikzcd}
X \arrow[r, dashed, "\rho"] \arrow[d, dashed, "\psi"'] & V \arrow[d, "\nu"] \\
\mathbb{P}(1,1,5) \arrow[r, dashed, "\zeta"'] & \mathbb{P}^2,
\end{tikzcd}
\tag{31.2}
$$

where ζ is a birational map.

Proof. Let $\mathcal{B}$ be the proper transform of $\mathcal{M}$ on Y. It follows from Theorem 2.2 that $\mathcal{B}\sim_{\mathbb{Q}} -kK_Y$, which implies that $\mathcal{B}$ lies in the fibres of η. This implies the existence of the diagram (31.2).

We may assume by Lemma 2.3 that $\mathbb{CS}\big(W,\frac{1}{k}\mathcal{H}\big)$ consists of the point $\widetilde{P}_1$.

Let $\pi\colon Z\to W$ be the weighted blow-up of $\widetilde{P}_1$ with weights $(1,2,3)$, E the exceptional divisor of π, $\mathcal{B}$ the proper transform of $\mathcal{M}$ on Z and $\tilde{G}$, $\tilde{F}$ the proper transforms on Z of the surfaces G, F, respectively. Arguing as in the proof of

Lemma 31.2, we see that the singularities of the log pair $\bigl(Z, \frac{1}{k}\mathcal{B}\bigr)$ are terminal. But $\mathcal{B} \sim_{\mathbb{Q}} -kK_Z$ by Theorem 2.2.

Let S_x, S_y, S_z, S_t and S_w be the proper transforms on Z of the surfaces cut out on X by the equations $x=0$, $y=0$, $z=0$, $t=0$ and $w=0$, respectively. Then

$$\begin{gathered} S_x \sim_{\mathbb{Q}} (\alpha\circ\beta\circ\pi)^*(-K_X) - \frac{1}{5}\,E - \frac{1}{8}\,\widetilde{G} - \frac{1}{4}\,\widetilde{F}, \\ S_y \sim_{\mathbb{Q}} (\alpha\circ\beta\circ\pi)^*(-K_X) - \frac{6}{5}\,E - \frac{1}{8}\,\widetilde{G} - \frac{1}{4}\,\widetilde{F}, \\ S_z \sim_{\mathbb{Q}} (\alpha\circ\beta\circ\pi)^*(-5K_X) - \frac{1}{5}\,E - \frac{13}{8}\,\widetilde{G} - \frac{9}{4}\,\widetilde{F}, \\ S_t \sim_{\mathbb{Q}} (\alpha\circ\beta\circ\pi)^*(-7K_X) - \frac{2}{5}\,E - \frac{7}{8}\,\widetilde{G} - \frac{3}{4}\,\widetilde{F}, \\ S_w \sim_{\mathbb{Q}} (\alpha\circ\beta\circ\pi)^*(-8K_X) - \frac{3}{5}\,E. \end{gathered} \tag{31.3}$$

The equivalences (31.3) imply that the functions y/x, zy/x^6, tzy^2/x^{14} and wz^2y^3/x^{21} are contained in the linear systems $|S_x|$ $|6S_x|$, $|14S_x|$ and $|21S_x|$, respectively. Therefore, the complete linear system $|-42K_Z|$ induces a birational map $\chi\colon Z \dashrightarrow \bar{X}$ such that $\bar{X}$ is a hypersurface in $\mathbb{P}(1,1,6,14,21)$ of degree 42. The base locus of $|-42K_Z|$ consists of an irreducible curve C such that the curve $\alpha\circ\beta\circ\pi(C)$ is cut out on X by the equations $x=y=0$. Therefore, it follows from [12] that there is an antiflip $\zeta\colon Z \dashrightarrow \bar{Z}$ in C, which implies that $-K_{\bar{Z}}$ is nef and big. The rational map ζ is a log flop with respect to the log pair $\bigl(Z, \frac{1}{k}\mathcal{B}\bigr)$. Therefore, we see that the singularities of the log pair $\bigl(\bar{Z}, \frac{1}{k}\mathcal{P}\bigr)$ are terminal, where $\mathcal{P}$ is the proper transform of $\mathcal{M}$ on Z. But this is impossible by Lemma 2.1. The proposition is proved.

Remark 31.6. We have constructed birational transformations of X into hypersurfaces in $\mathbb{P}(1,1,6,8,9)$, $\mathbb{P}(1,1,6,8,15)$ and $\mathbb{P}(1,1,6,14,21)$ of degrees 24, 30 and 42, respectively. The anticanonical models of the varieties U and W are hypersurfaces in $\mathbb{P}(1,1,5,7,13)$ and $\mathbb{P}(1,1,5,12,18)$ of degrees 26 and 36, respectively. Arguing as in the proof of Proposition 31.1, we see that, up to the action of the group $\mathrm{Bir}(X)$, there are no other non-trivial birational transformations of X into Fano threefolds with canonical singularities.

§ 32. The case $n = 48$: a hypersurface of degree 21 in $\mathbb{P}(1,2,3,7,9)$

We use the notation and assumptions of § 3. Let $n = 48$. Then X is a general hypersurface in $\mathbb{P}(1,2,3,7,9)$ of degree 21 whose singularities consist of a point P_1 that is a singularity of type $\frac{1}{2}(1,1,1)$, points P_2 and P_3 that are singularities of type $\frac{1}{3}(1,1,2)$ and a point P_4 that is a singularity of type $\frac{1}{9}(1,2,7)$. There is a commutative diagram

$$\begin{tikzcd} U \arrow[d, "\alpha"'] & W \arrow[l, "\beta"'] & Y \arrow[l, "\gamma"'] \arrow[d, "\eta"] \\ X \arrow[rr, dashed, "\psi"'] & & \mathbb{P}(1,2,3), \end{tikzcd}$$

where ψ is the natural projection, α is the weighted blow-up of P_4 with weights $(1,2,7)$, β is the weighted blow-up with weights $(1,2,5)$ of the singular point of U that is a quotient singularity of type $\frac{1}{7}(1,2,5)$ contained in the exceptional divisor of α, γ is the weighted blow-up with weights $(1,2,3)$ of the singular point of W that is a singularity of type $\frac{1}{5}(1,2,3)$ contained in the exceptional divisor of β, and η is an elliptic fibration.

Proposition 32.1. *The assertion of Theorem 1.10 holds for* $n=48$.

Proof. It follows from Theorem 3.3, Lemma 3.11 and Proposition 3.5 that $\mathbb{CS}\big(X,\frac{1}{k}\mathcal{M}\big)=\{P_4\}$. Let E be the α-exceptional divisor, $\mathcal{D}$ the proper transform of $\mathcal{M}$ on U and P_5, P_6 the singular points of U contained in the surface E that are singularities of types $\frac{1}{2}(1,1,1)$, $\frac{1}{7}(1,2,5)$, respectively. Then $E\cong\mathbb{P}(1,2,7)$, but $\mathcal{D}\sim_{\mathbb{Q}}-kK_U$ by Theorem 2.2.

Lemma 32.2. *The set* $\mathbb{CS}\big(U,\frac{1}{k}\mathcal{D}\big)$ *does not contain the point* P_5.

Proof. Suppose that $\mathbb{CS}\big(U,\frac{1}{k}\mathcal{D}\big)$ contains P_5. Let $\pi\colon Z\to U$ be the weighted blow-up of P_5 with weights $(1,1,1)$, G the exceptional divisor of the blow-up π and $\mathcal{B}$, $\mathcal{P}$ the proper transforms of the linear systems $\mathcal{M}$, $|-7K_X|$ on Z, respectively. Then $\mathcal{B}\sim_{\mathbb{Q}}-kK_Z$ by Theorem 2.2, but the base locus of $\mathcal{P}$ does not contain curves. Let H be a general divisor of $\mathcal{P}$. Then H is numerically effective. But

$$H\cdot B_1\cdot B_2=\biggl((\alpha\circ\pi)^*(-kK_X)-\frac{k}{9}\,\pi^*(E)-\frac{k}{2}\,G\biggr)^2 \times\biggl((\alpha\circ\pi)^*(-7K_X)-\frac{7}{9}\,\pi^*(E)-\frac{1}{2}\,G\biggr)=-\frac{1}{6}\,k^2,$$

where B_1 and B_2 are general surfaces in $\mathcal{B}$. The resulting contradiction completes the proof of the lemma.

It follows from Lemmas 2.1 and 2.3 that $\mathbb{CS}\big(U,\frac{1}{k}\mathcal{D}\big)=\{P_6\}$. Let F be the exceptional divisor of the blow-up β, $\mathcal{H}$ the proper transform of $\mathcal{M}$ on W and P_7, P_8 the singular points of W contained in F that are singularities of types $\frac{1}{2}(1,1,1)$, $\frac{1}{5}(1,2,3)$, respectively. Then $F\cong\mathbb{P}(1,2,5)$, but $\mathcal{H}\sim_{\mathbb{Q}}-kK_W$ by Theorem 2.2.

Lemma 32.3. *Suppose that* $P_8\in\mathbb{CS}\big(W,\frac{1}{k}\mathcal{H}\big)$. *Then there is a commutative diagram*

$$\begin{tikzcd} X \arrow[r, dashed, "\rho"] \arrow[d, dashed, "\psi"'] & V \arrow[d, "\nu"] \\ \mathbb{P}(1,2,3) \arrow[r, dashed, "\zeta"'] & \mathbb{P}^2, \end{tikzcd} \tag{32.1}$$

where ζ *is a birational map.*

Proof. Let S be the proper transform of a general surface in $\mathcal{M}$ on Y and Γ a generic fibre of η. Then $S\sim_{\mathbb{Q}}-kK_Y$ by Theorem 2.2, which implies that $S\cdot\Gamma=0$. Therefore, S lies in the fibres of η, which implies the existence of the commutative diagram (32.1).

By Lemmas 2.1 and 2.3, we may assume that $P_7 \in \mathbb{CS}\big(W, \frac{1}{k}\mathcal{H}\big)$.

Let $\pi\colon Z \to W$ be the weighted blow-up of P_7 with weights $(1,1,1)$, G the exceptional divisor of the blow-up π and $\mathcal{B}$ the proper transform of $\mathcal{M}$ on Z. Then $\mathcal{B} \sim_{\mathbb{Q}} -kK_Z$ by Theorem 2.2.

The hypersurface X can be given by the equation

$$w^2z + wf_{12}(x,y,z,t) + f_{21}(x,y,z,t) = 0 \subset \mathbb{P}(1,2,3,7,9) \cong \operatorname{Proj}\big(\mathbb{C}[x,y,z,t,w]\big),$$

where $\operatorname{wt}(x)=1$, $\operatorname{wt}(y)=2$, $\operatorname{wt}(z)=3$, $\operatorname{wt}(t)=7$, $\operatorname{wt}(w)=9$ and $f_i(x,y,z,t)$ is a quasi-homogeneous polynomial of degree i. Let $\mathcal{P}$ be the proper transform on Z of the pencil of surfaces cut out on X by the equations $\lambda x^3 + \mu z = 0$, where $(\lambda,\mu) \in \mathbb{P}^1$. Then the base locus of $\mathcal{P}$ consists of irreducible curves C, L_1 and L_2 such that $\alpha\circ\beta\circ\pi(C)$ is the curve cut out on X by the equations $x=z=0$, $\beta\circ\pi(L_1)$ is contained in the exceptional divisor E, $\beta\circ\pi(L_1)$ is the unique curve in the base locus of $\big|\mathcal{O}_{\mathbb{P}(1,2,7)}(1)\big|$, $\pi(L_2)$ is contained in F and $\pi(L_2)$ is the unique curve of $\big|\mathcal{O}_{\mathbb{P}(1,\,2,\,5)}(1)\big|$.

Let D be a general surface in $\mathcal{P}$, $\bar{E}$ and $\bar{F}$ the proper transforms of E and F on Z, respectively and S the proper transform on Z of the surface cut out on X by the equation $x=0$. Then

$$S\cdot D = C + L_1 + L_2, \qquad \bar{E}\cdot D = 3L_1, \qquad \bar{F}\cdot D = 3L_2,$$

the surface D is normal, and

$$\begin{aligned}
\bar{F} &\sim_{\mathbb{Q}} \pi^*(F) - \frac{1}{2}G,\\
\bar{E} &\sim_{\mathbb{Q}} (\beta\circ\pi)^*(E) - \frac{5}{7}\pi^*(F) - \frac{1}{2}G,\\
D &\sim_{\mathbb{Q}} (\alpha\circ\beta\circ\pi)^*(-3K_X) - \frac{3}{9}(\beta\circ\pi)^*(E) - \frac{3}{7}\pi^*(F) - \frac{3}{2}G,\\
S &\sim_{\mathbb{Q}} (\alpha\circ\beta\circ\pi)^*(-K_X) - \frac{1}{9}(\beta\circ\pi)^*(E) - \frac{1}{7}\pi^*(F) - \frac{1}{2}G.
\end{aligned} \tag{32.2}$$

The curves C, L_1 and L_2 are divisors on $\bar{D}$. It follows from the equivalences (32.2) that

$$C\cdot C = L_1\cdot L_1 = -\frac{1}{2}, \qquad L_2\cdot L_2 = -\frac{2}{5}, \qquad C\cdot L_1 = C\cdot L_2 = L_1\cdot L_2 = 0,$$

which implies that the intersection form of the curves C, L_1 and L_2 on D is negative definite. On the other hand, we have $B|_D \sim_{\mathbb{Q}} kC + kL_1 + kL_2$, where B is a general surface in $\mathcal{B}$. But this is impossible by Lemmas 2.9 and 2.7. The proposition is proved.

§ 33. The case $n = 49$: a hypersurface of degree 21 in $\mathbb{P}(1,3,5,6,7)$

We use the notation and assumptions of § 3. Let $n=49$. Then X is a general hypersurface in $\mathbb{P}(1,3,5,6,7)$ of degree 21 whose singularities consist of points P_1, P_2 and P_3 that are singularities of type $\frac{1}{3}(1,1,2)$, a point P_4 that is a singularity of type $\frac{1}{5}(1,2,3)$ and a point P_5 that is a singularity of type $\frac{1}{6}(1,1,5)$. There is a commutative diagram

$$\begin{tikzcd} & Y \arrow[dl, "\alpha"'] \arrow[dr, "\eta"] & \\ X \arrow[rr, dashed, "\psi"'] & & \mathbb{P}(1,3,5), \end{tikzcd}$$

where ψ is a projection, α is the blow-up of P_5 with weights $(1,1,5)$, and η is an elliptic fibration. There is a commutative diagram

$$\begin{tikzcd} & U \arrow[dl, "\beta"'] \arrow[dr, "\omega"] & \\ X \arrow[rr, dashed, "\xi"'] & & \mathbb{P}(1,3,6), \end{tikzcd}$$

where ξ is a projection, β is the blow-up of P_4 with weights $(1,2,3)$ and ω is an elliptic fibration.

Proposition 33.1. *Either there is a commutative diagram*

$$\begin{tikzcd} X \arrow[r, dashed, "\rho"] \arrow[d, dashed, "\psi"'] & V \arrow[d, "\nu"] \\ \mathbb{P}(1,3,5) \arrow[r, dashed, "\varphi"'] & \mathbb{P}^2 \end{tikzcd} \tag{33.1}$$

or there is a commutative diagram

$$\begin{tikzcd} X \arrow[r, dashed, "\rho"] \arrow[d, dashed, "\xi"'] & V \arrow[d, "\nu"] \\ \mathbb{P}(1,3,6) \arrow[r, dashed, "\sigma"'] & \mathbb{P}^2, \end{tikzcd} \tag{33.2}$$

where φ and σ are birational maps.

Proof. It follows from Theorem 3.3, Proposition 3.5 and Lemma 3.11 that $\mathbb{CS}\big(X, \frac{1}{k}\mathcal{M}\big) \subseteq \{P_4, P_5\}$.

Suppose that the set $\mathbb{CS}\big(X, \frac{1}{k}\mathcal{M}\big)$ contains the point P_4. Let $\mathcal{D}$ be the proper transform of $\mathcal{M}$ on U. Then $\mathcal{D} \sim_{\mathbb{Q}} -kK_U$ by Theorem 2.2. Intersecting a general surface in $\mathcal{D}$ with a generic fibre of ω, we see that $\mathcal{D}$ lies in the fibres of ω. This implies the existence of the commutative diagram (33.2). Similarly, we see that the commutative diagram (33.1) exists in the case when $P_5 \in \mathbb{CS}\big(X, \frac{1}{k}\mathcal{M}\big)$.

§ 34. The case $n = 51$: a hypersurface of degree 22 in $\mathbb{P}(1,1,4,6,11)$

We use the notation and assumptions of § 3. Let $n = 51$. Then X is a hypersurface of degree 22 in $\mathbb{P}(1,1,4,6,11)$ and the equality $-K_X^3 = 1/12$ holds. The singularities of X consist of a point P_1 that is a quotient singularity of type $\frac{1}{2}(1,1,1)$, a point P_2 that is a quotient singularity of type $\frac{1}{3}(1,1,4)$ and a point P_3 that is a quotient singularity of type $\frac{1}{6}(1,1,5)$.

There is a commutative diagram

$$\begin{tikzcd} U \arrow[d, "\alpha"'] & & Y \arrow[ll, "\beta"'] \arrow[d, "\eta"] \\ X \arrow[rr, dashed, "\psi"'] & & \mathbb{P}(1,1,4), \end{tikzcd}$$

where ψ is the natural projection, α is the weighted blow-up of P_3 with weights $(1,1,5)$, β is the weighted blow-up with weights $(1,1,4)$ of the singular point of U that is contained in the exceptional divisor of α, and η is an elliptic fibration.

There is a commutative diagram

$$\begin{tikzcd} & W \arrow[dl, "\gamma"'] \arrow[dr, "\omega"] & \\ X \arrow[rr, dashed, "\xi"'] & & \mathbb{P}(1,1,6), \end{tikzcd}$$

where ξ is a projection, γ is the blow-up of P_2 with weights $(1,1,3)$ and ω is an elliptic fibration.

Proposition 34.1. *Either there is a commutative diagram*

$$\begin{tikzcd} X \arrow[r, dashed, "\rho"] \arrow[d, dashed, "\psi"'] & V \arrow[d, "\nu"] \\ \mathbb{P}(1,1,4) \arrow[r, dashed, "\varphi"'] & \mathbb{P}^2 \end{tikzcd} \tag{34.1}$$

or there is a commutative diagram

$$\begin{tikzcd} X \arrow[r, dashed, "\rho"] \arrow[d, dashed, "\xi"'] & V \arrow[d, "\nu"] \\ \mathbb{P}(1,1,6) \arrow[r, dashed, "\sigma"'] & \mathbb{P}^2, \end{tikzcd} \tag{34.2}$$

where φ and σ are birational maps.

Proof. The fact that X is birationally superrigid implies that the singularities of the log pair $\big(X, \frac{1}{k}\mathcal{M}\big)$ are canonical.

Suppose that P_2 is contained in $\mathbb{CS}\big(X, \frac{1}{k}\mathcal{M}\big)$. Let $\mathcal{D}$ be the proper transform of $\mathcal{M}$ on W. Then $\mathcal{D} \sim_{\mathbb{Q}} -kK_W$ by Theorem 2.2, and this implies the existence of the commutative diagram (34.2).

We may assume that $\mathbb{CS}\big(X, \frac{1}{k}\mathcal{M}\big) = \{P_3\}$ by Theorem 3.3, Proposition 3.5 and Lemma 3.11.

Let $\mathcal{B}$ be the proper transform of $\mathcal{M}$ on U. Then $\mathcal{B} \sim_{\mathbb{Q}} -kK_U$ by Theorem 2.2, but the anticanonical divisor $-K_U$ is nef and big. It follows from Lemmas 2.1 and 2.3 that the set $\mathbb{CS}\big(U, \frac{1}{k}\mathcal{B}\big)$ contains the singular point of U that is contained in the exceptional divisor of α.

Let $\mathcal{H}$ be the proper transform of $\mathcal{M}$ on Y. Then $\mathcal{H} \sim_{\mathbb{Q}} -kK_Y$ by Theorem 2.2. Intersecting a general surface in $\mathcal{H}$ with a generic fibre of η, we see that $\mathcal{H}$ lies in the fibres of η. This implies the existence of the commutative diagram (34.1).

§ 35. The case $n = 56$: a hypersurface of degree 24 in $\mathbb{P}(1, 2, 3, 8, 11)$

We use the notation and assumptions of § 3. Let $n = 56$. Then X is a general hypersurface in $\mathbb{P}(1, 2, 3, 8, 11)$ of degree 24 whose singularities consist of points P_1, P_2 and P_3 that are singularities of type $\frac{1}{2}(1, 1, 1)$ and a point P_4 that is a singularity of type $\frac{1}{11}(1, 3, 8)$. The equality $-K_X^3 = 1/22$ holds.

There is a commutative diagram

$$\begin{tikzcd}
U \arrow[d, "\alpha"'] & W \arrow[l, "\beta"'] & Y \arrow[l, "\gamma"'] \arrow[d, "\eta"] \\
X \arrow[rr, dashed, "\psi"'] & & \mathbb{P}(1, 2, 3),
\end{tikzcd}$$

where ψ is the natural projection, α is the weighted blow-up of P_4 with weights $(1, 3, 8)$, β is the weighted blow-up with weights $(1, 3, 5)$ of the point of U that is a quotient singularity of type $\frac{1}{8}(1, 3, 5)$ contained in the exceptional divisor of α, γ is the weighted blow-up with weights $(1, 2, 3)$ of the point of W that is a quotient singularity of type $\frac{1}{5}(1, 2, 3)$ contained in the exceptional divisor of β, and η is an elliptic fibration.

Proposition 35.1. *The assertion of Theorem 1.10 holds for* $n = 56$.

Proof. It follows from Theorem 3.3, Lemma 3.11 and Proposition 3.5 that $\mathbb{CS}\big(X, \frac{1}{k}\mathcal{M}\big) = \{P_4\}$.

Let E be the exceptional divisor of α, $\mathcal{D}$ the proper transform of $\mathcal{M}$ on U and P_5, P_6 the singular points of U contained in E that are singularities of types $\frac{1}{3}(1, 1, 2)$, $\frac{1}{8}(1, 3, 5)$, respectively. Then $E \cong \mathbb{P}(1, 3, 8)$, and $\mathcal{D} \sim_{\mathbb{Q}} -kK_U$ by Theorem 2.2. The divisor $-K_U$ is nef and big. It follows from Lemmas 2.1 and 2.3 that one of the following assertions holds:

(i) the set $\mathbb{CS}\big(U, \frac{1}{k}\mathcal{D}\big)$ contains the point P_5;

(ii) the set $\mathbb{CS}\big(U, \frac{1}{k}\mathcal{D}\big)$ consists of the point P_6.

Lemma 35.2. *The set* $\mathbb{CS}\big(U, \frac{1}{k}\mathcal{D}\big)$ *does not contain the point* P_5.

Proof. We suppose that $\mathbb{CS}\big(U, \frac{1}{k}\mathcal{D}\big)$ contains P_5 and seek a contradiction.

Let $\pi\colon Z \to U$ be the weighted blow-up of P_5 with weights $(1, 1, 2)$, G the exceptional divisor of the blow-up π and $\mathcal{B}$, $\mathcal{P}$ the proper transforms of $\mathcal{M}$, $|-8K_X|$

on Z, respectively. Then $\mathcal{B} \sim_{\mathbb{Q}} -kK_Z$, but the base locus of $\mathcal{P}$ does not contain curves. In particular, a sufficiently general surface H in $\mathcal{P}$ is nef, but

$$H \cdot B_1 \cdot B_2 = \left((\alpha \circ \pi)^*(-kK_X) - \frac{k}{11}\,\pi^*(E) - \frac{k}{3}\,G\right)^2 \times \left((\alpha \circ \pi)^*(-8K_X) - \frac{8}{11}\,\pi^*(E) - \frac{2}{3}\,G\right) = 0,$$

where B_1 and B_2 are general surfaces in $\mathcal{B}$. But this is impossible by Corollary 2.6 because $H^3 > 0$.

Hence, the set $\mathbb{CS}\big(U, \frac{1}{k}\mathcal{D}\big)$ consists of the point P_6. Let F be the exceptional divisor of β, $\mathcal{H}$ the proper transform of $\mathcal{M}$ on W and P_7, P_8 the singular points of W contained in F that are singularities of types $\frac{1}{3}(1,1,2)$, $\frac{1}{5}(1,2,3)$, respectively. Then $\mathcal{H} \sim_{\mathbb{Q}} -kK_W$ by Theorem 2.2. It follows from Lemmas 2.1 and 2.3 that $\mathbb{CS}\big(W, \frac{1}{k}\mathcal{H}\big) = \{P_7\}$ or $P_8 \in \mathbb{CS}\big(W, \frac{1}{k}\mathcal{H}\big)$.

Arguing as in the proof of Lemma 32.3, we obtain the existence of a commutative diagram

$$\begin{tikzcd} X \arrow[r, dashed, "\rho"] \arrow[d, dashed, "\psi"'] & V \arrow[d, "\nu"] \\ \mathbb{P}(1,2,3) \arrow[r, dashed, "\zeta"'] & \mathbb{P}^2 \end{tikzcd}$$

in the case when $P_8 \in \mathbb{CS}\big(W, \frac{1}{k}\mathcal{H}\big)$, where ζ is a birational map. Thus, we may assume that $\mathbb{CS}\big(W, \frac{1}{k}\mathcal{H}\big) = \{P_7\}$.

Let $\pi\colon Z \to W$ be the weighted blow-up of P_7 with weights $(1,1,2)$, G the exceptional divisor of π and $\mathcal{B}$ the proper transform of $\mathcal{M}$ on Z. Then $\mathcal{B} \sim_{\mathbb{Q}} -kK_Z$ by Theorem 2.2.

Lemma 35.3. *The singularities of the log pair $\big(Z, \frac{1}{k}\mathcal{B}\big)$ are terminal.*

Proof. Suppose that the set $\mathbb{CS}\big(Z, \frac{1}{k}\mathcal{B}\big)$ is non-empty. Let P_9 be the singular point of the surface G. Then P_9 is a singularity of type $\frac{1}{2}(1,1,1)$ of Z, the set $\mathbb{CS}\big(Z, \frac{1}{k}\mathcal{B}\big)$ contains P_9 by Lemma 2.3, and $G \cong \mathbb{P}(1,1,2)$.

Let $\bar{\pi}\colon \bar{Z} \to Z$ be the weighted blow-up of P_9 with weights $(1,1,1)$ and $\bar{G}$ the exceptional divisor of $\bar{\pi}$. Take any divisor D on $\bar{Z}$ such that

$$D \sim_{\mathbb{Q}} -2K_{\bar{Z}} - (\beta \circ \pi \circ \bar{\pi})^*(16K_U) - (\pi \circ \bar{\pi})^*(18K_W).$$

Analyzing the base locus of the pencil $\big|-2K_{\bar{Z}}\big|$, we see that D is nef, but $D^3 > 0$. Thus, the divisor D is nef and big. But $D \cdot \bar{H}_1 \cdot \bar{H}_2 = 0$, where $\bar{H}_1$ and $\bar{H}_2$ are the proper transforms on $\bar{Z}$ of general surfaces in $\mathcal{M}$. But this is impossible by Corollary 2.6.

The hypersurface X can be given by the equation

$$w^2y + wf_{13}(x,y,z,t) + f_{24}(x,y,z,t) = 0 \subset \mathbb{P}(1,2,3,8,11) \cong \operatorname{Proj}\big(\mathbb{C}[x,y,z,t,w]\big),$$

where $\operatorname{wt}(x) = 1$, $\operatorname{wt}(y) = 2$, $\operatorname{wt}(z) = 3$, $\operatorname{wt}(t) = 8$, $\operatorname{wt}(w) = 11$ and $f_i(x,y,z,t)$ is a sufficiently general quasi-homogeneous polynomial of degree i. Let $\bar{E}$ and $\bar{F}$ be the proper transforms on Z of E and F, respectively, and let S_1, S_2, S_3 and S_8 be the proper transforms on Z of the surfaces cut out on X by the equations $x=0$, $y=0$, $z=0$ and $t=0$, respectively. Then

$$\begin{aligned} \bar{F} &\sim_{\mathbb{Q}} \pi^*(F)-\frac{1}{3}G,\\ \bar{E} &\sim_{\mathbb{Q}} (\beta\circ\pi)^*(E)-\frac{5}{8}\pi^*(F)-\frac{2}{3}G,\\ S_1 &\sim_{\mathbb{Q}} (\alpha\circ\beta\circ\pi)^*(-K_X)-\frac{1}{11}(\beta\circ\pi)^*(E)-\frac{1}{8}\pi^*(F)-\frac{1}{3}G,\\ S_2 &\sim_{\mathbb{Q}} (\alpha\circ\beta\circ\pi)^*(-2K_X)-\frac{13}{11}(\beta\circ\pi)^*(E)-\frac{5}{8}\pi^*(F)-\frac{2}{3}G,\\ S_3 &\sim_{\mathbb{Q}} (\alpha\circ\beta\circ\pi)^*(-3K_X)-\frac{3}{11}(\beta\circ\pi)^*(E)-\frac{3}{8}\pi^*(F)-\frac{1}{3}G,\\ S_8 &\sim_{\mathbb{Q}} (\alpha\circ\beta\circ\pi)^*(-8K_X)-\frac{8}{11}(\beta\circ\pi)^*(E). \end{aligned} \tag{35.1}$$

The base locus of $|-2K_Z|$ consists of irreducible curves C, L_1, L_2 and L_3 such that the curve $\alpha\circ\beta\circ\pi(C)$ is cut out on X by the equations $x=y=0$, the curve $\beta\circ\pi(L_1)$ is contained in E, the curve $\beta\circ\pi(L_1)$ is contained in $\left|\mathcal{O}_{\mathbb{P}(1,3,8)}(1)\right|$, the curve $\pi(L_2)$ is contained in F, the curve $\pi(L_2)$ is contained in $\left|\mathcal{O}_{\mathbb{P}(1,3,5)}(1)\right|$ and L_3 is contained in G and in $\left|\mathcal{O}_{\mathbb{P}(1,1,2)}(1)\right|$. Then

$$S_1\cdot D=C+2L_1+2L_2+L_3,\qquad \bar{E}\cdot D=2L_1,\qquad \bar{F}\cdot D=2L_2,\qquad G\cdot D=2L_3,$$

where D is a general surface in $|-2K_Z|$. It follows from the equivalences (35.1) that

$$-K_Z\cdot C=\frac{1}{10},\qquad -K_Z\cdot L_1=-\frac{1}{3},\qquad -K_Z\cdot L_2=-\frac{1}{10},\qquad -K_Z\cdot L_3=\frac{1}{2}.$$

The singularities of the log pair $(Z,\lambda|-2K_Z|)$ are log-terminal for some rational number $\lambda>1/2$ because X is birationally rigid. But the divisor $K_Z+\lambda|-2K_Z|$ has non-negative intersection with all curves on Z except L_1 and L_2. It follows from [12] that there is a birational map $\zeta\colon Z\dashrightarrow \overline{Z}$ which is an isomorphism in codimension 1, and the divisor $-K_{\overline{Z}}$ is numerically effective.

Let $\mathcal{P}$ be the proper transform of $\mathcal{M}$ on $\overline{Z}$. Then the singularities of the log pair $\left(\overline{Z},\frac{1}{k}\mathcal{P}\right)$ are terminal because those of the log pair $\left(Z,\frac{1}{k}\mathcal{B}\right)$ are, and ζ is a log flop with respect to the log pair $\left(Z,\frac{1}{k}\mathcal{B}\right)$. It follows from the equivalences (35.1) that $-K_Z\sim_{\mathbb{Q}} S_1$ and

$$\begin{aligned} S_1 &\sim_{\mathbb{Q}} (\alpha\circ\beta\circ\pi)^*(-K_X)-\frac{1}{11}\bar{E}-\frac{2}{11}\bar{F}-\frac{5}{11}G,\\ S_2 &\sim_{\mathbb{Q}} (\alpha\circ\beta\circ\pi)^*(-2K_X)-\frac{13}{11}\bar{E}-\frac{15}{11}\bar{F}-\frac{21}{11}G,\\ S_3 &\sim_{\mathbb{Q}} (\alpha\circ\beta\circ\pi)^*(-3K_X)-\frac{3}{11}\bar{E}-\frac{6}{11}\bar{F}-\frac{4}{11}G,\\ S_8 &\sim_{\mathbb{Q}} (\alpha\circ\beta\circ\pi)^*(-8K_X)-\frac{8}{11}\bar{E}-\frac{5}{11}\bar{F}-\frac{7}{11}G. \end{aligned} \tag{35.2}$$

The equivalences (35.2) imply that the rational functions y/x^2, zy/x^5 and ty^3/x^{14} are contained in the linear systems $|2S_1|$, $|5S_1|$ and $|14S_1|$, respectively. In particular, the complete linear system $|-70K_Z|$ induces a dominant rational map $Z \dashrightarrow \mathbb{P}(1,2,5,14)$, which implies that the divisor $-K_{\overline{Z}}$ is nef and big. But this is impossible by Lemma 2.1. The proposition is proved.

§ 36. The case $n = 58$: a hypersurface of degree 24 in $\mathbb{P}(1,3,4,7,10)$

We use the notation and assumptions of § 3. Let $n = 58$. Then X is a hypersurface of degree 24 in $\mathbb{P}(1,3,4,7,10)$ and the equality $-K_X^3 = 1/35$ holds. The singularities of X consist of points P_1, P_2, P_3 of types $\frac{1}{2}(1,1,1)$, $\frac{1}{7}(1,3,4)$, $\frac{1}{10}(1,3,7)$, respectively.

Proposition 36.1. *The assertion of Theorem 1.10 holds for $n = 58$.*

Proof. It follows from Theorem 3.3, Lemma 3.11 and Proposition 3.5 that $\mathbb{CS}\bigl(X, \frac{1}{k}\mathcal{M}\bigr) \subseteq \{P_2, P_3\}$.

There is a commutative diagram

$$\begin{tikzcd}
 & & Y \arrow[dl, "\gamma_4"'] \arrow[dr, "\gamma_2"] \arrow[drrr, "\eta"] & & & \\
 & U_{23} \arrow[dl, "\beta_3"'] \arrow[dr, "\beta_2"] & & U_{34} \arrow[dl, "\beta_4"'] & & \mathbb{P}(1,3,4), \\
U_2 \arrow[dr, "\alpha_2"'] & & U_3 \arrow[dl, "\alpha_3"'] & & & \\
 & X \arrow[uurrrr, dashed, "\psi"'] & & & &
\end{tikzcd}$$

where ψ is the natural projection, α_2 is the weighted blow-up of P_2 with weights $(1,3,4)$, α_3 is the weighted blow-up of P_3 with weights $(1,3,7)$, β_3 is the weighted blow-up with weights $(1,3,7)$ of the proper transform of P_3 on U_2, β_2 is the weighted blow-up with weights $(1,3,4)$ of the proper transform of P_2 on U_2, β_4 is the weighted blow-up with weights $(1,3,4)$ of the singular point of U_3 that is a quotient singularity of type $\frac{1}{7}(1,3,4)$ contained in the exceptional divisor of α_3, γ_2 is the weighted blow-up with weights $(1,3,4)$ of the proper transform of P_2 on U_{34}, γ_4 is the weighted blow-up with weights $(1,3,4)$ of the singular point of U_{23} that is a quotient singularity of type $\frac{1}{7}(1,3,4)$ contained in the exceptional divisor of β_3, and η is an elliptic fibration.

Lemma 36.2. *The set $\mathbb{CS}\bigl(X, \frac{1}{k}\mathcal{M}\bigr)$ contains the point P_3.*

Proof. Suppose that $\mathbb{CS}\bigl(X, \frac{1}{k}\mathcal{M}\bigr)$ does not contain P_3. Let $\mathcal{D}_2$ be the proper transform of $\mathcal{M}$ on U_2 and O, Q the singular points of U_2 contained in the exceptional divisor of α_2 that are singularities of types $\frac{1}{3}(1,1,2)$, $\frac{1}{4}(1,1,3)$, respectively. Then $\mathcal{D}_2 \sim_{\mathbb{Q}} -kK_{U_2}$ by Theorem 2.2.

Suppose that $O \in \mathbb{CS}\bigl(U_2, \frac{1}{k}\mathcal{D}_2\bigr)$. Let $\pi\colon W \to U_2$ be the weighted blow-up of O with weights $(1,1,2)$, $\mathcal{B}$ and $\mathcal{P}$ the proper transforms of $\mathcal{M}$ and $|-4K_X|$ on W, respectively, and S a general surface in $\mathcal{P}$. Then the base locus of $\mathcal{P}$ consists of an irreducible curve C such that $\alpha_2(C)$ is the base curve of $|-4K_X|$. Then

$\mathcal{B} \sim_{\mathbb{Q}} -kK_W$ by Theorem 2.2. The surface S is normal, the inequality $C^2 < 0$ holds on S and $\mathcal{B}|_S \sim_{\mathbb{Q}} kC$. Thus, it follows from Lemma 2.9 that

$$\mathrm{Supp}(S) \cap \mathrm{Supp}(B) = \mathrm{Supp}(C),$$

where B is a general surface in $\mathcal{B}$. But this is impossible by Lemma 2.7 because $\mathcal{B}$ is not composed of a pencil.

Thus, we see that $Q \in \mathbb{CS}\big(U_2, \frac{1}{k}\mathcal{D}_2\big)$ by Lemma 2.3 because $\mathbb{CS}\big(U_2, \frac{1}{k}\mathcal{D}_2\big) \neq \varnothing$ by Lemma 2.1.

Let $\zeta\colon U \to U_2$ be the weighted blow-up of Q with weights $(1,1,3)$, $\mathcal{H}$ the proper transform of $\mathcal{M}$ on U, H a general surface in $\mathcal{H}$ and D a general surface in $|-3K_U|$. Then D is normal and the base locus of $|-3K_U|$ consists of an irreducible curve Z such that $\alpha_2(Z)$ is the unique base curve of $|-3K_X|$. The equivalence $\mathcal{H}|_D \sim_{\mathbb{Q}} kZ$ holds by Theorem 2.2 and the inequality $Z^2 < 0$ holds on D. But this is impossible by Lemmas 2.7 and 2.9.

Let $\mathcal{D}_2$ and $\mathcal{D}_{23}$ be the proper transforms of $\mathcal{M}$ on U_2 and U_{23}, respectively. Arguing as in the proof of Lemma 36.2, we obtain the following corollaries.

Corollary 36.3. *Suppose that $\mathcal{D}_2 \sim_{\mathbb{Q}} -kK_{U_2}$. Then the set $\mathbb{CS}\big(U_2, \frac{1}{k}\mathcal{D}_2\big)$ does not contain subvarieties of U_2 that are contained in the exceptional divisor of α_2.*

Corollary 36.4. *Suppose that $\mathcal{D}_{23} \sim_{\mathbb{Q}} -kK_{U_{23}}$. Then the set $\mathbb{CS}\big(U_{23}, \frac{1}{k}\mathcal{D}_{23}\big)$ does not contain subvarieties of U_{23} that are contained in the exceptional divisor of β_2.*

Let $\mathcal{D}_3$ and $\mathcal{D}_{23}$ be the proper transforms of $\mathcal{M}$ on U_3 and U_{34}, respectively. Then $\mathcal{D}_3 \sim_{\mathbb{Q}} -kK_{U_3}$ by Theorem 2.2.

Lemma 36.5. *The set $\mathbb{CS}\big(X, \frac{1}{k}\mathcal{M}\big)$ contains the point P_2.*

Proof. Suppose that $P_2 \notin \mathbb{CS}\big(X, \frac{1}{k}\mathcal{M}\big)$. Then $\mathcal{D}_3 \sim_{\mathbb{Q}} -kK_{U_2}$ by Theorem 2.2, and this implies that the set $\mathbb{CS}\big(U_3, \frac{1}{k}\mathcal{D}_3\big)$ is non-empty by Lemma 2.1.

Let P_4 and P_5 be the singular points of U_3 contained in the exceptional divisor of α_3 that are quotient singularities of types $\frac{1}{7}(1,3,4)$ and $\frac{1}{3}(1,1,2)$, respectively. Then the set $\mathbb{CS}\big(U_3, \frac{1}{k}\mathcal{D}_3\big)$ contains either P_4 or P_5 by Lemma 2.3. It follows from Lemma 36.2 that $\mathbb{CS}\big(U_3, \frac{1}{k}\mathcal{D}_3\big)$ does not contain P_5.

Thus, $\mathbb{CS}\big(U_3, \frac{1}{k}\mathcal{D}_3\big)$ contains P_4. Then $\mathcal{D}_{34} \sim_{\mathbb{Q}} -kK_{U_{34}}$ by Theorem 2.2, and the set $\mathbb{CS}\big(U_{34}, \frac{1}{k}\mathcal{D}_{34}\big)$ is non-empty by Lemma 2.1 because $-K_{U_{34}}$ is nef and big.

Let P_6 and P_7 be the singular points of U_{34} contained in the exceptional divisor of β_4 that are quotient singularities of types $\frac{1}{3}(1,1,2)$ and $\frac{1}{4}(1,1,3)$, respectively. Then the set $\mathbb{CS}\big(U_{34}, \frac{1}{k}\mathcal{D}_{34}\big)$ contains either P_6 or P_7 by Lemma 2.3.

Suppose that $\mathbb{CS}\big(U_{34}, \frac{1}{k}\mathcal{D}_{34}\big)$ contains P_7. Let $\zeta\colon U \to U_{34}$ be the weighted blow-up of P_7 with weights $(1,1,3)$, $\mathcal{H}$ the proper transform of $\mathcal{M}$ on U, H a general surface in $\mathcal{H}$ and D a general surface in $|-3K_U|$. Then D is normal and the base locus of $|-3K_U|$ consists of an irreducible curve Z such that $\alpha_3 \circ \beta_4(Z)$ is the unique base curve of $|-3K_X|$. Moreover, the equivalence $\mathcal{H}|_D \sim_{\mathbb{Q}} kZ$ holds and $Z^2 < 0$ on D. But this is is impossible by Lemmas 2.7 and 2.9.

Therefore, the set $\mathbb{CS}\big(U_{34}, \frac{1}{k}\mathcal{D}_{34}\big)$ contains the point P_6.

The hypersurface X can be given by the quasi-homogeneous equation

$$w^2z + wf_{14}(x,y,z,t) + f_{24}(x,y,z,t) = 0 \subset \mathrm{Proj}\big(\mathbb{C}[x,y,z,t,w]\big),$$

where $\mathrm{wt}(x)=1$, $\mathrm{wt}(y)=3$, $\mathrm{wt}(z)=4$, $\mathrm{wt}(t)=7$, $\mathrm{wt}(w)=10$ and $f_i(x,y,z,w)$ is a quasi-homogeneous polynomial of degree i. Let $\mathcal{P}$ be a pencil consisting of the surfaces cut out on X by the equations $\lambda x^4+\mu z=0$, where $(\lambda,\mu)\in\mathbb{P}^1$. Then the base locus of $\mathcal{P}$ consists of the irreducible curve cut out on X by the equations $x=z=0$.

Let $\pi\colon W\to U_{34}$ be the weighted blow-up of P_6 with weights $(1,1,2)$, $\mathcal{B}$ the proper transform of $\mathcal{M}$ on W, $\mathcal{H}$ the proper transform of $\mathcal{P}$ on W, H a general surface in $\mathcal{H}$ and E, F, G the exceptional divisors of α_3, β_4, π, respectively. Then

$$H\sim_{\mathbb{Q}}(\alpha_3\circ\beta_4\circ\pi)^*(-4K_X)-\frac{1}{5}\,(\beta_4\circ\pi)^*(E)-\frac{4}{7}\,(\pi)^*(F)-\frac{1}{3}\,G,$$

the surface H is normal and the base locus of $\mathcal{H}$ consists of curves C and L such that $\alpha_3\circ\beta_4\circ\pi(C)$ is the unique base curve of $\mathcal{P}$ and $\beta_4\circ\pi(L)$ is the unique curve on the surface $E\cong\mathbb{P}(1,3,7)$ that is contained in $\bigl|\mathcal{O}_{\mathbb{P}(1,3,7)}(1)\bigr|$.

Let S be a surface in $|-K_W|$ and $\bar{E}$ the proper transform of E on W. Then $S\cdot H=C+L$ and $\bar{E}\cdot H=4L$. But

$$\bar{E}\sim_{\mathbb{Q}}(\beta_4\circ\pi)^*(E)-\frac{4}{7}\,(\pi)^*(F)-\frac{1}{3}\,G,$$

which implies that the intersection form of L and C on H is negative definite. On the other hand, the equivalence $\mathcal{B}|_H\sim_{\mathbb{Q}}kC+kL$ holds, and this is impossible by Lemmas 2.9 and 2.7.

Hence, we see that $\mathbb{CS}\bigl(X,\frac{1}{k}\mathcal{M}\bigr)=\{P_2,P_3\}$, and $\mathcal{D}_{23}\sim_{\mathbb{Q}}-kK_{U_{23}}$ by Theorem 2.2. It follows easily from Lemmas 2.1 and 2.3, the proof of Lemma 36.5 and Corollary 36.4 that the set $\mathbb{CS}\bigl(U_{23},\frac{1}{k}\mathcal{D}_{23}\bigr)$ contains the singular point of U_{23} that is a quotient singularity of type $\frac{1}{7}(1,3,4)$ contained in the exceptional divisor of β_3. This implies that the proper transform of $\mathcal{M}$ on Y lies in the fibres of η by Theorem 2.2. The proposition is proved.

§ 37. The case $n=64$: a hypersurface of degree 26 in $\mathbb{P}(1,2,5,6,13)$

We use the notation and assumptions of § 3. Let $n=64$. Then X is a general hypersurface in $\mathbb{P}(1,2,5,6,13)$ of degree 26. The singularities of X consist of points P_1, P_2, P_3, P_4 that are singularities of type $\frac{1}{2}(1,1,1)$, a point P_5 that is a singularity of type $\frac{1}{5}(1,2,3)$ and a point P_6 that is a singularity of type $\frac{1}{6}(1,1,5)$. There is a commutative diagram

$$\begin{tikzcd} & Y \arrow[dl, "\alpha"'] \arrow[dr, "\eta"] & \\ X \arrow[rr, dashed, "\psi"'] & & \mathbb{P}(1,2,5), \end{tikzcd}$$

where ψ is a projection, α is the blow-up of P_6 with weights $(1,1,5)$ and η is an elliptic fibration. There is a commutative diagram

$$\begin{tikzcd} & U \arrow[dl, "\beta"'] \arrow[dr, "\omega"] & \\ X \arrow[rr, dashed, "\xi"'] & & \mathbb{P}(1,1,3), \end{tikzcd}$$

where ξ is a projection, β is the blow-up of the point P_5 with weights $(1,2,3)$ and ω is an elliptic fibration.

Proposition 37.1. *Either there is a commutative diagram*

$$\begin{tikzcd} X \arrow[r, dashed, "\rho"] \arrow[d, dashed, "\psi"'] & V \arrow[d, "\nu"] \\ \mathbb{P}(1,2,5) \arrow[r, dashed, "\varphi"'] & \mathbb{P}^2 \end{tikzcd} \tag{37.1}$$

or there is a commutative diagram

$$\begin{tikzcd} X \arrow[r, dashed, "\rho"] \arrow[d, dashed, "\xi"'] & V \arrow[d, "\nu"] \\ \mathbb{P}(1,1,3) \arrow[r, dashed, "\sigma"'] & \mathbb{P}^2, \end{tikzcd} \tag{37.2}$$

where φ and σ are birational maps.

Proof. See the proof of Proposition 33.1.

§ 38. The case $n = 65$: a hypersurface of degree 27 in $\mathbb{P}(1,2,5,9,11)$

We use the notation and assumptions of § 3. Let $n = 65$. Then X is a general hypersurface of degree 27 in $\mathbb{P}(1,2,5,9,11)$ and the equality $-K_X^3 = 3/110$ holds. The singularities of X consist of a point P_1 that is a quotient singularity of type $\frac{1}{2}(1,1,1)$, a point P_2 that is a quotient singularity of type $\frac{1}{5}(1,1,4)$ and a point P_3 that is a quotient singularity of type $\frac{1}{11}(1,2,9)$. There is a commutative diagram

$$\begin{tikzcd} U \arrow[d, "\alpha"'] & W \arrow[l, "\beta"'] & Y \arrow[l, "\gamma"'] \arrow[d, "\eta"] \\ X \arrow[rr, dashed, "\psi"'] & & \mathbb{P}(1,2,5), \end{tikzcd}$$

where ψ is the natural projection, α is the weighted blow-up of P_3 with weights $(1,2,9)$, β is the weighted blow-up with weights $(1,2,7)$ of the point of U that is a singularity of type $\frac{1}{9}(1,2,7)$ contained in the α-exceptional divisor, γ is the weighted blow-up with weights $(1,2,5)$ of the singular point of type $\frac{1}{7}(1,2,5)$ contained in the β-exceptional divisor, and η is an elliptic fibration.

Proposition 38.1. *The assertion of Theorem 1.10 holds for $n = 65$.*

Proof. It follows from Theorem 3.3, Lemma 3.11 and Proposition 3.5 that $\mathbb{CS}\big(X, \frac{1}{k}\mathcal{M}\big) = \{P_3\}$.

Let E be the exceptional divisor of α, $\mathcal{D}$ the proper transform of $\mathcal{M}$ on U and P_4, P_5 the singular points of U contained in the surface E that are singularities of types $\frac{1}{2}(1,1,1)$, $\frac{1}{9}(1,2,7)$, respectively. Then $\mathcal{D} \sim_{\mathbb{Q}} -kK_U$ by Theorem 2.2. It follows from Lemmas 2.1 and 2.3 and the proof of Lemma 32.2 that $\mathbb{CS}\big(U, \frac{1}{k}\mathcal{D}\big) = \{P_5\}$.

Let F be the exceptional divisor of β, $\mathcal{H}$ the proper transform of $\mathcal{M}$ on W and P_6, P_7 the singular points of W contained in F that are singularities of types $\frac{1}{2}(1,1,1)$, $\frac{1}{7}(1,2,5)$, respectively. Then $\mathcal{H} \sim_{\mathbb{Q}} -kK_W$ by Theorem 2.2. It follows from Lemmas 2.1 and 2.3 that the set $\mathbb{CS}\big(U, \frac{1}{k}\mathcal{D}\big)$ contains either P_6 or P_7.

Remark 38.2. In the case when $\mathbb{CS}\big(W, \frac{1}{k}\mathcal{H}\big)$ contains P_7, it follows from Theorem 2.2 that there is a commutative diagram

$$\begin{tikzcd} X \arrow[r, dashed, "\rho"] \arrow[d, dashed, "\psi"'] & V \arrow[d, "\nu"] \\ \mathbb{P}(1,2,5) \arrow[r, dashed, "\zeta"'] & \mathbb{P}^2, \end{tikzcd}$$

where ζ is a birational map.

We may assume that $P_6 \in \mathbb{CS}\big(W, \frac{1}{k}\mathcal{H}\big)$. Note, that $E \cong \mathbb{P}(1,2,9)$ and $F \cong \mathbb{P}(1,2,7)$.

Let $\pi\colon Z \to W$ be the weighted blow-up of P_6 with weights $(1,1,1)$, G the exceptional divisor of π and $\mathcal{B}$ the proper transform of $\mathcal{M}$ on Z. Then $G \cong \mathbb{P}^2$, and $\mathcal{B} \sim_{\mathbb{Q}} -kK_Z$ by Theorem 2.2. The hypersurface X can be given by the equation

$$w^2z + wf_{16}(x,y,z,t) + f_{27}(x,y,z,t) = 0 \subset \mathbb{P}(1,2,5,9,11) \cong \mathrm{Proj}\big(\mathbb{C}[x,y,z,t,w]\big),$$

where $\mathrm{wt}(x) = 1$, $\mathrm{wt}(y) = 2$, $\mathrm{wt}(z) = 5$, $\mathrm{wt}(t) = 9$, $\mathrm{wt}(w) = 11$ and $f_i(x,y,z,t)$ is a quasi-homogeneous polynomial of degree i. Let $\bar{E}$ and $\bar{F}$ be the proper transforms of E and F on Z, respectively, and let $\mathcal{P}$ be the proper transform on Z of the pencil of surfaces cut out on X by the pencils $\lambda x^5 + \mu z = 0$, where $(\lambda,\mu) \in \mathbb{P}^1$.

The base locus of $\mathcal{P}$ consists of irreducible curves C, L_1, L_2, Δ_1, Δ_2 and Δ such that $\alpha\circ\beta\circ\pi(C)$ is cut out on X by the equations $x = z = 0$, $\beta\circ\pi(L_1)$ is contained in E, $\beta\circ\pi(L_1)$ is the unique curve in $\big|\mathcal{O}_{\mathbb{P}(1,2,9)}(1)\big|$, $\pi(L_1)$ is contained in F and in $\big|\mathcal{O}_{\mathbb{P}(1,2,7)}(1)\big|$, Δ_1 and Δ_2 are the lines on G cut out by $\bar{E}$ and $\bar{F}$, respectively, and Δ is a line on G different from Δ_1 and Δ_2.

Let D be a general surface in $\mathcal{P}$ and S the proper transform on Z of the surface cut out on X by the equation $x = 0$. Then

$$S\cdot D = C + L_1 + L_2, \qquad \bar{E}\cdot D = 5L_1 + \Delta_1, \qquad \bar{F}\cdot D = 5L_2 + \Delta_2$$

and the surface D is normal, and smooth in the neighbourhood of G. Then

$$\Delta_1\cdot\Delta_2 = \Delta_1\cdot L_2 = \Delta_2\cdot L_1 = 1, \quad \Delta_1\cdot C = \Delta_2\cdot C = 0, \quad \Delta_1^2 = \Delta_2^2 = -4 \tag{38.1}$$

on D. But

$$\begin{gathered} \bar{F} \sim_{\mathbb{Q}} \pi^*(F) - \frac{1}{2}G, \\ \bar{E} \sim_{\mathbb{Q}} (\beta\circ\pi)^*(E) - \frac{7}{9}\pi^*(F) - \frac{1}{2}G, \\ D \sim_{\mathbb{Q}} (\alpha\circ\beta\circ\pi)^*(-5K_X) - \frac{5}{11}(\beta\circ\pi)^*(E) - \frac{5}{9}\pi^*(F) - \frac{3}{2}G, \\ S \sim_{\mathbb{Q}} (\alpha\circ\beta\circ\pi)^*(-K_X) - \frac{1}{11}(\beta\circ\pi)^*(E) - \frac{1}{9}\pi^*(F) - \frac{1}{2}G. \end{gathered} \tag{38.2}$$

It follows from (38.1) and (38.2) that

$$C\cdot C=L_1\cdot L_1=-\frac{1}{2}\,,\qquad L_2\cdot L_2=-\frac{3}{7}\,,\qquad C\cdot L_1=C\cdot L_2=L_1\cdot L_2=0$$

on D. The intersection form of the curves C, L_1 and L_2 on D is negative definite. But $\mathcal{B}|_D\sim_{\mathbb{Q}}kC+kL_1+kL_2$, which is impossible by Lemmas 2.9 and 2.7. The proposition is proved.

§ 39. The case $n=68$: a hypersurface of degree 28 in $\mathbb{P}(1,3,4,7,14)$

We use the notation and assumptions of § 3. Let $n=68$. Then X is a hypersurface of degree 28 in $\mathbb{P}(1,3,4,7,14)$. The singularities of X consist of a point P_1 that is a singularity of type $\frac{1}{2}(1,1,1)$, a point P_2 that is a singularity of type $\frac{1}{3}(1,1,2)$ and points P_3 and P_4 that are singularities of type $\frac{1}{7}(1,3,4)$. The equality $-K_X^3=1/42$ holds. There is a commutative diagram

$$\begin{tikzcd}
 & Y \arrow[dl, "\gamma"'] \arrow[dr, "\delta"] \arrow[rrr, "\eta"] & & & \mathbb{P}(1,2,3), \\
U \arrow[dr, "\alpha"'] & & W \arrow[dl, "\beta"] & & \\
 & X \arrow[rrruu, dashed, "\psi"'] & & &
\end{tikzcd}$$

where ψ is a projection, α is the weighted blow-up of P_3 with weights $(1,3,4)$, β is the weighted blow-up of P_4 with weights $(1,3,4)$, γ is the weighted blow-up with weights $(1,3,4)$ of the proper transform of P_4 on U, δ is the weighted blow-up with weights $(1,3,4)$ of the proper transform of P_3 on W, and η is an elliptic fibration.

Proposition 39.1. *The assertion of Theorem 1.10 holds for* $n=68$.

Proof. It follows from Lemma 3.11 and Proposition 3.5 that $\mathbb{CS}\big(X,\frac{1}{k}\mathcal{M}\big)\subseteq\{P_3,P_4\}$. The desired assertion is obvious in the case when $\mathbb{CS}\big(X,\frac{1}{k}\mathcal{M}\big)=\{P_3,P_4\}$. Hence, we may assume that $P_4\notin\mathbb{CS}\big(X,\frac{1}{k}\mathcal{M}\big)$.

Let $\mathcal{B}$ be the proper transform of $\mathcal{M}$ on U and P_5, P_6 the singular points of U that are singularities of types $\frac{1}{3}(1,1,2)$, $\frac{1}{4}(1,1,3)$, respectively, contained in the exceptional divisor of α. It follows from Lemma 2.3 that the set $\mathbb{CS}\big(U,\frac{1}{k}\mathcal{B}\big)$ contains either P_5 or P_6.

Suppose that $P_6\in\mathbb{CS}\big(U,\frac{1}{k}\mathcal{B}\big)$. Let $\pi\colon W\to U$ be the weighted blow-up of P_6 with weights $(1,1,3)$ and $\mathcal{B}$ and $\mathcal{P}$ the proper transforms on W of $\mathcal{M}$ and $|-3K_X|$, respectively. Then $\mathcal{B}\sim_{\mathbb{Q}}-kK_W$ by Theorem 2.2 and the base locus of $\mathcal{P}$ consists of an irreducible curve Z such that $\alpha\circ\pi(Z)$ is the base curve of $|-3K_X|$.

Let S be a general surface in $\mathcal{P}$ and B a general surface in $\mathcal{B}$. Then S is normal and $Z^2<0$ on S. But $\mathcal{B}|_S\sim_{\mathbb{Q}}kZ$. Therefore, the support of the cycle $B\cdot S$ is contained in Z by Lemma 2.9, which is impossible by Lemma 2.7.

Hence, $\mathbb{CS}\big(U,\frac{1}{k}\mathcal{B}\big)$ contains P_5. Let $\zeta\colon Z\to U$ be the weighted blow-up of P_5 with weights $(1,1,2)$ and $\mathcal{D}$ and $\mathcal{H}$ the proper transforms on Z of $\mathcal{M}$ and $|-4K_X|$, respectively. Then $\mathcal{D}\sim_{\mathbb{Q}}-kK_Z$ by Theorem 2.2 and the base locus of $\mathcal{H}$ consists of an irreducible curve C such that $\alpha\circ\zeta(C)$ is the base curve of $|-4K_X|$.

Let H be a general surface in $\mathcal{H}$. Then $H\cdot C=0$ and $H^3>0$. Thus, the divisor H is nef and big. On the other hand, the equality $H\cdot D_1\cdot D_2=0$ holds, where D_1 and D_2 are general surfaces in $\mathcal{D}$. But this is impossible by Corollary 2.6.

§ 40. The case $n=74$: a hypersurface of degree 30 in $\mathbb{P}(1,3,4,10,13)$

We use the notation and assumptions of § 3. Let $n=74$. Then X is a hypersurface of degree 30 in $\mathbb{P}(1,3,4,10,13)$, the equality $-K_X^3=1/52$ holds and the singularities of X consist of a point P_1 that is a quotient singularity of type $\frac{1}{2}(1,1,1)$, a point P_2 that is a quotient singularity of type $\frac{1}{4}(1,1,3)$ and a point P_3 that is a quotient singularity of type $\frac{1}{13}(1,3,10)$.

There is a commutative diagram

$$\begin{tikzcd} U \arrow[d, "\alpha"'] & & W \arrow[ll, "\beta"'] & & Y \arrow[ll, "\gamma"'] \arrow[d, "\eta"] \\ X \arrow[rrrr, dashed, "\psi"'] & & & & \mathbb{P}(1,3,4), \end{tikzcd}$$

where ψ is the natural projection, α is the weighted blow-up of P_3 with weights $(1,3,10)$, β is the weighted blow-up with weights $(1,3,7)$ of the singular point of U that is a quotient singularity of type $\frac{1}{10}(1,3,7)$ contained in the exceptional divisor of α, γ is the weighted blow-up with weights $(1,3,4)$ of the singular point of W that is a quotient singularity of type $\frac{1}{7}(1,3,4)$ contained in the exceptional divisor of β, and η is an elliptic fibration.

Proposition 40.1. *The assertion of Theorem 1.10 holds for $n=74$.*

Proof. It follows from Theorem 3.3, Lemma 3.11 and Proposition 3.5 that $\mathbb{CS}\big(X,\frac{1}{k}\mathcal{M}\big)=\{P_3\}$.

Let E be the exceptional divisor of α, $\mathcal{D}$ the proper transform of $\mathcal{M}$ on U and P_4, P_5 the singular points of U contained in the divisor E that are quotient singularities of types $\frac{1}{3}(1,1,2)$, $\frac{1}{10}(1,3,7)$, respectively. Then $\mathcal{D}\sim_{\mathbb{Q}}-kK_U$ by Theorem 2.2.

Lemma 40.2. *The set $\mathbb{CS}\big(U,\frac{1}{k}\mathcal{D}\big)$ does not contain the point P_4.*

Proof. Suppose that $\mathbb{CS}\big(U,\frac{1}{k}\mathcal{D}\big)$ contains P_4. Let $\pi\colon Z\to U$ be the weighted blow-up of P_4 with weights $(1,1,2)$, G the exceptional divisor of π and $\mathcal{B}$ the proper transform of $\mathcal{M}$ on Z. Then $\mathcal{B}\sim_{\mathbb{Q}}-kK_Z$ by Theorem 2.2.

Let D be a divisor on Z such that the equivalence $D\sim_{\mathbb{Q}}-4K_Z-\pi^*(36K_U)$ holds. Analyzing the base locus of $|-4K_Z|$, we see that D is nef and big. But

$$D\cdot B_1\cdot B_2=\bigg((\alpha\circ\pi)^*(-kK_X)-\frac{k}{13}\,\pi^*(E)-\frac{k}{3}\,G\bigg)^2\big(-4K_Z-\pi^*(36K_U)\big)=0,$$

where B_1 and B_2 are general surfaces in $\mathcal{B}$. But this is impossible by Corollary 2.6.

Corollary 40.3. *The set $\mathbb{CS}\big(U,\frac{1}{k}\mathcal{D}\big)$ consists of the point P_5 by Lemmas 2.1 and 2.3.*

Let F be the exceptional divisor of β, $\mathcal{H}$ the proper transform of $\mathcal{M}$ on W and P_6, P_7 the singular points of W contained in F that are quotient singularities of types $\frac{1}{3}(1,1,2)$, $\frac{1}{7}(1,3,4)$, respectively. Then $\mathcal{H} \sim_{\mathbb{Q}} -kK_W$ by Theorem 2.2.

Suppose that the set $\mathbb{CS}\big(W, \frac{1}{k}\mathcal{H}\big)$ contains the point P_7. Arguing as in the proof of Lemma 32.3, we see that the assertion of Theorem 1.10 holds for X. Hence, we may assume that $\mathbb{CS}\big(W, \frac{1}{k}\mathcal{H}\big)$ does not contain P_7. Thus, $\mathbb{CS}\big(W, \frac{1}{k}\mathcal{H}\big)$ consists of the singular point P_6 by Lemmas 2.1 and 2.3.

Let $\pi\colon Z \to W$ be the weighted blow-up of P_6 with weights $(1,1,2)$, G the exceptional divisor of π and $\mathcal{B}$ the proper transform of $\mathcal{M}$ on Z. Then $\mathcal{B} \sim_{\mathbb{Q}} -kK_Z$ by Theorem 2.2.

Let D be a divisor on Z such that $D \sim_{\mathbb{Q}} -4K_Z - (\beta\circ\pi)^*(20K_U) - \pi^*(24K_W)$. Then D is nef and big. But

$$D \cdot B_1 \cdot B_2 = 0,$$

where B_1 and B_2 are general surfaces in $\mathcal{B}$. But this is impossible by Corollary 2.6. The proposition is proved.

§ 41. The case $n = 79$: a hypersurface of degree 33 in $\mathbb{P}(1,3,5,11,14)$

We use the notation and assumptions of § 3. Let $n = 79$. Then X is a general hypersurface of degree 33 in $\mathbb{P}(1,3,5,11,14)$ and the equality $-K_X^3 = 1/70$ holds. The singularities of X consist of points P_1 and P_2 that are singularities of types $\frac{1}{5}(1,1,4)$ and $\frac{1}{14}(1,3,11)$, respectively.

Proposition 41.1. *The assertion of Theorem 1.10 holds for $n = 79$.*

Proof. There is a commutative diagram

$$\begin{tikzcd}
U \arrow[d, "\alpha"'] & W \arrow[l, "\beta"'] & Y \arrow[l, "\gamma"'] \arrow[d, "\eta"] \\
X \arrow[rr, dashed, "\psi"'] & & \mathbb{P}(1,3,5),
\end{tikzcd}$$

where ψ is the natural projection, α is the weighted blow-up of P_2 with weights $(1,3,11)$, β is the weighted blow-up with weights $(1,3,8)$ of the singular point of type $\frac{1}{11}(1,3,8)$ contained in the exceptional divisor of α, γ is the weighted blow-up with weights $(1,3,5)$ of the singular point of type $\frac{1}{8}(1,3,5)$ contained in the exceptional divisor of β, and η is an elliptic fibration.

It follows from Theorem 3.3, Lemma 3.11 and Proposition 3.5 that $\mathbb{CS}\big(X, \frac{1}{k}\mathcal{M}\big) = \{P_2\}$.

Let E be the exceptional divisor of α, $\mathcal{D}$ the proper transform of $\mathcal{M}$ on U and P_3, P_4 the singular points of U contained in E that are quotient singularities of types $\frac{1}{3}(1,1,2)$, $\frac{1}{11}(1,3,8)$, respectively. Then $\mathcal{D} \sim_{\mathbb{Q}} -kK_U$. It follows from Lemmas 2.1 and 2.3 and the proof of Lemma 40.2 that $\mathbb{CS}\big(U, \frac{1}{k}\mathcal{D}\big) = \{P_4\}$.

Let F be the exceptional divisor of β, $\mathcal{H}$ the proper transform of $\mathcal{M}$ on W and P_5, P_6 the singular points of W contained in F that are quotient singularities of types $\frac{1}{3}(1,1,2)$, $\frac{1}{8}(1,3,4)$, respectively. Then $\mathcal{H} \sim_{\mathbb{Q}} -kK_W$ by Theorem 2.2.

In the case when the set $\mathbb{CS}\big(W, \frac{1}{k}\mathcal{H}\big)$ contains the point P_6, it follows easily from Theorem 2.2 that the assertion of Theorem 1.10 holds for X. Therefore, we may assume that $\mathbb{CS}\big(W, \frac{1}{k}\mathcal{H}\big)$ consists of the point P_5 by Lemmas 2.1 and 2.3.

Let $\pi\colon Z \to W$ be the weighted blow-up of P_5 with weights $(1,1,2)$, G the exceptional divisor of π and $\mathcal{B}$ the proper transform of $\mathcal{M}$ on Z. Then $\mathcal{B} \sim_{\mathbb{Q}} -kK_Z$ by Theorem 2.2. The hypersurface X can be given by the equation

$$w^2z + wf_{19}(x,y,z,t) + f_{33}(x,y,z,t) = 0 \subset \mathbb{P}(1,2,3,8,11) \cong \operatorname{Proj}\big(\mathbb{C}[x,y,z,t,w]\big),$$

where $\mathrm{wt}(x) = 1$, $\mathrm{wt}(y) = 3$, $\mathrm{wt}(z) = 5$, $\mathrm{wt}(t) = 11$, $\mathrm{wt}(w) = 14$ and $f_i(x,y,z,t)$ is a quasi-homogeneous polynomial of degree i. Let $\mathcal{P}$ be the linear system on X generated by the monomials x^{30}, y^{10}, z^6, t^2x^8, $t^2y^2x^2$, ty^6x and wtz, let $\mathcal{R}$ be the proper transform of $\mathcal{P}$ on Z and let R be a general surface in $\mathcal{R}$. Then R is nef and big because the base locus of $\mathcal{R}$ does not contain curves. But

$$R \sim_{\mathbb{Q}} (\alpha\circ\beta\circ\pi)^*(-30K_X) - \frac{30}{11}(\beta\circ\pi)^*(E) - \frac{8}{11}\pi^*(F) - \frac{2}{3}G,$$

which implies that $R\cdot B_1\cdot B_2 = 0$, where B_1 and B_2 are general surfaces in $\mathcal{B}$. This contradicts Corollary 2.6.

§ 42. The case $n = 80$: a hypersurface of degree 34 in $\mathbb{P}(1,3,4,10,17)$

We use the notation and assumptions of § 3. Let $n = 80$. Then X is a general hypersurface of degree 34 in $\mathbb{P}(1,3,4,10,17)$ whose singularities consist of a point P_1 that is a singularity of type $\frac{1}{2}(1,1,1)$, a point P_2 that is a singularity of type $\frac{1}{3}(1,1,2)$, a point P_3 that is a singularity of type $\frac{1}{4}(1,1,3)$ and a point P_4 that is a singularity of type $\frac{1}{10}(1,3,7)$. The equality $-K_X^3 = 1/60$ holds.

Proposition 42.1. *The assertion of Theorem 1.10 holds for $n = 80$.*

Proof. There is a commutative diagram

$$\begin{tikzcd} U \arrow[d, "\alpha"'] & Y \arrow[l, "\beta"'] \arrow[d, "\eta"] \\ X \arrow[r, dashed, "\psi"'] & \mathbb{P}(1,3,4), \end{tikzcd}$$

where ψ is the natural projection, α is the weighted blow-up of P_4 with weights $(1,3,7)$, β is the weighted blow-up with weights $(1,3,4)$ of the singular point of U that is a quotient singularity of type $\frac{1}{7}(1,3,4)$ contained in the exceptional divisor of α, and η is an elliptic fibration.

It follows from Theorem 3.3, Lemma 3.11 and Proposition 3.5 that $\mathbb{CS}\big(X, \frac{1}{k}\mathcal{M}\big) = \{P_4\}$.

Let $\mathcal{D}$ be the proper transform of $\mathcal{M}$ on U and P_5, P_6 the singular points of U that are quotient singularities of types $\frac{1}{3}(1,1,2)$, $\frac{1}{4}(1,1,3)$, respectively, contained in the exceptional divisor of α. Then $\mathcal{D} \sim_{\mathbb{Q}} -kK_U$ by Theorem 2.2. It follows from Lemmas 2.1 and 2.3 that either $P_5 \in \mathbb{CS}\big(U, \frac{1}{k}\mathcal{D}\big)$ or $P_6 \in \mathbb{CS}\big(U, \frac{1}{k}\mathcal{D}\big)$. In the latter case, the assertion of Theorem 1.10 holds for X by Theorem 2.2.

We may assume $P_5 \in \mathbb{CS}\big(U, \frac{1}{k}\mathcal{D}\big)$. The hypersurface X can be given by the quasi-homogeneous equation

$$t^3z + t^2f_{14}(x,y,z,w) + tf_{24}(x,y,z,w) + f_{34}(x,y,z,w) = 0 \subset \mathrm{Proj}\big(\mathbb{C}[x,y,z,t,w]\big),$$

where $\mathrm{wt}(x)=1$, $\mathrm{wt}(y)=3$, $\mathrm{wt}(z)=4$, $\mathrm{wt}(t)=10$, $\mathrm{wt}(w)=17$ and $f_i(x,y,z,w)$ is a general quasi-homogeneous polynomial of degree i. Let $\mathcal{P}$ be the pencil consisting of the surfaces cut out on X by the equations $\lambda x^4 + \mu z = 0$, where $(\lambda,\mu) \in \mathbb{P}^1$. Then the base locus of $\mathcal{P}$ consists of the irreducible curve cut out on X by the equations $x = z = 0$.

Let $\gamma\colon W \to U$ be the weighted blow-up of P_5 with weights $(1,1,2)$, $\mathcal{B}$ the proper transform of $\mathcal{M}$ on W, $\mathcal{H}$ the proper transform of $\mathcal{P}$ on W, D a sufficiently general surface in $\mathcal{H}$ and E, F the exceptional divisors of α, γ, respectively. Then the surface D is normal, the equivalences

$$D \sim_{\mathbb{Q}} -4K_W \sim_{\mathbb{Q}} (\alpha\circ\gamma)^*(-4K_X) - \frac{2}{5}\gamma^*(E) - \frac{4}{3}F$$

hold and the base locus of $\mathcal{H}$ consists of curves C and L such that $\alpha\circ\gamma(C)$ is the base curve of $\mathcal{P}$ and $\gamma(L)$ is the curve on $E \cong \mathbb{P}(1,3,7)$ that is contained in $\big|\mathcal{O}_{\mathbb{P}(1,3,7)}(1)\big|$.

Let S be the unique surface in $|-K_W|$ and $\bar{E}$ the proper transform of E on W. Then $S\cdot D = C + L$ and $\bar{E}\cdot D = 4L$. The equivalence $\bar{E} \sim_{\mathbb{Q}} \gamma^*(E) - \frac{2}{3}F$ holds, which implies that

$$C\cdot C = -\frac{1}{3}, \qquad C\cdot L = 0, \qquad L\cdot L = -\frac{2}{7},$$

on D. Hence, the intersection form of C and L on D is negative definite. But $\mathcal{B}|_D \sim_{\mathbb{Q}} kC + kL$, which is impossible by Lemmas 2.9 and 2.7.

§ 43. The case $n = 82$: a hypersurface of degree 36 in $\mathbb{P}(1,1,5,12,18)$

We use the notation and assumptions of § 3. Let $n = 82$. Then X is a general hypersurface of degree 36 in $\mathbb{P}(1,1,5,12,18)$, the equality $-K_X^3 = 1/30$ holds and the singularities of X consist of a point P_1 that is a quotient singularity of type $\frac{1}{5}(1,2,3)$ and a point P_2 that is a quotient singularity of type $\frac{1}{6}(1,1,5)$.

Proposition 43.1. *The assertion of Theorem 1.10 holds for* $n = 82$.

Proof. Suppose that $P_2 \in \mathbb{CS}\big(X, \frac{1}{k}\mathcal{M}\big)$. There is a commutative diagram

$$\begin{tikzcd} & U \arrow[dl, "\alpha"'] \arrow[dr, "\eta"] & \\ X \arrow[rr, dashed, "\psi"'] & & \mathbb{P}(1,1,5), \end{tikzcd}$$

where ψ is a projection, α is the weighted blow-up of P_2 with weights $(1,1,6)$ and η is an elliptic fibration. Let $\mathcal{D}$ be the proper transform of $\mathcal{M}$ on U. Then $\mathcal{D} \sim_{\mathbb{Q}} -kK_U$

by Theorem 2.2, which implies the existence of a commutative diagram

$$\begin{tikzcd} X \arrow[rr, dashed, "\rho"] \arrow[d, dashed, "\psi"'] & & V \arrow[d, "\nu"] \\ \mathbb{P}(1,1,5) \arrow[rr, dashed, "\sigma"'] & & \mathbb{P}^2, \end{tikzcd}$$

where σ is a birational map. Thus, it follows from Theorem 3.3, Lemma 3.11 and Proposition 3.5 that we may assume that the set $\mathbb{CS}\big(X, \frac{1}{k}\mathcal{M}\big)$ consists of the point P_1.

Let $\pi\colon W \to X$ be the weighted blow-up of P_1 with weights $(1,2,3)$ and $\mathcal{B}$ the proper transform of $\mathcal{M}$ on W. Then $\mathcal{B} \sim_{\mathbb{Q}} -kK_W$ by Theorem 2.2. The singularities of the exceptional divisor of π consist of the points Q and O that are quotient singularities of types $\frac{1}{2}(1,1,1)$ and $\frac{1}{3}(1,1,2)$ on W, respectively.

Suppose that the set $\mathbb{CS}\big(W, \frac{1}{k}\mathcal{B}\big)$ is non-empty. Then it contains either O or Q. Arguing as in the proof of Proposition 22.1, we see that $\mathbb{CS}\big(W, \frac{1}{k}\mathcal{B}\big)$ can contain neither O nor Q because $\mathcal{B}$ is not composed of a pencil.

Hence, the singularities of the log pair $(W, \frac{1}{k}\mathcal{B})$ are terminal. Arguing as in the proof of Proposition 29.1, we see that there is a birational map $\gamma\colon W \dashrightarrow Y$ such that γ is an antiflip, the divisor $-K_Y$ is nef and big and $|-rK_Y|$ induces a birational map $Y \dashrightarrow X'$ such that X' is a hypersurface in $\mathbb{P}(1,1,6,14,21)$ of degree 42 with canonical singularities (see [2], proof of Theorem 5.5.1), where $r \gg 0$.

Let $\mathcal{H}$ be the proper transform of $\mathcal{M}$ on Y. Then $\mathcal{H} \sim_{\mathbb{Q}} -kK_Y$ because γ is an isomorphism in codimension one. The map γ is a log flip with respect to $(W, \lambda\mathcal{B})$ for some rational number $\lambda > 1/k$. Thus, the singularities of the mobile log pair $\big(Y, \frac{1}{k}\mathcal{H}\big)$ are terminal, which is impossible by Lemma 2.1.

§ 44. Conclusion of the proof of Theorem 1.10

We use the notation and assumptions of § 3. In this section we complete the proof of Theorem 1.10.

Proposition 44.1. *Suppose that* $n \in \{21, 24, 33, 35, 41, 42, 46, 50, 54, 55, 61, 62, 63, 67, 69, 71, 76, 77, 83, 85, 91\}$. *Then there is a commutative diagram*

$$\begin{tikzcd} X \arrow[rr, dashed, "\rho"] \arrow[d, dashed, "\psi"'] & & V \arrow[d, "\nu"] \\ \mathbb{P}(1,a_1,a_2) \arrow[rr, dashed, "\sigma"'] & & \mathbb{P}^2, \end{tikzcd} \tag{44.1}$$

where ψ is the natural projection and σ is a birational map.

Proof. It follows from Theorem 3.3 and Lemma 3.11 that the set $\mathbb{CS}\big(X, \frac{1}{k}\mathcal{M}\big)$ consists of singular points of X. We shall prove the existence of the diagram (44.1) case by case.

Case $n = 21$. The variety X is a general hypersurface in $\mathbb{P}(1,1,2,4,7)$ of degree 14, the equality $-K_X^3 = 1/4$ holds and the singularities of X consist of points O_1 and O_2

that are quotient singularities of type $\frac{1}{2}(1,1,1)$ and a point P that is a quotient singularity of type $\frac{1}{4}(1,1,3)$. It follows from Proposition 3.5 that $\mathbb{CS}\big(X,\frac{1}{k}\mathcal{M}\big)=\{P\}$. Let $\mathcal{D}$ be the proper transform of $\mathcal{M}$ on U. Then $\mathcal{D}\sim_{\mathbb{Q}} -kK_U$ by Theorem 2.2, and it follows from Lemmas 2.1 and 2.3 that the set $\mathbb{CS}\big(U,\frac{1}{k}\mathcal{D}\big)$ contains the singular point of U that is contained in the exceptional divisor of α. The existence of the commutative diagram (44.1) now follows from Theorem 2.2 because there is a commutative diagram

$$\begin{tikzcd} U \arrow[d, "\alpha"'] & & W \arrow[ll, "\beta"'] \arrow[d, "\eta"] \\ X \arrow[rr, dashed, "\psi"'] & & \mathbb{P}(1,1,2), \end{tikzcd}$$

where ψ is the natural projection, α is the weighted blow-up of P with weights $(1,1,3)$, β is the weighted blow-up with weights $(1,1,2)$ of the singular point of U that is a singularity of type $\frac{1}{3}(1,1,2)$, and η is an elliptic fibration.

Case $n=24$. The variety X is a general hypersurface in $\mathbb{P}(1,1,2,5,7)$ of degree 15. The equality $-K_X^3=3/14$ holds. The singularities of X consist of a point P_1 that is a quotient singularity of type $\frac{1}{2}(1,1,1)$ and a point P_2 that is a quotient singularity of type $\frac{1}{7}(1,2,5)$. There is a commutative diagram

$$\begin{tikzcd} U \arrow[d, "\alpha"'] & W \arrow[l, "\beta"'] & & Y \arrow[ll, "\gamma"'] \arrow[d, "\eta"] \\ X \arrow[rrr, dashed, "\psi"'] & & & \mathbb{P}(1,1,2), \end{tikzcd}$$

where α is the weighted blow-up of P with weights $(1,2,5)$, β is the weighted blow-up with weights $(1,2,3)$ of the singular point of U that is a singularity of type $\frac{1}{5}(1,2,3)$, γ is the weighted blow-up with weights $(1,1,2)$ of the singular point of W that is a quotient singularity of type $\frac{1}{3}(1,1,2)$, and η is an elliptic fibration.

It follows from Proposition 3.5 that $\mathbb{CS}\big(X,\frac{1}{k}\mathcal{M}\big)=\{P_2\}$.

Arguing as in the proof of Proposition 22.1, we see that $\mathcal{D}\sim_{\mathbb{Q}} -kK_W$, where $\mathcal{D}$ is the proper transform of $\mathcal{M}$ on W and the set $\mathbb{CS}\big(W,\frac{1}{k}\mathcal{D}\big)$ does not contain subvarieties of W that are not contained in the exceptional divisor of β. We can apply the arguments in the proof of Proposition 22.1 to the log pair $(W,\frac{1}{k}\mathcal{D})$ to prove that the set $\mathbb{CS}\big(W,\frac{1}{k}\mathcal{D}\big)$ contains the singular point of W that is a singularity of type $\frac{1}{3}(1,1,2)$. This implies the existence of the commutative diagram (44.1) by Theorem 2.2.

Case $n=33$. The variety X is a general hypersurface in $\mathbb{P}(1,2,3,5,7)$ of degree 17. The equality $-K_X^3=17/210$ holds. The singularities of X consist of a point P_1 that is a quotient singularity of type $\frac{1}{2}(1,1,1)$, a point P_2 that is a quotient singularity of type $\frac{1}{3}(1,1,2)$, a point P_3 that is a quotient singularity of type $\frac{1}{5}(1,2,3)$ and a point P_4 that is a singularity of type $\frac{1}{7}(1,2,5)$. There is a commutative diagram

$$\begin{tikzcd}
 & & Y \arrow[dl, "\gamma_5"'] \arrow[dr, "\gamma_3"] \arrow[drr, "\eta"] & & \\
 & U_{34} \arrow[dl, "\beta_4"'] \arrow[dr, "\beta_3"] & & U_{45} \arrow[dl, "\beta_5"'] & \mathbb{P}(1,2,3), \\
U_3 \arrow[dr, "\alpha_3"'] & & U_2 \arrow[dl, "\alpha_4"'] & & \\
 & X \arrow[uurrr, dashed, "\psi"'] & & &
\end{tikzcd}$$

where α_3 is the weighted blow-up of P_3 with weights $(1,2,3)$, α_4 is the weighted blow-up of P_4 with weights $(1,2,5)$, β_4 is the weighted blow-up with weights $(1,2,5)$ of the proper transform of P_4 on U_3, β_3 is the weighted blow-up with weights $(1,2,3)$ of the proper transform of P_3 on U_4, β_5 is the weighted blow-up with weights $(1,2,3)$ of the singular point of the variety U_4 that is a quotient singularity of type $\frac{1}{5}(1,2,3)$ contained in the exceptional divisor of α_4, γ_3 is the weighted blow-up with weights $(1,2,3)$ of the proper transform of P_3 on U_{45}, γ_5 is the weighted blow-up with weights $(1,2,3)$ of the singular point of U_{34} that is a quotient singularity of type $\frac{1}{5}(1,2,3)$ contained in the exceptional divisor of β_4, and η is an elliptic fibration.

Arguing as in the proof of Proposition 18.1, we see that $\mathbb{CS}\big(X, \frac{1}{k}\mathcal{M}\big) = \{P_3, P_4\}$.

Let $\mathcal{D}_{34}$ be the proper transform of $\mathcal{M}$ on U_{34} and let $\bar{P}_5$ and $\bar{P}_6$ be the singular points of U_{34} that are quotient singularities of types $\frac{1}{5}(1,2,3)$ and $\frac{1}{2}(1,1,1)$, respectively, contained in the exceptional divisor of β_4. Then $\mathcal{D}_{34} \sim_{\mathbb{Q}} -kK_{U_{34}}$ by Theorem 2.2. It follows from Lemma 2.3 and the proof of Proposition 18.1 that $\bar{P}_5 \in \mathbb{CS}\big(U_{34}, \frac{1}{k}\mathcal{D}_{34}\big)$. The existence of the commutative diagram (44.1) now follows from Theorem 2.2.

Case $n = 35$. The variety X is a general hypersurface in $\mathbb{P}(1,1,3,5,9)$ of degree 18 and the equality $-K_X^3 = 2/15$ holds. The singularities of X consist of points P_1 and P_2 that are singularities of type $\frac{1}{3}(1,1,2)$ and a point P_3 that is a singularity of type $\frac{1}{5}(1,1,4)$. There is a commutative diagram

$$\begin{tikzcd}
U \arrow[d, "\alpha"'] & W \arrow[l, "\beta"'] \arrow[d, "\eta"] \\
X \arrow[r, dashed, "\psi"'] & \mathbb{P}(1,1,3),
\end{tikzcd}$$

where α is the blow-up of P_3 with weights $(1,1,4)$, β is the blow-up with weights $(1,1,3)$ of the point of U that is a singularity of type $\frac{1}{4}(1,1,3)$, and η is an elliptic fibration.

It follows from Theorem 3.3, Lemma 3.11 and Proposition 3.5 that $\mathbb{CS}\big(X, \frac{1}{k}\mathcal{M}\big) = \{P_3\}$.

Let $\mathcal{D}$ be the proper transform of $\mathcal{M}$ on U. Then $\mathcal{D} \sim_{\mathbb{Q}} -kK_U$ by Theorem 2.2. It follows from Lemma 2.3 that the set $\mathbb{CS}\big(U, \frac{1}{k}\mathcal{D}\big)$ contains the singular point of U

that is contained in the exceptional divisor of α. The existence of the diagram (44.1) follows from Theorem 2.2.

Case $n = 41$. The variety X is a general hypersurface in $\mathbb{P}(1,1,4,5,10)$ of degree 20 whose singularities consist of a point O that is a singularity of type $\frac{1}{2}(1,1,1)$ and points P_1 and P_2 that are singularities of type $\frac{1}{5}(1,1,4)$.

There is a commutative diagram

$$\begin{tikzcd}
 & Y \arrow[dl, "\gamma"'] \arrow[dr, "\delta"] \arrow[rrr, "\eta"] & & & \mathbb{P}(1,1,4), \\
U \arrow[dr, "\alpha"'] & & W \arrow[dl, "\beta"] & & \\
 & X \arrow[rrruu, dashed, "\psi"'] & & &
\end{tikzcd}$$

where α is the weighted blow-up of P_1 with weights $(1,1,4)$, β is the weighted blow-up of P_2 with weights $(1,1,4)$, γ is the weighted blow-up with weights $(1,1,4)$ of the proper transform of P_2 on U, δ is the weighted blow-up with weights $(1,1,4)$ of the proper transform of P_1 on W, and η is an elliptic fibration.

It follows from Proposition 3.5 that $\mathbb{CS}\big(X, \frac{1}{k}\mathcal{M}\big) \subseteq \{P_1, P_2\}$. Arguing as in the proof of Proposition 13.1, we see that the diagram (44.1) exists.

Case $n = 42$. The variety X is a hypersurface in $\mathbb{P}(1,2,3,5,10)$ of degree 20. The equality $-K_X^3 = 1/15$ holds. The singularities of X consist of points P_1, P_2, P_3 that are quotient singularities of type $\frac{1}{2}(1,1,1)$, a point P_4 that is a singularity of type $\frac{1}{3}(1,1,2)$ and points P_5, P_6 that are singularities of type $\frac{1}{5}(1,2,3)$.

There is a commutative diagram

$$\begin{tikzcd}
 & Y \arrow[dl, "\gamma"'] \arrow[dr, "\delta"] \arrow[rrr, "\eta"] & & & \mathbb{P}(1,2,3), \\
U \arrow[dr, "\alpha"'] & & W \arrow[dl, "\beta"] & & \\
 & X \arrow[rrruu, dashed, "\psi"'] & & &
\end{tikzcd}$$

where α is the weighted blow-up of P_5 with weights $(1,2,3)$, β is the weighted blow-up of P_6 with weights $(1,2,3)$, γ is the weighted blow-up of the proper transform of P_6 on U with weights $(1,2,3)$, δ is the weighted blow-up of the proper transform of P_5 on W with weights $(1,2,3)$, and η is an elliptic fibration.

It follows from Proposition 3.5 that $\mathbb{CS}\big(X, \frac{1}{k}\mathcal{M}\big) \subseteq \{P_5, P_6\}$. The existence of the diagram (44.1) follows from Theorem 2.2 in the case when $\mathbb{CS}\big(X, \frac{1}{k}\mathcal{M}\big) = \{P_5, P_6\}$. Thus, we may assume that $\mathbb{CS}\big(X, \frac{1}{k}\mathcal{M}\big) = \{P_5\}$.

Let $\mathcal{B}$ be the proper transform of $\mathcal{M}$ on U and O, Q the singular points of U that are quotient singularities of types $\frac{1}{3}(1,1,2)$, $\frac{1}{2}(1,1,1)$, respectively, contained in the exceptional divisor of α. Arguing as in the proof of Lemma 21.3, we see that $\mathbb{CS}\big(U, \frac{1}{k}\mathcal{B}\big) = \{Q\}$.

Let $\zeta\colon Z \to U$ be the weighted blow-up of Q with weights $(1,1,1)$, $\mathcal{D}$ the proper transform of $\mathcal{M}$ on Z and $\mathcal{H}$ the proper transform of $|-3K_X|$ on Z. Then

$\mathcal{D} \sim_{\mathbb{Q}} -kK_Z$ by Theorem 2.2. The base locus of $\mathcal{H}$ consists of an irreducible curve C such that $\alpha \circ \zeta(C)$ is the base curve of $|-3K_X|$.

Let S be a general surface in $\mathcal{H}$. Then S is normal and $C^2 < 0$ on S. But the equivalence $\mathcal{D}|_S \sim_{\mathbb{Q}} kC$ holds, which is impossible by Lemmas 2.7 and 2.9.

Case $n = 45$. The variety X is a general hypersurface in $\mathbb{P}(1,3,4,5,8)$ of degree 20 whose singularities consist of a point P_1 of type $\frac{1}{3}(1,1,2)$, points P_2 and P_3 of type $\frac{1}{4}(1,1,3)$ and a point P_4 of type $\frac{1}{8}(1,3,5)$.

There is a commutative diagram

$$\begin{tikzcd} U & & Y \arrow[ll, "\beta"'] \arrow[d, "\eta"] \\ X \arrow[from=u, "\alpha"'] \arrow[rr, dashed, "\psi"'] & & \mathbb{P}(1,3,4), \end{tikzcd}$$

where α is the weighted blow-up of P_4 with weights $(1,3,5)$, β is the weighted blow-up with weights $(1,2,3)$ of the singular point of U that is a quotient singularity of type $\frac{1}{5}(1,2,3)$ contained in the exceptional divisor of α, and η is an elliptic fibration.

It follows from Proposition 3.5 that $\mathbb{CS}\big(X, \frac{1}{k}\mathcal{M}\big) = \{P_4\}$.

Let $\mathcal{B}$ be the proper transform of $\mathcal{M}$ on U, $\bar{P}_5$ the singular point of U that is a quotient singularity of type $\frac{1}{5}(1,2,3)$ contained in the exceptional divisor of α and $\bar{P}_6$ the singular point of U that is a quotient singularity of type $\frac{1}{2}(1,1,1)$ contained in the exceptional divisor of α. Then it follows from Lemma 2.3 that $\mathbb{CS}\big(U, \frac{1}{k}\mathcal{B}\big) \cap \{\bar{P}_5, \bar{P}_6\} \neq \varnothing$.

Arguing as in the proof of Proposition 30.1, we see that $\bar{P}_5 \in \mathbb{CS}\big(U, \frac{1}{k}\mathcal{B}\big)$, which implies the existence of the commutative diagram (44.1) by Theorem 2.2.

Case $n = 46$. The variety X is a general hypersurface in $\mathbb{P}(1,1,3,7,10)$ of degree 21. The equality $-K_X^3 = 1/10$ holds. The singularities of X consist of a point P that is a singularity of type $\frac{1}{10}(1,3,7)$. There is a commutative diagram

$$\begin{tikzcd} U \arrow[d, "\alpha"'] & W \arrow[l, "\beta"'] & Y \arrow[l, "\gamma"'] \arrow[d, "\eta"] \\ X \arrow[rr, dashed, "\psi"'] & & \mathbb{P}(1,1,3), \end{tikzcd}$$

where α is the blow-up of P with weights $(1,3,7)$, β is the blow-up with weights $(1,3,4)$ of the singular point of U that is a singularity of type $\frac{1}{7}(1,3,4)$, γ is the blow-up with weights $(1,1,3)$ of the singular point of W that is a singularity of type $\frac{1}{4}(1,1,3)$, and η is an elliptic fibration.

It follows from Proposition 3.5 that $\mathbb{CS}\big(X, \frac{1}{k}\mathcal{M}\big) = \{P\}$.

Let $\mathcal{D}$ be the proper transform of $\mathcal{M}$ on W. Arguing as in the proof of Proposition 22.1, we see that $\mathcal{D} \sim_{\mathbb{Q}} -kK_W$ and the set $\mathbb{CS}\big(W, \frac{1}{k}\mathcal{D}\big)$ does not contain subvarieties of W that are not contained in the exceptional divisor of β. The set $\mathbb{CS}\big(W, \frac{1}{k}\mathcal{D}\big)$ is non-empty because the divisor $-K_W$ is nef and big. Therefore, we can apply the arguments in the proof of Proposition 22.1 to the log pair $\big(W, \frac{1}{k}\mathcal{D}\big)$, which implies that the set $\mathbb{CS}\big(W, \frac{1}{k}\mathcal{D}\big)$ contains the singular point of W that is a singularity of type $\frac{1}{4}(1,1,3)$. The existence of the commutative diagram (44.1) is implied by Theorem 2.2.

Case $n = 50$. The variety X is a general hypersurface in $\mathbb{P}(1,1,3,7,11)$ of degree 22 and the inequality $-K_X^3 = 2/21$ holds. The singularities of X consist of a point P_1 that is a quotient singularity of type $\frac{1}{3}(1,1,2)$ and a point P_2 that is a quotient singularity of type $\frac{1}{7}(1,3,4)$. There is a commutative diagram

$$\begin{tikzcd} U \arrow[d, "\alpha"'] & & W \arrow[ll, "\beta"'] \arrow[d, "\eta"] \\ X \arrow[rr, dashed, "\psi"'] & & \mathbb{P}(1,1,3), \end{tikzcd}$$

where α is the blow-up of P_2 with weights $(1,3,7)$, β is the blow-up with weights $(1,1,3)$ of the singular point of U that is a singularity of type $\frac{1}{4}(1,1,3)$ and η is an elliptic fibration.

It follows from Proposition 3.5 that $\mathbb{CS}\big(X, \frac{1}{k}\mathcal{M}\big) = \{P_2\}$. Arguing as in the proof of Proposition 22.1, we see that the diagram (44.1) exists.

Case $n = 54$. The variety X is a general hypersurface in $\mathbb{P}(1,1,6,8,9)$ of degree 24 whose singularities consist of a point P_1 type $\frac{1}{2}(1,1,1)$, a point P_2 of type $\frac{1}{3}(1,1,2)$ and a point P_3 of type $\frac{1}{9}(1,1,8)$. The equality $-K_X^3 = 1/18$ holds.

There is a commutative diagram

$$\begin{tikzcd} U \arrow[d, "\alpha"'] & W \arrow[l, "\beta"'] & Y \arrow[l, "\gamma"'] \arrow[d, "\eta"] \\ X \arrow[rr, dashed, "\psi"'] & & \mathbb{P}(1,1,6), \end{tikzcd}$$

where α is the blow-up of P_3 with weights $(1,1,8)$, β is the blow-up with weights $(1,1,7)$ of the singular point of U that is a singularity of type $\frac{1}{8}(1,1,7)$, γ is the blow-up with weights $(1,1,3)$ of the singular point of W that is a singularity of type $\frac{1}{7}(1,1,6)$, and η is an elliptic fibration.

It follows from Proposition 3.5 that $\mathbb{CS}\big(X, \frac{1}{k}\mathcal{M}\big) = \{P_3\}$. Arguing as in the proof of Proposition 14.1, we see that the diagram (44.1) exists.

Case $n = 55$. The variety X is a general hypersurface in $\mathbb{P}(1,2,3,7,12)$ of degree 24 whose singularities consist of points P_1, P_2 of type $\frac{1}{2}(1,1,1)$, points P_3, P_4 of type $\frac{1}{3}(1,1,2)$ and a point P_5 of type $\frac{1}{7}(1,2,5)$. There is a commutative diagram

$$\begin{tikzcd} U \arrow[d, "\alpha"'] & & W \arrow[ll, "\beta"'] \arrow[d, "\eta"] \\ X \arrow[rr, dashed, "\psi"'] & & \mathbb{P}(1,2,3), \end{tikzcd}$$

where α is the blow-up of P_5 with weights $(1,2,5)$, β is the blow-up with weights $(1,2,3)$ of the singular point of U that is a singularity of type $\frac{1}{5}(1,2,3)$ and η is an elliptic fibration.

It follows from Proposition 3.5 that $\mathbb{CS}\big(X, \frac{1}{k}\mathcal{M}\big) = \{P_5\}$.

Let $\mathcal{B}$ be the proper transform of $\mathcal{M}$ on U and let P_6 and P_7 be the singular points of U that are singularities of types $\frac{1}{2}(1,1,1)$ and $\frac{1}{5}(1,2,3)$, respectively, contained in the exceptional divisor of α. Then $\mathbb{CS}\big(U, \frac{1}{k}\mathcal{B}\big) \subseteq \{P_6, P_7\}$ by Lemmas 2.1, 2.3 and 2.4.

The existence of the diagram (44.1) follows from Theorem 2.2 in the case when $P_7 \in \mathbb{CS}\big(U, \frac{1}{k}\mathcal{B}\big)$. Hence, we may assume that $\mathbb{CS}\big(U, \frac{1}{k}\mathcal{B}\big)$ consists of the point P_6.

The hypersurface X can be given by the equation

$$t^3 z + t^2 f_{10}(x,y,z,w) + t f_{17}(x,y,z,w) + f_{24}(x,y,z,w) = 0 \subset \operatorname{Proj}\big(\mathbb{C}[x,y,z,t,w]\big),$$

where $\mathrm{wt}(x)=1$, $\mathrm{wt}(y)=2$, $\mathrm{wt}(z)=3$, $\mathrm{wt}(t)=7$, $\mathrm{wt}(w)=12$ and $f_i(x,y,z,w)$ is a sufficiently general quasi-homogeneous polynomial of degree i. Let $\gamma\colon W \to U$ be the weighted blow-up of P_6 with weights $(1,1,1)$, $\mathcal{H}$ the proper transform of $\mathcal{M}$ on W, $\mathcal{P}$ the proper transform on W of the pencil of surfaces cut out on X by the equations $\lambda x^3 + \mu z = 0$, where $(\lambda, \mu) \in \mathbb{P}^1$, and D a sufficiently general surface in $\mathcal{P}$. Then the base locus of $\mathcal{P}$ consists of irreducible curves C, L and Δ such that $\alpha \circ \gamma(C)$ is the base curve of $|-3K_X|$, $\gamma(L)$ is contained in the exceptional divisor of α and Δ is contained in the exceptional divisor of γ.

The surface D is normal. The intersection form of the curves L and C on the surface D is negative definite. But $\mathcal{H}|_D \sim_{\mathbb{Q}} kC + kL$, which is impossible by Lemmas 2.9 and 2.7.

Case $n = 61$. The variety X is a general hypersurface in $\mathbb{P}(1,4,5,7,9)$ of degree 25. The equality $-K_X^3 = 5/252$ holds. The singularities of X consist of a point P_1 that is a quotient singularity of type $\frac{1}{4}(1,1,3)$, a point P_2 that is a quotient singularity of type $\frac{1}{7}(1,2,5)$ and a point P_3 that is a quotient singularity of type $\frac{1}{9}(1,4,5)$. There is a commutative diagram

$$\begin{tikzcd}
 & Y \arrow[dl, "\beta_3"'] \arrow[dr, "\beta_2"] \arrow[rrr, "\eta"] & & & \\
U_2 \arrow[dr, "\alpha_2"'] & & U_3 \arrow[dl, "\alpha_3"] & & \mathbb{P}(1,4,5), \\
 & X \arrow[urrr, dashed, "\psi"'] & & &
\end{tikzcd}$$

where α_2 is the weighted blow-up of P_2 with weights $(1,2,5)$, α_3 is the weighted blow-up of P_3 with weights $(1,4,5)$, β_2 is the weighted blow-up with weights $(1,2,5)$ of the proper transform of P_2 on U_3, β_3 is the weighted blow-up with weights $(1,4,5)$ of the proper transform of P_3 on U_2, and η is an elliptic fibration.

It follows from Proposition 3.5 that $\mathbb{CS}\big(X, \frac{1}{k}\mathcal{M}\big) \subseteq \{P_2, P_3\}$. Arguing as in the proof of Lemma 36.2, we see that $\mathbb{CS}\big(X, \frac{1}{k}\mathcal{M}\big) = \{P_2, P_3\}$. The existence of the diagram (44.1) follows from Theorem 2.2.

Case $n = 62$. The variety X is a hypersurface in $\mathbb{P}(1,1,5,7,13)$ of degree 26. The equality $-K_X^3 = 2/35$ holds. The singularities of X consist of a point P_1 that is a quotient singularity of type $\frac{1}{5}(1,2,3)$ and a point P_2 that is a singularity of type $\frac{1}{7}(1,1,6)$.

There is a commutative diagram

$$\begin{tikzcd} U \arrow[d, "\alpha"'] & & Y \arrow[ll, "\beta"'] \arrow[d, "\eta"] \\ X \arrow[rr, dashed, "\psi"'] & & \mathbb{P}(1,1,5), \end{tikzcd}$$

where α is the weighted blow-up of P_2 with weights $(1,1,6)$, β is the weighted blow-up with weights $(1,1,5)$ of the singular point of U that is a quotient singularity of type $\frac{1}{6}(1,1,5)$, and η is an elliptic fibration.

It follows from Proposition 3.5 that $\mathbb{CS}\big(X, \frac{1}{k}\mathcal{M}\big) \subseteq \{P_1, P_2\}$. Arguing as in the proof of Proposition 31.1, we see that the commutative diagram (44.1) exists.

Case $n = 63$. The variety X is a general hypersurface in $\mathbb{P}(1,2,3,8,13)$ of degree 26. The singularities of X consist of points P_1, P_2, P_3 that are quotient singularities of type $\frac{1}{2}(1,1,1)$, a point P_4 that is a quotient singularity of type $\frac{1}{3}(1,1,2)$ and a point P_5 that is a singularity of type $\frac{1}{8}(1,3,5)$. The equality $-K_X^3 = 1/24$ holds. There is a commutative diagram

$$\begin{tikzcd} U \arrow[d, "\alpha"'] & & Y \arrow[ll, "\beta"'] \arrow[d, "\eta"] \\ X \arrow[rr, dashed, "\psi"'] & & \mathbb{P}(1,2,3), \end{tikzcd}$$

where α is the weighted blow-up of P_5 with weights $(1,3,5)$, β is the weighted blow-up with weights $(1,2,3)$ of the singular point of U that is a quotient singularity of type $\frac{1}{5}(1,2,3)$ contained in the exceptional divisor of α, and η is an elliptic fibration.

It follows from Proposition 3.5 that $\mathbb{CS}\big(X, \frac{1}{k}\mathcal{M}\big) = \{P_5\}$.

Let $\mathcal{B}$ be the proper transform of $\mathcal{M}$ on U and P_6, P_7 the singular points of U that are quotient singularities of types $\frac{1}{5}(1,2,3)$, $\frac{1}{2}(1,1,1)$, respectively, contained in the exceptional divisor of α. Then it follows from Lemma 2.3 that the set $\mathbb{CS}\big(U, \frac{1}{k}\mathcal{B}\big)$ contains either P_6 or P_7.

Suppose that $\mathbb{CS}\big(U, \frac{1}{k}\mathcal{B}\big)$ contains P_7. Let $\gamma\colon W \to U$ be the weighted blow-up of P_7 with weights $(1,1,1)$ and let $\mathcal{H}$ and $\mathcal{P}$ be the proper transforms of the linear system $\mathcal{M}$ and the pencil $|-2K_X|$ on W, respectively. Then the base locus of $\mathcal{P}$ consists of irreducible curves C and L such that $\alpha \circ \gamma(C)$ is the unique curve in the base locus of $|-2K_X|$ and $\gamma(L)$ is contained in the exceptional divisor of α. The intersection form of L and C on D is negative definite. On the other hand, the equivalence $\mathcal{H}|_D \sim_{\mathbb{Q}} kC + kL$ holds, which is impossible by Lemmas 2.9 and 2.7.

Therefore, the set $\mathbb{CS}\big(U, \frac{1}{k}\mathcal{B}\big)$ contains the point P_6. The assertion of Theorem 2.2 implies the existence of the commutative diagram (44.1).

Case $n = 67$. The variety X is a general hypersurface in $\mathbb{P}(1,1,4,9,14)$ of degree 28. The singularities of X consist of a point P_1 that is a quotient singularity of type $\frac{1}{2}(1,1,1)$ and a point P_2 that is a quotient singularity of type $\frac{1}{9}(1,4,5)$. The equality $-K_X^3 = 1/18$ holds. It follows from Proposition 3.5 that $\mathbb{CS}\big(X, \frac{1}{k}\mathcal{M}\big) = \{P_2\}$. There is a commutative diagram

$$\begin{tikzcd} U \arrow[d, "\alpha"'] & & W \arrow[ll, "\beta"'] \arrow[d, "\eta"] \\ X \arrow[rr, dashed, "\psi"'] & & \mathbb{P}(1,1,4), \end{tikzcd}$$

where α is the weighted blow-up of P_2 with weights $(1,4,5)$, β is the weighted blow-up with weights $(1,1,4)$ of the singular point of U that is a quotient singularity of type $\frac{1}{5}(1,1,4)$, and η is an elliptic fibration. Arguing as in the proof of Proposition 22.1, we see that the commutative diagram (44.1) exists.

Case $n = 69$. The variety X is a general hypersurface in $\mathbb{P}(1,4,6,7,11)$ of degree 28. The singularities of X consist of points P_1, P_2 that are quotient singularities of type $\frac{1}{2}(1,1,1)$ and points P_3, P_4 that are quotient singularities of types $\frac{1}{6}(1,1,5)$, $\frac{1}{11}(1,4,7)$, respectively. The equality $-K_X^3 = 1/66$ holds. There is a commutative diagram

$$\begin{tikzcd} U \arrow[d, "\alpha"'] & & W \arrow[ll, "\beta"'] \arrow[d, "\eta"] \\ X \arrow[rr, dashed, "\psi"'] & & \mathbb{P}(1,3,4), \end{tikzcd}$$

where α is the weighted blow-up of P_4 with weights $(1,4,7)$, β is the weighted blow-up with weights $(1,3,4)$ of the singular point of type $\frac{1}{7}(1,3,4)$, and η is an elliptic fibration. It follows from Proposition 3.5 that $\mathbb{CS}\big(X, \frac{1}{k}\mathcal{M}\big) = \{P_4\}$.

Let $\mathcal{B}$ be the proper transform of $\mathcal{M}$ on U and let P_5 and P_6 be the singular points of U that are quotient singularities of types $\frac{1}{7}(1,3,4)$ and $\frac{1}{4}(1,1,3)$, respectively, contained in the exceptional divisor of α. Then it follows from Lemmas 2.1 and 2.3 that the set $\mathbb{CS}\big(U, \frac{1}{k}\mathcal{B}\big)$ contains either P_5 or P_6. Arguing as in the proof of Lemma 28.5, we see that $\mathbb{CS}\big(U, \frac{1}{k}\mathcal{B}\big)$ does not contain P_6. Thus, it contains P_5. The existence of the commutative diagram (44.1) follows from Theorem 2.2.

Case $n = 71$. The variety X is a general hypersurface in $\mathbb{P}(1,1,6,8,15)$ of degree 30 whose singularities consist of points P_1, P_2 and P_3 that are singularities of types $\frac{1}{2}(1,1,1)$, $\frac{1}{3}(1,1,2)$ and $\frac{1}{8}(1,1,7)$, respectively. The equality $-K_X^3 = 1/24$ holds.

It follows from Proposition 3.5 that $\mathbb{CS}\big(X, \frac{1}{k}\mathcal{M}\big) = \{P_3\}$. There is a commutative diagram

$$\begin{tikzcd} U \arrow[d, "\alpha"'] & & W \arrow[ll, "\beta"'] \arrow[d, "\eta"] \\ X \arrow[rr, dashed, "\psi"'] & & \mathbb{P}(1,1,6), \end{tikzcd}$$

where α is the weighted blow-up of P_3 with weights $(1,1,7)$, β is the weighted blow-up with weights $(1,1,6)$ of the singular point of U that is contained in the exceptional divisor of α, and η is an elliptic fibration.

Let $\mathcal{D}$ be the proper transform of $\mathcal{M}$ on U. Then $\mathcal{D} \sim_{\mathbb{Q}} -kK_U$ by Theorem 2.2 and it follows from Lemmas 2.1 and 2.3 that the set $\mathbb{CS}\big(U, \frac{1}{k}\mathcal{D}\big)$ contains the singular point of U that is contained in the exceptional divisor of α. The existence of the commutative diagram (44.1) is implied by Theorem 2.2.

Case $n = 76$. The variety X is a general hypersurface in $\mathbb{P}(1,5,6,8,11)$ of degree 30 whose singularities consist of points P_1, P_2 and P_3 that are singularities of types $\frac{1}{2}(1,1,1)$, $\frac{1}{8}(1,3,5)$ and $\frac{1}{11}(1,5,6)$, respectively. The equality $-K_X^3 = 1/88$ holds. There is a commutative diagram

$$\begin{tikzcd}
 & Y \arrow[dl, "\beta_3"'] \arrow[dr, "\beta_2"] \arrow[rrr, "\eta"] & & & \mathbb{P}(1,5,6), \\
U_2 \arrow[dr, "\alpha_2"'] & & U_3 \arrow[dl, "\alpha_3"] & & \\
 & X \arrow[rrruu, dashed, "\psi"'] & & &
\end{tikzcd}$$

where α_2 is the weighted blow-up of P_2 with weights $(1,3,5)$, α_3 is the weighted blow-up of P_3 with weights $(1,5,6)$, β_2 is the weighted blow-up of the proper transform of P_2 on U_3 with weights $(1,3,5)$, β_3 is the weighted blow-up of the proper transform of P_3 on U_2 with weights $(1,5,6)$, and η is an elliptic fibration.

Arguing as in the proofs of Lemmas 29.3 and 36.2, we see that $\mathbb{CS}\big(X, \frac{1}{k}\mathcal{M}\big) = \{P_2, P_3\}$. The existence of the commutative diagram (44.1) follows from Theorem 2.2.

Case $n = 77$. The variety X is a general hypersurface in $\mathbb{P}(1,2,5,9,16)$ of degree 32. The singularities of X consist of points P_1, P_2 that are quotient singularities of type $\frac{1}{2}(1,1,1)$, a point P_3 that is a quotient singularity of type $\frac{1}{5}(1,1,4)$ and a point P_4 that is a quotient singularity of type $\frac{1}{9}(1,2,7)$. The equality $-K_X^3 = 1/45$ holds. There is a commutative diagram

$$\begin{tikzcd}
U \arrow[d, "\alpha"'] & Y \arrow[l, "\beta"'] \arrow[d, "\eta"] \\
X \arrow[r, dashed, "\psi"'] & \mathbb{P}(1,2,5),
\end{tikzcd}$$

where α is the blow-up of P_4 with weights $(1,2,7)$, β is the blow-up with weights $(1,2,5)$ of the singular point of U that is a singularity of type $\frac{1}{7}(1,2,5)$, and η is an elliptic fibration.

It follows from Proposition 3.5 that $\mathbb{CS}\big(X, \frac{1}{k}\mathcal{M}\big) = \{P_4\}$.

Let $\mathcal{B}$ be the proper transform of $\mathcal{M}$ on U and let P_5 and P_6 be the singular points of U that are singularities of types $\frac{1}{7}(1,2,5)$ and $\frac{1}{2}(1,1,1)$, respectively, contained in the exceptional divisor of α. It follows from Lemmas 2.1 and 2.3 that the set $\mathbb{CS}\big(U, \frac{1}{k}\mathcal{B}\big)$ contains either P_5 or P_6.

Suppose that $\mathbb{CS}\big(U, \frac{1}{k}\mathcal{B}\big)$ contains P_6. Let $\gamma\colon W \to U$ be the weighted blow-up of P_5 with weights $(1,1,1)$, let $\mathcal{H}$ and $\mathcal{D}$ be the proper transforms of $\mathcal{M}$ and $|-16K_X|$ on W, respectively, let D be a general surface in $\mathcal{D}$ and let H_1 and H_2 be general surfaces in $\mathcal{H}$. Then the base locus of $\mathcal{D}$ does not contain curves. In particular, the divisor D is nef. But $D \cdot H_1 \cdot H_1 < 0$.

Therefore, the set $\mathbb{CS}\big(U, \frac{1}{k}\mathcal{B}\big)$ contains the point P_5. The assertion of Theorem 2.2 now implies the existence of the commutative diagram (44.1).

Case $n = 83$. The variety X is a general hypersurface in $\mathbb{P}(1,3,4,11,18)$ of degree 36. The singularities of X consist of a point P_1 that is a quotient singularity of type $\frac{1}{2}(1,1,1)$, points P_2 and P_3 that are quotient singularities of type $\frac{1}{3}(1,1,2)$ and a point P_4 that is a quotient singularity of type $\frac{1}{11}(1,4,7)$. The equality $-K_X^3 = 1/66$ holds. There is a commutative diagram

$$\begin{tikzcd} U \arrow[d, "\alpha"'] & & W \arrow[ll, "\beta"'] \arrow[d, "\eta"] \\ X \arrow[rr, dashed, "\psi"'] & & \mathbb{P}(1,3,4), \end{tikzcd}$$

where α is the weighted blow-up of P_4 with weights $(1,4,7)$, β is the weighted blow-up with weights $(1,3,4)$ of the singular point of U that is a quotient singularity of type $\frac{1}{7}(1,3,4)$, and η is an elliptic fibration.

It follows from Proposition 3.5 that $\mathbb{CS}\big(X, \frac{1}{k}\mathcal{M}\big) = \{P_4\}$.

Let $\mathcal{B}$ be the proper transform of $\mathcal{M}$ on U and let P_5 and P_6 be the singular points of U that are singularities of types $\frac{1}{7}(1,3,4)$ and $\frac{1}{4}(1,1,3)$, respectively, contained in the exceptional divisor of α. Then it follows from Lemmas 2.1 and 2.3 that the set $\mathbb{CS}\big(U, \frac{1}{k}\mathcal{B}\big)$ contains either P_5 or P_6.

Arguing as in the proof of Proposition 25.1, we see that $\mathbb{CS}\big(U, \frac{1}{k}\mathcal{B}\big)$ does not contain P_6. Thus, it contains the point P_5. The existence of the diagram (44.1) follows from Theorem 2.2.

Case $n = 85$. The variety X is a general hypersurface in $\mathbb{P}(1,3,5,11,19)$ of degree 38. The singularities of X consist of a point P_1 that is a quotient singularity of type $\frac{1}{3}(1,1,2)$, a point P_2 that is a quotient singularity of type $\frac{1}{5}(1,1,4)$ and a point P_3 that is a quotient singularity of type $\frac{1}{11}(1,3,8)$. The equality $-K_X^3 = 2/165$ holds.

There is a commutative diagram

$$\begin{tikzcd} U \arrow[d, "\alpha"'] & & Y \arrow[ll, "\beta"'] \arrow[d, "\eta"] \\ X \arrow[rr, dashed, "\psi"'] & & \mathbb{P}(1,3,5), \end{tikzcd}$$

where α is the weighted blow-up of P_3 with weights $(1,3,8)$, β is the weighted blow-up with weights $(1,3,5)$ of the singular point of U that is a singularity of type $\frac{1}{8}(1,3,5)$, and η is an elliptic fibration.

It follows from Proposition 3.5 that $\mathbb{CS}\big(X, \frac{1}{k}\mathcal{M}\big) = \{P_3\}$.

Let $\mathcal{B}$ be the proper transform of $\mathcal{M}$ on U and let P_4 and P_5 be the singular points of U that are quotient singularities of types $\frac{1}{8}(1,3,5)$ and $\frac{1}{3}(1,1,2)$, respectively, contained in the exceptional divisor of α. Then it follows from Lemmas 2.1 and 2.3 that the set $\mathbb{CS}\big(U, \frac{1}{k}\mathcal{B}\big)$ contains either P_4 or P_5.

Suppose that $\mathbb{CS}\big(U, \frac{1}{k}\mathcal{B}\big)$ contains P_5. Let $\gamma\colon W \to U$ be the weighted blow-up of P_5 with weights $(1,1,2)$, let $\mathcal{H}$ and $\mathcal{D}$ be the proper transforms of $\mathcal{M}$ and $|-19K_X|$ on W, respectively, let D be a general surface in $\mathcal{D}$ and let H_1 and H_2 be general surfaces in $\mathcal{H}$. Then the base locus of $\mathcal{D}$ does not contain curves. Thus, the divisor D is nef. In particular, the inequality $D\cdot H_1\cdot H_1 \geqslant 0$ holds. But $D\cdot H_1\cdot H_1 = -2k^2/15$, which is a contradiction.

Hence, the set $\mathbb{CS}\big(U, \frac{1}{k}\mathcal{B}\big)$ contains the singular point P_4. The existence of the commutative diagram (44.1) follows from Theorem 2.2.

Case $n=91$. The variety X is a general hypersurface in $\mathbb{P}(1,4,5,13,22)$ of degree 44. The singularities of X consist of a point P_1 that is a quotient singularity of type $\frac{1}{2}(1,1,1)$, a point P_2 that is a quotient singularity of type $\frac{1}{5}(1,2,3)$ and a point P_3 that is a quotient singularity of type $\frac{1}{13}(1,4,9)$. The equality $-K_X^3 = 1/130$ holds.

There is a commutative diagram

$$\begin{tikzcd} U \arrow[d, "\alpha"'] & & W \arrow[ll, "\beta"'] \arrow[d, "\eta"] \\ X \arrow[rr, dashed, "\psi"'] & & \mathbb{P}(1,4,5), \end{tikzcd}$$

where α is the weighted blow-up of P_3 with weights $(1,4,9)$, β is the weighted blow-up with weights $(1,4,5)$ of the singular point of U that is a quotient singularity of type $\frac{1}{9}(1,4,5)$, and η is an elliptic fibration.

It follows from Proposition 3.5 that $\mathbb{CS}\big(X, \frac{1}{k}\mathcal{M}\big) = \{P_3\}$.

Let $\mathcal{B}$ be the proper transform of $\mathcal{M}$ on U and let P_4 and P_5 be the singular points of U that are quotient singularities of types $\frac{1}{9}(1,4,5)$ and $\frac{1}{4}(1,1,3)$, respectively, contained in the exceptional divisor of α. Then it follows from Lemmas 2.1 and 2.3 that $\mathbb{CS}\big(U, \frac{1}{k}\mathcal{B}\big)$ contains either P_4 or P_5.

Arguing as in the proof of Proposition 25.1, we see that $\mathbb{CS}\big(U, \frac{1}{k}\mathcal{B}\big)$ does not contain P_5. Thus, it contains the singular point P_4. The existence of the commutative diagram (44.1) follows from Theorem 2.2.

This completes the proof of Theorem 1.10.

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