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Embedded flat tori in the unit 3-sphere

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1. Introduction.

Let S^3 be the unit hypersphere in the 4-dimensional Euclidean space E^4 given by $\sum_{i=1}^4 x_i^2=1$. For each θ with $0<\theta<\pi/2$, we consider a surface M_θ in S^3 defined by

$$x_1^2+x_2^2=\cos^2\theta , \qquad x_3^2+x_4^2=\sin^2\theta .$$

The surface M_θ, which is called a Clifford torus in S^3, can be viewed as an embedded flat torus in S^3. There are many other examples of embedded flat tori in S^3. Let $p: S^3\to S^2$ be the Hopf fibration, and let γ be a simple closed curve in S^2. Then it is known [4] that the inverse image $p^{-1}(\gamma)$ is an embedded flat torus in S^3. Note that $p^{-1}(\gamma)$ is foliated by great circles of S^3, and so it satisfies the *antipodal symmetry*, i.e., it is invariant under the antipodal map of S^3. Recently the author [2] obtained another example of embedded flat tori in S^3. Although this example contains no great circle of S^3, it also satisfies the antipodal symmetry. In this paper we show that the antipodal symmetry holds for all embedded flat tori in S^3. In other words, we prove the following theorem.

THEOREM 1.1. *If $f: M\to S^3$ is an isometric embedding of a flat torus M into S^3, then the image $f(M)$ is invariant under the antipodal map of S^3.*

REMARK. In Theorem 1.1 the word "embedding" cannot be replaced by "immersion". In fact, Theorem 4.4 says that there exists a flat torus M and an isometric immersion $f: M\to S^3$ such that the image $f(M)$ is not invariant under the antipodal map of S^3. However the author does not know the answer to the following question: *For every isometric immersion f of a flat torus M into S^3, does there exist a pair of points p and q in M such that $f(p)$ and $f(q)$ are antipodal points of S^3?*

The outline of this paper is as follows. Let $SU(2)$ be the group of all 2×2 unitary matrices with determinant 1. Then $SU(2)$, endowed with a bi-invariant metric, is isometric to S^3. Using the group structure on S^3, we define a double covering $p_2: S^3 \to US^2$, where US^2 denotes the unit tangent bundle of S^2. The double covering p_2 satisfies $p_2(a)=p_2(-a)$ for all $a \in S^3$. For each regular curve γ in S^2, define a curve $\hat{\gamma}$ in US^2 by $\hat{\gamma}=\dot{\gamma}/\|\dot{\gamma}\|$. In Section 2 we study the behavior of a curve c in S^3 satisfying the relation $p_2(c)=\hat{\gamma}$.

In Section 3 we explain a method for constructing all the flat tori in S^3. A pair $\Gamma=(\gamma_1, \gamma_2)$ of periodic regular curves $\gamma_i: \boldsymbol{R} \to S^2$ is said to be a *periodic admissible pair* if the geodesic curvature of γ_1 is greater than that of γ_2 and some auxiliary conditions are satisfied. For each periodic admissible pair $\Gamma=(\gamma_1, \gamma_2)$, using the group structure on S^3, we define an immersion $F_\Gamma: \boldsymbol{R}^2 \to S^3$ by

$$F_\Gamma(s_1, s_2) = c_1(s_1) \cdot c_2(s_2)^{-1}, \tag{1.1}$$

where c_i denotes a lift of $\hat{\gamma}_i$ with respect to p_2. The immersion F_Γ induces a flat Riemannian metric g_Γ on $\boldsymbol{R}^2$. Define $G(\Gamma)$ to be the group of all diffeomorphisms ρ of $\boldsymbol{R}^2$ satisfying $F_\Gamma \circ \rho = F_\Gamma$. Then we obtain a flat torus $M_\Gamma = (\boldsymbol{R}^2, g_\Gamma)/G(\Gamma)$ and an isometric immersion $f_\Gamma: M_\Gamma \to S^3$ such that $f_\Gamma \circ \pi = F_\Gamma$, where π denotes the canonical projection of $\boldsymbol{R}^2$ onto M_Γ. Note that the immersion f_Γ is *primitive*, i.e., the identity map of M_Γ is the only diffeomorphism $\varphi: M_\Gamma \to M_\Gamma$ satisfying $f_\Gamma \circ \varphi = f_\Gamma$. Conversely, we show that if $f: M \to S^3$ is a primitive isometric immersion of a flat torus M into S^3, then there exists a periodic admissible pair Γ such that f and f_Γ are congruent (Theorem 3.1).

For each periodic admissible pair $\Gamma=(\gamma_1, \gamma_2)$, the group $G(\Gamma)$ can be identified with a lattice in $\boldsymbol{R}^2$. In Section 4 we study generators of the lattice $G(\Gamma)$. Let $l_i > 0$ be the minimum period of γ_i, and let $I(\gamma_i)$ be the element of the homology group $H_1(US^2)$ represented by the closed curve $\hat{\gamma}_i | [0, l_i]$. Note that $H_1(US^2) \cong \boldsymbol{Z}_2$ and

$$c_i(s+l_i) = -c_i(s) \qquad \text{if} \quad I(\gamma_i)=1, \tag{1.2}$$

where c_i denotes a lift of $\hat{\gamma}_i$ with respect to p_2. We show that generators of $G(\Gamma)$ can be written in terms of l_i and $I(\gamma_i)$ (Theorem 4.1).

In Section 5 we study asymptotic curves of embedded flat tori in S^3, and prove Theorem 1.1. Let M be a flat torus isometrically embedded in S^3 with a unit normal vector field ξ. We consider a unit speed asymptotic curve $c: \boldsymbol{R} \to M$. Then there exists a positive number l such that $c(s+l)=c(s)$ and $c|[0, l]$ is a simple closed curve (Theorem 5.1). Let $\alpha = c|[0, l]$, and let α^+ be a λ curve in $S^3 - M$ obtained by pushing α a very small amount along the unit normal vector field ξ. Then we show that the linking number of α and α^+ is odd (Theorem 5.2). To establish Theorem 5.2 we need a lemma which is stated in Section 2 without proof. The proof of this lemma will be given in Section 6.

We now sketch the proof of Theorem 1.1. Let $\Gamma=(\gamma_1, \gamma_2)$ be a periodic admissible pair such that $f_\Gamma: M_\Gamma \to S^3$ is an embedding. Consider a disk $D \subset M_\Gamma$

and define a knot K in S^3 by setting $K=f_\Gamma(\partial D)$. Since K is unknotted, the Arf invariant of K vanishes, i.e., $\mathrm{Arf}(K)=0$. We set $V=f_\Gamma(M_\Gamma-D)$. Then V is a Seifert surface of K, and so $\mathrm{Arf}(K)$ can be computed by using a canonical basis of the homology group $H_1(V)$. Unless $I(\gamma_1)=I(\gamma_2)=1$, it follows from Theorem 4.1 that a canonical basis of $H_1(V)$ can be represented by asymptotic curves of M_Γ. So Theorem 5.2 implies $\mathrm{Arf}(K)=1$. This shows that the periodic admissible pair $\Gamma=(\gamma_1, \gamma_2)$ must satisfy $I(\gamma_1)=I(\gamma_2)=1$. Therefore it follows from (1.1) and (1.2) that the image of the embedding f_Γ is invariant under the antipodal map of S^3. Hence the assertion of Theorem 1.1 follows from Theorem 3.1.

In the final section we compute the Gauss map of the immersion $F_\Gamma: \boldsymbol{R}^2 \to S^3$. The result of this computation will help us to understand the relation between the construction explained in Section 3 and another one which was established in the recent works of Weiner [7], [8].

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2. Preliminaries.

Let $SU(2)$ be the group of all 2×2 unitary matrices with determinant 1. Its Lie algebra $\mathfrak{su}(2)$ consists of all 2×2 skew Hermitian matrices of trace 0. The adjoint representation Ad of $SU(2)$ is given by $\mathrm{Ad}(a)x=a\cdot x\cdot a^{-1}$, where $a\in SU(2)$ and $x\in\mathfrak{su}(2)$. For $x, y\in\mathfrak{su}(2)$, we set $\langle x, y\rangle=-(1/2)\,\mathrm{trace}\,(xy)$. Then $\langle\,,\rangle$ is a positive definite inner product on $\mathfrak{su}(2)$ which is invariant under Ad. We set

$$e_1=\begin{bmatrix} 0 & \sqrt{-1} \\ \sqrt{-1} & 0 \end{bmatrix}, \quad e_2=\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \quad e_3=\begin{bmatrix} \sqrt{-1} & 0 \\ 0 & -\sqrt{-1} \end{bmatrix}.$$

Then $\{e_1, e_2, e_3\}$ is an orthonormal basis of $\mathfrak{su}(2)$. Note that

$$[e_1, e_2]=2e_3, \qquad [e_2, e_3]=2e_1, \qquad [e_3, e_1]=2e_2,$$

where $[\,,]$ denotes the Lie bracket on $\mathfrak{su}(2)$. Let E_i be a left invariant vector field on $SU(2)$ which corresponds to e_i. We endow $SU(2)$ with a Riemannian metric and an orientation such that $\{E_1, E_2, E_3\}$ is a positive orthonormal frame field. Then $SU(2)$ is a Riemannian manifold isometric to the unit 3-sphere S^3. Henceforth, we identify S^3 with $SU(2)$.

Let S^2 be the unit 2-sphere in $\mathfrak{su}(2)$ given by $S^2=\{x\in\mathfrak{su}(2): \|x\|=1\}$, and let $p: S^3\to S^2$ be the Hopf fibration defined by $p(a)=\mathrm{Ad}(a)e_3$. Note that the fibers of the Hopf fibration p coincide with the integral curves of the vector field E_3. We identify US^2, the unit tangent bundle of S^2, with a subset of $\mathfrak{su}(2)\times\mathfrak{su}(2)$ in the usual way, i.e.,

$$US^2 = \{(x, y) : \|x\| = \|y\| = 1,\ \langle x, y\rangle = 0\},$$

and the canonical projection $p_1 : US^2 \to S^2$ is given by $p_1(x, y) = x$.

LEMMA 2.1. *Let $\gamma(s)$ be a curve in S^2 defined by $\gamma(s) = p(\exp(sv))$, where $v = a_1e_1 + a_2e_2$. For $i = 1, 2$, let $\xi_i(s) = (\gamma(s), \mathrm{Ad}(\exp(sv))e_i) \in US^2$. Then ξ_i is a parallel vector field along γ.*

PROOF. We set $u_i(s) = \mathrm{Ad}(\exp(sv))e_i$. Then $u_i'(s) = \mathrm{Ad}(\exp(sv))[v, e_i]$. Since $[v, e_i] = te_3$ for some t, we obtain $u_i'(s) = t\gamma(s)$. This shows the assertion of Lemma 2.1. Q. E. D.

Define a map $p_2 : S^3 \to US^2$ by

$$p_2(a) = (\mathrm{Ad}(a)e_3,\ \mathrm{Ad}(a)e_1). \tag{2.1}$$

Then $p = p_1 \circ p_2$, and p_2 is a double covering such that $p_2(a) = p_2(-a)$ for all $a \in S^3$. We consider a regular curve $\gamma(s)$ in S^2. Its geodesic curvature $k(s)$ is given by

$$k(s) = \langle \gamma''(s),\ J(\gamma'(s))\rangle / \|\gamma'(s)\|^3,$$

where $J : T_xS^2 \to T_xS^2$ denotes a complex structure given by $J(v) = [x, v]/2$. We define a curve $\hat{\gamma}(s)$ in US^2 by

$$\hat{\gamma}(s) = (\gamma(s),\ \gamma'(s)/\|\gamma'(s)\|). \tag{2.2}$$

Since p_2 is a covering, there exists a curve $c(s)$ in S^3 such that $p_2(c(s)) = \hat{\gamma}(s)$.

LEMMA 2.2. $\dot{c}(s) = (1/2)\|\gamma'(s)\|\{E_2(c(s)) + k(s)E_3(c(s))\}$.

PROOF. We set $\dot{c}(s) = \sum_{i=1}^3 f_i(s)E_i(c(s))$. Then we obtain

$$\frac{d}{ds}\{\mathrm{Ad}(c)e_j\} = \mathrm{Ad}(c)\left[\sum_{i=1}^3 f_ie_i,\ e_j\right]. \tag{2.3}$$

Since $p_2(c) = \hat{\gamma}$, we have (2.4)-(2.6).

$$\frac{d}{ds}\{\mathrm{Ad}(c)e_3\} = \|\gamma'\|\ \mathrm{Ad}(c)e_1, \tag{2.4}$$

$$\left\langle \frac{d}{ds}\{\mathrm{Ad}(c)e_1\},\ J(\gamma')\right\rangle = k\|\gamma'\|^2, \tag{2.5}$$

$$J(\gamma') = \|\gamma'\|\ \mathrm{Ad}(c)e_2. \tag{2.6}$$

By (2.3) and (2.4) we see that $\|\gamma'\|e_1 = -2f_1e_2 + 2f_2e_1$, and so $f_1 = 0$ and $f_2 = \|\gamma'\|/2$. Furthermore it follows from (2.3), (2.5) and (2.6) that $f_3 = k\|\gamma'\|/2$.
Q. E. D.

Let $\gamma : \boldsymbol{R} \to S^2$ be a regular curve with a period $l > 0$, and let $c : \boldsymbol{R} \to S^3$ be a lift of $\hat{\gamma}$ with respect to the covering p_2. We define ω to be the element of the homology group $H_1(US^2)$ represented by the closed curve $\hat{\gamma}|[0, l]$. Note that US^2 is homeomorphic to the real projective space P^3, and so $H_1(US^2)\cong \boldsymbol{Z}_2$. Since p_2 is a double covering with $p_2(a)=p_2(-a)$, it is easy to see that

$$c(s+l)=\begin{cases} c(s) & \text{if } \omega=0\,, \\ -c(s) & \text{if } \omega=1\,. \end{cases} \tag{2.7}$$

We now assume that $c(s+l)=c(s)$ and $c|[0, l]$ is a simple closed curve. Let $b=c|[0, l]$, and let φ^t be the 1-parameter group of diffeomorphisms of S^3 generated by the vector field E_3. Since $\dot{b}(s)$ and $E_3(b(s))$ are linearly independent, there exists a positive number δ such that $\varphi^t(b)$ does not intersect b for all t with $0<t\leqq\delta$. Define $\mathrm{lk}(b, E_3)$ to be the linking number $\mathrm{lk}(b, \varphi^\delta(b))$, which does not depend on the choice of δ. We refer the reader to [**6**, p. 132] for the definition of the linking numbers. The following lemma, which will be proved in Section 6, plays an important role in the proof of Theorem 5.2.

LEMMA 2.3. $\mathrm{lk}(b, E_3)\equiv 1 \pmod 2$.

3. Construction of flat tori in S^3.

In this section we explain a method for constructing flat tori in S^3 established in [2]. Let $\Gamma=(\gamma_1, \gamma_2)$ be a pair of regular curves $\gamma_i: \boldsymbol{R}\to S^2$. The pair Γ is said to be an *admissible pair* if it satisfies the following conditions (3.1)-(3.3).

$$\hat{\gamma}_i(0)=(e_3, e_1)\,, \tag{3.1}$$

$$\|\gamma_i'\|^2(1+k_i^2)=4\,, \tag{3.2}$$

$$k_1(s_1)>k_2(s_2) \qquad \text{for all } (s_1, s_2)\in \boldsymbol{R}^2\,, \tag{3.3}$$

where k_i denotes the geodesic curvature of γ_i. Let $\Gamma=(\gamma_1, \gamma_2)$ be an admissible pair. For each i, it follows from (2.1) and (3.1) that there exists a curve $c_i: \boldsymbol{R}\to S^3$ such that $p_2(c_i)=\hat{\gamma}_i$ and $c_i(0)=\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}$. Using the group structure on S^3, we define a map $F_\Gamma: \boldsymbol{R}^2\to S^3$ by

$$F_\Gamma(s_1, s_2)=c_1(s_1)\cdot c_2(s_2)^{-1}\,. \tag{3.4}$$

It follows from [**2**, Lemma 3.8 and Theorem 4.2] that F_Γ is a FAT. Here we recall the following definition.

DEFINITION. An immersion $F: \boldsymbol{R}^2\to S^3$ is said to be a *FAT* if F induces a flat Riemannian metric g on $\boldsymbol{R}^2$ and

$$g\left(\frac{\partial}{\partial s_i}, \frac{\partial}{\partial s_i}\right)=1, \qquad h\left(\frac{\partial}{\partial s_i}, \frac{\partial}{\partial s_i}\right)=0 \qquad (i=1, 2),$$

where h denotes the second fundamental form of F.

Let g_Γ be a flat Riemannian metric on $\boldsymbol{R}^2$ induced by F_Γ, and let $G(\Gamma)$ be a group defined by

$$G(\Gamma)=\{\rho\in \mathrm{Diff}\,(\boldsymbol{R}^2): F_\Gamma\circ\rho=F_\Gamma\},$$

where Diff $(\boldsymbol{R}^2)$ denotes the group of all diffeomorphisms of $\boldsymbol{R}^2$. Then we obtain a 2-dimensional flat Riemannian manifold $M_\Gamma=(\boldsymbol{R}^2, g_\Gamma)/G(\Gamma)$ and an isometric immersion $f_\Gamma: M_\Gamma\to S^3$ such that $f_\Gamma\circ\pi_\Gamma=F_\Gamma$, where π_Γ denotes the canonical projection of $\boldsymbol{R}^2$ onto M_Γ. It is easy to see that the immersion $f_\Gamma: M_\Gamma\to S^3$ is primitive. Here we give the following definition.

DEFINITION. Let X and Y be smooth manifolds, and let $f: X\to Y$ be an immersion. The immersion f is said to be *primitive* if the identity map of X is the only diffeomorphism $\varphi: X\to X$ such that $f\circ\varphi=f$.

Since F_Γ is a FAT, it follows from [2, Theorem 2.3] that the group $G(\Gamma)$ consists of parallel translations of $\boldsymbol{R}^2$, and so M_Γ is orientable. Furthermore it follows from [2, Theorem 5.1] that M_Γ is compact if and only if Γ is periodic. Here $\Gamma=(\gamma_1, \gamma_2)$ is said to be *periodic* if γ_1 and γ_2 are periodic. So we see that a periodic admissible pair Γ induces a flat torus M_Γ and a primitive isometric immersion $f_\Gamma: M_\Gamma\to S^3$. Conversely, we obtain the following theorem.

THEOREM 3.1. *Let $f: M\to S^3$ be a primitive isometric immersion of a flat torus M into S^3. Then there exists a periodic admissible pair Γ such that f and f_Γ are congruent, i.e., there exists an isometry A of S^3 which satisfies $A\circ f=f_\Gamma\circ\rho$ for some diffeomorphism $\rho: M\to M_\Gamma$.*

PROOF. It follows from [3] that there exists a covering $T: \boldsymbol{R}^2\to M$ such that $f\circ T$ is a FAT. Then it follows from [2, Theorem 4.3] that there exists an admissible pair Γ such that $F_\Gamma=A\circ f\circ T$ for some isometry A of S^3. It is easy to see that the covering transformation group of T is contained in $G(\Gamma)$. Hence there exists a covering $\rho: M\to M_\Gamma$ such that $\rho\circ T=\pi_\Gamma$. This implies that M_Γ is compact, and so the admissible pair Γ is periodic. Since the fundamental group of M_Γ is isomorphic to the abelian group $G(\Gamma)$, the covering ρ is normal. Furthermore the covering transformation group of ρ is trivial because f is primitive and $A\circ f=f_\Gamma\circ\rho$. Hence the covering ρ must be a diffeomorphism. Q. E. D.

COROLLARY 3.2. *If $f: M\to S^3$ is an isometric embedding of a flat torus M*

into S^3, then there exists a periodic admissible pair Γ such that f and f_Γ are congruent.

We conclude this section with two lemmas.

LEMMA 3.3 ([**2**, Lemma 5.5]). *Let $\Gamma=(\gamma_1, \gamma_2)$ be an admissible pair, and let $(l_1, l_2)\in\boldsymbol{R}^2$. If the immersion $F_\Gamma: \boldsymbol{R}^2\to S^3$ satisfies $F_\Gamma(s_1, s_2)=F_\Gamma(s_1+l_1, s_2+l_2)$, then $\gamma_i(s+l_i)=\gamma_i(s)$ for $i=1, 2$.*

LEMMA 3.4. *Let $\Gamma=(\gamma_1, \gamma_2)$ be an admissible pair, and let η be a unit normal vector field along F_Γ. Then η is orthogonal to E_2 and E_3 along the curve $F_\Gamma(s, 0)$.*

PROOF. Let $c_i: \boldsymbol{R}\to S^3$ be a curve such that $p_2(c_i)=\hat{\gamma}_i$ and $c_i(0)=\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}$. By (3.4) we obtain

$$\partial_1 F_\Gamma(s, 0)=\dot{c}_1(s)\,, \qquad \partial_2 F_\Gamma(s, 0)=\{L_{c_1(s)}\}_*(-\dot{c}_2(0))\,,$$

where $\partial_i=\partial/\partial s_i$. So the assertion of Lemma 3.4 follows from Lemma 2.2.

Q. E. D.

4. Generators of $G(\Gamma)$.

Let Γ be a periodic admissible pair. Since the group $G(\Gamma)$ consists of parallel translations of $\boldsymbol{R}^2$ and the quotient space $\boldsymbol{R}^2/G(\Gamma)$ is compact, the group $G(\Gamma)$ can be identified with a lattice in $\boldsymbol{R}^2$ in the natural way. In this section we study generators of the lattice $G(\Gamma)$. Let $\gamma: \boldsymbol{R}\to S^2$ be a periodic regular curve, and let $l>0$ be the minimum period of γ. Recall the curve $\hat{\gamma}: \boldsymbol{R}\to US^2$ given by (2.2), and define $I(\gamma)$ to be the element of the homology group $H_1(US^2)$ represented by the closed curve $\hat{\gamma}|[0, l]$.

THEOREM 4.1. *Let $\Gamma=(\gamma_1, \gamma_2)$ be a periodic admissible pair, and let $l_i>0$ be the minimum period of γ_i. Then the lattice $G(\Gamma)$ has the following generators:*

(1) $(l_1, 0)$, $(0, l_2)$ *if* $I(\gamma_1)=0$, $I(\gamma_2)=0$,

(2) $(2l_1, 0)$, $(0, l_2)$ *if* $I(\gamma_1)=1$, $I(\gamma_2)=0$,

(3) $(l_1, 0)$, $(0, 2l_2)$ *if* $I(\gamma_1)=0$, $I(\gamma_2)=1$,

(4) (l_1, l_2), $(l_1, -l_2)$ *if* $I(\gamma_1)=1$, $I(\gamma_2)=1$.

PROOF. Let $c_i: \boldsymbol{R}\to S^3$ be a curve such that $p_2(c_i)=\hat{\gamma}_i$ and $c_i(0)=\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}$. Then it follows from (2.7) that

$$c_i(s+l_i)=\begin{cases} c_i(s) & \text{if } I(\gamma_i)=0\,, \\ -c_i(s) & \text{if } I(\gamma_i)=1\,. \end{cases} \tag{4.1}$$

Suppose that $I(\gamma_1)=I(\gamma_2)=0$. Then it follows from (3.4) and (4.1) that

$$F_\Gamma(s_1+l_1,\ s_2)=F_\Gamma(s_1,\ s_2)=F_\Gamma(s_1,\ s_2+l_2).$$

Hence the lattice $G(\Gamma)$ contains $(l_1, 0)$ and $(0, l_2)$. Let x_1 and x_2 be real numbers such that

$$x_1(l_1,\ 0)+x_2(0,\ l_2)\in G(\Gamma).$$

Then Lemma 3.3 implies $\gamma_i(s+x_il_i)=\gamma_i(s)$. So it follows from the definition of l_i that x_1 and x_2 are integers. This proves (1).

Suppose that $I(\gamma_1)=1$ and $I(\gamma_2)=0$. Then it follows from (3.4) and (4.1) that

$$F_\Gamma(s_1+2l_1,\ s_2)=F_\Gamma(s_1,\ s_2)=F_\Gamma(s_1,\ s_2+l_2).$$

Hence the lattice $G(\Gamma)$ contains $(2l_1, 0)$ and $(0, l_2)$. Let x_1 and x_2 be real numbers such that

$$x_1(2l_1,\ 0)+x_2(0,\ l_2)\in G(\Gamma). \tag{4.2}$$

To establish (2) it is sufficient to show that x_1 and x_2 are integers. We may assume that $0\leqq x_i<1$. It follows from (4.2) and Lemma 3.3 that $2x_1l_1$ and x_2l_2 are periods of γ_1 and γ_2, respectively. So it follows from the definition of l_i that $2x_1$ and x_2 are integers. Hence $(x_1, x_2)=(0, 0)$ or $(1/2, 0)$. We now consider the case $(x_1, x_2)=(1/2, 0)$. By (4.2) we have $(l_1, 0)\in G(\Gamma)$, and so $F_\Gamma(s_1+l_1, s_2)=F_\Gamma(s_1, s_2)$. However it follows from (3.4), (4.1) and the assumption $I(\gamma_1)=1$ that

$$F_\Gamma(s_1+l_1,\ s_2)=-c_1(s_1)\cdot c_2(s_2)^{-1}=-F_\Gamma(s_1,\ s_2).$$

Hence we have $F_\Gamma=-F_\Gamma$, which is a contradiction. So we have $x_1=x_2=0$. This proves (2). The assertion (3) is proved in the same way.

Suppose that $I(\gamma_1)=I(\gamma_2)=1$. Then it follows from (3.4) and (4.1) that

$$F_\Gamma(s_1+l_1,\ s_2\pm l_2)=F_\Gamma(s_1,\ s_2).$$

Hence the lattice $G(\Gamma)$ contains (l_1, l_2) and $(l_1, -l_2)$. Let x_1 and x_2 be real numbers such that

$$x_1(l_1,\ l_2)+x_2(l_1,\ -l_2)\in G(\Gamma). \tag{4.3}$$

To establish (4) it is sufficient to show that x_1 and x_2 are integers. We may assume that $0\leqq x_i<1$. It follows from (4.3) and Lemma 3.3 that $(x_1+x_2)l_1$ and $(x_1-x_2)l_2$ are periods of γ_1 and γ_2, respectively. So it follows that x_1+x_2 and x_1-x_2 are integers. Hence $(x_1, x_2)=(0, 0)$ or $(1/2, 1/2)$. We now consider the case $x_1=x_2=1/2$. By (4.3) we have $(l_1, 0)\in G(\Gamma)$, and so $F_\Gamma(s_1+l_1, s_2)=F_\Gamma(s_1, s_2)$. However it follows from $I(\gamma_1)=1$ that $F_\Gamma(s_1+l_1, s_2)=-F_\Gamma(s_1, s_2)$. Hence we have $F_\Gamma=-F_\Gamma$, which is a contradiction. This proves (4). Q. E. D.

In the rest of this section we construct a flat torus in S^3 which does not satisfy the antipodal symmetry. Let $\gamma: \boldsymbol{R}\to S^2$ be a periodic regular curve defined by

$$\gamma(\theta) = \frac{x(\theta)e_1+y(\theta)e_2+e_3}{\sqrt{x(\theta)^2+y(\theta)^2+1}},$$

where $x(\theta)=R(\theta)\sin\theta$, $y(\theta)=3/2-R(\theta)\cos\theta$ and $R(\theta)=1/2+\cos\theta$. The minimum period of γ is equal to 2π and $\gamma|[0, 2\pi]$ has exactly one self-intersection. Therefore $I(\gamma)=0$. We now introduce a function $\theta(s)$ by the following relation.

$$s = \frac{1}{2}\int_0^{\theta(s)} \|\gamma'\| \sqrt{1+k^2}\,d\theta\,,$$

where k denotes the geodesic curvature of γ. Furthermore define a periodic regular curve $\gamma_1: \boldsymbol{R}\to S^2$ by $\gamma_1(s)=\gamma(\theta(s))$. Then it is not difficult to see the following lemma.

LEMMA 4.2. *Let k_1 be the geodesic curvature of γ_1, and let l be a positive number with $\theta(l)=2\pi$. Then*

(1) $\hat{\gamma}_1(0)=(e_3, e_1)$,

(2) $\|\gamma_1'\|^2(1+k_1^2)=4$,

(3) $k_1>0$,

(4) *l is the minimum period of γ_1 and $I(\gamma_1)=0$,*

(5) *$\langle\gamma_1(s), e_2\rangle>0$ unless s/l is an integer.*

Let Φ be an orientation reversing linear isometry of $\mathfrak{su}(2)$ such that $\Phi(e_1)=e_1$, $\Phi(e_2)=-e_2$ and $\Phi(e_3)=e_3$. Define a periodic regular curve $\gamma_2: \boldsymbol{R}\to S^2$ by $\gamma_2(s)=\Phi(\gamma_1(s))$. Since k_2, the geodesic curvature of γ_2, satisfies $k_2(s)=-k_1(s)$, it follows from Lemma 4.2 (1)-(3) that $\Gamma=(\gamma_1, \gamma_2)$ is a periodic admissible pair. We set $E=\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}$. Then

LEMMA 4.3. *$F_\Gamma(s_1, s_2)\neq -E$ for all $(s_1, s_2)\in\boldsymbol{R}^2$.*

PROOF. Suppose that there exists $(t_1, t_2)\in\boldsymbol{R}^2$ such that $F_\Gamma(t_1, t_2)=-E$. Then it follows from (3.4) that $c_1(t_1)=-c_2(t_2)$, where c_i denotes a curve in S^3 such that $p_2(c_i)=\hat{\gamma}_i$ and $c_i(0)=E$. Since $p_2(a)=p_2(-a)$, we obtain $\gamma_1(t_1)=\gamma_2(t_2)$. So it follows from Lemma 4.2 (5) that there exist integers n_1 and n_2 such that $t_1=n_1 l$ and $t_2=n_2 l$. By (4.1) and Lemma 4.2 (4) we obtain $c_i(t_i)=c_i(n_i l)=c_i(0)=E$. Hence $F_\Gamma(t_1, t_2)=E$ which is a contradiction. Q.E.D.

Since $F_\Gamma(0, 0)=E$, the image of the immersion $f_\Gamma: M_\Gamma\to S^3$ contains E. On the other hand, Lemma 4.3 shows that $-E$ is not contained in the image of f_Γ. Therefore we obtain the following theorem.

THEOREM 4.4. *There exists a flat torus M and an isometric immersion $f: M \to S^3$ such that the image $f(M)$ is not invariant under the antipodal map of S^3.*

5. Asymptotic curves of embedded flat tori in S^3.

In this section we study asymptotic curves of flat tori isometrically embedded in S^3, and prove Theorem 1.1. We now recall the notion of asymptotic curves. Let $f: M \to S^3$ be an isometric immersion of a flat surface M into S^3. A curve $c(s)$ in M is said to be an *asymptotic curve* of the immersion f if it satisfies the equation $h(\dot{c}, \dot{c})=0$, where h denotes the second fundamental form of f. For each point of the flat surface M, there are exactly two asymptotic curves through the point.

EXAMPLE. Let Γ be an admissible pair, and let f_Γ be the isometric immersion of the flat surface $M_\Gamma=(\boldsymbol{R}^2, g_\Gamma)/G(\Gamma)$ into S^3 defined by the relation $f_\Gamma \circ \pi_\Gamma = F_\Gamma$, where π_Γ denotes the canonical projection of $\boldsymbol{R}^2$ onto M_Γ. Consider two curves a_1 and a_2 in M_Γ given by $a_1(s)=\pi_\Gamma(s, 0)$ and $a_2(s)=\pi_\Gamma(0, s)$. Since F_Γ is a FAT, the curves a_1 and a_2 are unit speed asymptotic curve of f_Γ.

THEOREM 5.1. *Let $f: M \to S^3$ be an isometric immersion of a flat torus M into S^3, and let $c: \boldsymbol{R} \to M$ be a unit speed asymptotic curve of f. Then there exists a positive number l such that $c(s+l)=c(s)$ and $c|[0, l]$ is a simple closed curve.*

PROOF. It follows from [**2**, Theorem A] that c is periodic. So there exists a positive number l which is the minimum period of c. Suppose that $c|[0, l]$ is not a simple closed curve. Then we may assume that there exists a positive number $l'<l$ such that $c(0)=c(l')$. It follows from [**3**] that there exists a covering $T: \boldsymbol{R}^2 \to M$ such that $f \circ T$ is a FAT. Since $c(s)$ is a unit speed asymptotic curve of f, we may assume that $c(s)=T(s, 0)$. Then $T(0, 0)=T(l', 0)$, and so there exists a covering transformation ρ of T such that $\rho(0, 0)=(l', 0)$. Since $f \circ T \circ \rho = f \circ T$, it follows from [**2**, Theorem 2.3] that $\rho(s_1, s_2)=(s_1+l', s_2)$. Hence $c(s+l')=T(\rho(s, 0))=T(s, 0)=c(s)$. This contradicts the definition of l. Q. E. D.

Let $f: M \to S^3$ be an isometric embedding of a flat torus M into S^3, and let ξ be a unit normal vector field along the embedding f. For each $t \in \boldsymbol{R}$, using the exponential map $\mathrm{Exp}: TS^3 \to S^3$, we define a map $f^t: M \to S^3$ by $f^t(x)=\mathrm{Exp}(t\xi(x))$. Since $f^0=f$, there exists a positive number δ such that the map $(x, t) \mapsto f^t(x)$, restricted on $M \times [0, \delta]$, is an embedding. For each simple closed curve $a: [0, l] \to M$, we denote by $\mathrm{lk}(f(a), f^\delta(a))$ the linking number of $f(a)$ and $f^\delta(a)$.

THEOREM 5.2. *Let $a:[0, l]\to M$ be a simple closed curve. If a is a unit speed asymptotic curve of the embedding f, then* $\mathrm{lk}(f(a), f^{\delta}(a))\equiv 1 \pmod 2$.

PROOF. There exists a covering $T: \boldsymbol{R}^2\to M$ such that $f\circ T$ is a FAT. Since $a:[0, l]\to M$ is a unit speed asymptotic curve of f, we may assume that $a(s)=T(s, 0)$ for $0\leqq s\leqq l$. Then Theorem 5.1 implies that

$$T(s+l, 0)=T(s, 0) \qquad \text{for all } s\in\boldsymbol{R}. \tag{5.1}$$

By [**2**, Theorem 4.3] there exists an admissible pair $\Gamma=(\gamma_1, \gamma_2)$ such that $F_\Gamma=A\circ f\circ T$ for some isometry A of S^3, where F_Γ denotes a FAT defined by (3.4). Let $c_1: \boldsymbol{R}\to S^3$ be a lift of $\hat{\gamma}_1$ with respect to p_2 such that $c_1(0)=\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}$. Since $c_1(s)=F_\Gamma(s, 0)$, it follows from (5.1) that $c_1(s+l)=c_1(s)$ for all s. We set $b=c_1|[0, l]$. Then $b=A(f(a))$, and so b is a simple closed curve. Hence Lemma 2.3 implies

$$\mathrm{lk}(b, E_3)\equiv 1 \pmod 2. \tag{5.2}$$

We consider a family of simple closed curves $b^t:[0, l]\to S^3$ $(0\leqq t\leqq\delta)$ defined by $b^t=A(f^t(a))$. Then

$$\mathrm{lk}(f(a), f^t(a))=\pm\mathrm{lk}(b, b^t) \qquad \text{for } 0<t\leqq\delta. \tag{5.3}$$

Define a map $F_\Gamma^t: \boldsymbol{R}^2\to S^3$ by $F_\Gamma^t=A\circ f^t\circ T$, and set $\eta(s_1, s_2)=dF_\Gamma^t(s_1, s_2)/dt|_{t=0}$. Then it follows from the definition of f^t that η is a unit normal vector field along F_Γ such that $F_\Gamma^t(s_1, s_2)=\mathrm{Exp}(t\eta(s_1, s_2))$. Since $b^t(s)=F_\Gamma^t(s, 0)$ for $0\leqq s\leqq l$, we obtain

$$b^t(s)=\mathrm{Exp}(t\eta(s, 0)) \qquad \text{for } 0\leqq s\leqq l. \tag{5.4}$$

For each $\theta\in\boldsymbol{R}$, let $X_\theta(s)$ be a vector field along $c_1(s)$ defined by

$$X_\theta(s)=(\cos\theta)\eta(s, 0)+(\sin\theta)E_3(c_1(s)).$$

Note that $X_\theta(s+l)=X_\theta(s)$. For $0\leqq t\leqq\delta$, define a closed curve $b_\theta^t:[0, l]\to S^3$ by $b_\theta^t(s)=\mathrm{Exp}(tX_\theta(s))$. It follows from Lemmas 2.2 and 3.4 that $X_\theta(s)$ and $\dot{c}_1(s)$ are linearly independent, and so there exists a positive number $\varepsilon<\delta$ such that b does not intersect b_θ^t for all $(\theta, t)\in\boldsymbol{R}\times(0, \varepsilon]$. Therefore we have

$$\mathrm{lk}(b, b_0^\varepsilon)=\mathrm{lk}(b, b_{\pi/2}^\varepsilon).$$

Since $b_{\pi/2}^t(s)=\mathrm{Exp}(tE_3(b(s)))$, it follows from the above relation that $\mathrm{lk}(b, b_0^\varepsilon)=\mathrm{lk}(b, E_3)$. Furthermore it follows from (5.4) that $b_0^\varepsilon=b^\varepsilon$. So (5.3) implies

$$\mathrm{lk}(f(a), f^\delta(a))=\mathrm{lk}(f(a), f^\varepsilon(a))=\pm\mathrm{lk}(b, E_3).$$

Hence (5.2) implies the assertion of Theorem 5.2. Q. E. D.

THEOREM 5.3. *Let $\Gamma=(\gamma_1, \gamma_2)$ be a periodic admissible pair. If the immersion $f_\Gamma: M_\Gamma\rightarrow S^3$ is an embedding, then $I(\gamma_1)=I(\gamma_2)=1$.*

PROOF. It is sufficient to show that the following three cases (1)-(3) are impossible.

(1) $I(\gamma_1)=0,\ I(\gamma_2)=0$,

(2) $I(\gamma_1)=1,\ I(\gamma_2)=0$,

(3) $I(\gamma_1)=0,\ I(\gamma_2)=1$.

We first assume (1). Let $l_i>0$ be the minimum period of γ_i. Then it follows from Theorem 4.1 that the lattice $G(\Gamma)$ is generated by $(l_1, 0)$ and $(0, l_2)$. Let π_Γ be the canonical projection of $\boldsymbol{R}^2$ onto $M_\Gamma=\boldsymbol{R}^2/G(\Gamma)$, and let D be a disk of M_Γ given by $D=\pi_\Gamma(\tilde{D})$, where $\tilde{D}$ is a domain of $\boldsymbol{R}^2$ such that

$$\tilde{D}=\left(\frac{1}{3}l_1, \frac{2}{3}l_1\right)\times\left(\frac{1}{3}l_2, \frac{2}{3}l_2\right).$$

We consider a knot K in S^3 defined by $K=f_\Gamma(\partial D)$. Since K is unknotted, it follows that $\mathrm{Arf}(K)=0$, where $\mathrm{Arf}(K)$ denotes the Arf invariant of the knot K. We refer the reader to [**1**, Chapter 10] for the definition of $\mathrm{Arf}(K)$. Let ξ be a unit normal vector field along the embedding f_Γ. For each $t\in\boldsymbol{R}$, we define a map $f_\Gamma^t: M_\Gamma\rightarrow S^3$ by $f_\Gamma^t(x)=\mathrm{Exp}(t\xi(x))$. Since $f_\Gamma^0=f_\Gamma$, there exists a positive number δ such that the map $(x, t)\mapsto f_\Gamma^t(x)$ is an embedding on $M_\Gamma\times[0, \delta]$. We now compute $\mathrm{Arf}(K)$ using a Seifert surface of K. Let V be a Seifert surface of K defined by $V=f_\Gamma(M_\Gamma-D)$, and let a_1 and a_2 be simple closed curves in M_Γ given by

$$a_1(s)=\pi_\Gamma(s, 0) \qquad \text{for } 0\leqq s\leqq l_1,$$

$$a_2(s)=\pi_\Gamma(0, s) \qquad \text{for } 0\leqq s\leqq l_2.$$

Then $\{f_\Gamma(a_1), f_\Gamma(a_2)\}$ represents a canonical basis of the homology group $H_1(V)$. Hence it follows from [**5**] that

$$\mathrm{Arf}(K)\equiv v_{11}v_{22} \pmod 2, \tag{5.5}$$

where $v_{ij}=\mathrm{lk}(f_\Gamma(a_i), f_\Gamma^\delta(a_j))$. Since a_i is a unit speed asymptotic curve of f_Γ, it follows from Theorem 5.2 that v_{ii} is odd. Therefore (5.5) implies $\mathrm{Arf}(K)=1$. This is a contradiction, and so (1) is impossible. By the same way we see that (2) and (3) are impossible. Q.E.D.

THEOREM 5.4. *Let $f: M\rightarrow S^3$ be an isometric embedding of a flat torus M into S^3. If $c: \boldsymbol{R}\rightarrow M$ is a unit speed asymptotic curve of f, then $f(c)$ is invariant under the antipodal map of S^3.*

PROOF. By Corollary 3.2 there exists a periodic admissible pair $\Gamma=(\gamma_1, \gamma_2)$

such that f and f_Γ are congruent. So, without loss of generality, we may assume that $M=M_\Gamma$ and $f=f_\Gamma$. Let $c_i\colon \boldsymbol{R}\to S^3$ be a curve such that $p_2(c_i)=\hat{\gamma}_i$ and $c_i(0)=\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}$. Since f_Γ is an embedding, Theorem 5.3 implies $I(\gamma_1)=I(\gamma_2)=1$. Hence it follows from (2.7) that

$$c_i(s+l_i)=-c_i(s) \qquad (i=1, 2), \tag{5.6}$$

where l_i denotes the minimum period of γ_i. Let $\tilde{c}$ be a lift of the curve c with respect to the covering $\pi_\Gamma\colon \boldsymbol{R}^2\to M_\Gamma$. Then $\tilde{c}$ is a unit speed asymptotic curve of the immersion $F_\Gamma\colon \boldsymbol{R}^2\to S^3$. Since F_Γ is a FAT, the curve $\tilde{c}$ satisfies

$$\tilde{c}(s)=(u_1\pm s,\ u_2) \quad \text{or} \quad \tilde{c}(s)=(u_1,\ u_2\pm s),$$

where $(u_1, u_2)=\tilde{c}(0)$. We first consider the case $\tilde{c}(s)=(u_1\pm s, u_2)$. Since $F_\Gamma=f_\Gamma\circ\pi_\Gamma$, we have

$$f_\Gamma(c(s))=F_\Gamma(\tilde{c}(s))=c_1(u_1\pm s)\cdot c_2(u_2)^{-1}.$$

Hence (5.6) implies $f_\Gamma(c(s+l_1))=-f_\Gamma(c(s))$. Similarly we have $f_\Gamma(c(s+l_2))=-f_\Gamma(c(s))$ when $\tilde{c}(s)=(u_1, u_2\pm s)$. This completes the proof of Theorem 5.4.

Q. E. D.

As an immediate consequence of Theorem 5.4, we obtain Theorem 1.1.

6. Proof of Lemma 2.3.

In this section we prove Lemma 2.3. A curve $\gamma\colon [0, 2\pi]\to S^2$ is called a *closed regular curve* if there exists a regular curve $\tilde{\gamma}\colon \boldsymbol{R}\to S^2$ such that $\tilde{\gamma}(s+2\pi)=\tilde{\gamma}(s)$ and $\gamma=\tilde{\gamma}\,|\,[0, 2\pi]$. Let Ω^0 be the set of all closed regular curves $\gamma\colon [0, 2\pi]\to S^2$ such that a lift of the curve $\hat{\gamma}\colon [0, 2\pi]\to US^2$ with respect to the covering p_2 is a simple closed curve. For each $\gamma\in\Omega^0$, we set

$$\mathrm{lk}(\gamma)=\mathrm{lk}(c,\ E_3),$$

where c denotes a lift of $\hat{\gamma}$ with respect to p_2. Note that $\mathrm{lk}(\gamma)$ does not depend on the choice of c. Then Lemma 2.3 is equivalent to the following assertion.

$$\mathrm{lk}(\gamma)\equiv 1 \pmod 2 \qquad \text{for all } \gamma\in\Omega^0. \tag{6.1}$$

To establish (6.1) we need several lemmas.

LEMMA 6.1. *Let* $\gamma\colon [0, 2\pi]\to S^2$ *be a closed regular curve defined by* $\gamma(s)=(\cos 2s)e_1+(\sin 2s)e_2$. *Then* $\gamma\in\Omega^0$ *and* $\mathrm{lk}(\gamma)=1$.

PROOF. Let $c\colon [0, 2\pi]\to S^3$ be a lift of $\hat{\gamma}$ with respect to p_2. Since γ is a geodesic and $\|\gamma'\|=2$, it follows from Lemma 2.2 that $\dot{c}=E_2(c)$. Hence c is a unit speed geodesic in S^3. In particular, c is a simple closed curve, and so

$\gamma\in\Omega^0$. We now compute $\mathrm{lk}(\gamma)$. Let φ^t be the 1-parameter group of diffeomorphisms of S^3 generated by E_3. Since $\varphi^t(c)$ does not intersect c for $0<t\leqq 1$, we obtain $\mathrm{lk}(\gamma)=\mathrm{lk}(c, E_3)=\mathrm{lk}(c, \varphi^1(c))$. We set

$$v_1(s)=(\cos s)E_3(c(s))-(\sin s)E_1(c(s)),$$

$$v_2(s)=(\sin s)E_3(c(s))+(\cos s)E_1(c(s)).$$

Let $X=S^3-\mathrm{Image}(c)$, and let $T:[0, 2\pi]\times[0, 2\pi]\rightarrow X$ be a map defined by

$$T(\theta, s)=\mathrm{Exp}((\cos\theta)v_1(s)+(\sin\theta)v_2(s)),$$

where $\mathrm{Exp}: TS^3\rightarrow S^3$ denotes the exponential map. We consider simple closed curves $a_1(\theta)=T(\theta, 0)$ and $a_2(s)=T(2\pi, s)$. Since $\varphi^1(c(s))=\mathrm{Exp}(E_3(c(s)))=T(s, s)$, it follows that $[\varphi^1(c)]=[a_1]+[a_2]$ in the homology group $H_1(X)$. Hence we obtain

$$\mathrm{lk}(\gamma)=\mathrm{lk}(c, \varphi^1(c))=\mathrm{lk}(c, a_1)+\mathrm{lk}(c, a_2).$$

It is easy to see that $\mathrm{lk}(c, a_1)=1$. Since v_1 is parallel along the geodesic c, there exists a totally geodesic 2-sphere Σ in S^3 such that c is contained in Σ and v_1 is tangent to Σ along c. Then the curve a_2 is contained in Σ, and so $\mathrm{lk}(c, a_2)=0$. Q.E.D.

LEMMA 6.2. *Let* $\gamma_t:[0, 2\pi]\rightarrow S^2$ $(0\leqq t\leqq 1)$ *be a smooth 1-parameter family of closed regular curves such that* $\gamma_t\in\Omega^0$ *for all* t. *Then* $\mathrm{lk}(\gamma_0)=\mathrm{lk}(\gamma_1)$.

PROOF. By the assumption there exists a smooth 1-parameter family of simple closed curves $c_t:[0, 2\pi]\rightarrow S^3$ $(0\leqq t\leqq 1)$ such that $p_2(c_t)=\hat{\gamma}_t$. Then there exists a positive number δ such that $\mathrm{lk}(c_t, E_3)=\mathrm{lk}(c_t, \varphi^\delta(c_t))$ for all t. Hence $\mathrm{lk}(c_t, E_3)$ does not depend on t. Q.E.D.

We define Ω^* to be the set of all closed regular curves $\gamma:[0, 2\pi]\rightarrow S^2$ such that γ is self-transversal and all of its self-intersections are double points.

LEMMA 6.3. *For each* $\alpha\in\Omega^0$, *there exists* $\beta\in\Omega^*\cap\Omega^0$ *such that* $\mathrm{lk}(\alpha)=\mathrm{lk}(\beta)$.

PROOF. Considering a sufficiently small deformation of α, if necessary, we may assume that $\alpha(s)\neq\alpha(\pi)$ for $s\neq\pi$. We choose $q_0\in S^2$ which is not contained in the image of α, and set

$$A=\{x\in\mathfrak{su}(2):\langle x, q_0\rangle=0\},$$

$$S^+=\{v\in S^2:\langle v, q_0\rangle>0\}.$$

For each $v\in S^+$, define $\pi_v:\mathfrak{su}(2)\rightarrow A$ to be the parallel projection in the direction of v. Let $f: A\rightarrow S^2-\{q_0\}$ be a diffeomorphism, and let $\gamma:[0, 2\pi]\rightarrow A$ be a closed regular curve given by $\gamma=f^{-1}(\alpha)$. Since $\gamma(s)\neq\gamma(\pi)$ for $s\neq\pi$, there exists a simple closed regular curve $c:[0, 2\pi]\to\mathfrak{su}(2)$ such that $\pi_{q_0}(c)=\gamma$. For each $v\in S^+$, define a curve $\alpha_v:[0, 2\pi]\to S^2$ by $\alpha_v=f(\pi_v(c))$. Note that $\alpha_{q_0}=\alpha$. Since $\alpha\in\Omega^0$, there exists a neighborhood W of q_0 in S^+ such that $\alpha_v\in\Omega^0$ for all $v\in W$. The set of all $v\in S^+$ with self-transversal $\pi_v(c)$ is dense in S^+. So there exists $q_1\in W$ such that α_{q_1} is self-transversal. Let $v:\boldsymbol{R}\to W$ be a smooth curve such that $v(0)=q_0$ and $v(1)=q_1$. Consider a smooth 1-parameter family of closed regular curves α_t in S^2 defined by $\alpha_t=\alpha_{v(t)}$. Since $\alpha_t\in\Omega^0$ for $0\leqq t\leqq 1$, it follows from Lemma 6.2 that $\mathrm{lk}(\alpha)=\mathrm{lk}(\alpha_0)=\mathrm{lk}(\alpha_1)$. Since α_1 is self-transversal, a small deformation of α_1 implies the existence of $\beta\in\Omega^*\cap\Omega^0$ such that $\mathrm{lk}(\alpha_1)=\mathrm{lk}(\beta)$.

Q. E. D.

For each closed regular curve $\gamma:[0, 2\pi]\to S^2$, we denote by $[\hat{\gamma}]$ the element of the homology group $H_1(US^2)$ represented by the closed curve $\hat{\gamma}:[0, 2\pi]\to US^2$. By (2.7) it is easy to see the following lemma.

LEMMA 6.4. *Let $\gamma:[0, 2\pi]\to S^2$ be a closed regular curve such that $\hat{\gamma}$ is a simple closed curve. Then $\gamma\in\Omega^0$ if and only if $[\hat{\gamma}]=0$.*

A closed regular curve $\gamma:[0, 2\pi]\to S^2$ is said to be *self-contact free* if $\hat{\gamma}$ is a simple closed curve. For $\alpha, \beta\in\Omega^*$, we say that α and β are *R*1*-related* if there exists a smooth 1-parameter family of self-contact free closed regular curves $\gamma_t:[0, 2\pi]\to S^2$ $(0\leqq t\leqq 1)$ such that $\alpha=\gamma_0$ and $\beta=\gamma_1$.

LEMMA 6.5. *Let $\alpha, \beta\in\Omega^*$ which are R1-related. If $\alpha\in\Omega^0$, then $\beta\in\Omega^0$ and* $\mathrm{lk}(\alpha)=\mathrm{lk}(\beta)$.

PROOF. Let $\gamma_t:[0, 2\pi]\to S^2$ $(0\leqq t\leqq 1)$ be a smooth 1-parameter family of self-contact free closed regular curves such that $\alpha=\gamma_0$ and $\beta=\gamma_1$. Since $\alpha\in\Omega^0$, it follows from (2.7) that $[\hat{\gamma}_t]=[\hat{\gamma}_0]=0$. So Lemma 6.4 implies that $\gamma_t\in\Omega^0$ for all t. Hence $\beta\in\Omega^0$, and it follows from Lemma 6.2 that $lk(\alpha)=lk(\beta)$. Q.E.D.

We now introduce another relation on Ω^*. Let B denote the set of all $(u_1, u_2)\in\boldsymbol{R}^2$ such that $|u_i|\leqq 1$ $(i=1, 2)$. Define $\Phi: B\to S^2$ by

$$\Phi(u_1, u_2)=(\cos\rho)e_3+\frac{\sin\rho}{\rho}(u_1e_1+u_2e_2),$$

where $\rho=\sqrt{u_1^2+u_2^2}$. We consider a closed regular curve $\gamma:[0, 2\pi]\to S^2$ which satisfies

$$\text{(6.2)}\qquad \begin{cases} \gamma(s)=\Phi(s-1, -(s-1)^2) & \text{if } 0\leqq s\leqq 2, \\ \gamma(s)=\Phi(s-4, 0) & \text{if } 3\leqq s\leqq 5, \\ \gamma(s)\notin\Phi(B) & \text{otherwise.} \end{cases}$$

Let $h:\boldsymbol{R}\to[0, 1/9]$ be a smooth function such that $h(x)=0$ for $|x|\geqq 2/3$ and

$h(x)=1/9$ for $|x|\leqq 1/3$. Define a smooth 1-parameter family of closed regular curves $\gamma_t:[0, 2\pi]\to S^2$ $(-1\leqq t\leqq 1)$ by setting

$$\gamma_t(s)=\begin{cases}\Phi(s-1, -(s-1)^2+h(s-1)t) & \text{if } 0\leqq s\leqq 2,\\ \gamma(s) & \text{otherwise.}\end{cases} \tag{6.3}$$

Since $[\hat{\gamma}_1]=[\hat{\gamma}_{-1}]$, it follows from Lemma 6.4 that

$$\gamma_1\in\Omega^*\cap\Omega^0 \quad \text{if and only if } \gamma_{-1}\in\Omega^*\cap\Omega^0 . \tag{6.4}$$

For $\alpha, \beta\in\Omega^*$, we say that α and β are *R2-related* if there exists a closed regular curve $\gamma:[0, 2\pi]\to S^2$ with the condition (6.2) and the family γ_t given by (6.3) satisfies $(\gamma_1, \gamma_{-1})=(\alpha, \beta)$ or (β, α).

LEMMA 6.6. *Let $\alpha, \beta\in\Omega^*$ which are R2-related. If $\alpha\in\Omega^0$, then $\beta\in\Omega^0$ and* $\mathrm{lk}(\alpha)\equiv\mathrm{lk}(\beta) \pmod 2$.

PROOF. By (6.4) we may assume that there exists a closed regular curve $\gamma:[0, 2\pi]\to S^2$ with the condition (6.2) and the family γ_t $(-1\leqq t\leqq 1)$ defined by (6.3) satisfies $\gamma_1=\alpha$ and $\gamma_{-1}=\beta$. Since $\gamma_1\in\Omega^*\cap\Omega^0$, it follows from the definition of γ_t that

$$\gamma_t\in\Omega^*\cap\Omega^0 \quad \text{for } t\neq 0 . \tag{6.5}$$

To establish Lemma 6.6 it is sufficient to show the following assertion.

$$\mathrm{lk}(\gamma_1)-\mathrm{lk}(\gamma_{-1})=0 \quad\text{or}\quad -2. \tag{6.6}$$

If $\gamma_0\in\Omega^0$, then it follows from (6.5) and Lemma 6.2 that $\mathrm{lk}(\gamma_1)=\mathrm{lk}(\gamma_{-1})$. So we may assume $\gamma_0\notin\Omega^0$. Since $\hat{\gamma}_0(1)=(e_3, e_1)\in US^2$, there exists a smooth 1-parameter family of closed curves $c_t:[0, 2\pi]\to S^3$ $(-1\leqq t\leqq 1)$ such that $p_2(c_t)=\hat{\gamma}_t$ and $c_0(1)=\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}$. Since $\hat{\gamma}_0(1)=\hat{\gamma}_0(4)$, it follows from the assumption $\gamma_0\notin\Omega^0$ that

$$c_0(1)=c_0(4)=\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}. \tag{6.7}$$

Let $\tilde{B}$ denote the set of all $(x, y, z)\in\boldsymbol{R}^3$ such that $|x|\leqq 1/2$, $|y|\leqq 1/2$ and $|z|<\pi/2$. Define $\tilde{\Phi}:\tilde{B}\to S^3$ by

$$\tilde{\Phi}(x, y, z)=\exp(xe_1+ye_2)\exp(ze_3) .$$

Then $\tilde{\Phi}$ carries $\tilde{B}$ diffeomorphically onto $\tilde{\Phi}(\tilde{B})$, and satisfies

$$p(\tilde{\Phi}(x, y, z))=\Phi(2y, -2x) , \tag{6.8}$$

where $p:S^3\to S^2$ denotes the Hopf fibration given in Section 2. For $i=1, 2$, we define V_i to be the vector field on $\Phi(B)$ such that $V_i(\Phi(0, 0))=\partial\Phi(0, 0)/\partial u_i$ and V_i is parallel along every geodesic through the point $\Phi(0, 0)$. Then it follows from Lemma 2.1 that

$$p_2(\tilde{\Phi}(x, y, z)) = (\cos 2z)V_1(m)+(\sin 2z)V_2(m), \tag{6.9}$$

where $m=p(\tilde{\Phi}(x, y, z))$. Let $\theta(s, t)$ be a smooth function defined on $[0, 2]\times[-1, 1]$ such that

$$\dot{\gamma}_t(s) = (\cos\theta(s, t))V_1+(\sin\theta(s, t))V_2, \tag{6.10}$$

and $\theta(1, 0)=0$. It is easy to see that $|\theta(t, s)|<\pi$. For $-1\leqq t\leqq 1$, define $\tilde{c}_t$: $[0, 2]\to\tilde{B}$ by $\tilde{c}_t=(x_t, y_t, z_t)$, where

$$x_t(s) = \{(s-1)^2-h(s-1)t\}/2,$$
$$y_t(s) = (s-1)/2,$$
$$z_t(s) = \theta(s, t)/2.$$

Then it follows from (6.7)-(6.10) that

$$c_t(s) = \begin{cases} \tilde{\Phi}(\tilde{c}_t(s)) & \text{if } 0\leqq s\leqq 2, \\ \tilde{\Phi}(0, (s-4)/2, 0) & \text{if } 3\leqq s\leqq 5, \\ c_0(s) & \text{otherwise.} \end{cases} \tag{6.11}$$

Note that $s=1\pm 1/3$ are the only zeros of the function $x_1(s)$ and that

$$z_1(2/3)>0, \qquad z_1(4/3)<0. \tag{6.12}$$

Let $\tilde{a}:[0, 2]\to\tilde{B}$ be the straight line from $\tilde{c}_0(0)$ to $\tilde{c}_0(2)$, and let $\tilde{b}:[0, 2]\to\tilde{B}$ be a piecewise smooth curve given by

$$\tilde{b}(s) = \begin{cases} (-s, -1/2, 0) & \text{if } 0\leqq s\leqq 1/2, \\ (-1/2, s-1, 0) & \text{if } 1/2\leqq s\leqq 3/2, \\ (s-2, 1/2, 0) & \text{if } 3/2\leqq s\leqq 2. \end{cases}$$

For $t=\pm 1$, we define closed curves $a_t:[0, 4]\to S^3$ by

$$a_t(s) = \begin{cases} \tilde{\Phi}(\tilde{c}_t(s)) & \text{if } 0\leqq s\leqq 2, \\ \tilde{\Phi}(\tilde{a}(4-s)) & \text{if } 2\leqq s\leqq 4. \end{cases}$$

Furthermore define closed curves $b:[0, 3]\to S^3$ and $c:[0, 2\pi]\to S^3$ by

$$b(s) = \begin{cases} \tilde{\Phi}(0, s-1/2, 0) & \text{if } 0\leqq s\leqq 1, \\ \tilde{\Phi}(\tilde{b}(3-s)) & \text{if } 1\leqq s\leqq 3, \end{cases}$$

$$c(s)=\begin{cases}\tilde{\Phi}(\bar{a}(s)) & \text{if } 0\leqq s\leqq 2\,,\\ \tilde{\Phi}(\bar{b}(s-3)) & \text{if } 3\leqq s\leqq 5\,,\\ c_0(s) & \text{otherwise.}\end{cases}$$

We now compute $\mathrm{lk}(\gamma_1)$ and $\mathrm{lk}(\gamma_{-1})$. Let δ be a positive number such that $\varphi^t(c_1)$ (resp. $\varphi^t(c_{-1})$) does not intersect c_1 (resp. c_{-1}) for all $0<t\leqq\delta$. Define D_1 to be the union of the images of the curves a_1, b and c. By the definition of $\tilde{\Phi}$, we see that $\varphi^t(\tilde{\Phi}(x, y, z))=\tilde{\Phi}(x, y, z+t)$. So we may assume that D_1 does not intersect D_1^δ, where $D_1^\delta=\varphi^\delta(D_1)$. Then it follows from (6.11) that $[c_1^\delta]=[a_1^\delta]+[b^\delta]+[c^\delta]$ in the homology group $H_1(S^3-D_1)$, where $c_1^\delta=\varphi^\delta(c_1)$, $a_1^\delta=\varphi^\delta(a_1)$, $b^\delta=\varphi^\delta(b)$ and $c^\delta=\varphi^\delta(c)$. Hence we obtain

$$\mathrm{lk}(c_1, c_1^\delta)=\mathrm{lk}(c_1, a_1^\delta)+\mathrm{lk}(c_1, b^\delta)+\mathrm{lk}(c_1, c^\delta)\,.$$

Since $[c_1]=[a_1]+[b]+[c]$ in $H_1(S^3-D_1^\delta)$, it follows that

$$\begin{aligned}\mathrm{lk}(c_1, c_1^\delta)=&\ \mathrm{lk}(a_1, a_1^\delta)+\mathrm{lk}(b, a_1^\delta)+\mathrm{lk}(c, a_1^\delta)\\ &+\mathrm{lk}(a_1, b^\delta)+\mathrm{lk}(b, b^\delta)+\mathrm{lk}(c, b^\delta)\\ &+\mathrm{lk}(a_1, c^\delta)+\mathrm{lk}(b, c^\delta)+\mathrm{lk}(c, c^\delta)\,.\end{aligned}$$

It is easy to see that $\mathrm{lk}(a_1, b^\delta)=\mathrm{lk}(b, a_1^\delta)=\mathrm{lk}(a_1, b)$ and the other terms on the right hand side are equal to zero, except for $\mathrm{lk}(c, c^\delta)$. Hence

$$\mathrm{lk}(\gamma_1)=\mathrm{lk}(c, c^\delta)+2\mathrm{lk}(a_1, b)\,.$$

By the same way we have

$$\mathrm{lk}(\gamma_{-1})=\mathrm{lk}(c, c^\delta)+2\mathrm{lk}(a_{-1}, b)\,.$$

Since $\tilde{\Phi}$ is orientation preserving, it follows from (6.12) that $\mathrm{lk}(a_1, b)=\mathrm{lk}(\tilde{\Phi}^{-1}(a_1), \tilde{\Phi}^{-1}(b))=-1$. On the other hand, $\mathrm{lk}(a_{-1}, b)=0$ since $x_{-1}(s)>0$. This implies (6.6). Q. E. D.

For $\alpha, \beta\in\Omega^*$, we write $\alpha\sim\beta$ if there exists a sequence $\alpha_0, \alpha_1, \cdots, \alpha_n$ in Ω^* such that $\alpha=\alpha_0$ and $\beta=\alpha_n$ and that α_{i-1} and α_i are R1 (or R2)-related for $1\leqq i\leqq n$.

LEMMA 6.7. *Let $\alpha, \beta\in\Omega^*$. If $[\hat{\alpha}]=[\hat{\beta}]=0$, then $\alpha\sim\beta$.*

PROOF. For each $\gamma\in\Omega^*$, we denote by $\#\gamma$ the number of double points of γ. We choose $\gamma_0, \gamma_1\in\Omega^*$ such that $\#\gamma_0=0$ and $\#\gamma_1=1$. Then it follows that $\alpha\sim\gamma_0$ or $\alpha\sim\gamma_1$. If $\alpha\sim\gamma_0$, we see that $[\hat{\alpha}]=[\hat{\gamma}_0]=1$. So the assumption implies that $\alpha\sim\gamma_1$. Similarly we obtain $\beta\sim\gamma_1$. Q. E. D.

We now prove (6.1). By Lemma 6.3 we may assume that $\gamma\in\Omega^*\cap\Omega^0$. It follows from Lemmas 6.1 and 6.3 that there exists $\beta\in\Omega^*\cap\Omega^0$ such that $\mathrm{lk}(\beta)=1$. Since $[\hat{\gamma}]=[\hat{\beta}]=0$, it follows from Lemma 6.7 that $\gamma\sim\beta$. By Lemmas 6.5 and 6.6 we obtain $\mathrm{lk}(\gamma)\equiv\mathrm{lk}(\beta) \pmod 2$. Hence $\mathrm{lk}(\gamma)\equiv 1 \pmod 2$.

7. Gauss maps.

As explained in Section 3, the author [2] established a method for constructing all the flat tori in S^3. Recently Weiner [8] obtained another method which is based on the study of the Gauss maps of flat tori in S^3. In this section, for each admissible pair Γ, we compute the Gauss map of the immersion $F_\Gamma: \boldsymbol{R}^2\to S^3$. The result of this computation will help us to understand the relation between the two methods.

First, we explain the Gauss maps of immersed surfaces in S^3. Let $F: \boldsymbol{R}^2\to S^3$ be an immersion, and let η be a unit normal vector field along F given by

$$\eta=\partial_1 F\times\partial_2 F/\|\partial_1 F\times\partial_2 F\|,$$

where $\times$ denotes the usual vector product on each tangent space of S^3 defined by using the metric and the orientation of S^3. Since $S^3=SU(2)$, we obtain two maps $\alpha, \beta: \boldsymbol{R}^2\to S^2\subset\mathfrak{su}(2)$ by setting $\alpha=(R_F^{-1})_*\eta$ and $\beta=(L_F^{-1})_*\eta$. We call α (resp. β) the *right* (resp. *left*) *Gauss map* of the immersion F. We now consider another Gauss map. Let $\boldsymbol{H}$ denote the set of all quaternions

$$x=x_1+x_2 i+x_3 j+x_4 k\,.$$

Then $\boldsymbol{H}$ can be viewed as a 4-dimensional oriented Euclidean vector space by setting $\{1, i, j, k\}$ as a positive orthonormal basis of $\boldsymbol{H}$. Identifying each complex number $x_1+x_2\sqrt{-1}$ with the quaternion $x_1+x_2 i$, we define a map $\varphi: S^3\to\boldsymbol{H}$ by

$$\varphi\left(\begin{bmatrix} g_1 & g_2 \\ -\bar{g}_2 & \bar{g}_1\end{bmatrix}\right)=g_1+g_2 j\,.$$

The map φ is an isometric embedding and it satisfies

$$\|\varphi(a)\|=1 \qquad \text{for all } a\in S^3, \tag{7.1}$$

$$\varphi(ab)=\varphi(a)\varphi(b) \qquad \text{for all } a, b\in S^3. \tag{7.2}$$

Using the map φ, we identify S^3 with the set of all quaternions of unit norm. Then, as in [8], the immersion $F: \boldsymbol{R}^2\to S^3\subset\boldsymbol{H}$ induces a map $G: \boldsymbol{R}^2\to G_{2,4}$ which is called the Gauss map of F. Here $G_{2,4}$ is the Grassmannian of oriented 2-dimensional subspaces of $\boldsymbol{H}$. The Grassmannian $G_{2,4}$ can be viewed as the set of all unit decomposable 2-vectors in $\wedge^2\boldsymbol{H}$, and the Gauss map G is given by

$$G=\partial_1\tilde{F}\wedge\partial_2\tilde{F}/\|\partial_1\tilde{F}\wedge\partial_2\tilde{F}\|, \tag{7.3}$$

where $\tilde{F}=\varphi\circ F$.

We now study the relation between the maps α, β and G. Let $*: \wedge^2\boldsymbol{H}\to\wedge^2\boldsymbol{H}$ be the Hodge star, and let E_+ (resp. E_-) be the $+1$ (resp. -1) eigenspace of $*$. Then we have the orthogonal decomposition $\wedge^2\boldsymbol{H}=E_+\oplus E_-$. The projections $p_\pm: \wedge^2\boldsymbol{H}\to E_\pm$ are given by $p_\pm(v)=(v\pm *v)/2$ for all $v\in\wedge^2\boldsymbol{H}$.

LEMMA 7.1. *Let x be a quaternion of unit norm, and let R_x and L_x denote linear transformations of $\boldsymbol{H}$ defined by $R_x(y)=yx$ and $L_x(y)=xy$. Then*

(1) $p_+\circ(\wedge^2R_x)=p_+$,

(2) $p_-\circ(\wedge^2L_x)=p_-$.

PROOF. Since R_x is an orientation preserving linear isometry, the operator $\wedge^2R_x$ commutes with the Hodge star $*$. Hence we obtain $p_+\circ(\wedge^2R_x)=(\wedge^2R_x)\circ p_+$. On the other hand, a straightforward calculation shows that

$$x\wedge ix+jx\wedge kx=1\wedge i+j\wedge k\,,$$

$$x\wedge jx+kx\wedge ix=1\wedge j+k\wedge i\,,$$

$$x\wedge kx+ix\wedge jx=1\wedge k+i\wedge j\,.$$

This implies $(\wedge^2R_x)\circ p_+=p_+$, and so we obtain (1). The assertion (2) is proved in the same way. Q.E.D.

Let $\varphi_\pm: \mathfrak{su}(2)\to E_\pm$ be linear maps defined by the following relations.

$$\varphi_\pm(e_1)=\sqrt{2}\,p_\pm(\varphi_*e_2\wedge\varphi_*e_3)\,,$$

$$\varphi_\pm(e_2)=\sqrt{2}\,p_\pm(\varphi_*e_3\wedge\varphi_*e_1)\,,$$

$$\varphi_\pm(e_3)=\sqrt{2}\,p_\pm(\varphi_*e_1\wedge\varphi_*e_2)\,,$$

where $\{e_1, e_2, e_3\}$ is a positive orthonormal basis of $\mathfrak{su}(2)$ given in Section 2. It is easy to see that φ_+ and φ_- are linear isometries. We set $G_\pm=p_\pm\circ G$. Then we obtain

LEMMA 7.2.

$$G_+=\frac{1}{\sqrt{2}}\varphi_+(\alpha)\,,\qquad G_-=\frac{1}{\sqrt{2}}\varphi_-(\beta)\,.$$

PROOF. For $i=1, 2$, define $a_i: \boldsymbol{R}^2\to\mathfrak{su}(2)$ by $a_i=(R_F^{-1})_*\partial_iF$. Then it follows from (7.2) that $\partial_i\tilde{F}=\varphi_*(R_F)_*a_i=R_{\tilde{F}}(\varphi_*a_i)$, and so

$$\partial_1\tilde{F}\wedge\partial_2\tilde{F}=(\wedge^2R_{\tilde{F}})(\varphi_*a_1\wedge\varphi_*a_2)\,.$$

By (7.1) the immersion $\tilde{F}: \boldsymbol{R}^2\to\boldsymbol{H}$ satisfies $\|\tilde{F}(s_1, s_2)\|=1$. Therefore Lemma 7.1 implies

(7.4) $$p_+(\partial_1\tilde{F}\wedge\partial_2\tilde{F}) = p_+(\varphi_*a_1\wedge\varphi_*a_2).$$

On the other hand, it follows from the definition of φ_+ that

(7.5) $$\varphi_+(a_1\times a_2) = \sqrt{2}\,p_+(\varphi_*a_1\wedge\varphi_*a_2).$$

Since $\alpha=a_1\times a_2/\|a_1\times a_2\|$ and $\|\partial_1\tilde{F}\wedge\partial_2\tilde{F}\|=\|a_1\times a_2\|$, it follows from (7.3)-(7.5) that $G_+=(1/\sqrt{2})\varphi_+(\alpha)$. By the same way we obtain $G_-=(1/\sqrt{2})\varphi_-(\beta)$. Q.E.D.

Note that in [8] the maps G_+ and G_- are denoted by G_1 and G_2, respectively. We now consider an admissible pair $\Gamma=(\gamma_1, \gamma_2)$, and compute the Gauss map of the immersion F_Γ.

LEMMA 7.3. *Let α (resp. β) be the right (resp. left) Gauss map of the immersion F_Γ. Then*

(1) $\alpha(s_1, s_2)=\gamma_1'(s_1)/\|\gamma_1'(s_1)\|$,

(2) $\beta(s_1, s_2)=\gamma_2'(s_2)/\|\gamma_2'(s_2)\|$.

PROOF. For simplicity, we set $F=F_\Gamma$. By (3.4) we obtain

(7.6) $$F(s_1, s_2) = c_1(s_1)\cdot c_2(s_2)^{-1},$$

where c_i denotes a curve in S^3 such that $p_2(c_i)=\gamma_i$ and $c_i(0)=\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}$. Define $v_i: \boldsymbol{R}\to\mathfrak{su}(2)$ by setting $v_i(s)=(L_{c_i(s)}^{-1})_*\dot{c}_i(s)$. Then it follows from (7.6) that

$$(R_F^{-1})_*\partial_1 F = \mathrm{Ad}\,(c_1(s_1))v_1(s_1), \qquad (R_F^{-1})_*\partial_2 F = -\mathrm{Ad}\,(c_1(s_1))v_2(s_2).$$

Hence

(7.7) $$\alpha(s_1, s_2) = \frac{-\mathrm{Ad}\,(c_1(s_1))(v_1(s_1)\times v_2(s_2))}{\|v_1(s_1)\times v_2(s_2)\|}.$$

By Lemma 2.2 we see that $v_i=\|\gamma_i'\|(e_2+k_ie_3)/2$, where k_i denotes the geodesic curvature of γ_i. This shows

(7.8) $$v_1(s_1)\times v_2(s_2) = \frac{1}{4}\|\gamma_1'(s_1)\|\|\gamma_2'(s_2)\|(k_2(s_2)-k_1(s_1))e_1.$$

Since $k_1(s_1)>k_2(s_2)$, it follows from (7.7) and (7.8) that $\alpha(s_1, s_2)=\mathrm{Ad}\,(c_1(s_1))e_1$. By (2.1) and (2.2) we also have $\mathrm{Ad}\,(c_1(s_1))e_1=\gamma_1'(s_1)/\|\gamma_1'(s_1)\|$. This implies (1). The assertion (2) is proved in the same way. Q.E.D.

Combining Lemmas 7.2 and 7.3, we obtain the following theorem.

THEOREM 7.4. *Let $\Gamma=(\gamma_1, \gamma_2)$ be an admissible pair, and let G be the Gauss map of the immersion $F_\Gamma: \boldsymbol{R}^2\to S^3$. Then*

(1) $G_+(s_1, s_2)=\frac{1}{\sqrt{2}}\varphi_+(\gamma_1'(s_1)/\|\gamma_1'(s_1)\|)$,

(2) $G_-(s_1, s_2)=\frac{1}{\sqrt{2}}\varphi_-(\gamma_2'(s_2)/\|\gamma_2'(s_2)\|)$.

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