Embedded minimal surfaces of finite topology

William H. Meeks III*   Joaquín Pérez †

October 7, 2009

Abstract

In this paper we prove that any complete, embedded minimal surface M in $\mathbb{R}^3$ with finite topology and compact boundary (possibly empty) is conformally a compact Riemann surface $\overline{M}$ with boundary punctured in a finite number of interior points and that M can be represented in terms of meromorphic data on its conformal completion $\overline{M}$. In particular, we demonstrate that M is a minimal surface of finite type and describe how this property permits a classification of the asymptotic behavior of M.

Mathematics Subject Classification: Primary 53A10, Secondary 49Q05, 53C42
Key words and phrases: Minimal surface, helicoid with handles, infinite total curvature, flux vector, minimal surface of finite type, asymptotic behavior of a minimal annulus.

1 Introduction.

Based on work of Colding and Minicozzi [9], Meeks and Rosenberg [19] proved that a properly embedded, simply-connected minimal surface in $\mathbb{R}^3$ is a plane or a helicoid. On the last page of their paper, they described how their proof of the uniqueness of the helicoid could be modified to prove:

Any nonplanar, properly embedded minimal surface M in $\mathbb{R}^3$ with one end, finite topology and infinite total curvature satisfies the following properties:

1. *M is conformally a compact Riemann surface $\overline{M}$ punctured at a single point.*
2. *M is asymptotic to a helicoid.*
3. *M can be expressed analytically in terms of meromorphic data on $\overline{M}$.*

*This material is based upon work for the NSF under Award No. DMS - 0703213. Any opinions, findings, and conclusions or recommendations expressed in this publication are those of the authors and do not necessarily reflect the views of the NSF.

†Research partially supported by a Spanish MEC-FEDER Grant no. MTM2007-61775 and a Regional J. Andalucía Grant no. P06-FQM-01642.

A rigorous proof of the above statement has been recently given by Bernstein and Breiner [1].

In our survey [14], we outlined the proof by Meeks and Rosenberg of the uniqueness of the helicoid and at the end of this outline we mentioned how some difficult parts of the proof could be simplified using some results of Colding and Minicozzi in [5], and referred the reader to the present paper for details. Actually, we will consider the more general problem of describing the asymptotic behavior, conformal structure and analytic representation of an annular end of any *complete,* injectively immersed[1] minimal surface M in $\mathbb{R}^3$ with compact boundary and finite topology. Although not explicitly stated in the paper [10] by Colding and Minicozzi, the results contained there imply that such an M is *properly embedded*[2] in $\mathbb{R}^3$; we also remark that Meeks, Pérez and Ros [16] proved the following more general result: If M is a complete, injectively immersed minimal surface of finite genus, compact boundary and a countable number of ends in $\mathbb{R}^3$, then M is proper (this follows from Theorem 1.4 in [16] with the observation that every limit end of such an M is a simple limit end since M has countably many ends). We will use this properness property of M in the proof of Theorem 1.1 below. Since properly embedded minimal annuli $E \subset \mathbb{R}^3$ with compact boundary and finite total curvature are conformally punctured disks, are asymptotic to the ends of planes and catenoids and have a well understood and simple analytic description in terms of meromorphic data on their conformal completion, we will focus our attention on the case that the minimal annulus has infinite total curvature.

Theorem 1.1 *Let $E \subset \mathbb{R}^3$ be a complete, embedded minimal annulus with infinite total curvature and compact boundary. Then, the following properties hold:*

1. *E is properly embedded in $\mathbb{R}^3$.*
2. *E is conformally diffeomorphic to a punctured disk.*
3. *After replacing E by a subend and applying a suitable homothety and rigid motion to E, then:*
   - (a) *The holomorphic height differential $dh = dx_3 + i dx_3^*$ of E is $dh = (1 + \frac{\lambda}{z-\mu})\, dz$, defined on $D(\infty, R) = \{z \in \mathbb{C} \mid R \leq |z|\}$ for some $R > 0$, where $\lambda \in \mathbb{R}$ and $\mu \in \mathbb{C}$. In particular, dh extends meromorphically across infinity with a double pole.*
   - (b) *The stereographic projection $g\colon D(\infty, R) \to \mathbb{C} \cup \{\infty\}$ of the Gauss map of E can be expressed as $g(z) = e^{iz+f(z)}$ for some holomorphic function f in $D(\infty, R)$ with $f(\infty) = 0$.*
   - (c) *E is asymptotic to the end of a helicoid if and only if it has zero flux (in particular, the number λ in the previous item 3(a) vanishes in this case).*

Since complete embedded minimal surfaces with finite topology are properly embedded in $\mathbb{R}^3$[10] and properly embedded minimal surfaces with more than one end and finite topology have finite total

[1] That is, M has no self-intersections and its intrinsic topology may or may not agree with the subspace topology as a subset of $\mathbb{R}^3$.

[2] By this we mean that the intrinsic topology on M coincides with the subpace topology and the intersection of M with every closed ball in $\mathbb{R}^3$ is compact in M.

curvature [11], then we have the following immediate corollary (which also appears in [1]). Note also that the zero flux condition in item 3(c) of Theorem 1.1 holds by Cauchy's theorem when M has just one end.

Corollary 1.2 *Suppose that $M \subset \mathbb{R}^3$ is a complete, embedded minimal surface with finite topology. Then M is conformally a compact Riemann surface $\overline{M}$ punctured in a finite number of points and M can be described analytically in terms of meromorphic data $\frac{dg}{g}, dh$ on $\overline{M}$, where g and dh denote the (stereographically projected) Gauss map and the height differential of M, after a suitable rotation in $\mathbb{R}^3$. Furthermore, M has bounded Gaussian curvature and each of its ends is asymptotic in the C^k topology (for any k) to a plane or a half-catenoid (if M has finite total curvature) or to a helicoid (if M has infinite total curvature).*

In order to state the next result, we need the following notation. Given $R > 0$, let $C(R) = \{(x_1, x_2, x_3) \mid x_1^2 + x_2^2 \leq R^2\}$. A multigraph over $D(\infty, R) = \{x_3 = 0\} \cap [\mathbb{R}^3 - \text{Int}(C(R))]$ is the graph $\Sigma = \{(re^{i\theta}, u(r,\theta)\} \subset \mathbb{C} \times \mathbb{R} \equiv \mathbb{R}^3$ of a function $u = u(r,\theta)$ defined on the universal cover $\widetilde{D}(\infty, R) = \{(r,\theta) \mid r \geq R,\ \theta \in \mathbb{R}\}$ of $D(\infty, R)$.

The next theorem is a consequence of Theorem 7.1 below, and provides a more detailed description of the geometry of a complete, embedded minimal annulus E in $\mathbb{R}^3$. The image in Figure 1 describes how the flux vector $(a, 0, -b)$ of E influences its geometry.

Theorem 1.3 (Asymptotics of embedded minimal annular ends) *Given $a \geq 0$ and $b \in \mathbb{R}$, there exist a positive number $R = R_E$ and a properly embedded minimal annulus $E = E_{a,b} \subset \mathbb{R}^3$ with compact boundary and flux vector $(a, 0, -b)$ along its boundary, such that the following statements hold.*

1. *$E_{a,b} - C(R)$ consists of two disjoint multigraphs Σ_1, Σ_2 over $D(\infty, R)$ of smooth functions $u_1, u_2 \colon \widetilde{D}(\infty, R) \to \mathbb{R}$ such that their gradients satisfy $\nabla u_i(r,\theta) \to 0$ as $r \to \infty$ and the separation function $w(r,\theta) = u_1(r,\theta) - u_2(r,\theta)$ between both multigraphs converges to π as $r + |\theta| \to \infty$. Furthermore for θ fixed and $i = 1, 2$,*
$$\lim_{r\to\infty} \frac{u_i(r,\theta)}{\log(\log(r))} = \frac{b}{2\pi}. \tag{1}$$
2. *The translated surfaces $E_{a,b} + (0, 0, -2\pi n - \frac{b}{2\pi}\log n)$ (resp. $E_{a,b} + (0, 0, 2\pi n - \frac{b}{2\pi}\log n)$) converge as $n \to \infty$ to a vertical helicoid H_T (resp. H_B) such that $H_B = H_T + (0, a/2, 0)$. Note that this last equation together with item 1 imply that for different values of a, b, the related surfaces $E_{a,b}$ are not asymptotic after a rigid motion and homothety.*
3. *The annuli $E_{0,b}$ are each invariant under reflection across the x_3-axis l and $l \cap E_{0,b}$ contains two infinite rays.*
4. *Every complete, embedded minimal annulus in $\mathbb{R}^3$ with compact boundary and infinite total curvature is asymptotic (up to a rotation and homothety by some $\lambda \in \mathbb{R} - \{0\}$) to exactly one of the surfaces $E_{a,b}$.*

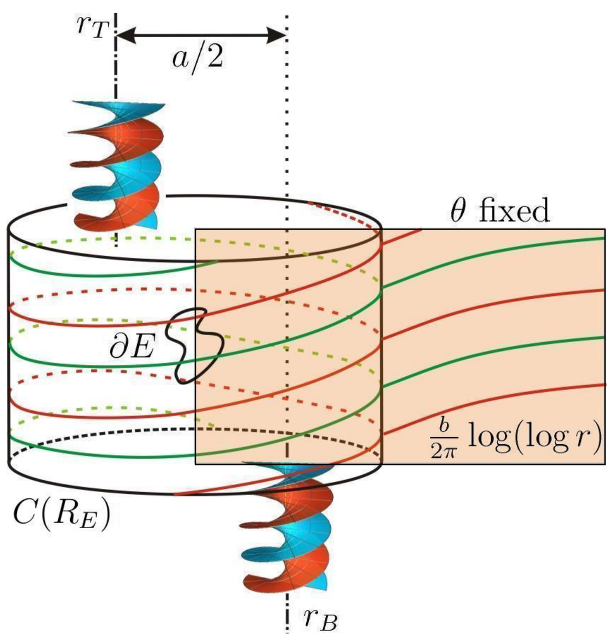


Figure 1: The embedded annulus E with flux vector $(a, 0, -b)$ (see Theorem 1.3) has the following description. Outside the cylinder $C(R_E)$, E consists of two horizontal multigraphs with asymptotic spacing π between them. The translated surfaces $E + (0, 0, -2\pi n - \frac{b}{2\pi}\log n)$ (resp. $E + (0, 0, 2\pi n - \frac{b}{2\pi}\log n)$) converge as $n \to \infty$ to a vertical helicoid H_T (resp. H_B) such that $H_B = H_T + (0, a/2, 0)$ (in the picture, r_T, r_B refer to the axes of H_T, H_B). The intersection of $E - C(R_E)$ with a vertical halfplane bounded by the x_3-axis consists of an infinite number of curves, each being a graph of a function $u(r)$ that satisfies the property $\frac{u(r)}{\log(\log r)}$ converges to $\frac{b}{2\pi}$ as the radial distance r to the x_3-axis tends to ∞.

As said above, Bernstein and Breiner [1] have given a proof of Corollary 1.2 in the infinite total curvature setting, which is just Theorem 1.1 in the special case that the annulus E is the end of a complete, embedded minimal surface M with finite topology, one end and empty boundary. Our proof of Theorem 1.1 and the proof of Bernstein and Breiner in the above special case, apply arguments that Meeks and Rosenberg [19] used to understand the asymptotic behavior of an E with zero flux, and to show that such an E has finite type[3]. Once E is proven to have finite type, then Meeks and Rosenberg, as well as Bernstein and Breiner, appeal to the deep results of Hauswirth, Pérez and Romon [12] on the geometry of embedded, minimal annular ends of finite type to complete the proof. In our proof of Theorem 1.3, we essentially avoid reference to the results in [12] by using self-contained arguments, some of which have a more geometric, less analytic nature than those in [12]. Central to all of these proofs are the results of Colding and Minicozzi [3, 5, 7, 8, 9, 10].

We refer the interested reader to our paper [16] with Antonio Ros on the embedded Calabi-Yau problem; more precisely, on the general structure of simple limit ends of complete, injectively immersed minimal surfaces $M \subset \mathbb{R}^3$ of finite genus and compact boundary, which together with Theorems 1.1 and 1.3 in this paper, improve our understanding of the conformal structure, asymptotic behavior and analytic description of the ends of properly embedded minimal surfaces in $\mathbb{R}^3$ which have compact boundary and finite genus. The main outstanding problem related to the embedded Calabi-Yau problem is whether every complete, injectively immersed minimal surface in $\mathbb{R}^3$ with finite genus, compact boundary and an infinite number of ends, must have only a countable number of ends, since such minimal surfaces are always properly embedded in $\mathbb{R}^3$ with exactly one or two limit ends; see [16] for further discussion and partial results on this problem.

2 Weierstrass representation of a minimal surface of finite type.

Recall that the Gauss map of an immersed minimal surface M in $\mathbb{R}^3$ can be viewed as a meromorphic function $g\colon M \to \mathbb{C} \cup \{\infty\}$ on the underlying Riemann surface, after stereographic projection from the north pole of the unit sphere. Furthermore, the harmonicity of the third coordinate function x_3 of M lets us define (at least locally) its harmonic conjugate function x_3^*; hence, the *height differential*[4] $dh = dx_3 + i dx_3^*$ is a holomorphic differential on M. As usual, we will refer to the pair (g, dh) as the *Weierstrass data* of M, and the minimal immersion $X\colon M \to \mathbb{R}^3$ can be expressed up to translation by $X(p_0)$, $p_0 \in M$, by

$$X(p) = \Re \int_{p_0}^{p} \left(\frac{1}{2}\left(\frac{1}{g} - g\right), \frac{i}{2}\left(\frac{1}{g} + g\right), 1 \right) dh, \tag{2}$$

where $\Re$ stands for real part [13, 21]. It is well-known (see Osserman [20] for details) that this procedure has a converse, namely if M is a Riemann surface, $g\colon M \to \mathbb{C} \cup \{\infty\}$ a meromorphic function and dh

[3]See Definition 2.1 for the notion of a minimal surface of finite type.

[4]The height differential might not be exact since x_3^* need not be globally well-defined on M. Nevertheless, the notation dh is commonly accepted and we will also make use of it here.

a holomorphic one-form on M, such that the zeros of dh coincide with the poles and zeros of g with the same order, and for any closed curve $\gamma \subset M$, the following equalities hold:

$$\overline{\int_\gamma g\,dh} = \int_\gamma \frac{dh}{g}, \qquad \Re \int_\gamma dh = 0, \tag{3}$$

(where $\overline{z}$ denotes the complex conjugate of $z \in \mathbb{C}$), then the map $X\colon M \to \mathbb{R}^3$ given by (2) is a conformal minimal immersion with Weierstrass data (g, dh).

Minimal surfaces M which satisfy the property of having *finite type,* as described in the next definition, can be defined analytically from data obtained by integrating certain meromorphic one-forms on M, which moreover extend meromorphically to the conformal completion of M.

Definition 2.1 (Finite Type) A minimal immersion $X\colon M \to \mathbb{R}^3$ is said to have *finite type* if it satisfies the following two properties.

1. The underlying Riemann surface to M is conformally diffeomorphic to a compact Riemann surface $\overline{M}$ with (possibly empty) compact boundary, punctured in a finite nonempty set $\mathcal{E} \subset \text{Int}(\overline{M})$.

2. Given an end $e \in \mathcal{E}$ of M, there exists a rotation of the surface in space such that if (g, dh) is the Weierstrass data of M after this rotation, then the meromorphic one-forms $\frac{dg}{g}$ and dh extend across the puncture e to meromorphic one-forms on a neighborhood of e in $\overline{M}$.

Therefore, complete immersed minimal surfaces with finite total curvature are of finite type, and the basic example of a surface with finite type but infinite total curvature is the helicoid ($M = \mathbb{C}$, $g(z) = e^{iz}$, $dh = dz$). Finite type minimal surfaces were first studied by Rosenberg [22]; we note that his definition of finite type differs somewhat from our definition above. An in-depth study of *embedded* minimal annular ends of finite type with exact height differentials was made by Hauswirth, Pérez and Romon in [12]. Some of the results in this paper extend the results in [12] to the case where the height differential might possibly be not exact.

3 The conformal type, height differential and Gauss map of embedded minimal annuli.

Throughout this section, we will fix a complete, embedded minimal surface $M \subset \mathbb{R}^3$ with infinite total curvature, where M is an annulus diffeomorphic to $\mathbb{S}^1 \times [0, \infty)$. By the main result in Colding and Minicozzi [10] (which holds in this setting of compact boundary), M is properly embedded in $\mathbb{R}^3$. Our goal in this section is to prove that after a rotation and a replacement of M by a subend, M is conformally diffeomorphic to $D(\infty, 1)$, its height differential extends meromorphically across the puncture ∞ with a double pole, and the meromorphic Gauss map $g\colon D(\infty, 1) \to \mathbb{C} \cup \{\infty\}$ of M has the form $g(z) = z^k e^{H(z)}$, where $k \in \mathbb{Z}$ and H is holomorphic in $D(\infty, 1)$.

In Theorem 1.4 of [17], Meeks, Pérez and Ros proved that a complete embedded minimal surface with compact boundary fails to have quadratic decay of curvature[5] if and only if it does not have finite total curvature. Since we are assuming that M has infinite total curvature, then it does not have quadratic decay of curvature.

Let $\{\lambda_n\}_n \subset \mathbb{R}^+$ be a sequence with $\lambda_n \searrow 0$ as $n \to \infty$. We want to analyze the structure of the limit of (a subsequence of) the $\lambda_n M$. In our setting with compact boundary we need to be careful at this point, since compactness results as in Colding and Minicozzi [9, 3] cannot be applied directly. We claim that the sequence of surfaces $\{\lambda_n M\}_n$ has locally positive injectivity radius in $\mathbb{R}^3 - \{\vec{0}\}$ (this means that for every $q \in \mathbb{R}^3 - \{\vec{0}\}$, there exists $\varepsilon_q > 0$ and $n_q \in \mathbb{N}$ such that for $n > n_q$, the injectivity radius functions of the surfaces $\lambda_n M$ restricted to $\{x \in \mathbb{R}^3 \mid |x-q| < \varepsilon_q\} \cap M_n$ form a sequence of functions which is uniformly bounded away from zero). This claim holds since otherwise (after extracting a subsequence) we can find a sequence of pairwise disjoint embedded geodesic loops $\gamma_k' \subset \{x \in \mathbb{R}^3 \mid \mid |x-q| < \frac{1}{k}\} \cap (\lambda_k M)$ with at most one cusp each and with the lengths of these loops tending to zero as k tends to infinity. These curves γ_k' produce related pairwise disjoint embedded geodesic loops γ_k on M with at most one cusp, and the sequence $\{\gamma_k\}_k$ diverges in M. By the Gauss-Bonnet formula, the curves γ_k are homotopically nontrivial and hence parallel (since M is an annulus). Therefore, the Gauss-Bonnet formula implies that M has finite total curvature, which is a contradiction.

Since the surface M does not have quadratic decay of curvature, there exists a divergent sequence of points $\{p_n\}_n \subset M$ such that the absolute value of the Gaussian curvature of M at p_n divided by $\lambda_n = \|p_n\|$ diverges to ∞. Applying item 7 of Theorem 1.5 of [18] to the countable closed set $W = \{\vec{0}\}$ and to the sequence $\{\lambda_n M\}_n$, we conclude that after extracting a subsequence, the surfaces $\lambda_n M$ converge as $n \to \infty$ to a foliation $\mathcal{F}$ of $\mathbb{R}^3$ by parallel planes and the convergence is C^1 away from one or two straight lines orthogonal to the planes in $\mathcal{F}$; the case of two orthogonal lines does not occur since limiting connection loops between the forming lines of the limit parking garage structure arise from pairwise disjoint simple closed geodesics on M which is impossible by the Gauss-Bonnet formula. Also note that under these homothetic shrinkings of M, the boundary of $\lambda_n M$ collapses into the origin $\vec{0}$, and the singular set of convergence of the $\lambda_n M$ to $\mathcal{F}$ must be a line passing through $\vec{0}$. Furthermore, this limit foliation $\mathcal{F}$ is easily seen to be independent of the sequence of positive numbers λ_n (see for example, [6, 4, 9]), so after a rotation of the initial surface we may assume that the planes in this foliation are horizontal.

It follows that there is a solid vertical hyperboloid $\mathcal{H}$ with axis being the x_3-axis, such that $M - \mathcal{H}$ consists of two multigraphs over their projections to the (x_1, x_2)-plane P.

Remark 3.1 *The arguments above use the uniqueness of the helicoid (when applying Theorem 1.5 in [18]). However, for a simply-connected surface $\widehat{M}$ without boundary (not an annulus M with boundary), the sentence before this remark and the existence of multigraphs outside of $\mathcal{H}$ follows directly from the main results in [9], without reference to the discussion in the previous paragraphs. We make this*

[5] A properly embedded surface $M \subset \mathbb{R}^3$ has *quadratic decay of curvature* if there is a constant $C > 0$ such that the absolute curvature function $|K_M|$ of M satisfies $|K_M(q)| \leq \frac{C}{R(q)^2}$ for all $q \in M$, where $R = \sqrt{x_1^2 + x_2^2 + x_3^2}$ is the radial distance function.

remark here to warn the reader that we will be applying the next proposition, which depends on the existence of multigraphs outside of the hyperboloid $\mathcal{H}$, in the proof of Theorem 4.1 on the uniqueness of the helicoid.

By work of Colding-Minicozzi, any embedded minimal multigraph with a large number of sheets contains a submultigraph which can be approximated by the multigraph of a helicoid with an additional logarithmic term. This approximation result, which appears as Corollary 14.3 in Colding and Minicozzi [5], together with estimates for the decay of the separation between consecutive sheets of a multigraph (equations (18.1) in [5] or (5.9) in [7], which in turn follow from Proposition 5.5 in [7], and from Proposition II.2.12 and Corollary II.3.7 in [6]), are sufficient to prove that each of the two multigraphs G_1, G_2 in $M - \mathcal{H}$ contains infinite submultigraphs G_1', G_2', respectively given by functions $u^1(\rho, \theta)$, $u^2(\rho, \theta)$, such that $\frac{\partial u^i}{\partial \theta}(\rho, \theta) > 0$ for $i = 1, 2$ (resp. < 0 when the multigraphs G_1', G_2' spiral downward in a counter clockwise manner); furthermore the boundary curves of the subgraphs G_1', G_2' can be assumed to lie on the boundary of another hyperboloid $\mathcal{H}'$. This observation concerning $\frac{\partial u^i}{\partial \theta}(\rho, \theta) > 0$ was also made by Bernstein and Breiner using the same arguments, see Proposition 3.3 in [2]. We refer the reader to their paper for details.

With the above discussion in mind, we are ready to describe the main result of this section. Given $r, h > 0$, let $C(r) = \{(x_1, x_2, x_3) \mid x_1^2 + x_2^2 \leq r\}$ denote the vertical solid cylinder of radius r whose axis is the x_3-axis, and let $C(r, h) = C(r) \cap \{|x_3| \leq h\}$ be the compact sub-cylinder of height $2h$ centered at the origin.

Proposition 3.2 *After a rotation of M so that the foliation $\mathcal{F}$ is horizontal, then:*

1. *There is a subend M' of M such that each horizontal plane $P_t = x_3^{-1}(t)$ in $\mathcal{F}$ intersects M' transversely in either a proper curve at height t or in two proper arcs, each with its single end point on the boundary of M'. More specifically, there exists a solid cylinder $C(r, h)$ with $\partial M \subset \mathrm{Int}(C(r, h))$ whose top and bottom disks each intersects M transversely in a compact arc and whose side $\{x_1^2 + x_2^2 = r^2, |x_3| \leq h\}$ intersects M transversely in two compact spiraling arcs $\alpha^1(\theta)$, $\alpha^2(\theta)$ with $\frac{d\alpha^i}{d\theta} > 0$, $i = 1, 2$ (after a possible replacement of M by $-M$), where θ is the multi-valued angle parameter over the circle $\partial C(r, h) \cap P_0$, and so that $M' = M - \mathrm{Int}(C(r, h))$ is the desired subend.*

2. *The end M' is conformally diffeomorphic to $D(\infty, 1)$.*

3. *There exists $k \in \mathbb{Z}$ and a holomorphic function H on $D(\infty, 1)$ such that the stereographically projected Gauss map of M' is $g(z) = z^k e^{H(z)}$ on $D(\infty, 1)$. Furthermore, the height differential $dh = dx_3 + i dx_3^*$ of M' extends meromorphically to $D(\infty, 1) \cup \{\infty\}$ with a double pole at ∞.*

Proof. We first construct the cylinder $C(r, h)$. Choose r and h large enough so that $\partial M \subset \mathrm{Int}(C(r, h))$ and so that the side $\{(x_1, x_2, x_3) \mid x_1^2 + x_2^2 = r^2\} \cap C(r, h)$ intersects the multigraphs G_1', G_2', transversely in two compact spiraling arcs $\alpha^1(\theta)$, $\alpha^2(\theta)$ with $\frac{d\alpha^i}{d\theta} > 0$, $i = 1, 2$ (we may assume this sign after

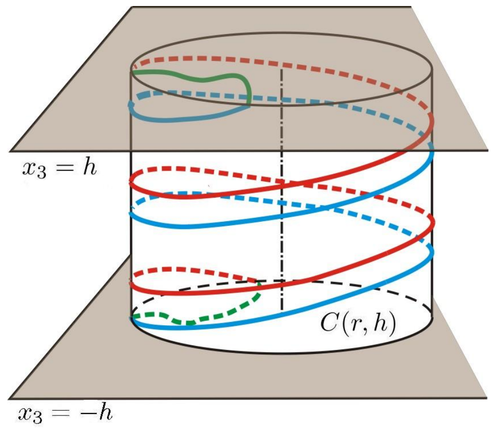


Figure 2: x_3 is monotonic along the two spiraling curves in $M' \cap [\partial C(r,h) \cap \{0 < x_3 < h\}]$ (red, blue), and constant along the top and bottom arcs in $M' \cap [\partial C(r,h) - \{0 < x_3 < h\}]$ (green).

replacing M by $-M$). Note that by increasing the radius r and height h, the sets $M \cap C(r,h)$ define a compact exhaustion of M (since M is proper). These r, h are taken so that $r \gg h$ in order to insure that the side $\{(x_1, x_2, x_3) \mid x_1^2 + x_2^2 = r^2\} \cap C(r,h)$ of $C(r,h)$ always intersects the multigraphs G_1', and G_2' in the above spiraling-type arcs. Take r, h large enough so that the component of $M \cap C(r,h)$ which contains ∂M, also contains the spiraling-type arcs in its boundary. In particular, the top and bottom disks of $C(r,h)$ each intersects M transversely in two compact arcs $\beta^T \subset P_h$, $\beta^B \subset P_{-h}$ by the convex hull property, see Figure 2. Hence, $M' = M - \mathrm{Int}(C(r,h))$ is a annular subend of M. Furthermore, each of the planes P_{-h}, P_h intersects M' in a connected proper curve which contains β^B, β^T, respectively; this is because each of these proper curves intersects each of the cylinders $C(r', h)$, $r' \geq r$, at exactly two points. By our choices of G_1', and G_2', each plane P_t, $t \in (-h, h)$, intersects M' in a pair of proper arcs each with one end point in $\partial M'$. Since each of the planes P_t, $|t| > h$, intersects each of the multigraphs G_1', and G_2' transversely in a single proper arc with end point in the boundary of the respective multigraph, elementary Morse theory and the convex hull property imply that these planes each intersects M' transversely in a single proper curve. Since M' intersects some horizontal plane in a connected proper curve, then the proposition follows directly from Corollary 1.2 in [15]. □

4 Uniqueness of the helicoid.

In this section we will apply Proposition 3.2 to simplify the original proof by Meeks and Rosenberg of the uniqueness of the helicoid among properly embedded simply-connected, nonflat minimal surfaces in $\mathbb{R}^3$. The proof we give below only uses the one-sided curvature estimates by Colding and Minicozzi

(Theorem 0.2 in [9]), Proposition 3.2 in this paper, and some arguments taken from the end of [19]. Since the uniqueness of the helicoid given in [19] is not used in the proof by Colding and Minicozzi that completeness implies properness for embedded, finite topology minimal surfaces, then we can replace the hypothesis of properness by the weaker one of completeness in the above statement of uniqueness of the helicoid.

Theorem 4.1 *If $M \subset \mathbb{R}^3$ is a complete, embedded, simply-connected minimal surface, then M is either a plane or a helicoid.*

Proof. Assume M is not flat. By Proposition 3.2, the conformal type of M is $\mathbb{C}$ and its height differential $dh = dx_3 + idx_3^*$ extends meromophically to ∞ with a double pole. Therefore $dh = \lambda\, dz$ for some $\lambda \in \mathbb{C}^*$, which means that $h = x_3 + ix_3^*$ gives a global conformal parametrization of M. In the sequel, we will use this parametrization from $\mathbb{C}$ into M, so that $h(z) = z$. Since $dh = dz$ has no zeros, we conclude that the Gauss map g misses $0, \infty$ on the whole surface M. Since M is simply-connected, then g lifts through the natural exponential map $e^w\colon \mathbb{C} \to \mathbb{C}^*$ and thus, $g(z) = e^{H(z)}$ for some entire function H.

The next argument shows that H is a linear function of z, which clearly implies that M is a helicoid (recall that the only associate surface[6] to the helicoid which is embedded as a mapping is the helicoid itself). Suppose H is not a linear function and we will obtain a contradiction. Applying Sard's theorem to the third component N_3 of the spherical Gauss map of M, we deduce that there exists a latitude $\gamma \subset \mathbb{S}^2$ arbitrarily close to the horizontal equator, such that $g^{-1}(\gamma)$ does not contain any branch point of N_3. Therefore, $g^{-1}(\gamma)$ consists of a proper collection Γ of smooth curves. Note that there are no compact curves in Γ since M is simply-connected and g misses $0, \infty$. Also note that $H|_\Gamma$ takes values along a certain line $l \subset \mathbb{C}$ parallel to the imaginary axis, and the restriction of H to every component of Γ parametrizes monotonically an arc in the line l. Hence, Picard's theorem implies that either H is a polynomial of degree $m \geq 2$ and Γ has $m \geq 2$ components or Γ has an infinite number of components (equivalently, H is entire with an essential singularity at ∞).

The remainder of the proof breaks up into Cases A, B below. Each of these cases uses results of Colding and Minicozzi from [9] on the geometry of M. By the one-sided curvature estimates of Colding and Minicozzi [9], there exists a solid vertical hyperboloid of revolution $\mathcal{H} \subset \mathbb{R}^3$ such that $M - \mathcal{H}$ consists of two infinitely sheeted multigraphs whose gradient is less than one. In particular, the curves in Γ, when considered to be in the surface $M \subset \mathbb{R}^3$, are contained in $\mathcal{H}$. In Case B below where H is a polynomial of degree $m \geq 2$ and Γ has $m \geq 2$ components, we may assume that at least one of the two ends of $\mathcal{H}$, say the top end, contains at least two of the ends of curves in Γ. In Case A where Γ is an infinite proper collection of curves inside $\mathcal{H}$, then at least one of the two ends of $\mathcal{H}$, say the top end, must contain an infinite number of ends of curves in Γ. Thus in Case A for every n, there exists a large $T = T(n) > 0$ such that every plane $P_t = \{x_3 = t\}$ with $t > T(n)$ intersects at least n of the curves in Γ.

[6] An associate surface to M with Weierstrass data (g, dh) is any of the (in general, multivalued) immersions with Weierstrass data $(g, e^{i\theta}dh)$, where $\theta \in [0, 2\pi)$.

Case A: Γ has an infinite number of components (equivalently, H is entire with an essential singularity at ∞).
Using the rescaling argument in the proof of Proposition 5.2 (page 754) of [19], we can produce a blow-up limit $\widetilde{M} = \lim_k \lambda_k(M - p_k)$ of M which is a properly embedded, simply-connected minimal surface with infinite total curvature and bounded Gaussian curvature ($\widetilde{M}$ is obtained in such a way that the intersection of $\widetilde{M}$ with $\{x_3 = 0\}$ corresponds to the limit of intersections of M with a sequence of planes P_{t_n} with $t_n \to \infty$). The new surface $\widetilde{M}$ has the same appearance as that of M with two infinite "horizontal" multigraphs, in particular if $\widetilde{g}$ is the Gauss map of $\widetilde{M}$, then $\widetilde{g}^{-1}(\gamma)$ consists of a collection of curves which is contained in the related solid hyperboloid $\widetilde{\mathcal{H}}$ for the new surface. In particular, there are only a finite number of components of $\widetilde{g}^{-1}(\gamma)$ that intersect $\{x_3 = 0\}$. By the extension property in [6], the related horizontal multigraphs in the domains $p_k + \frac{1}{\lambda_k}\left[\widetilde{M} \cap (\mathbb{R}^3 - \widetilde{\mathcal{H}})\right]$ near p_k extend with small gradient all the way to infinity in $\mathbb{R}^3$. Therefore, each of the infinitely many components of Γ must intersect the plane $\{x_3 = x_3(p_k)\}$ in the compact disk $p_k + \frac{1}{\lambda_k}\left[\{x_3 = 0\} \cap \widetilde{\mathcal{H}}\right]$ for k sufficiently large. In particular, we find that $\widetilde{g}^{-1}(\gamma) \cap \{x_3 = 0\}$ must be an infinite set which is a contradiction. This contradiction implies that Case A does not occur.

Case B: H is a polynomial of degree $m \geq 2$ and Γ has m components.
Again using the rescaling argument in the proof of Proposition 5.2 of [19], we produce a simply-connected minimal surface $\widetilde{M} = \lim_k \lambda_k(M - p_k)$ with infinite total curvature and bounded Gaussian curvature, with the property that the intersection of $\widetilde{M}$ with $\{x_3 = 0\}$ corresponds to the limit of intersections of M with a sequence of planes P_{t_n} with $t_n \to \infty$. As in Case A above, $\widetilde{M}$ has the same appearance as that of M with two infinite "horizontal" multigraphs, and if $\widetilde{g}$ denotes the Gauss map of $\widetilde{M}$, then $\widetilde{g}^{-1}(\gamma)$ consists of a finite number of curves, each of which is contained in the related solid hyperboloid $\widetilde{\mathcal{H}}$ for $\widetilde{M}$. The same arguments as in the beginning of this proof of Theorem 4.1 demonstrate that the conformal type of $\widetilde{M}$ is $\mathbb{C}$ and $\widetilde{M}$ can be conformally parametrized in $\mathbb{C}$ with height differential dz and Gauss map $\widetilde{g}(z) = e^{\widetilde{H}(z)}$, for some entire function $\widetilde{H}$.

We now show how the fact that $\widetilde{M}$ has bounded Gaussian curvature gives that $\widetilde{H}$ is linear (and so, $\widetilde{M}$ is a helicoid): Since the arguments in Case A above can be applied to $\widetilde{M}$, we conclude that $\widetilde{H}$ cannot have an essential singularity at $z = \infty$ and thus, $\widetilde{H}$ is a polynomial. Suppose that $\widetilde{H}$ has degree strictly greater than one, and we will get a contradiction. As $\widetilde{g}(z)$ is an entire function with an essential singularity at ∞ and $\widetilde{g}$ misses $0, \infty$ (by the open mapping property since g also omits $0, \infty$), then $\widetilde{g}$ takes any value in $\mathbb{C} - \{0\}$ infinitely often in any punctured neighborhood of $z = \infty$. Thus, there exists a sequence $z_k \in \mathbb{C}$ diverging to ∞ as $k \to \infty$, such that $\widetilde{g}(z_k) = 1$ for all k. Recall that the Gaussian curvature $\widetilde{K}$ of $\widetilde{M}$ in terms of its Weierstrass data $\left(\widetilde{g}(z) = e^{\widetilde{H}(z)}, dz\right)$ is given by

$$\widetilde{K} = -16|\widetilde{H}'|^2 \frac{|\widetilde{g}|^4}{(1 + |\widetilde{g}|^2)^4};$$

hence $K(z_k) = -|\widetilde{H}'(z_k)|^2$ tends to $-\infty$ as $k \to \infty$, which is a contradiction. This contradiction proves that $\widetilde{H}$ is linear and so, $\widetilde{M}$ is a vertical helicoid.

Finally, the extension property in [6] implies that the related horizontal multigraphs in the domains $p_k + \frac{1}{\lambda_k}\left[\widetilde{M} \cap (\mathbb{R}^3 - \widetilde{\mathcal{H}})\right]$ near p_k extend all the way to infinity. In particular, $\widetilde{g}^{-1}(\gamma)$ also contains at least $m \geq 2$ components but a helicoid contains only one such component. This is a contradiction which completes the proof of the theorem. $\square$

The next result is an immediate consequence of Theorem 4.1 and the discussion before the statement of Proposition 3.2; also see the related rescaling argument in the proof of Proposition 5.2 of [19].

Corollary 4.2 *Let $M \subset \mathbb{R}^3$ be a complete, embedded minimal annulus and $X\colon D(R,\infty) \to \mathbb{R}^3$ be a conformal parametrization of M with Weierstrass data $\left(g(z) = z^k e^{H(z)}, dh = dx_3 + i dx_3^*\right)$, where $H(z)$ is holomorphic and dh has no zeroes in $D(R,\infty)$ and extends holomorphically across ∞ with a double pole at ∞. Then there exist sequences of points $\{p_n^+\}_{n\in\mathbb{N}}$, $\{p_n^-\}_{n\in\mathbb{N}}$ in $M \cap \mathcal{H}$ and positive numbers $\{\lambda_n^+, \lambda_n^-\}_{n\in\mathbb{N}}$ such that:*

1. $\lim_{n\to\infty} x_3(p_n^+) = \infty$ *and* $\lim_{n\to\infty} x_3(p_n^-) = -\infty$.
2. *The surfaces $\lambda_n(M - p_n^+)$ converge smoothly on compact subsets of $\mathbb{R}^3$ as $n \to \infty$ to a vertical helicoid $\mathbf{H} \subset \mathbb{R}^3$ which contains the x_3 and the x_2-axes.*
3. *For all $n \in \mathbb{N}$, $g(p_n^+) = g(p_n^-) = 1 \in \mathbb{C}$.*

5 The Weierstrass representation of embedded minimal surfaces of finite topology.

In this section we describe an improvement on the analytic description of a complete injective minimal immersion of $\mathbb{S}^1 \times [0,\infty)$ into $\mathbb{R}^3$ which was given in Proposition 3.2. This improvement is given in the next theorem, which implies that M has finite type. Theorem 4 in [12] states that a complete, embedded minimal annular end of finite type with Weierstrass data as given in the next theorem and with zero flux, is asymptotic to the end of a vertical helicoid. We will prove this last fact again, independently, later with the stronger result described in Theorem 7.1. Note that if M has zero flux, then the number λ in item 2 below is zero and so, the next theorem will complete the proof of Theorem 1.1 except for item 3(c); this item will follow from item 7 of Theorem 7.1.

Theorem 5.1 *Suppose $X\colon M \to \mathbb{R}^3$ is a complete minimal embedding of an annulus M diffeomorphic to $\mathbb{S}^1 \times [0,\infty)$. After a homothety and rotation, M contains a subend M' with the following Weierstrass data:*

1. *$g(z) = e^{iz+f(z)}$ where $f\colon D(\infty,R) \cup \{\infty\} \to \mathbb{C}$ is holomorphic and $f(\infty) = 0$, where R is some positive number.*

2. $dh = (1 + \frac{\lambda}{z-\mu})\, dz$ *where* $\lambda \in \mathbb{R}$ *and* $\mu \in \mathbb{C}$.

Proof. We will identify M with its image $X(M)$. After a rotation of M and a replacement by a subend, we may assume by Proposition 3.2 that M is the image of a conformal harmonic embedding $X\colon D(\infty, 1) \to \mathbb{R}^3$, the meromorphic Gauss map of X is equal to $g(z) = z^k e^{H(z)}$ where $k \in \mathbb{Z}$ and $H\colon D(\infty, 1) \to \mathbb{C}$ is some holomorphic function, and the holomorphic height differential of X is the one-form $dh = dx_3 + i dx_3^*$ which has no zeroes and which extends to $D(\infty, 1) \cup \{\infty\}$ with a double pole at infinity.

Assertion 5.2 *In the situation above,* $k = 0$*, i.e.* $g(z) = e^{H(z)}$.

Proof. Since $g(z) = z^k e^{H(z)}$, then k is the winding number of $g|_{\partial D(\infty,1)}$ in $\mathbb{C} - \{0\}$. We will show that the winding number of $g|_{\partial D(\infty,1)}$ is zero by proving that there exists a simple closed curve $\Gamma \subset D(\infty, 1)$ which is the boundary of an end representative of $D(\infty, 1)$ and such that $g|_\Gamma$ has winding number 0; since such a simple closed curve Γ is homotopic to $\partial D(\infty, 1)$, it will follow that the winding number of $g|_{\partial D(\infty,1)}$ also vanishes.

By Proposition 3.2, there exist $r', h' > 0$ such that $M' = M - \text{Int}(C(r', h'))$ is a subend of M, each horizontal plane $P_t = \{x_3 = t\}$ intersects M' transversely in either a proper curve (if $|t| \geq h'$) or in two proper arcs, each with its single end point on the boundary of M' (if $|t| < h'$). By Corollary 4.2 there exists some point $p^- \in M \cap \mathcal{H}$ with $x_3(p^-)$ large and negative such that $g(p^-) = 1$ and after homothety and translation, M has the appearence of a vertical helicoid around p^-. By the extension property in [6], the two resulting multigraphs in M around p^- extend indefinitely sideways as very flat multigraphs. Consider the intersection of M with the tangent plane $T_{p^-}M$. This intersection is an analytic 1-complex which contains an almost horizontal proper smooth curve α_B passing through p^-, which we may assume is globally parameterized by its x_2-coordinate. In this case, α_B near p^- plays the role of the x_2-axis contained in the helicoid $\mathbf{H}$ appearing in Corollary 4.2. It follows that close to p^-, the argument of $g(z)$ on α_B can be chosen to be close to zero, see Figure 3. Since the two horizontal multigraphs around p^- extend horizontally all to way to infinity in $\mathbb{R}^3$, our earlier discussion before the statement of Proposition 3.2 implies that the related multigraphing functions u_1^-, u_2^- with polar coordinates centered at p^-, satisfy $\frac{\partial u_i^-}{\partial \theta} > 0$ for $i = 1, 2$ when restricted to $\alpha_B - \{p^-\}$, even at points far from the forming helicoid at p^-. Thus, if we consider the argument of the complex number $g(p^-)$ to lie in $(-\frac{\pi}{2}, \frac{\pi}{2})$, we conclude that the argument of $g(\alpha_B(t))$ also lies in $(-\frac{\pi}{2}, \frac{\pi}{2})$ for all $t \in \mathbb{R}$. A similar discussion shows that there is a point $p^+ \in M$ with $x_3(p^+)$ being a large positive number and a related almost horizontal curve $\alpha_T \subset T_{p^+}M \cap M$ passing through p^+ and such that the argument of $g(z)$ along α_T also takes values in the interval $(2\pi j - \frac{\pi}{2}, 2\pi j + \frac{\pi}{2})$ for all $t \in \mathbb{R}$ when we consider the argument of $g(p^+) = 1$ to lie in the interval $(2\pi j - \frac{\pi}{2}, 2\pi j + \frac{\pi}{2})$ for some $j \in \mathbb{N}$.

Recall from the discussion just before Proposition 3.2 that the hyperboloid $\mathcal{H}'$ was chosen so that in the complement of $\mathcal{H}'$, M consists of two multigraphs G_1', G_2', respectively given by functions $u^1(r, \theta)$, $u^2(r, \theta)$, such that $\frac{\partial u^i}{\partial \theta}(r, \theta) > 0$, $i = 1, 2$. Now can choose r' sufficiently large so that:

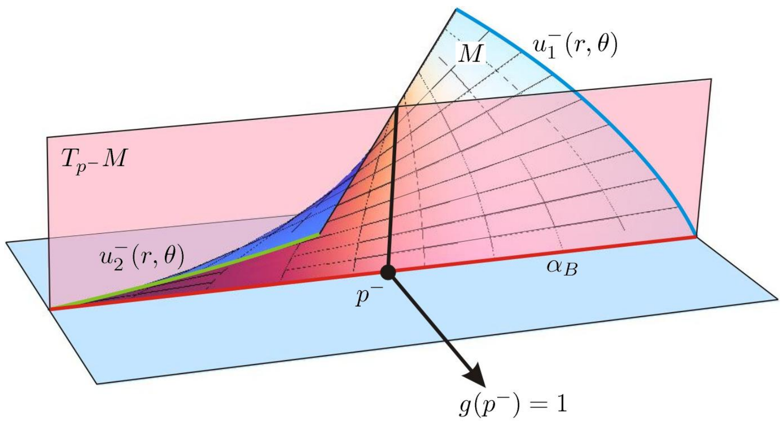


Figure 3: Around p^-, M has the appearance of a translated and homothetically shrunk vertical helicoid, made of two multigraphs $u_i^-(r,\theta)$, $i = 1, 2$.

- The portion of $\alpha_T \cup \alpha_B$ outside of the solid vertical cylinder $C(r') = \{x_1^2 + x_2^2 \leq (r')^2\}$ consists of four connected arcs in $G_1' \cup G_2'$, and the portion of $\alpha_T \cup \alpha_B$ inside $C(r')$ consists of two compact arcs $\alpha_T^{r'}$, $\alpha_B^{r'}$.
- $\partial C(r') \cap (G_1' \cup G_2')$ contains two connected spiraling arcs β_1, β_2, each of which connects one end point of $\alpha_T^{r'}$ with one end point of $\alpha_B^{r'}$.

We define the simple closed curve $\Gamma = \alpha_B^{r'} \cup \beta_1 \cup \alpha_T^{r'} \cup \beta_2 \subset M$. Note that Γ bounds a subend of M, and consider Γ to be contained in the parametrization domain $D(\infty, 1)$; here we are assuming that Γ can be parameterized by a mapping $\gamma\colon [0,4] \to \Gamma$ where $\alpha_B^{r'}$ has end points $\gamma(0), \gamma(1)$, $\gamma(1), \gamma(2)$ are the end points of β_1, $\gamma(2), \gamma(3)$ are the end points of $\alpha_T^{r'}$ and finally β_2 has end points $\gamma(3), \gamma(4) = \gamma(0)$, see Figure 4 left. Notice that the condition $\frac{\partial u^i}{\partial \theta}(r, \theta) > 0$ on points of Γ on $\beta_1 \cup \beta_2$ implies that the argument of $g(z)$ along β_1, which we can choose to have an initial value at $\gamma(1) = \beta_1 \cap \alpha_B^{r'}$ in the interval $(-\frac{\pi}{2}, \frac{\pi}{2})$, has its value at $\gamma(2)$ in $(2\pi n - \frac{\pi}{2}, 2\pi n + \frac{\pi}{2})$, where n is the number of θ-revolutions that β_1 makes around the x_3-axis (when we assume that the Gauss map of M is upward pointing along β_1). Similarly the argument of $g(z)$ along β_2, which has an initial value in $(2\pi n - \frac{\pi}{2}, 2\pi n + \frac{\pi}{2})$, has its ending value in $(-\frac{\pi}{2}, \frac{\pi}{2})$; we are using here the earlier observation that if the argument of $g(z)$ at one end point of $\alpha_T^{r'}$ or $\alpha_B^{r'}$ lies in $(2\pi m - \frac{\pi}{2}, 2\pi m + \frac{\pi}{2})$ for some $m \in \mathbb{N}$, then the other end point of the same α-curve also lies in $(2\pi m - \frac{\pi}{2}, 2\pi m + \frac{\pi}{2})$. It follows now that the winding number of $g|_\Gamma$ is zero, which completes the proof of the assertion. $\square$

We continue with the proof of Theorem 5.1. So far we have deduced that M can be parameterized by $D(\infty, 1)$ with Weierstrass data $g(z) = e^{H(z)}$, $H(z)$ holomorphic on $D(\infty, 1)$ and $dh = dx_3 + i dx_3^*$, where dh has no zeroes and extends to $D(\infty, 1) \cup \{\infty\}$ with a double pole at infinity. We claim that

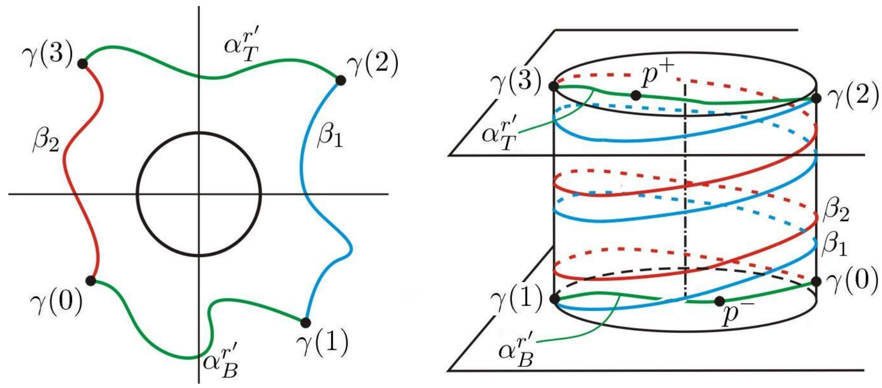


Figure 4: The closed curve $\Gamma = \alpha_B^{r'} \cup \beta_1 \cup \alpha_T^{r'} \cup \beta_2$ is homotopic to the boundary of $D(\infty, 1)$. Note that $\alpha_T^{r'}, \alpha_T^{r'}$ are contained in vertical planes parallel to the (x_2, x_3)-plane.

$H(z) = P(z) + f(z)$, where $P(z)$ is a polynomial of degree one and $f(z)$ extends to a holomorphic function on $D(\infty, 1) \cup \{\infty\}$ with $f(\infty) = 0$. The first case we consider, Case A, is when $H(z)$ has an essential singularity at ∞. This case cannot occur by the same arguments as those given in the proof of Theorem 4.1 in the similar Case A. The second case we consider, Case B, is when $H(z) = P(z) + f(z)$ where $P(z)$ is a polynomial of degree $m \geq 2$ and $f(\infty) = 0$. In this case the arguments of the related Case B in the proof of Theorem 4.1 show that this case cannot occur either. This proves the claim that $H(z) = cz + d + f(z)$ where $c, d \in \mathbb{C}$ and $f(\infty) = 0$. Since M is assumed to have infinite total curvature, its Gauss map takes on almost all values of $\mathbb{S}^2$ infinitely often, which means that $c \neq 0$.

Finally we will obtain the Weierstrass data that appears in the statement of Theorem 5.1. Since dh has a double pole at infinity, then after a change of variables of the type $z \mapsto zT(z)$ for a suitable holomorphic function $T(z)$ in a neighborhood of ∞ with $T(\infty) \in \mathbb{C} - \{0\}$, we can write $dh = (A_0 + \frac{\lambda}{w})\, dw$, where $A_0 \in \mathbb{C} - \{0\}$ and $-2\pi\lambda \in \mathbb{R}$ is the vertical component of the flux of M along its boundary. After a homothety of M in $\mathbb{R}^3$ and a suitable rotation $w \mapsto e^{i\theta} w$ in the parameter domain, we can prescribe A_0 in $\mathbb{C} - \{0\}$. Since a rotation of M in $\mathbb{R}^3$ around the x_3-axis does not change dh but multiplies g by a unitary complex number, then we can assume that the Weierstrass data of M are $g(w) = e^{cw+d+f(w)}$ and $dh = (-ic + \frac{\lambda}{w})\, dw$. Finally, the linear change of variables $cw + d = i\xi$ produces $g(\xi) = e^{i\xi + f_1(\xi)}$, $dh = (1 + \frac{\lambda}{\xi + id})\, d\xi$ where $f_1(\xi) = f(w)$. After choosing $R > 0$ sufficiently large, we have the desired Weierstrass data on $D(\infty, R)$. This concludes the proof of the theorem. $\square$

6 The analytic construction of the examples $E^{a,b}$.

In this section we will construct examples $E^{a,b}$ of complete *immersed* minimal annuli which depend on parameters $a, b \in [0, \infty)$ so that their flux vector is $(a, 0, -b)$. These examples will be used to construct the related *embedded* canonical examples $E_{a,b}$ referred to in Theorem 1.3. Our construction of the examples discussed in this section will be based on the classical Weierstrass representation. We first review how we can deduce the periods and fluxes of a potential minimal annulus from its Weierstrass representation.

Consider a meromorphic function g and a holomorphic one-form dh on the punctured disk $D(\infty, R)$ centered at ∞, for some $R > 0$. Suppose that $(|g| + |g|^{-1})|dh|$ has no zeros in $D(\infty, R)$. The period and flux along $\partial_R = \{|z| = R\}$ of the (possibly multivalued) unbranched minimal immersion $X\colon D(\infty, R) \to \mathbb{R}^3$ associated to the Weierstrass data (g, dh) are given by

$$\text{Per} + i\,\text{Flux} = \frac{1}{2}\left(\int_{\partial_R}\frac{dh}{g} - \int_{\partial_R} g\,dh, i\int_{\partial_R}\frac{dh}{g} + i\int_{\partial_R} g\,dh, 0\right) + \left(0, 0, \int_{\partial_R} dh\right) \in \mathbb{C}^3.$$

If we assume that the period of X vanishes, i.e. (3) holds, then the above formula gives

$$\text{Flux} = \left(-\Im\int_{\partial_R} g\,dh, \Re\int_{\partial_R} g\,dh, \Im\int_{\partial_R} dh\right) \equiv \left(i\int_{\partial_R} g\,dh, \Im\int_{\partial_R} dh\right) \in \mathbb{C}\times\mathbb{R}, \tag{4}$$

where $\Im$ denotes imaginary part and we have identified $\mathbb{R}^3$ with $\mathbb{C}\times\mathbb{R}$ by $(a_1, a_2, a_3) \equiv (a_1 + ia_2, a_3)$.

It is clear that the choice $g(z) = e^{iz}$, $dh = dz$ produces the end of a vertical helicoid. In this section, we will consider explicit expressions for (g, dh) such that

1. $g(z) = e^{iz+f(z)}$ where $f\colon D(\infty, R) \cup \{\infty\} \to \mathbb{C}$ is holomorphic and $f(\infty) = 0$.
2. $dh = (1 + \frac{\lambda}{z-\mu})\,dz$ where $\lambda \in \mathbb{R}$ and $\mu \in \mathbb{C}$,

in terms of certain parameters defining $f(z)$ and μ and we will prove that these parameters can be adjusted so that the period problem for (g, dh) is solved, thereby defining a complete immersed minimal annulus $E^{a,b}$; furthermore, the flux vector of the resulting minimal immersion cover is $F = (a, 0, -b)$ and this vector can be chosen for every $a \geq 0$, $b \in \mathbb{R}$. In the particular case $a = b = 0$, we choose $f = \lambda = 0$ hence $E^{0,0}$ is the end of a vertical helicoid. The argument for the construction of the remaining annuli $E^{a,b}$ breaks up into two cases, depending on whether the flux vector F is vertical or not.

Case 1: The flux is not vertical.

Consider the following particular choice of g, dh:

$$g(z) = t\,e^{iz}\frac{z - A}{z}, \qquad dh = \left(1 + \frac{B}{z}\right)dz, \qquad z \in D(\infty, R), \tag{5}$$

where $t > 0$, $A \in \mathbb{C} - \{0\}$, $B \in \mathbb{R}$ and $R > |A|$ are to be determined. Note that g can rewritten as $g(z) = t\,e^{iz+f(z)}$ where $f(z) = \log\frac{z-A}{z}$, which is univalued and holomorphic in $D(\infty, R)$ since $R > |A|$. Furthermore, $f(\infty) = 0$. Also note that $\int_{\partial_R} dh = -2\pi i\,B$ is purely imaginary since $B \in \mathbb{R}$ (here ∂_R is oriented as boundary of $D(\infty, R)$). Therefore, the period problem for (g, dh) given by equation (3) reduces to the horizontal component. To study this horizontal period problem, we first compute $\int_{\partial_R} g\,dh$ as the residue of a meromorphic one-form in $\{|z| \leq R\}$, obtaining

$$\int_{\partial_R} g\,dh = 2\pi i t\,[A - B + iAB]\,. \tag{6}$$

Arguing analogously with $\int_{\partial_R} \frac{dh}{g}$ we have

$$\int_{\partial_R} \frac{dh}{g} = -2\pi i \frac{A+B}{te^{iA}}. \tag{7}$$

Hence the period condition (3) reduces to the equation

$$t\,[A(1 + i\,B) - B] = \frac{\overline{A} + B}{t} e^{i\,\overline{A}}. \tag{8}$$

Next we will show that the period problem (8) can be solved in the parameters t, A, B with arbitrary values of the flux vector different from vertical (the case of vertical nonzero flux will be treated in Case 2 below, with a different choice of g, dh).

Let $A = x + iy$ with $x, y \in \mathbb{R}$, and rewrite equation (8) in the following two ways:

$$t^2 e^{-y}[A(1 + iB) - B] = (\overline{A} + B)e^{ix} \tag{9}$$

$$t^2 e^{-y}[(x - By - B) + i(Bx + y)] = [(x + B) - iy]e^{ix}. \tag{10}$$

We fix $B \geq 0$ (resp. $B < 0$) and for $y \in \mathbb{R}$, define the following related $\mathbb{C}$-valued curves:

$$L(x) = (x - By - B) + i(Bx + y), \qquad R(x) = [(x + B) - iy]e^{ix}, \tag{11}$$

that contain the information on the arguments of the complex numbers of the left and right hand sides of equation (10), provided they are nonzero. Note that these curves depend on the parameter y.

Also note that

$$|L(x)|^2 = (1 + B^2)(x^2 + y^2) + B^2 - 2Bx \geq (1 + B^2)x^2 + B^2 - 2Bx. \tag{12}$$

Since our goal is to prove that $t > 0$, $A \in \mathbb{C} - \{0\}$ can be chosen so that the period problem (8) is solved for a given $B \geq 0$ (resp. $B < 0$); in the sequel we will assume $x = \Re(A)$ is larger (resp. more negative) than some particular values. Assume that $x > 0$ (resp. $x < 0$), in particular, (12) gives that $L(x) \neq 0$. We will write $L(x)$ in polar coordinates as $L(x) = r_L(x)e^{i\theta_L(x)}$, where $\theta_L(x)$ is the argument of $L(x)$ with values taken in the interval $(-\pi, \pi]$ (resp. $(0, 2\pi]$). Note that if $x \geq 1$ (resp. $x \leq -1$), then $L(x)$ cannot lie in the closed third (resp. fourth) quadrant, i.e. $\theta_L(x) \in (-\frac{\pi}{2}, \pi)$ (resp. $\theta_L(x) \in (0, \frac{3\pi}{2})$): if $B = 0$ this is trivial; if $B > 0$ and $x - By - B \leq 0$, then $Bx + y \geq Bx + \frac{1}{B}(x - B) \geq B + \frac{1}{B} - 1 > 0$

(if $B < 0$ and $x - By - B \geq 0$ one can argue in a similar way). Viewing $L(x)$ as a curve in $\mathbb{R}^2$ with parameter $x \geq 1$ if $B \geq 0$ (resp. with $x \leq -1$ if $B < 0$), we have:

$$\left|\frac{\partial \theta_L}{\partial x}(x)\right| \leq \frac{|L'(x)|}{|L(x)|} = \frac{|1+iB|}{|L(x)|} \overset{(12)}{\leq} \frac{\sqrt{1+B^2}}{\sqrt{(1+B^2)x^2 + B^2 - 2Bx}}. \tag{13}$$

On the other hand, note that $R(x) = 0$ if and only if $x = -B$ and $y = 0$. Since we are assuming $B \geq 0$ and $x \geq 1$ (resp. $B < 0$ and $x \leq -1$), then $R(x)$ cannot vanish and thus, we can we write $R(x) = r_R(x)e^{\theta_R(x)}$ in polar coordinates, where $\theta_R(x) = x + \theta_1(x)$ and $\theta_1(x) = \arg((x+B) - iy)$, which lies in the interval $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ if $B \geq 0$ (resp. in the interval $\left(\frac{\pi}{2}, \frac{3\pi}{2}\right)$ if $B < 0$). Therefore,

$$\frac{\partial \theta_R}{\partial x}(x) = 1 + \frac{\partial \theta_1}{\partial x}(x) = 1 + \frac{y}{(B+x)^2 + y^2} \geq 1 + \min_{y \in \mathbb{R}} \frac{y}{(B+x)^2 + y^2} = 1 - \frac{1}{2|B+x|}. \tag{14}$$

Comparing the expression $m = m(B, x) = 1 - \frac{1}{2|B+x|}$ in the right hand side of (14) with the one at the right hand side of (13), we deduce directly the following property.

Assertion 6.1 *Given $B \geq 0$ (resp. $B < 0$), there exists $x_0 = x_0(B) \geq 1$ (resp. $x_0 \leq -1$) such that if $x > x_0$ (resp. if $x < x_0$), then $\frac{\partial \theta_L}{\partial x}(x) < \frac{\partial \theta_R}{\partial x}(x)$.*

We are now ready to find $A \in \mathbb{C}$ (given $B \in \mathbb{R}$) such that the complex numbers $L(x), R(x)$ in equation (8) have the same argument. To do this, we will restrict ourselves to the case $B \geq 0$; the case $B < 0$ can be argued similarly.

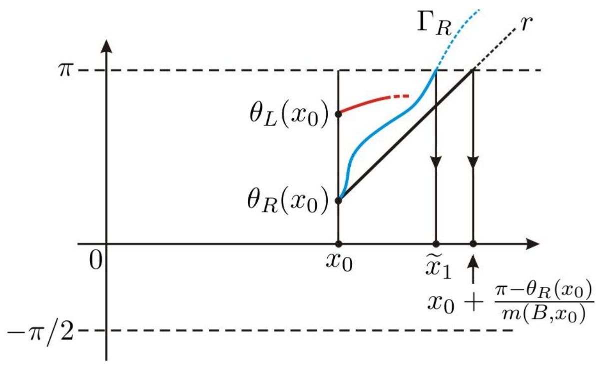
Assertion 6.2 *Given $B \geq 0$, we have one of the following two possibilities.*

(A) *If $\theta_R(x_0) \leq \theta_L(x_0)$ (here $x_0 = x_0(B)$ refers to the value obtained in Assertion 6.1), then there exists a unique point $x_1 = x_1(B, y) \in I = \left[x_0, x_0 + \frac{\pi - \theta_R(x_0)}{m(B, x_0)}\right)$ such that $\theta_R(x_1) = \theta_L(x_1)$.*

(B) *If $\theta_R(x_0) \geq \theta_L(x_0)$, then there exists a unique point $x_1 = x_1(B, y) \in I = \left(x_0 - \frac{\theta_R(x_0) + \frac{\pi}{2}}{m(B, x_0)}, x_0\right]$ such that $\theta_R(x_1) = \theta_L(x_1)$.*

Furthermore, the function $(B, y) \mapsto x_1(B, y)$ is continuous.

Proof. Assume we are in case **(A)**, i.e. $\theta_R(x_0) \leq \theta_L(x_0)$. First note that by (14), the parametrized curve Γ_R given by $x \in [x_0, +\infty) \mapsto (x, \theta_R(x))$ lies entirely above the half-line r that starts at $(x_0, \theta_R(x_0))$ with slope $m(B, x_0)$ (this slope can be assumed arbitrarily close to 1 by taking x_0 large enough; note also that θ_R restricted to the half-open interval $[x_0, +\infty)$ is injective). By comparison of Γ_R with r, we easily conclude that θ_R must reach the value π at some point $\widetilde{x}_1 > x_0$ which is not greater than $x_0 + \frac{\pi - \theta_R(x_0)}{m(B, x_0)}$, see Figure 5 left. Since the range of $x \in [x_0, \infty) \mapsto \theta_L(x)$ is $(-\frac{\pi}{2}, \pi)$, the intermediate value theorem applied to the function $\theta_R - \theta_L$ implies that there is at least one point $x_1 \in [x_0, \widetilde{x}_1) \subset I$



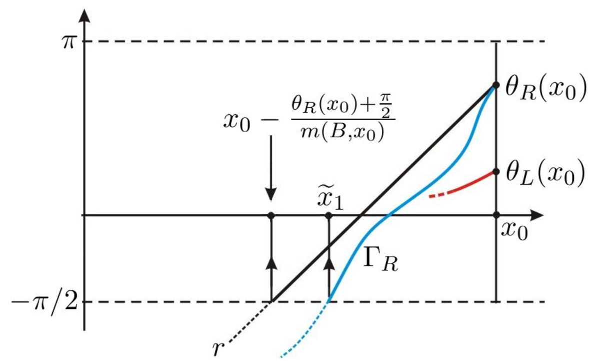


Figure 5: Left: Case **(A)** of Assertion 6.2; the half-line r has slope $m(B, x_0)$ very close to 1. The red and blue curves must intersect. Right: Case **(B)** of Assertion 6.2.

such that $\theta_R(x_1) = \theta_L(x_1)$. Since $(\theta_R - \theta_L)'(x) > 0$ for all $x > x_0$ by Assertion 6.1, then the point $x_1 = x_1(B, y)$ is unique.

Case **(B)** can be proved with similar reasoning as in the last paragraph, using Figure 5 right instead of Figure 5 left (note that we need to apply Assertion 6.1 for values of x less than x_0, which can be done by taking x_0 large enough). Finally, the continuous dependence of x_1 with respect to B, y follows from the same type of dependence for the data $\theta_L, \theta_R, x_0, m(B, x_0)$. This finishes the proof of the assertion. $\square$

Fix $B \in \mathbb{R}$. Given $y \in \mathbb{R}$, consider the value $x_1 = x_1(B, y)$ appearing in Assertion 6.2 (provided that $B \geq 0$; this assertion has a counterpart in the case $B < 0$) and let $A = x_1 + iy$. Since the left and right hand sides of equation (10) (with $x = x_1$ and $t > 0$ to be determined) are not zero and the arguments of the complex numbers on each of the sides are the same, then there exists a unique $t = t(B, y) \in (0, \infty)$ varying continuously in B, y so that equation (10) holds. Therefore, the period problem (8) is solved for the Weierstrass data (5) given by the values $B, A = x_1 + iy$, t. Let $E(B, y)$ denote the related minimally immersed annulus in $\mathbb{R}^3$ defined by these Weierstrass data. Let $F(B, y)$ denote the length of the horizontal flux of $E(B, y)$, which can be identified with the integral

$$F(B, y) = \left| \int_{\partial_R} g dh \right| = 2\pi t \left| (x_1 - By - B) + (Bx_1 + y)i \right|. \tag{15}$$

By equation (10) and the fact that $y \in \mathbb{R} \mapsto x_1(B, y)$ is bounded for B fixed (by Assertion 6.2 if $B \geq 0$ and by its counterpart if $B < 0$), we see that $y \in \mathbb{R} \mapsto t(B, y)$ essentially grows exponentially to $+\infty$ as $y \to +\infty$ and essentially decays exponentially to 0 as $y \to -\infty$. Hence, from (15) we also find:

$$\lim_{y \to +\infty} F(B, y) = +\infty, \qquad \lim_{y \to -\infty} F(B, y) = 0.$$

Since $y \in \mathbb{R} \mapsto F(B, y)$ is continuous, the intermediate value theorem proves the next assertion.

Assertion 6.3 *Given $B \in \mathbb{R}$, the minimally immersed annuli $\{E(B, y) \mid y \in \mathbb{R}\}$ defined above attain all possible positive lengths for the horizontal component of their fluxes (the vertical component of their flux is always equal to $-2\pi B$). Hence, by the axiom of choice, for each $a > 0$ and $b \in \mathbb{R}$, there exists $y = y(a, b) \in \mathbb{R}$ such that $E(\frac{b}{2\pi}, y(a, b))$ has vertical component of its flux equal to $-b$ and the length of the horizontal component of its flux is equal to a. After a rotation of $E(\frac{b}{2\pi}, y(a, b))$ around the x_3-axis, we obtain the desired example $E^{a,b}$ with flux vector $(a, 0, -b)$.*

Case 2: The flux vector is vertical.

We now complete the construction of the canonical examples $E^{0,b}$, as complete immersed minimal annuli with infinite total curvature and flux vector $(0, 0, -b)$ for any $b \in \mathbb{R} - \{0\}$. Fix $B \in \mathbb{R} - \{0\}$ and consider the following choices for g, dh:

$$g(z) = e^{iz} \frac{z - A}{z - \overline{A}}, \qquad dh = \left(1 + \frac{B}{z}\right) dz, \quad z \in D(\infty, R), \tag{16}$$

where $A \in \mathbb{C} - \{0\}$ and $R > |A|$. Our goal is to show that given $B \in \mathbb{R} - \{0\}$, there exists A as before so that the Weierstrass data given by (16) solve the period problem (3) and define a complete immersed minimal annulus with infinite total curvature and flux vector $(0, 0, -2\pi B)$. Similarly as in Case 1, we can write g as $g(z) = e^{iz+f(z)}$ where $f(z) = \log \frac{z-A}{z-\overline{A}}$, which is univalent and holomorphic in $D(\infty, R)$ because $R > |A|$. The period problem (3) for (g, dh) reduces in this vertical flux case to

$$\int_{\partial_R} g\, dh = \int_{\partial_R} \frac{1}{g}\, dh = 0. \tag{17}$$

We first compute $\int_{\partial_R} g\, dh$ in terms of residues of a meromorphic one-form in $\{|z| \leq R\}$ which we impose to be zero:

$$\int_{\partial_R} g\, dh = -2\pi i \left[e^{i\overline{A}} \left(\overline{A} - A + B - \frac{AB}{\overline{A}} \right) + \frac{AB}{\overline{A}} \right] = 0. \tag{18}$$

A direct computation shows that the integral $\int_{\partial_R} \frac{1}{g}\, dh$ is the complex conjugate of the expression involving A, B in equation (18). Hence it suffices to find, given $B \in \mathbb{R} - \{0\}$, a nonzero complex number A solving (18).

Multiplying equation (18) by $\overline{A}$ and dividing by -2π yields:

$$e^{i\overline{A}} \left(\overline{A}^2 - |A|^2 + B(\overline{A} - A) \right) = -AB \tag{19}$$

Letting $A = x + iy$, we obtain

$$2ye^y e^{ix} \left[y + i(x + B) \right] = B \left(x + iy \right). \tag{20}$$

After fixing $y>0$ and letting x vary in the range $\{1<x<\infty\}$ when $B>0$ (resp. fix $y>0$ and let x vary in $\{-\infty<x<-1\}$ if $B<0$), we have that $1<\frac{\partial\theta_L}{\partial x}(x)<2$ and $\frac{\partial\theta_R}{\partial x}(x)<0$, where θ_L and θ_R denote the argument of the left and right hand sides of equation (20). Also note that we can consider branches of $\theta_R(x), \theta_L(x)$ so that

- If $B>0$, then $0<\theta_R(x)<\frac{\pi}{2}$ for the range of values of x and y that we are considering and $\theta_L(2\pi-\frac{\pi}{2})\in(-\frac{\pi}{2},0)$.
- If $B<0$, then $\frac{\pi}{2}<\theta_R(x)<\pi$ for the range of values of x and y that we are considering and $\theta_L(\frac{\pi}{2}-2\pi)\in(0,\frac{\pi}{2})$.

We define the interval

$$J=\begin{cases}(2\pi-\frac{\pi}{2},2\pi+\frac{\pi}{2}) & \text{if } B>0,\\ (\frac{\pi}{2}-2\pi,-\frac{\pi}{2}) & \text{if } B<0.\end{cases}$$

It follows that given $B\in\mathbb{R}-\{0\}$, for each $y>0$ there is a unique $x=x(y)\in J$ such that the arguments of both sides of (20) are equal for $A=x(y)+iy$. Note that the norm of the left hand side of (20), considered now to be a positive function of y, varies continuously with limit value 0 as y tends to 0^+ and $+\infty$ as $y\to\infty$, and this norm grows essentially exponentially to $+\infty$ as $y\to\infty$. On the other hand, the norm of the right hand side of (20) is bounded away from zero and grows essentially linearly in y as y tends to $+\infty$. Thus, for any initial value $B\in\mathbb{R}-\{0\}$ there exists $y\in\mathbb{R}^+$ and the corresponding $x=x(y)\in J$ such that the left and right hand sides of (20) are equal. This completes the proof of Case 2.

We can summarize this case in the next assertion. The proof of the last statement of this assertion follows from the observation that when the flux vector is $(0,0,-b)$, then the Weierstrass data of the corresponding immersed minimal annulus $E^{0,b}$ is given by equation (16) with $B=\frac{b}{2\pi}$. It is easy to check that the conformal map $z\overset{\Phi}{\mapsto}\overline{z}$ in the parameter domain $D(R,\infty)$ of $E^{0,b}$ satisfies $g\circ\Phi=1/\overline{g}$, $\Phi^*dh=\overline{dh}$. Hence, after translating the surface so that the image of the point $R\in D(R,\infty)$ lies on the x_3-axis, we deduce that Φ produces an isometry of $E^{0,b}$ which extends to a 180$^{\text{o}}$-rotation of $\mathbb{R}^3$ around the x_3-axis.

Assertion 6.4 *For any $b\in\mathbb{R}-\{0\}$, there exists a constant $A\in\{x+iy\mid x\in J,\ y\in\mathbb{R}^+\}$ depending on b such that the corresponding Weierstrass data in (16) (with $B=\frac{b}{2\pi}$) define a minimally immersed annulus $E^{0,b}$ with flux vector $(0,0,-b)$. Furthermore, the conformal map $z\to\overline{z}$ in the parameter domain $D(R,\infty)$ of $E^{0,b}$ extends to an isometry of $\mathbb{R}^3$ which is a rotation around the x_3-axis.*

Finally, we can join Cases 1 and 2 to conclude the following main result of this section.

Assertion 6.5 *For each $a\geq 0$ and $b\in\mathbb{R}$ there exists a complete, minimally immersed annulus $E^{a,b}$ defined on some $D(\infty,R)$, $R=R(a,b)>0$ by the Weierstrass data:*

1. *$g(z)=e^{iz+f(z)}$ where $f(z)$ is a holomorphic function in $D(\infty,R)\cup\{\infty\}$ with $f(\infty)=0$,*

2. $dh = \left(1 + \frac{1}{2\pi}\frac{b}{z-\mu}\right) dz$ *for some* $\mu \in \mathbb{C}$.

Furthermore, the flux vector of $E^{a,b}$ *along* $\{|z| = R\}$ *is* $(a, 0, -b)$ *and when* $a = b = 0$*, we choose* $f(z) = 0$.

7 The proof of Theorem 1.3 and the embeddedness of the canonical examples $E_{a,b}$.

Up to this point in this paper, we have focused on the Weierstrass representation of the surfaces under consideration. The actual surfaces themselves for given Weierstrass data are only determined up to a translation, because they depend on integration of analytic forms after making a choice of a base point z_0 in the parameter domain $\mathcal{D} \subset \mathbb{C}$. The parametrization of the surfaces is calculated as a path integral where the path begins at the base point. In the proof of the next theorem the reader should remember that different choices of base points give rise to translations of the image minimal annulus that we are considering; the image point of the base point is always the origin $(0, 0, 0)$.

Theorem 1.3 stated in the introduction follows from the material described in the last section together with items 1, 5, 6, 7, 8 and 9 of the next theorem. Theorem 1.1 is also implied by the next statement. Recall that the remaining item 3(c) to be proved in Theorem 1.1 follows immediately from item 7 below and the fact that $E_{0,0}$ can be taken to be the end of the vertical helicoid with Weierstrass data given by $g(z) = e^{iz}$, $dh = dz$. In item 6 below, we use the notation introduced just before the statement of Theorem 1.3.

The reader will find it useful to refer to Figure 1 in the Introduction for a visual geometric interpretation of some qualities of the complete, embedded minimal annulus E appearing in the next statement.

Theorem 7.1 *Suppose* E *is a minimally immersed annulus in* $\mathbb{R}^3$ *which is the image of a conformal immersion* $X\colon D(\infty, R') \to \mathbb{R}^3$*, with flux vector* $(a, 0, -2\pi\lambda)$ *for some* $a \geq 0$ *and* $\lambda \in \mathbb{R}$*, whose Weierstrass data* (g, dh) *on* $D(\infty, R')$ *are given by*

- $g(z) = e^{iz+f(z)}$ *where* $f(z)$ *is a holomorphic at* ∞ *with* $f(\infty) = 0$*;*
- $dh = \left(1 + \frac{\lambda}{z-\mu}\right) dz$ *where* $\mu \in \mathbb{C}$.

Then:

1. *For some* $R_1 \geq R'$*,* X *restricted to* $D(\infty, R_1)$ *is injective. In particular, each canonical example* $E^{d,e}$ *described in Assertion 6.4 contains an end* $E_{d,e}$ *which is a complete, properly embedded minimal annulus.*

2. *The Gaussian curvature function* K *of* X *satisfies* $\limsup_{R\to\infty} \left|K|_{X(D(\infty,R))}\right| = 1$. *In particular,* K *is bounded.*

3. *There exist* $(x_T, y_T), (x_B, y_B) \in \mathbb{R}^2$ *such that the image curves* $X([R', \infty)), X((-\infty, -R'])$ *are asymptotic to the vertical half lines* $r_T = \{(x_T, y_T, t) \mid t \in [0, \infty)\}$, $r_B = \{(x_B, y_B, t) \mid t \in (-\infty, 0]\}$ *respectively.*

4. $(x_T, y_T) - (x_B, y_B) = (0, -\frac{a}{2})$.

5. *The sequences of translated surfaces* $X_n = E + (0, 0, -2\pi n - \lambda \log n)$ *and* $Y_n = E + (0, 0, 2\pi n - \lambda \log n)$ *converge respectively to right handed vertical helicoids* H_T, H_B *where* r_T *is contained in the axis of* H_T *and* r_B *is contained in the axis of* H_B*. Furthermore,* $H_B = H_T + (0, \frac{a}{2}, 0)$.

6. *After a fixed translation of* E*, assume that*

$$(x_T, y_T) + (x_B, y_B) = (0, 0) \quad \textit{and} \quad x_3(z) = x + \lambda \log |z - \mu|,$$

   where x_3 *is the third coordinate function of* X *and* $z = x + iy$.

   (a) *There exists an* $R_E > 1$ *such that* $E - C(R_E)$ *consists of two disjoint multigraphs* Σ_1, Σ_2 *over* $D(\infty, R_E) \subset \mathbb{R}^2$ *of smooth functions* $u_1, u_2 \colon \widetilde{D}(\infty, R_E) \to \mathbb{R}$ *whose gradients satisfy* $\nabla u_i(r, \theta) \to 0$ *as* $r \to \infty$.

   (b) *Consider the multigraphs* $v_1, v_2 \colon \widetilde{D}(\infty, R_E) \to \mathbb{R}$ *defined by*

$$\begin{aligned} v_1(r,\theta) = & \left(\theta - \tfrac{\pi}{2}\right) + \tfrac{\lambda}{2} \log \left[\left(\theta - \tfrac{\pi}{2}\right)^2 + \log^2 r \right], \\ v_2(r,\theta) = & \left(\theta + \tfrac{\pi}{2}\right) + \tfrac{\lambda}{2} \log \left[\left(\theta + \tfrac{\pi}{2}\right)^2 + \log^2 r \right]. \end{aligned}$$

   Then, for each $n \in \mathbb{N}$*, there exists an* $R_n > R_E$ *such that when restricted to* $\widetilde{D}(\infty, R_n)$*, the multigraphing functions* u_i, v_i *satisfy* $|u_i - v_i| < \frac{1}{n}$*. In particular, for* θ *fixed and* $i = 1, 2$*, we have* $\lim_{r\to\infty} \frac{u_i(r,\theta)}{\log(\log(r))} = \lambda$ *(see Figure 1 and note that* $b = 2\pi\lambda$ *with* $\lambda > 0$ *there).*

   (c) *Furthermore:*

      i. *When* $a = 0$*, then the multigraphing functions* u_i, v_i *on the domain* $\widetilde{D}(\infty, R_E)$ *satisfy* $|u_i - v_i| \to 0$ *as* $r + |\theta| \to \infty$.

      ii. *The separation function* $w(r, \theta) = u_1(r, \theta) - u_2(r, \theta)$ *between both multigraphs converges to* π *as* $r + |\theta| \to \infty$.

7. *Suppose that* $X_2 \colon D(\infty, R_2) \to \mathbb{R}^3$ *is another conformal minimal immersion with the same flux vector* $(a, 0, -2\pi\lambda)$ *as* X *and Weierstrass data* (g_2, dh_2) *given by*

   (a) $g_2(z) = e^{iz + f_2(z)}$ *where* $f_2(z)$ *is holomorphic at* ∞ *with* $f_2(\infty) = 0$;

   (b) $dh_2 = \left(1 + \frac{\lambda}{z - \mu_2}\right) dz$ *where* $\mu_2 \in \mathbb{C}$.

Then, there exists a vector $\tau \in \mathbb{R}^3$ such that the $X_2(D(\infty, R_2)) + \tau$ is asymptotic to $E = X(D(\infty, R_1))$. In particular, for some translation vector $\widehat{\tau} \in \mathbb{R}^3$, $E + \widehat{\tau}$ is asymptotic to the canonical example $E_{a,b}$, where $b = 2\pi\lambda$. Note that if E and $E_{a,b}$ are each normalized by translation as in the first sentence of item 6, then E is asymptotic to $E_{a,b}$.

8. *Given $(a, b) \neq (a', b') \in \{(x_1, x_2) \mid x_1 \in [0, \infty),\ x_2 \in \mathbb{R}\}$, then, after any homothety and a rigid motion applied to $E_{a,b}$, the image surface is never asymptotic to $E_{a',b'}$.*

9. *Each of the canonical examples $E_{0,b}$ for $b \in \mathbb{R}$ is invariant under the 180^{o}-rotation around the x_3-axis l and $l \cap E_{0,b}$ contains two infinite rays.*

Proof. We will follow the ordering 2→3→4→5→6→7→8→9→1 when proving the items in the statement of Theorem 7.1.

As f is holomorphic in a neighborhood of ∞ with $f(\infty) = 0$, the expansion of f has only negative powers of z, so we can find $C > 0$ depending only on f such that

$$|f(z)| \leq C|z|^{-1}, \qquad |f'(z)| \leq C|z|^{-2} \qquad\qquad \text{for any } z \in D(\infty, R'). \tag{21}$$

Throughout the proof, we will assume that $|C| \geq R'$. We will frequently refer to these two simple estimates for $f(z)$ and $f'(z)$.

We first prove item 2. The expression of the absolute Gaussian curvature $|K|$ of X in terms of its Weierstrass data is

$$|K| = \frac{16}{(|g| + |g|^{-1})^4} \frac{|dg/g|^2}{|dh|^2} = \frac{1}{[\cosh \Re(iz + f(z)]^4} \frac{|i + f'(z)|^2}{\left|1 + \frac{\lambda}{z-\mu}\right|^2}.$$

Since f is holomorphic in a neighborhood of ∞ with $f(\infty) = 0$, the last right hand side is bounded from above, which easily implies item 2.

We next demonstrate item 3. Let $\gamma(r) = r$, $r \geq R'$, be a parametrization of the portion of the positive real axis in $D(\infty, R')$. We will check that $X \circ \gamma$ is asymptotic to a vertical half-line pointing up, which will be r_T (the case of r_B follows from the same arguments, using $\gamma(r) = -r$). From the Weierstrass data of X, we have that for given real numbers T, R with $R' < R < T$:

$$(x_1 + ix_2)(T) - (x_1 + ix_2)(R) = \frac{1}{2}\left(\overline{\int_R^T \frac{1}{g}\,dh} - \int_R^T g\,dh\right)$$

$$= \frac{1}{2}\left(\overline{\int_R^T e^{-it-f(t)}\left(1 + \frac{\lambda}{t-\mu}\right)dt} - \int_R^T e^{it+f(t)}\left(1 + \frac{\lambda}{t-\mu}\right)dt\right)$$

$$= \frac{1}{2}\int_R^T e^{it}\left[e^{-\overline{f(t)}}\left(1 + \frac{\lambda}{t-\overline{\mu}}\right) - e^{f(t)}\left(1 + \frac{\lambda}{t-\mu}\right)\right]dt. \tag{22}$$

Expanding the expression between brackets in the last displayed equation as a series in the variable t, we find

$$e^{-\overline{f(t)}}\left(1+\frac{\lambda}{t-\overline{\mu}}\right)-e^{f(t)}\left(1+\frac{\lambda}{t-\mu}\right)=-\frac{2\Re(C)}{t}+\mathcal{O}(t^{-2}),$$

where $f(t)=\frac{C}{t}+\mathcal{O}(t^{-2})$ and $\mathcal{O}(t^{-2})$ denotes a function such that $t^2\mathcal{O}(t^{-2})$ is bounded as $t\to\infty$. Hence, (22) yields

$$(x_1+ix_2)(T)-(x_1+ix_2)(R)=-\Re(C)\int_R^T\frac{e^{it}}{t}\,dt+\int_R^T e^{it}\mathcal{O}(t^{-2})\,dt$$

Note that $\left|\int_R^T e^{it}t^{-2}\,dt\right|\leq\int_R^T t^{-2}dt=R^{-1}-T^{-1}$ which converges to R^{-1} as $T\to+\infty$. On the other hand, taking R,T of the form $R=\pi m$, $T=\pi n$ for large positive integers $m<n$, we have

$$\int_R^T\frac{e^{it}}{t}\,dt=\left(\sum_{k=1}^{n-m}\int_{\pi(m+k-1)}^{\pi(m+k)}\frac{\cos t}{t}dt\right)+i\left(\sum_{k=1}^{n-m}\int_{\pi(m+k-1)}^{\pi(m+k)}\frac{\sin t}{t}dt\right)$$

Both parenthesis in the last expression are partial sums of alternating series of the type $\sum_k(-1)^k a_k$ where $a_k\in\mathbb{R}^+$ converges monotonically to zero as $k\to\infty$. Therefore, both series converge and we conclude that $(x_1+ix_2)(T)-(x_1+ix_2)(R)$ converges to a complex number as $T\to\infty$ (independent of R). This proves item 3 of the theorem.

We next prove item 4 which states that $(x_T,y_T)-(x_B,y_B)=(0,-\frac{a}{2})$. Item 4 is equivalent to:

$$\lim_{r\to+\infty}\left[(x_1+ix_2)(r)-(x_1+ix_2)(-r)\right]=-\frac{ia}{2}. \tag{23}$$

To see that (23) holds, we use computations modeled on those found in Section 5 of [12]. With this aim, take $r>R'$ and consider the boundary ∂S_r of the square $S_r\subset\mathbb{C}$ with vertices $r+ir, r-ir, -r-ir, -r+ir$. Its boundary ∂S_r can be written as $\Lambda_r-\overline{\Lambda_r}$, where Λ_r is the polygonal arc with vertices $-r,-r+ir,r+ir,r$ (the orientations on ∂S_r and Λ_r are chosen so that both lists of vertices are naturally well-ordered, see Figure 6). As $\{|z|=R'\}$ and ∂S_r bound a compact domain in the surface, it follows that the horizontal component of the flux of X along ∂S_r equals $(a,0)$. By equation (4), this is equivalent to the following equality:

$$\int_{\Lambda_r}\frac{dh}{g}=\int_{\overline{\Lambda_r}}\frac{dh}{g}+ia. \tag{24}$$

Hence,

$$(x_1+ix_2)(r)-(x_1+ix_2)(-r)=\int_{\Lambda_r}d(x_1+ix_2)=\frac{1}{2}\left(\overline{\int_{\Lambda_r}\frac{dh}{g}}-\int_{\Lambda_r}g\,dh\right)$$

$$\overset{(24)}{=}\frac{1}{2}\left(\overline{\int_{\overline{\Lambda_r}}\frac{dh}{g}}-ia-\int_{\Lambda_r}g\,dh\right)=\frac{1}{2}\left(I_2-I_1\right)+\frac{1}{2}\left(\widehat{I}_2-\widehat{I}_1\right)-\frac{ia}{2},$$

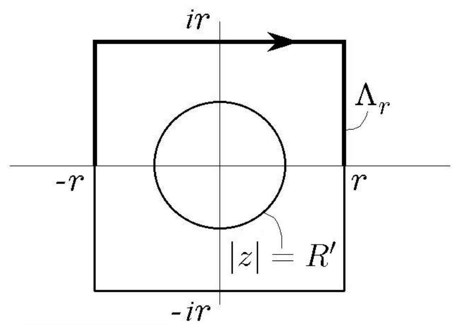


Figure 6: The square $\Lambda_2 - \overline{\Lambda_r}$ is homologous to the circle $\{|z| = R'\}$ in $D(\infty, R')$.

where $I_1 = \int_{\Lambda_r} g\, dz$, $I_2 = \int_{\overline{\Lambda_r}} \overline{g}^{-1}\, d\overline{z}$, $\widehat{I}_1 = \int_{\Lambda_r} g\, \frac{\lambda}{z-\mu} dz$ and $\widehat{I}_2 = \int_{\overline{\Lambda_r}} \overline{g}^{-1}\, \overline{\frac{\lambda}{z-\mu} dz}$. We now compute and estimate the norm of each of these four integrals separately. We first calculate I_1 and I_2.

$$
\begin{aligned}
I_1 &= \int_0^r g(-r+iv)i\, dv + \int_{-r}^r g(u+ir)\, du - \int_0^r g(r+iv)i\, dv \\
&= e^{-r} \int_{-r}^r e^{iu+f(u+ir)}\, du + i \int_0^r e^{-v} \left(e^{-ir+f(-r+iv)} - e^{ir+f(r+iv)} \right) dv.
\end{aligned}
$$

If we take r such that $\frac{r}{2\pi}$ is integer, then I_1 becomes

$$
I_1 = e^{-r} \int_{-r}^r e^{iu+f(u+ir)}\, du + i \int_0^r e^{-v} \left(e^{f(-r+iv)} - e^{f(r+iv)} \right) dv.
$$

and a similar computation for I_2 gives

$$
I_2 = e^{-r} \int_{-r}^r e^{iu-\overline{f(u-ir)}} du + i \int_0^r e^{-v} \left(e^{-\overline{f(-r-iv)}} - e^{-\overline{f(r-iv)}} \right) dv.
$$

Thus, both I_1 and I_2 are sum of two integrals and each one of these four integrals can be estimated using (21) as follows:

$$
\begin{aligned}
\left| e^{-r} \int_{-r}^r e^{iu+f(u+ir)}\, du \right| &\leq e^{-r} \int_{-r}^r e^{\Re(f(u+ir))}\, du \\
&\leq e^{-r} \int_{-r}^r e^{C|u+ir|^{-1}}\, du \leq e^{-r} \int_{-r}^r e^{Cr^{-1}}\, du = 2re^{-r+Cr^{-1}},
\end{aligned}
$$

and analogously, $\left| e^{-r} \int_{-r}^r e^{iu-\overline{f(u-ir)}}\, du \right| \leq 2re^{-r+Cr^{-1}}$, while

$$
\left| i \int_0^r e^{-v} \left(e^{f(-r+iv)} - e^{f(r+iv)} \right) dv \right| \leq \int_0^r e^{-v} \left| e^{f(-r+iv)} - e^{f(r+iv)} \right| dv.
$$

Using (21), it is straightforward to check that both $e^{f(-r+iv)}$ and $e^{f(r+iv)}$ can be expressed as $1+\mathcal{O}(r^{-1})$, hence the last expression is of the type $\int_0^r e^{-v}\mathcal{O}(r^{-1})\,dv \leq C_1 r^{-1}\,(1-e^{-r})$, for a constant $C_1>0$ independent of r. Similarly, $\left|i\int_0^r e^{-v}\left(e^{-\overline{f(-r-iv)}} - e^{-\overline{f(r-iv)}}\right)dv\right| \leq C_1 r^{-1}\,(1-e^{-r})$. Since the integrands of $\widehat{I}_1$, $\widehat{I}_2$ differ from the respective integrand of I_1, I_2 by a product with $\frac{\lambda}{z-\mu}$ and since $|\frac{\lambda}{z-\mu}|<1$ for $|z|$ sufficiently large, then $\widehat{I}_1 < I_1$, $\widehat{I}_2 < I_2$, when $|z|$ is large. In summary,

$$\left|(x_1+ix_2)(r)-(x_i+ix_2)(-r)+\frac{ia}{2}\right| \leq 2\left(|I_2|+|I_1|\right) \leq 2\left[2re^{-r+Cr^{-1}} + C_1 r^{-1}\left(1-e^{-r}\right)\right],$$

which tends to zero as $r\to+\infty$. This completes the proof that equation (23) holds and so proves item 4 of the theorem.

Next we prove item 5. For $n\in\mathbb{Z}$, $2\pi n > R'$, let $D(n)=\{z\in\mathbb{C}\mid |z-2\pi n|\leq 2\pi n - R'\}$ be the disk of radius $2\pi n - R'$ centered at $2\pi n$, and note that $D(n)\subset D(\infty,R')$. For $|n|$ large, consider the immersion $\widetilde{X}_n\colon D(n)\to\mathbb{R}^3$ with the same Weierstrass data as X restricted to the disk $D(n)$, and with base point $2\pi n$ (in particular, $\widetilde{X}_n$ is a translation of a portion of X). The minimal immersions $\widetilde{X}_n\colon D(n)\to\mathbb{R}^3$ converge uniformly to the right handed vertical helicoid H with Weierstrass representation data $g(z)=e^{iz}$, $dh=dz$, $z\in\mathbb{C}$, and base point $\widetilde{z}_0=0$. After letting $z_0\in D(\infty,R')$ denote the base point of X, then the third coordinate functions x_3 of X and $\widetilde{x}_{3,n}$ of $\widetilde{X}_n$ are related by

$$\begin{aligned}
x_3(z)-2\pi n+\Re(z_0) &= \Re\int_{z_0}^{z}\left(1+\frac{\lambda}{z-\mu}\right)dz - 2\pi n + \Re(z_0)\\
&= \Re(z)+\lambda\log\left|\frac{z-\mu}{z_0-\mu}\right| - 2\pi n\\
&= \Re\int_{2\pi n}^{z}\left(1+\frac{\lambda}{z-\mu}\right)dz + \lambda\log\left|\frac{2\pi n-\mu}{z_0-\mu}\right|\\
&= \widetilde{x}_{3,n}(z)+\lambda\log\left|\frac{2\pi n-\mu}{z_0-\mu}\right|\\
&= \widetilde{x}_{3,n}(z)+\lambda\log n+\lambda\log\left|\frac{2\pi-\frac{\mu}{n}}{z_0-\mu}\right|.
\end{aligned}$$

Therefore, the minimal surfaces $X_n = E+(0,0,-2\pi n-\lambda\log n)$ converge as $n\to\infty$ to a translated image $H_T := H+\tau$ of the helicoid H, where $\tau = (0,0,-\Re(z_0)+\lambda\log\frac{2\pi}{|z_0-\mu|})\in\mathbb{R}^3$. Clearly, $r_T = r_T+(0,0,-2\pi n-\lambda\log n)$ is contained in H_T. Similar computations give that the surfaces $E+(0,0,2\pi n-\lambda\log n)$ converge as $n\to\infty$ to a translated image $H_B := H+\tau'$ of H such that $x_3(\tau)=x_3(\tau')$. Now item 4 gives that $\tau-\tau'=(0,-\frac{a}{2},0)$, and item 5 holds.

Next we prove item 6. Throughout the proof of this item we will let $\mathbb{C}^+$ and $\mathbb{C}^-$ denote respectively the upper and lower open half planes in the complex plane $\mathbb{C}$, and $\Delta(n)=\{z=x+iy\in\mathbb{C}\mid |y|\geq n\}$, where $n\in\mathbb{N}$. First translate E as required in the first sentence of item 6 of the theorem, and use the same notation X for the translated immersion which parameterizes E. Next observe that for some fixed large positive integer n_0 and for each $n\in\mathbb{N}$ with $n\geq n_0$, we have

- $\partial D(\infty, R')$ is contained in the horizontal strip $\mathbb{C} - \Delta(n)$.
- The normal vectors to $X(\Delta(n))$ are contained in the intrinsic $\frac{1}{2}$-neighborhood of $(0,0,\pm 1)$ in the sphere $\mathbb{S}^2$.
- As $n \to \infty$, the Gaussian image of $X(\Delta(n))$ is contained in smaller and smaller neighborhoods of $(0,0,\pm 1)$ in $\mathbb{S}^2$, which converge as sets to the set $\{(0,0,\pm 1)\}$ in the limit.

From the Weierstrass representation of X we conclude that for $z \in \mathbb{C}$ with $|z|$ very large,

$$\begin{aligned}
2(x_1 + ix_2)(z) &= \overline{\int^z \frac{dh}{g}} - \int^z g\,dh \\
(z = x + iy) \qquad &= e^y \int^z e^{iu - \overline{f(u+iy)}}\left(1 + \tfrac{\lambda}{u - iy - \overline{\mu}}\right) du - ie^{ix} \int^z e^{v - \overline{f(x+iv)}}\left(1 + \tfrac{\lambda}{x - iv - \overline{\mu}}\right) dv \\
&- e^{-y} \int^z e^{iu + f(u+iy)}\left(1 + \tfrac{\lambda}{u + iy - \mu}\right) du - ie^{ix} \int^z e^{-v + f(x+iv)}\left(1 + \tfrac{\lambda}{x + iv - \mu}\right) dv \\
&= \begin{cases} -2ie^y e^{ix} + o(e^y) & \text{if } y > 0, \\ 2ie^{|y|} e^{ix} + o(e^{|y|}) & \text{if } y < 0, \end{cases}
\end{aligned} \tag{25}$$

where given $t > 0$, $o(e^t)$ denotes a complex valued function of z such that $e^{-t} o(e^t)$ converges to the complex number 0 as $t \to +\infty$. Hence, for $n \in \mathbb{N}$ large, we have $|(x_1 + ix_2)(z)| < 2e^n$ whenever $|y| \leq n$, from where $X(\mathbb{C} - \Delta(n)) \subset C(2e^n)$. After choosing $R_E = 2e^{n_0}$ with $n_0 \in \mathbb{N}$ large, the remaining properties of item 6(a) are easily seen to hold.

Let Σ_1 (resp. Σ_2) correspond to the multigraph over the universal cover $\widetilde{D}(\infty, R_E)$ of $D(\infty, R_E)$ which lies in $X[D(\infty, R') \cap \mathbb{C}^+] - \mathrm{Int}[C(R_E)]$ (resp. in $X[D(\infty, R') \cap \mathbb{C}^-] - \mathrm{Int}[C(R_E)]$). Observe that we will use the positive integer n_0 in the following paragraphs.

We next obtain the asymptotic results described in item 6(b) for the multigraphs Σ_1, Σ_2 described in the last paragraph, with respective graphing functions $u_1(r,\theta)$, $u_2(r,\theta)$ defined on $\widetilde{D}(\infty, R_E)$. For each integer $k > 1$, $\theta_0 \in \mathbb{S}^1 = \mathbb{R}/\langle 2\pi \rangle$ and $r_0 > R_E$, consider the polar rectangle (see Figure 7)

$$S(k, r_0, \theta_0) = \left\{ re^{i\theta} : \left| \frac{r}{r_0} - 1 \right| \leq \frac{1}{k}, \quad |\theta - \theta_0| \leq \frac{2\pi}{k} \right\}.$$

We want to estimate the placement and size of the components in the z-plane of the preimage through $x_1 + ix_2$ of $S(k, r_0, \theta_0)$ when $r_0 > 0$ is very large. Taking modulus in (25) we deduce that for r_0 and k large and if $z \in (x_1 + ix_2)^{-1}[S(k, r_0, \theta_0)] \cap \mathbb{C}^+$, then

$$\left| e^y \frac{|-ie^{ix} + e^{-y} o(e^y)|}{r_0} - 1 \right| \leq \frac{1}{k}.$$

In particular if $y \sim \log r_0$ is large enough, then the last inequality implies $\left| \frac{e^y}{r_0} - 1 \right| \leq \frac{2}{k}$, i.e.

$$|y - \log r_0| \leq \min\left\{ -\log(1 - \frac{2}{k}), \log(1 + \frac{2}{k}) \right\} < \frac{3}{k} \tag{26}$$

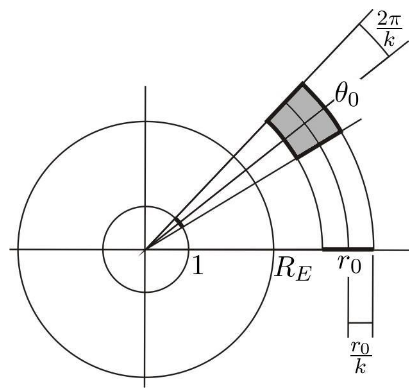


Figure 7: The polar rectangle $S(k, r_0, \theta_0)$ is the shaded region.

for k large. Now taking arguments in (25) we have

$$\left|\arg\left(-ie^{ix} + e^{-y}o(e^y)\right) - \theta_0\right| \leq \frac{2\pi}{k}.$$

Using (26) and reasoning similarly as before we conclude that for r_0 large enough, $|\arg(-ie^{ix}) - \theta_0| \leq \frac{4\pi}{k}$; hence

$$x \text{ is } \frac{4\pi}{k}\text{-close to the set } \left(\theta_0 + \frac{\pi}{2}\right) + 2\pi\mathbb{Z} \subset \mathbb{R}. \tag{27}$$

From (26) and (27) we deduce that $(x_1 + ix_2)^{-1}\left[S(k, r_0, \theta_0)\right] \cap \mathbb{C}^+$ consists of an infinite collection $\Gamma = \Gamma(k, r_0, \theta_0)$ of (topological) disk components in $\mathbb{C}^+$, each of which is $\frac{4\pi}{k}$-close to one of the points in the set $\left\{\left[\left(\theta_0 + \frac{\pi}{2}\right) + 2\pi\mathbb{Z}\right] + i\log r_0\right\} \subset \mathbb{C}$.

Recall that by our previous translation normalization of E, we have

$$x_3(z) = x + \frac{\lambda}{2}\log\left[(x - a_1)^2 + (y - a_2)^2\right], \tag{28}$$

where $z = x + iy$ and $\mu = a_1 + ia_2$. It follows from (27) and (28) that for r_0, k large, the image under X of each of the disk components in Γ is $\frac{8\pi}{k}$-close to one of the points in the set

$$\left\{\left(r_0e^{i\theta_0}, v_1\left(r_0, \theta_0 + \frac{\pi}{2} + 2\pi m\right)\right) \mid m \in \mathbb{Z}\right\} \subset \mathbb{C} \times \mathbb{R} \equiv \mathbb{R}^3.$$

Since for r_0 large the multigraph v_1 is closer and closer to being horizontal on each of the liftings of $S(k, r_0, \theta_0)$ to $\widetilde{D}(\infty, R_E)$, then we have that the function u_1 which defines the multigraph Σ_1 and v_1 satisfy $|u_1 - v_1| < \frac{16\pi}{k}$ on each of the liftings of of $S(k, r_0, \theta_0)$ to $\widetilde{D}(\infty, R_E)$. Clearly, all these estimates do not depend on θ_0 but only on the choice of r_0 large enough (equivalently, on the choice of $y_0 \sim \log r_0$ large). Therefore, we have that for every $k \in \mathbb{N}$, there exists $R_n > R_E$ such that $|u_1 - v_1| < \frac{1}{n}$ in

$\widetilde{D}(\infty, R_n)$. Similar arguments allow us to exchange $\mathbb{C}^+$ by $\mathbb{C}^-$ and u_1, v_1 by u_2, v_2 respectively. This finishes the proof of item 6(b).

By item 4 and our translation normalization, if $a = 0$ then the rays r_T and r_B are contained in the x_3-axis and so, on the domain $\widetilde{D}(\infty, R_E)$, the multigraphing functions u_i, v_i satisfy $|u_i - v_i| \to 0$ as $r + |\theta| \to \infty$. Regardless of the condition $a = 0$, the facts that the separation function between v_1 and v_2 has an asymptotic value of π together with the estimates in item 6(b) and item 5 imply that the separation function $w(r,\theta) = u_1(r,\theta) - u_2(r,\theta)$ between u_1 and u_2 converges to π as $r + |\theta| \to \infty$. Now the proof of item 6 is complete.

Next we prove item 7. Suppose that $X_2 \colon D(\infty, R_2) \to \mathbb{R}^3$ is another conformal minimal immersion with the same flux vector $(a, 0, -2\pi\lambda)$ as X and with Weierstrass data (g_2, dh_2) as described in item 7. Since X, X_2 have the same flux vector, then item 4 together with item 6 imply that we can find fixed translations $Y_1 = E + \tau_1$ and $Y_2 = X_2(D(\infty, R_2)) + \tau_2$ of $E = X(D(\infty, R_1))$ and $X_2(D(\infty, R_2))$ respectively which include the property that Y_1, Y_2 each has the same half-axes projections $(x_T, y_T) = (0, -\frac{a}{4})$, $(x_B, y_B) = (0, \frac{a}{4})$, and by transitivity we have that given any $n \in \mathbb{N}$, there is a large $R_n > 0$ such that each of the two multigraphs of $Y_1 - C(R_n)$ is $\frac{2}{n}$-close to the corresponding multigraph of $Y_2 - C(R_n)$ in $\widetilde{D}(\infty, R_n)$. Since the vertical fluxes of Y_1, Y_2 are equal, then item 5 implies that $Y_1 \cap C(R_n)$ is asymptotic to $Y_2 \cap C(R_n)$. Therefore, Y_1 and Y_2 are asymptotic in $\mathbb{R}^3$, and item 7 is proved.

Item 8 follows from the asymptotic description of an example E with flux vector $(a, 0, -b)$ given in 6 in terms of the flux components a, $b = 2\pi\lambda$. The symmetry property in item 9 follows directly from Assertion 6.4. Finally, item 1 on the embeddedness of some end of E follows immediately from the asymptotic embeddedness information described in items 5 and 6, together with the next straightforward observation: For any pair $\widetilde{u}_1, \widetilde{u}_2$ of multigraphs defined on $\widetilde{D}(\infty, R_1)$ (here R_1 is defined in item 6(b)), each of which is 1-close respectively to the restricted multigraphs v_1, v_2 given in item 6(b), then the union of the graphs of $\widetilde{u}_1, \widetilde{u}_2$ when viewed to lie in $\mathbb{R}^3$ defines an embedded surface with two components (for this observation to hold, note that equation (25) implies that for $x \in \mathbb{R}$ fixed, $e^{-y}\left[(x_1 + ix_2)(x + iy) + (x_1 + ix_2)(x - iy)\right] \to 0$ as $y \to \infty$). $\square$

William H. Meeks, III at profmeeks@gmail.com
Mathematics Department, University of Massachusetts, Amherst, MA 01003

Joaquín Pérez at jperez@ugr.es
Department of Geometry and Topology, University of Granada, Granada, Spain

References

[1] J. Bernstein and C. Breiner. Conformal structure of minimal surfaces with finite topology. Preprint available at http://arxiv.org/abs/0810.4478v1.

[2] J. Bernstein and C. Breiner. Helicoid-like minimal disks and uniqueness. Preprint available at http://arxiv.org/abs/0802.1497.

[3] T. H. Colding and W. P. Minicozzi II. The space of embedded minimal surfaces of fixed genus in a 3-manifold V; Fixed genus. Preprint math.DG/0509647 (2005).

[4] T. H. Colding and W. P. Minicozzi II. Minimal annuli with and without slits. *Journal of Simplectic Geometry*, 1(1):47–61, 2001. MR1959578, Zbl 1087.53008.

[5] T. H. Colding and W. P. Minicozzi II. An excursion into geometric analysis. In *Surveys of Differential Geometry IX - Eigenvalues of Laplacian and other geometric operators*, pages 83–146. International Press, edited by Alexander Grigor'yan and Shing Tung Yau, 2004. MR2195407, Zbl 1076.53001.

[6] T. H. Colding and W. P. Minicozzi II. The space of embedded minimal surfaces of fixed genus in a 3-manifold I; Estimates off the axis for disks. *Ann. of Math.*, 160:27–68, 2004. MR2119717, Zbl 1070.53031.

[7] T. H. Colding and W. P. Minicozzi II. The space of embedded minimal surfaces of fixed genus in a 3-manifold II; Multi-valued graphs in disks. *Ann. of Math.*, 160:69–92, 2004. MR2119718, Zbl 1070.53032.

[8] T. H. Colding and W. P. Minicozzi II. The space of embedded minimal surfaces of fixed genus in a 3-manifold III; Planar domains. *Ann. of Math.*, 160:523–572, 2004. MR2123932, Zbl 1076.53068.

[9] T. H. Colding and W. P. Minicozzi II. The space of embedded minimal surfaces of fixed genus in a 3-manifold IV; Locally simply-connected. *Ann. of Math.*, 160:573–615, 2004. MR2123933, Zbl 1076.53069.

[10] T. H. Colding and W. P. Minicozzi II. The Calabi-Yau conjectures for embedded surfaces. *Ann. of Math.*, 167:211–243, 2008. MR2373154, Zbl 1142.53012.

[11] P. Collin. Topologie et courbure des surfaces minimales de $\mathbb{R}^3$. *Ann. of Math. (2)*, 145–1:1–31, 1997. MR1432035, Zbl 886.53008.

[12] L. Hauswirth, J. Pérez, and P. Romon. Embedded minimal ends of finite type. *Transactions of the AMS*, 353:1335–1370, 2001. MR1806738, Zbl 0986.53005.

[13] H. B. Lawson, Jr. *Lectures on Minimal Submanifolds*. Publish or Perish Press, Berkeley, 1980. MR0576752, Zbl 0434.53006.

[14] W. H. Meeks III and J. Pérez. The classical theory of minimal surfaces. Preprint, available at `http://www.ugr.es/local/jperez/papers/papers.htm`.

[15] W. H. Meeks III and J. Pérez. Finite type annular ends for harmonic functions. Preprint, available at `http://www.ugr.es/local/jperez/papers/papers.htm`.

[16] W. H. Meeks III, J. Pérez, and A. Ros. The embedded Calabi-Yau conjectures for finite genus. Preprint, available at `http://www.ugr.es/local/jperez/papers/papers.htm`.

[17] W. H. Meeks III, J. Pérez, and A. Ros. Local removable singularity theorems for minimal and H-laminations. Preprint, available at `http://www.ugr.es/local/jperez/papers/papers.htm`.

[18] W. H. Meeks III, J. Pérez, and A. Ros. Structure theorems for singular minimal laminations. Preprint, available at `http://www.ugr.es/local/jperez/papers/papers.htm`.

[19] W. H. Meeks III and H. Rosenberg. The uniqueness of the helicoid. *Ann. of Math.*, 161:723–754, 2005. MR2153399, Zbl 1102.53005.

[20] R. Osserman. Global properties of minimal surfaces in E^3 and E^n. *Ann. of Math.*, 80(2):340–364, 1964. MR0179701, Zbl 0134.38502.

[21] R. Osserman. *A Survey of Minimal Surfaces*. Dover Publications, New York, 2nd edition, 1986. MR0852409, Zbl 0209.52901.

[22] H. Rosenberg. Minimal surfaces of finite type. *Bull. Soc. Math. France*, 123:351–354, 1995. MR1373739, Zbl 0848.53004.