Anais da Academia Brasileira de Ciências (2005) 77(2): 183–199
(Annals of the Brazilian Academy of Sciences)
ISSN 0001-3765
www.scielo.br/aabc

Embedded positive constant r-mean curvature hypersurfaces in $M^m \times \mathbf{R}$

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Manuscript received on September 20, 2004; accepted for publication on January 14, 2005; contributed by HAROLD ROSENBERG*

ABSTRACT

Let M be an m-dimensional Riemannian manifold with sectional curvature bounded from below. We consider hypersurfaces in the $(m + 1)$-dimensional product manifold $M \times \mathbf{R}$ with positive constant r-mean curvature. We obtain height estimates of certain compact vertical graphs in $M \times \mathbf{R}$ with boundary in $M \times \{0\}$. We apply this to obtain topological obstructions for the existence of some hypersurfaces. We also discuss the rotational symmetry of some embedded complete surfaces in $\mathbf{S}^2 \times \mathbf{R}$ of positive constant 2-mean curvature.

Key words: product manifold, hypersurface, r-mean curvature.

1 INTRODUCTION

If $\overline{M}^{m+1}$ is an $(m+1)$-dimensional oriented Riemannian manifold and Σ^m is a hypersurface in $\overline{M}$, the r-mean curvature of Σ, denoted by H_r, is the weighted r'th symmetric function of the second fundamental form (see Definition 2.1). The hypersurfaces with constant r-mean curvature include those of constant mean curvature, and of constant Gauss-Kronecker curvature. In some situations, for example, $\overline{M} = \mathbb{R}^{m+1}$, $\mathbb{S}^{m+1}$, $\mathbb{H}^{m+1}(-1)$, a hypersurface of constant r-mean curvature is a critical point for a certain variational problem (See the related works in (Reilly 1973), (Rosenberg 1993), (Barbosa and Colares 1997) and (Elbert 2002)).

Heinz discovered that a compact graph Σ in $\mathbb{R}^{m+1}$ with zero boundary values, of constant mean curvature $H \neq 0$, is at most a height $\frac{1}{H}$ from its boundary. A hemisphere in $\mathbb{R}^{m+1}$ of radius $\frac{1}{H}$ shows this estimate is optimal. Using Alexandrov reflection techniques, it follows that a

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compact embedded hypersurface with constant mean curvature $H \neq 0$ and boundary contained in $x_{m+1} = 0$, is at most a distance $\frac{2}{H}$ from the hyperplane $x_{m+1} = 0$.

Height estimates have been obtained by the second author (Rosenberg 1993) for compact graphs Σ in $\mathbb{R}^{m+1}$ with zero boundary values and with a higher order positive constant r-mean curvature. He also obtained such height estimates in space forms. For example, in $\mathbb{R}^3$, such a graph of positive constant Gauss curvature can rise at most $\frac{1}{\sqrt{H_2}}$ from the plane containing the boundary. In general, the maximum height is $\frac{1}{(H_r)^{\frac{1}{r}}}$. For $r = 1$, this result in hyperbolic space $\mathbb{H}^{m+1}(-1)$ was proved by Korevaar, Kusner, Meeks and Solomon (Korevaar et al. 1992). We refer the reader to (Rosenberg 1993) for applications of these height estimates.

In (Hoffman et al. 2005), Hoffman, Lira and Rosenberg obtained height estimates for some vertical graphs in $M^2 \times \mathbb{R}$ of nonzero constant mean curvature, where M is a Riemannian surface.

In this paper, we will consider orientable hypersurfaces in an $(m + 1)$-dimensional oriented product manifold $M \times \mathbb{R}$ with positive constant r-mean curvature. Here M is an m-dimensional Riemannian manifold with sectional curvature bounded below. The typical models are when $M = \mathbb{R}^m, \mathbb{S}^m, \mathbb{H}^m(-1)$. We will obtain height estimates for such compact vertical graphs with boundary in $M \times \{0\}$. A significant difference between this case and the previous cases we discussed is the nature of the linearized operator. It is no longer a divergence form operator and we do not have a flux formula. We prove that,

THEOREM 1.1. (**Th.** 4.1**).** *Let M be an m-dimensional oriented Riemannian manifold. Let Σ be a compact vertical graph in the $(m+1)$-dimensional product manifold $M \times \mathbb{R}$ with positive constant H_r, for some $1 \leq r \leq m$, with boundary in $M \times \{0\}$. Let $h : \Sigma \to \mathbb{R}$ denote the height function of Σ.*

(i) *If the sectional curvature of M satisfies $K \geq 0$, then on Σ,*

$$|h| \leq H_r^{-\frac{1}{r}}; \tag{1.1}$$

(ii) *When $r = 2$, if the sectional curvature of M satisfies $K \geq -\tau$ ($\tau > 0$), and $H_2 > \tau$, then on Σ,*

$$|h| \leq \frac{\sqrt{H_2}}{H_2 - \tau}. \tag{1.2}$$

(iii) *When $r = 1$, if the sectional curvature of M satisfies $K \geq -\tau$ ($\tau > 0$), and $H_1^2 > \frac{m-1}{m}\tau$, then on Σ,*

$$|h| \leq \frac{H_1}{H_1^2 - \frac{m-1}{m}\tau}. \tag{1.3}$$

We give some applications of Theorem 1.1 using the Alexandrov reflection technique. First we give the extrinsic vertical diameter estimate of compact embedded surfaces in $M \times \mathbb{R}$ with positive constant r-mean curvature (under the same curvature assumptions as in Theorem 1.1. See Th.4.2); Secondly, we prove that if M is an m-dimensional compact Riemannian manifold and Σ is a noncompact properly embedded hypersurface in $M^m \times \mathbb{R}$ with positive constant r-mean curvature (under the same curvature assumptions as in Theorem 1.1), then the number of ends of Σ is not one (Th.4.3). Thus we give a topological obstruction for the existence of such a hypersurface. For example, if Σ is a noncompact properly embedded hypersurface in $\mathbb{S}^m \times \mathbb{R}$ of positive constant r-mean curvature for $1 \leq r \leq m$, then the number of ends of Σ is more than one (Cor.4.1).

In this paper, we also consider the properties of symmetry of certain embedded hypersurfaces of positive constant r-mean curvature. Alexandrov (Alexandrov 1962) showed an embedded constant mean curvature hypersurface in the Euclidean space must be a standard round hypersurface. The symmetric properties of hypersurfaces (with or without boundary) in the Euclidean space and the other space forms with constant mean curvature or higher order constant r-mean curvatures have been studied in varied degrees. We discuss the rotational symmetry of a complete embedded surface in $\mathbb{S}^2 \times \mathbb{R}$ of positive constant 2-mean curvature and obtain that,

THEOREM 1.2. (**Th.** 5.1**; Cor.** 5.1**).** *Let $\mathbb{S}^2$ be the unit sphere in $\mathbb{R}^3$ and $\mathbb{D}$ be the open upper hemisphere in $\mathbb{S}^2$. Let Σ be a complete orientable embedded surface in $\mathbb{S}^2 \times \mathbb{R}$ with $H_2 =$ constant $>$ 0. If $\Sigma \subset \mathbb{D} \times \mathbb{R}$, then Σ has the following properties:*

(i) *Σ is topologically a sphere; and*

(ii) *Σ is a compact surface of revolution about $o' \times \mathbb{R}$, $o' \in \mathbb{D}$, that is, Σ is foliated by round circles in $\mathbb{D} \times \{t\}$, the centers of which are $o' \times t$, where $t \in (h_{min}, h_{max})$ and h_{min}, h_{max} are the maximum and minimum of the height of Σ respectively. Moreover, there exists a horizontal level $\mathbb{D} \times \{t_0\}$, $t_0 = \frac{h_{min}+h_{max}}{2}$, such that Σ divides into two (upper and lower) symmetric parts; and*

(iii) *Let $\phi_0 \in \left[0, \frac{\pi}{2}\right)$ denote the polar angle between the north pole of the open upper hemisphere $\mathbb{D}$ and o'. Then $H_2 > -\frac{1}{2 \log \sin \phi_0}$ (if $\phi_0 = 0$, $H_2 > 0$) and the generating curve of the rotational surface Σ can be denoted by $\lambda(u) = (\phi(u), t(u))$ where $-1 \leq u \leq 1$, and ϕ denotes the polar coordinate (about o') of $\mathbb{S}^2$, and:*

$$\phi = \cos^{-1} \exp\left(-\frac{1-u^2}{2H_2}\right), \tag{1.4}$$

$$t = -\frac{1}{H_2} \int_1^u \frac{\sqrt{1-u^2}}{\sqrt{\exp\left(\frac{1-u^2}{H_2}\right) - 1}} du + C, \tag{1.5}$$

where C is a real constant.

We conjecture a compact immersed surface in $\mathbb{S}^2 \times \mathbb{R}$ of positive constant 2-mean curvature is rotationally symmetric about a vertical line. Moreover, such a surface in $\mathbb{H}^2 \times \mathbb{R}$ is rotational when it is topologically a sphere.

2 PRELIMINARIES

Let $\overline{M}^{m+1}$ be an $(m+1)$-dimensional oriented Riemannian manifold and let Σ^m be an m-dimensional orientable Riemannian manifold. Suppose $x : \Sigma \to \overline{M}$ is an isometric immersion. We choose a unit normal field N to Σ and define the shape operator A associated with the second fundamental form of Σ, i.e., for any $p \in \Sigma$,

$$A : T_p\Sigma \to T_p\Sigma, \langle A(X), Y\rangle = -\left\langle \overline{\nabla}_X N, Y\right\rangle, X, Y \in T_p\Sigma,$$

where $\overline{\nabla}$ is the Riemannian connection in $\overline{M}$.

Let $\lambda_1, \dots, \lambda_m$ denote the eigenvalues of A. The r-th symmetric function of $\lambda_1, \dots, \lambda_r$, denoted by S_r, is defined as

$$\begin{aligned} S_0 &= 1, \\ S_r &= \sum_{i_1<\cdots<i_r} \lambda_{i_1}\cdots\lambda_{i_r}, 1 \le r \le m, \\ S_r &= 0, r > m. \end{aligned} \tag{2.1}$$

Definition 2.1. *With the above notations, $H_r = \frac{1}{C_m^r} S_r, r = 1, \dots, m$, is called the r-mean curvature of x.*

Particularly, $H_1 = H$ is the mean curvature; H_m is the Gauss-Kronecker curvature. H_2 is , modulo a constant, the scalar curvature of Σ, when the ambient space $\overline{M}^{m+1}$ is Einstein (Elbert 2002).

We also introduce endomorphisms of $T(\Sigma)$, the Newton transformations, defined by

$$T_0 = I,$$
$$T_r = S_r I - S_{r-1}A + \cdots + (-1)^r A^r, r = 1, \dots, m.$$

It is obvious that $T_r, r = 0, 1, \dots, m$ are symmetric linear operators and $T_r = S_r I - AT_{r-1}$.

Let $e_1, \dots, e_m$ be the principal directions corresponding respectively to the principal curvatures $\lambda_1, \dots, \lambda_m$. For $i = 1, \dots, m$, let A_i denote the restriction of the transformation A to the $(m-1)$-dimensional subspace normal to e_i, and let $S_r(A_i)$ denote the r-symmetric function associated to A_i.

One has the following properties of T_r and S_r.

Proposition 2.1. *For $0 \le r \le m$, $1 \le i \le m$,*

(i) $T_r(e_i) = \dfrac{\partial S_{r+1}}{\partial \lambda_i} e_i = S_r(A_i)$;

(ii) $(m-r)S_r = \text{trace}(T_r) = \sum_{i=1}^{m} S_r(A_i)$;

(iii) $(r+1)S_{r+1} = \mathrm{trace}(AT_r) = \sum_{i=1}^{m} \lambda_i S_r(A_i)$;

(iv) $H_1^2 \geq H_2$.

Proposition 2.1 can been verified directly.

PROPOSITION 2.2. *For each* $r(1 \leq r \leq m)$, *if* $H_1, H_2, \ldots, H_r$ *are nonnegative, then*

(i) $H_1 \geq H_2^{\frac{1}{2}} \geq H_3^{\frac{1}{3}} \geq \ldots \geq H_r^{\frac{1}{r}}$;

(ii) $H_{r-1}H_{r+1} \leq H_r^2$;

(iii) $H_1 H_r \geq H_{r+1}$, *where,* $H_0 = 1$, $H_{m+1} = 0$.

See the proof of (i) and (ii) of Proposition 2.2 in (Hardy et al. 1989. p.52). The inequality in (iii) can be obtained from (ii).

Given a function f in $C^2(\Sigma)$ for $p \in \Sigma$, the linear operator Hessian of f is defined as $\mathrm{Hess} f(X) = \nabla_X(\nabla f)$, $X \in T_p\Sigma$, where ∇ is the induced connection on Σ.

With T_r and Hess, we can define a differential operator L_r as follows:

DEFINITION 2.2. *Given* $f \in C^2(\Sigma)$, $0 \leq r \leq m$,

$$L_r(f) = \mathrm{tr}\left(T_r \mathrm{Hess} f\right). \tag{2.2}$$

Given a local coordinate frame $\left\{\frac{\partial}{\partial x_i}\right\}$ of Σ at p, by direct computation, we have locally the expression of L_r,

$$L_r f(p) = \sum_{i,j}\sum_{k,l} g^{ik} t_{kl} g^{lj} \frac{\partial^2 f}{\partial x^i \partial x^j} - \sum_{i,j,k,l} g^{ik} t_{kl} g^{lj} \Gamma_{ij}^s \frac{\partial f}{\partial x^s},$$

where

$$g_{ij} = \left\langle \frac{\partial f}{\partial x^i}, \frac{\partial f}{\partial x_j} \right\rangle, G = (g_{ij}), G^{-1} = (g^{ij}), t_{ij} = T_r\left(\frac{\partial f}{\partial x^i}, \frac{\partial f}{\partial x_j}\right), \Gamma_{ij}^s$$

are the connection coefficients of ∇.

From the above local expression, we know that the linear operator L_r is elliptic if and only if T_r is positive definitive. Clearly, $L_0 = \mathrm{tr}\left(\mathrm{Hess} f\right) = \mathrm{div}(\nabla f)$ is elliptic.

In this paper, the ambient space which we study is an $(m+1)$-dimensional product manifold $M^m \times \mathbb{R}$, where M is an m-dimensional Riemannian manifold. Σ will be a hypersurface in $M \times \mathbb{R}$ with positive constant r-mean curvature, i.e, $H_r = \mathrm{constant} > 0$.

We use t to denote the last coordinate in $M \times \mathbb{R}$.

The height function, denoted by h, of Σ in $M \times \mathbb{R}$ is defined as the restriction of the projection $t : M \times \mathbb{R} \to \mathbb{R}$ to Σ , i.e., if $p \in \Sigma$, $x(p) \in M \times \{t\}$, then $h(p) = t$. We have $\frac{\partial}{\partial t} = \overline{\nabla} h$.

3 THE ELLIPTIC PROPERTIES OF THE OPERATOR L_r

The following Prop. 3.1 is known (cf. Elbert 2002. Lemma 3.10). For completeness, we give its proof.

PROPOSITION 3.1. *Let $\overline{M}^{m+1}$ be an $(m+1)-$dimensional oriented Riemannian manifold and let Σ^m be a connected $m-$dimensional orientable Riemannian manifold. Suppose $x : \Sigma \to \overline{M}$ is an isometric immersion. If $H_2 > 0$, then the operator L_1 is elliptic.*

PROOF. Since L_1 is elliptic if and only if T_1 is positive definite, we will prove that T_1 is positive definite.

By $H_2 > 0$ and $H_1^2 \geq H_2$, H_1 is nonzero and has the same sign on Σ. We may choose the normal N such that H_1 is positive. So $S_1 = mH_1 > 0$.

By $S_1^2 = \sum \lambda_i^2 + 2S_2 > \lambda_i^2$, we have $S_1 > \lambda_i$. Then $S_1(A_i) = S_1 - \lambda_i > 0$. Since $T_1 e_i = S_1(A_i)e_i$ (Prop. 2.1 (i)), T_1 is positive definite. $\square$

In Prop. 3.2, we will prove the ellipticity of L_r when the hypersurface Σ with positive r-mean curvature has an elliptic or parabolic point.

PROPOSITION 3.2. *Let $\overline{M}^{m+1}$ be an $(m+1)-$dimensional oriented Riemannian manifold and let Σ^m be a connected $m-$dimensional orientable Riemannian manifold (with or without boundary). Suppose $x : \Sigma \to \overline{M}$ is an isometric immersion with $H_r > 0$ for some $1 \leq r \leq m$. If there exists an interior point p of Σ such that all the principle curvatures at p are nonnegative, then for all $1 \leq j \leq r-1$, the operator L_j is elliptic, and the j-mean curvature H_j is positive.*

PROOF. Since L_j is elliptic if and only if T_j is positive definite, it is sufficient to prove that the eigenvalues of T_j are positive, that is $S_j(A_i) > 0$ on Σ, for all $1 \leq j \leq r-1$, $1 \leq i \leq m$.

In the following proof, the ranges of j and i are $1 \leq j \leq r-1$, $1 \leq i \leq m$ respectively.

Since $S_r > 0$ and the principal curvatures of x at p are nonnegative, we have that at p, at least r principal curvatures are positive. For simplicity, we suppose that $\lambda_1, \lambda_2, \ldots, \lambda_r$ are positive.

Then, by direct verification, at p, $S_j(A_i) > 0$.

By continuity, there exists an open intrinsic ball $B(p) \subset \Sigma$ with center p such that the functions $S_j(A_i) > 0$ on $B(p)$.

For any $q \in \Sigma$, since Σ is connected, there exists a path $\gamma(t)(t \in [0,1])$ in Σ, joining p to q with $\gamma(0) = p$ and $\gamma(1) = q$.

Define $J = \{t \in [0,1] | S_j(A_i) > 0 \quad \text{on} \quad \gamma|_{[0,t]}\}$. Let $t_0 = \sup J$.

Note $S_j(A_i) > 0$ on $B(p)$ so $t_0 > 0$. By continuity, at t_0, $S_j(A_i) \geq 0$.

We will prove that $S_j(A_i) > 0$ at t_0 and hence $t_0 \in J$.

We first show that $S_{r-1}(A_i) > 0$ at t_0. Otherwise, there exists $1 \leq i \leq r-1$ such that $S_{r-1}(A_i) = 0$ at t_0. For this i, by $S_r = \lambda_i S_{r-1}(A_i) + S_r(A_i)$, we have $S_r(A_i) = S_r > 0$ at t_0. So, at t_0, $S_j(A_i) \geq 0$, $1 \leq j \leq r-1$ and $S_r(A_i) > 0$. By (ii) in Prop. 2.2, at t_0, the following inequality holds,

$$H_1(A_i) \geq H_2(A_i)^{\frac{1}{2}} \geq \ldots \geq H_{r-1}(A_i)^{\frac{1}{r-1}} \geq H_r(A_i)^{\frac{1}{r}}.$$

Then at t_0, $H_{r-1}(A_i) > 0$, i.e., $S_{r-1}(A_i) > 0$, which is a contradiction. Thus, we have $S_{r-1}(A_i) > 0$ at t_0.

Next, by $S_{r-1}(A_i) > 0$ and $H_1(A_i) \geq H_2(A_i)^{\frac{1}{2}} \geq \ldots \geq H_{r-1}(A_i)^{\frac{1}{r-1}}$ at t_0, we have that $S_j(A_i) > 0$ at t_0. Hence $t_0 \in J$.

If $t_0 < 1$, by continuity, there exists an open intrinsic ball $B(\gamma(t_0))$ of center $\gamma(t_0)$ such that $S_j(A_i) > 0$ on $B(\gamma(t_0))$, which contradicts our choice of $t_0 = \sup J$. Hence, $t_0 = 1$.

So we obtain that at q, $S_j(A_i) > 0$. Hence the L_j, for all $1 \leq j \leq r-1$ are elliptic.

By Prop. 2.1 (ii), H_j (for all $1 \leq j \leq r-1$) are positive. □

4 THE HEIGHT ESTIMATES OF THE VERTICAL GRAPH

In this section, we will consider hypersurfaces in an $(m+1)-$dimensional product manifold $M^m \times \mathbb{R}$ of positive constant r-mean curvature.

LEMMA 4.1. *Let M be an $m-$dimensional oriented Riemannian manifold and let Σ be an immersed orientable hypersurface in $M \times \mathbb{R}$ (with or without boundary). Then*

$$L_{r-1}(h) = rS_r n, \tag{4.1}$$

where $1 \leq r \leq m+1$, h denotes the height function of Σ, and $n = \left\langle N, \frac{\partial}{\partial t}\right\rangle$.

PROOF. Fix $p \in \Sigma$. Let $\{e_i\}$ be a geodesic orthonormal frame of Σ at p. So at p, $\nabla_{e_i} e_j(p) = 0$. We can assume $\{e_i\}$ are the principal directions at p, that is, $Ae_i(p) = \lambda_i e_i$, where λ_i are the eigenvalues (principal curvatures) of A at p (this frame can been obtained by rotating $\{e_i\}$).

Observe that the vertical translation in $M \times \mathbb{R}$ is an isometry. Hence $\overline{\nabla} h = \frac{\partial}{\partial t}$ is a Killing vector field, and $\overline{\nabla}_{e_i} \overline{\nabla} h = 0$, for $i = 1, \ldots, m$.

We have

$$\begin{aligned} \operatorname{Hess}(h)(e_i) &= \nabla_{e_i} \nabla h = [\overline{\nabla}_{e_i}(\nabla h)]^T \\ &= [\overline{\nabla}_{e_i}(\overline{\nabla} h - \langle \overline{\nabla} h, N\rangle N)]^T \\ &= -[\overline{\nabla}_{e_i}(\langle \overline{\nabla} h, N\rangle N)]^T, \end{aligned}$$

where the T denotes the tangent part of the vector of $T(M \times \mathbb{R})$ to Σ.

Then

$$\begin{aligned} \langle \operatorname{Hess}(h)(e_i), e_i\rangle (p) &= -\left\langle \langle \overline{\nabla} h, N\rangle \overline{\nabla}_{e_i} N, e_i\right\rangle (p) \\ &= n \langle \lambda_i e_i, e_i\rangle (p) = n\lambda_i \end{aligned}$$

Hence

$$\begin{aligned} L_{r-1}(h)(p) &= \sum_i \langle e_i, T_{r-1}\operatorname{Hess} h(e_i)\rangle (p) \\ &= \sum_i n(p)\lambda_i T_{r-1}(e_i) \\ &= n(p)\operatorname{tr}(AT_{r-1}) = n(p) r S_r. \end{aligned} \tag{4.2}$$

Since rS_rn is independent of the choice of the frame, we obtain that, on Σ

$$L_{r-1}(h)(p) = rS_rn. \qquad (4.3\,\square)$$

In order to prove the following Lemma 4.2, we first recall some known results.

Let $\Sigma \subset \overline{M}^{m+1}$ be an oriented hypersurface. Let D be a compact domain of Σ. Consider a variation of D, denoted by ϕ. $\phi : (-\epsilon, \epsilon) \times \overline{D} \to M \times \mathbb{R}$, $\epsilon > 0$, such that for each $s \in (-\epsilon, \epsilon)$, the map $\phi_s : \{s\} \times \overline{D} \to M \times \mathbb{R}$, $\phi_s(p) = \phi(s, p)$ is an immersion, and $\phi_0 = \overline{D}$.

Let $A_t(p)$ be the shape operator of $\Sigma(t)$ at p and for $0 \leq r \leq m$, let $S_r(t)(p)$ be the r-th symmetric function of the eigenvalues of $A_t(p)$. We are interested in the first variation of S_r. To calculate this one may differentiate the equation $rS_r = \mathrm{tr}(T_{r-1}A)$. We refer the reader to (Rosenberg 1993) for details of this calculation in space forms. In general ambient spaces, Elbert (Elbert 2002. Proposition 3.2) proved the following result.

Set $E_s(p) = \frac{\partial \phi}{\partial s}(p, s)$ and $f_s =< E_s, N_s >$, where N_s is the unit normal to $\phi_s(D)$. We have

$$\begin{aligned} \frac{\partial}{\partial s} S_r(s) &= L_{r-1}(f_s) + f_s(S_1S_r - (r+1)S_{r+1}) \\ &+ f_s\mathrm{tr}(T_{r-1}\overline{R}_N) + E_s^T(S_r), \end{aligned} \qquad (4.4)$$

where $\overline{R}_N$ is defined as $\overline{R}_N(X) = \overline{R}(N, X)N$, $\overline{R}$ the curvature operator of $\overline{M}$, and E_s^T denotes the tangent part of E_s.

LEMMA 4.2. *With M and Σ as in Lemma 4.1, assume H_r of Σ is constant, for some r, $1 \leq r \leq m+1$. Then, on Σ,*

$$L_{r-1}(n) = -n(S_1S_r - (r+1)S_{r+1} + \mathrm{tr}(T_{r-1}\overline{R}_N)). \qquad (4.5)$$

PROOF. We may choose the variation $\phi_s : (-\epsilon, \epsilon) \times D \subset (-\epsilon, \epsilon) \times \Sigma \to M \times \mathbb{R}$ given by vertical translation of $M \times \mathbb{R}$, $(x, t_0) \to (x, t_0 + s)$, $(x, t_0) \in M \times R$.

Under this variation, $S_r(s)$ is constant, i.e., $\frac{\partial}{\partial s}S_r(s) = 0$. Also by the hypothesis $S_r =$ constant on Σ, we have $E_s^T(S_r) = 0$, for $s = 0$. Hence, when $s = 0$, we have $f_0 = \langle E_0, N_0 \rangle = \left\langle \frac{\partial}{\partial t}, N \right\rangle = n$ and

$$L_{r-1}(n) + n(S_1S_r - (r+1)S_{r+1} + \mathrm{tr}(T_{r-1}\overline{R}_N)) = 0. \qquad (4.6)$$

Thus (4.6) holds on Σ. $\square$

We will express $\mathrm{tr}(T_1\overline{R}_N)$ using the sectional curvature of M in order to prove Theorem 4.1.

Given $p = (x, t) \in M \times \mathbb{R}$. For $X \in T_pM \times \mathbb{R}$, let X^h denote the horizontal component of X. If $\{e_i\}$ denotes a geodesic frame of Σ at p and N denotes the unit normal to Σ at p, we have, by direct calculation,

$$\overline{R}(e_i, N, e_i, N) = \overline{R}(e_i^h, N^h, e_i^h, N^h) = K(e_i^h, N^h) \times |e_i^h \wedge N^h|^2,$$

where K is the sectional curvature of M.

If e_i are also the principal directions of A. We have $T_r e_i = S_r(A_i)e_i$. Then

$$\begin{aligned}\operatorname{tr}(T_r\overline{R}_N)(p) &= \sum_i \left\langle e_i, T_r\overline{R}_N(e_i)\right\rangle(p)\\ &= \sum_i S_r(A_i)\overline{R}(N, e_i, N, e_i)(p) \qquad (4.7)\\ &= \sum_i S_r(A_i)K(e_i^h, N^h)|e_i^h \wedge N^h|^2(p).\end{aligned}$$

THEOREM 4.1. *Let M be an m-dimensional oriented Riemannian manifold. Let Σ be a compact vertical graph in the product manifold $M \times \mathbb{R}$ with boundary in $M \times \{0\}$ and with positive constant H_r, for some $1 \le r \le m$. Let $h : \Sigma \to \mathbb{R}$ denote the height function of Σ.*

(i) *If the sectional curvature of M satisfies $K \ge 0$, then on Σ,*

$$|h| \le H_r^{-\frac{1}{r}}; \qquad (4.8)$$

(ii) *When $r = 2$, if the sectional curvature of M satisfies $K \ge -\tau$ ($\tau > 0$), and $H_2 > \tau$, then on Σ,*

$$|h| \le \frac{\sqrt{H_2}}{H_2 - \tau}. \qquad (4.9)$$

(iii) *When $r = 1$, if the sectional curvature of M satisfies $K \ge -\tau$ ($\tau > 0$), and $H_1^2 > \frac{m-1}{m}\tau$, then on Σ,*

$$|h| \le \frac{H_1}{H_1^2 - \frac{m-1}{m}\tau}. \qquad (4.10)$$

PROOF. At a highest point, all the principal curvatures have the same sign. Since we assume that $H_r > 0$, we know that at this point, all the principal curvatures are nonnegative and the unit normal N to Σ is downward pointing. Since Σ is a vertical graph, we may choose the smooth unit normal field N to Σ to be downward pointing (i.e.,$n = \langle N, \frac{\partial}{\partial t}\rangle \le 0$ on Σ). Hence we can apply Prop. 3.1 and Prop. 3.2 to obtain that, L_{r-1} is elliptic, and $H_i (1 \le i \le r-1)$ are positive.

Define $\varphi = ch + n$ on Σ, where c is a positive constant to be determined. On $\partial\Sigma$, $\varphi = n \le 0$.

Since L_{r-1} is an elliptic operator, we have, by the maximum principle, that if $L_{r-1}\varphi \ge 0$, then $\varphi \le 0$ on Σ. Then $h \le -\frac{n}{c} \le \frac{1}{c}$.

Now we will choose c such that $L_{r-1}\varphi \ge 0$. By Lemma 4.1 and 4.2,

$$L_{r-1}(\varphi) = crS_r n - (S_1 S_r - (r+1)S_{r+1} + \operatorname{tr}(T_{r-1}\overline{R}_N))n.$$

So $L_{r-1}(\varphi) \ge 0$ is equivalent to

$$-rS_r c + S_1 S_r - (r+1)S_{r+1} + \operatorname{tr}(T_{r-1}\overline{R}_N) \ge 0.$$

If $K \geq a$ $(a \leq 0)$, we have

$$\begin{aligned}
\mathrm{tr}(T_{r-1}\overline{R}_N)(p) &= \sum_i S_{r-1}(A_i)K(e_i^h, N^h)|e_i^h \wedge N^h|^2 \\
&\geq a\sum_i S_{r-1}(A_i)|e_i^h \wedge N^h|^2 \\
&\geq a\sum_i S_{r-1}(A_i) \\
&= (m-r+1)aS_{r-1}. \quad \text{(by Prop. 2.1)}
\end{aligned}$$

So we may choose c such that

$$-rS_rc + S_1S_r - (r+1)S_{r+1} + (m-r+1)aS_{r-1} \geq 0.$$

When $H_1, \dots, H_r$ are nonnegative, $H_1H_r \geq H_{r+1}$, that is, $\frac{m-r}{m}S_1S_r \geq (r+1)S_{r+1}$. So it is sufficient to choose c such that

$$-rS_rc + \frac{r}{m}S_1S_r + (m-r+1)aS_{r-1} \geq 0.$$

Then, $c \leq \frac{S_1}{m} + \frac{(m-r+1)a}{r} \cdot \frac{S_{r-1}}{S_r} = H_1 + a \cdot \frac{H_{r-1}}{H_r}$.

(i) Take $a = 0$ and choose $c \leq H_r^{\frac{1}{r}}$. Hence $h \leq H_r^{-\frac{1}{r}}$.

(ii) Take $a = -\tau(\tau > 0)$. Then $c \leq H_1 - \tau \cdot \frac{H_{r-1}}{H_r}$.
When $r = 2$, $c \leq H_1 - \tau \cdot \frac{H_1}{H_2} = H_1(1 - \frac{\tau}{H_2})$.
By $H_1 \geq \sqrt{H_2}$, we may choose $c \leq \sqrt{H_2}\left(1 - \frac{\tau}{H_2}\right) = \frac{H_2-\tau}{\sqrt{H_2}}$. Hence $h \leq \frac{\sqrt{H_2}}{H_2-\tau}$.

(iii) When $K \geq a(a < 0)$ and $r = 1$, we have a better estimate. Note

$$\begin{aligned}
\mathrm{tr}(T_0\overline{R}_N)(p) &= \sum_i K(e_i^h, N^h)|e_i^h \wedge N^h|^2 \\
&\geq a\sum_i |e_i^h \wedge N^h|^2 \\
&= a(m-1)|N^h|^2 \\
&\geq (m-1)a.
\end{aligned}$$

Similar to the above, we may choose $c \leq \frac{S_1}{m} + \frac{(m-1)a}{S_1} = H_1 + a \cdot \frac{(m-1)}{mH_1}$. Take $a = -\tau(\tau > 0)$. We have $h \leq \frac{H_1}{H_1^2 - \frac{m-1}{m}\tau}$. □

Remark 4.1. The height estimate in Theorem 4.1 is sharp. Consider a hemisphere of the unitary round sphere $\mathbb{S}^m$ in $\mathbb{R}^{m+1}$. It is a vertical graph on $\mathbb{R}^m = \mathbb{R}^m \times \{0\}$ of $H_r = 1$ with boundary $\mathbb{S}^{m-1} \times \{0\}$ and has the maximum height 1.

REMARK 4.2. In the case $m = 2$ and $r = 1$, (i) and (iii) of Theorem 4.1 was proved in (Hoffman et al. 2005).

THEOREM 4.2. *Let M be an m-dimensional Riemannian manifold and let Σ be a compact orientable embedded hypersurface in $M \times \mathbb{R}$ with $H_r = constant > 0$, for some $1 \leq r \leq m$. Then Σ is symmetric about some horizontal surface $M \times \{t_0\}$, $t_0 \in \mathbb{R}$. Moreover,*

(i) *If the sectional curvature of M satisfies $K \geq 0$, then the extrinsic vertical diameter of Σ is no more than $2H_r^{-\frac{1}{r}}$;*

(ii) *When $r = 2$, if the sectional curvature of M satisfies $K \geq -\tau$ and $H_2 > \tau (\tau > 0)$, then the extrinsic vertical diameter of Σ is no more than $\frac{2\sqrt{H_2}}{H_2 - \tau}$.*

(iii) *When $r = 1$, if the sectional curvature of M satisfies $K \geq -\tau$ $(\tau > 0)$, and $H_1^2 > \frac{m-1}{m}\tau$, then the extrinsic vertical diameter of Σ is no more than $\frac{2H_1}{H_1^2 - \frac{m-1}{m}\tau}$.*

PROOF. We will prove that Σ has a horizontal surface of symmetry $M(t_0) = M \times \{t_0\}$, for some $t_0 \in \mathbb{R}$.

Note that vertical translation is an isometry, as well as reflection through each horizontal $M(t)$. Hence we can use the Alexandrov reflection technique. One comes down from above Σ with the horizontal surfaces $M(t)$. For t slightly smaller than the highest value of the height of Σ, the part of Σ above $M(t)$ is a vertical graph of bounded gradient over a domain in $M(t)$. The symmetry through $M(t)$ of this part of Σ, is below $M(t)$, contained in the domain bounded by Σ, and meets Σ only along the boundary. One continues to do these reflections through the $M(t)$ translated downwards, until a first accident occurs. This plane is a plane of symmetry of Σ, and the part of Σ above this plane is a graph with zero boundary values. This proves the Theorem. $\square$

Remark 4.3. Consider the case $m = 2$ and $r = 2$ in Theorem 4.1 or Theorem 4.2. From the Gauss equation, we obtain that the intrinsic sectional curvature of Σ has a positive lower bound H_2. So Bonnet-Myers theorem yields the intrinsic diameter estimate of Σ and hence the vertical height or the extrinsic vertical diameter estimate. That is, (a) when Σ is a vertical graph with zero boundary values, a geodesic on Σ from the highest point to $\partial\Sigma$ (and hence the vertical height) is at most $\frac{\pi}{\sqrt{H_2}}$; (b) when Σ is a compact embedded surface, a geodesic on Σ from the highest point to the lowest point (and hence extrinsic vertical diameter) is at most $\frac{\pi}{\sqrt{H_2}}$. But our estimates (a) $\frac{1}{\sqrt{H_2}}$ and (b) $\frac{2}{\sqrt{H_2}}$ are better. Moreover our estimate works for any dimension m and any r, for which Bonnet-Myers theorem indeed doesn't apply.

THEOREM 4.3. *Let M be an m-dimensional compact Riemannian manifold and let Σ be a noncompact properly embedded hypersurface in $\mathbb{M} \times \mathbb{R}$ with $H_r = constant > 0$, for some $1 \leq r \leq m$. If the sectional curvature of M and the r-mean curvature of Σ satisfy the conditions in (i), (ii) or (iii) of Theorem 4.1, then the number of ends of Σ is not one.*

PROOF. Suppose on the contrary, that Σ has exactly one end E. Since Σ is properly embedded, E must go up or down, but not both. Assume E goes down. Then Σ has a highest point so we can do Alexandrov reflection coming down from above Σ with horizontal surfaces $M(t)$.

Since Σ is not invariant by symmetry in any $M(t)$, there is no first point of contact of the symmetry of the part of Σ above $M(t)$, with the part of Σ below $M(t)$. But by the Alexandrov reflection technique, the part of Σ above each $M(t)$ is always a vertical graph. This contradicts the height estimate for such vertical graphs. $\square$

COROLLARY 4.1. *Let Σ be a noncompact properly embedded hypersurface in $\mathbb{S}^m \times \mathbb{R}$ with $H_r =$ constant > 0, for some $1 \leq r \leq m$. Then the number of ends of Σ is not one.*

5 ROTATIONAL SYMMETRY OF SOME SURFACES IN $\mathbb{S}^2 \times \mathbb{R}$ OF CONSTANT 2-MEAN CURVATURE

If M is the unit sphere $\mathbb{S}^2$ in $\mathbb{R}^3$, an embedded surface in $\mathbb{S}^2 \times \mathbb{R}$ of positive constant 2-mean curvature has another symmetry under some restriction. If we demand the vertical projection of such a surface on $\mathbb{S}^2 \times \{0\}$ is contained in an open hemisphere, then it is also rotational about a vertical line parallel to the vertical $\mathbb{R}$-axis.

We now give some definitions which are modifications of the related concepts in $\mathbb{R}^3$ (cf. Hopf 1983. p.147-148).

Let $\mathbb{D}$ be the open hemisphere in $\mathbb{S}^2$ and $\partial\mathbb{D}$ be the boundary of $\mathbb{D}$. Let $\{p, -p\}$ denote a pair of antipodal points on $\partial\mathbb{D}$, and γ denote a semi-circle on $\overline{\mathbb{D}}$ joining p and $-p$. We call the surface $P = \gamma \times \mathbb{R}$ in $\mathbb{S}^2 \times \mathbb{R}$ a vertical geodesic strip.

DEFINITION 5.1. *For a point $q \in \mathbb{D} \times \mathbb{R}$, a point $q' \in \mathbb{D} \times \mathbb{R}$ is called a point of symmetry of q about a vertical geodesic strip $P = \gamma \times \mathbb{R}$ if, for a geodesic l passing through q and perpendicular to P, we have $q' \in l$, lies on the opposite part of l divided by P, and $dist(q, P) = dist(q', P)$. This means q' is the image of q by the isometry of $S \times \mathbb{R}$ which is reflection of each $S(t)$ through $\gamma \times \{t\}$.*

A vertical geodesic strip $P = \gamma \times \mathbb{R}$ in $\mathbb{S}^2 \times \mathbb{R}$ is called a vertical strip of symmetry for a set $W \subset \mathbb{S}^2 \times \mathbb{R}$ if, for every point $q \in W$, its point of symmetry q' is also in W.

We will obtain the following:

Theorem 5.1. *Let $\mathbb{S}^2$ be the unit sphere in $\mathbb{R}^3$ and $\mathbb{D}$ be an open hemisphere in $\mathbb{S}^2$. Let Σ be a complete oriented embedded surface in $\mathbb{S}^2 \times \mathbb{R}$ with $H_2 =$ constant > 0. If $\Sigma \subset \mathbb{D} \times \mathbb{R}$, then Σ has the following properties:*

(i) *Σ is topologically a sphere; and*

(ii) *Σ is a surface of revolution about a vertical line parallel to the vertical $\mathbb{R}$-axis, that is, Σ is foliated by round circles in $\mathbb{D} \times \{t\}$, the centers of which are the same point on $\mathbb{D}$ modulo t, where $t \in (h_{min}, h_{max}) \subset \mathbb{R}$, and h_{min}, h_{max} are the maximum and minimum of vertical height of Σ respectively; and*

(iii) *there exists* $t_0 = \frac{h_{min}+h_{max}}{2}$ *such that* $\mathbb{D} \times \{t_0\}$ *divides* Σ *into two (upper and lower) symmetric parts.*

PROOF. By the Gauss equation, the sectional curvature of Σ is bounded below by a positive constant, hence Σ is compact by the Bonnet-Myers theorem. By the Gauss-Bonnet theorem, Σ is topologically a sphere. By Corollary 4.2, we know there exists $t_0 = \frac{h_{\min}+h_{\max}}{2}$ such that the horizontal level $\mathbb{D} \times \{t_0\}$ divides Σ into two (upper and lower) symmetric parts.

We prove Σ is rotational in two steps 1) and 2).

1) Σ has a vertical strip of symmetry about every pair of antipodal points.

Fix a pair of antipodal points $\{p, -p\}$ on $\partial\mathbb{D}$. Then $\partial\mathbb{D}$ is divided into two semi-circles joining p and $-p$. Let θ denote the rotational angle from one semi-circle to the other ($0 \leq \theta \leq \pi$), and $\gamma(\theta)$ denote the semi-circle on $\mathbb{D}$ joining p and $-p$, with the rotational angle θ.

Since $\Sigma \subset \mathbb{D} \times \mathbb{R}$ and Σ is compact, the vertical strip $P(\theta) = \gamma(\theta) \times \mathbb{R}$ is disjoint from Σ for sufficiently small θ. Note the rotation on $\mathbb{S}^2$ is an isometry. We may do the Alexander reflection through $P(\theta)$, as θ moves from 0 to π, to obtain a vertical strip of symmetry $P(\theta_0)$ such that Σ is divided into two graphs of symmetry over a domain of $P(\theta_0)$.

By the arbitrariness of p on $\partial\mathbb{D}$, we have proved 1).

2) Since any two such vertical strips of symmetry intersect in a line parallel to the vertical $\mathbb{R}$-axis, it is sufficient to prove all of these lines coincide.

Choose a horizontal surface $\mathbb{D}(t) = \mathbb{D} \times \{t\}$ whose intersection with Σ has at least two points.

Fix this t. Let $\Sigma(t) = \mathbb{D}(t) \cap \Sigma$. Since Σ is embedded and compact, $\Sigma(t)$ is a simple closed curve on $\mathbb{D}(t)$. Any vertical strip of symmetry $P = \gamma \times \mathbb{R}$ of Σ corresponds to a semi-circle of symmetry $\gamma(t) = \gamma \times \{t\}$ of $\Sigma(t)$ on $\mathbb{D}(t)$. By 1), we know that every pair of antipodal points $\{p, -p\}$ on $\partial\mathbb{D}(t)$ determines a semi-circle of symmetry γ of $\Sigma(t)$.

We will look into the relation of all the semi-circles of symmetry $\{\gamma(t)\}$. For simplification of notation, we omit t in $\gamma(t)$. Fix a semi-circle of symmetry γ_0 on $\mathbb{D}(t)$. Let γ_1 be any other one, and β denote the angle between γ_0 and γ_1 (we may choose the direction of the angle such that, if the rotation from the oriented γ_0 to the oriented γ_1 is anti-clockwise, β is positive). Then the reflection of $\Sigma(t)$ by γ_1, followed by a reflection of $\Sigma(t)$ by γ_0, corresponds to a rotation of $\Sigma(t)$ through an angle 2β about the intersection of γ_0 and γ_1, and leaves $\Sigma(t)$ invariant. By this property of rotation and the arbitrariness of $p \in \partial\mathbb{D}$, we know all of the rotations of γ are about the same point. Hence $\Sigma(t)$ contains a circle and thus must be this circle. Thus we have proved that all of γ intersect at the same point, and $\Sigma(t)$ is rotational about this point on $\mathbb{D}(t)$.

Therefore we have proved that Σ is a surface of revolution (rotational about a vertical line parallel to the $\mathbb{R}$-axis). $\square$

In the following, we discuss complete (hence compact) smooth surfaces of revolution, about a vertical line, in $\mathbb{S}^2 \times \mathbb{R}$ of positive constant 2-mean curvature. We will give the parameterized equation of such surfaces.

Given $o' \in \mathbb{S}^2$. Then $\{o'\} \times \mathbb{R}$ is a vertical line. Let Σ denote a complete surface of revolution about $\{o'\} \times \mathbb{R}$, in $\mathbb{S}^2 \times \mathbb{R}$ (that is to say that Σ is rotational about the vertical line $\{o'\} \times \mathbb{R}$).

Let $p \in \mathbb{S}^2 \times \mathbb{R}$, and let (ϕ, θ, t) $(0 \leq \phi \leq \pi, 0 \leq \theta \leq 2\pi)$ denote the local coordinate of p where t denotes the $\mathbb{R}$-coordinate of $\mathbb{R}$ and (ϕ, θ) denotes the spherical coordinate of $\mathbb{S}^2$ about o', that is, ϕ and θ are the polar coordinate and azimuthal coordinate respectively. Then $p = (\sin\phi\cos\theta, \sin\phi\sin\theta, \cos\phi, t)$.

The generating curve $\lambda(s)$ of the rotational surface Σ is $\lambda(s) = (\phi(s), t(s))$, where s is the arc-length parameter.

Σ can be denoted by $(\sin\phi(s)\cos\theta, \sin\phi(s)\sin\theta, \cos\phi(s), t(s))$.

Take the unit normal N of Σ as $N = (t'\cos\phi\cos\theta, t'\cos\phi\sin\theta, -t'\sin\phi, -\phi')$. Then the two principle curvatures of Σ are

$$\lambda_1 = t''\phi' - t'\phi'', \lambda_2 = t'\cot\phi, \tag{5.1}$$

and the 2-mean curvature of Σ is

$$H_2 = (t'\cot\phi)(\phi' t'' - t'\phi''). \tag{5.2}$$

Since s is the arc-length parameter, $t'^2 + \phi'^2 = 1$. Then $t't'' + \phi'\phi'' = 0$. So

$$H_2 = -\phi''(\cot\phi) = \text{const.} > 0. \tag{5.3}$$

Suppose y is the lowest point of Σ. We may choose s positive with $y = \lambda(0)$. We have $t'(0) = 0$. From $\lambda_2 = t'\cot\phi$, we know that $\phi = 0$, or π at y. Without loss of generality, we assume that $\phi = 0$ at y. This implies $y = o' \times \{t_y\}$ for some $t_y \in \mathbb{R}$.

Since $H_2 > 0, \phi \neq \frac{\pi}{2}$. Hence the domain of ϕ is contained either in $\left[0, \frac{\pi}{2}\right)$ or in $\left(\frac{\pi}{2}, \pi\right]$. Since we assume $\phi(0) = 0$, we have $\phi \in \left[0, \frac{\pi}{2}\right)$. This means that Σ stays in $\mathbb{D} \times \mathbb{R}$, where $\mathbb{D}$ is the open hemisphere of $\mathbb{S}^2$ of the center o'.

Since $y = \lambda(0)$ is the lowest point, we have $t'(s) > 0$ and hence $\lambda_2(s) > 0$ for sufficiently small positive s. But λ_2 must have the same sign, so $t'(s) > 0$ for all s. Hence t increases as s increases and Σ must be embedded.

Suppose $z = \lambda(s_1)$ is the highest point Σ. Also we have $\phi = 0$ at z (i.e., $z = o' \times \{t_z\}$ for some $t_z \in \mathbb{R}$.

Hence the domain of s is $[0, s_1]$, and $\phi(0) = \phi(s_1) = 0$; $t'(0) = t'(s_1) = 0$; $\phi'(0) = 1$, $\phi'(s_1) = -1$.

Since $H_2 = -\phi''(\cot\phi)$, we have $\phi'' < 0$. This implies ϕ' decreases from $\phi(0) = 1$ to $\phi'(s_1) = -1$. So as s increases from 0 to s_1, ϕ first increases from 0 to $\phi_{\max}$ then decreases from $\phi_{\max}$ to 0.

By equation (5.3),

$$\phi'^2 = 2H_2 \log\cos\phi + C_1, \tag{5.4}$$

where C_1 is a constant to be determined by the boundary conditions.

By $\phi(0) = 0$ and $\phi'(0) = 1$, we have $C_1 = 1$. So

$$\phi'^2 = 2H_2 \log\cos\phi + 1. \tag{5.5}$$

We have $\phi_{\min} = 0$, $\phi_{\max} = \phi|_{\phi'=0} = \cos^{-1}\exp\left(-\frac{1}{2H_2}\right)$.

Let $u = \phi'(|u| \le 1)$. We will give the equation of the generating curve λ.

By equation (5.5), we have

$$\phi = \cos^{-1}\exp\left(-\frac{1-u^2}{2H_2}\right), \quad (-1 \le u \le 1). \tag{5.6}$$

Since $u = \phi'$, $\frac{du}{ds} = \phi'' = -\frac{H_2}{\cot\phi}$, we have

$$\frac{dt}{du} = t' \cdot \frac{ds}{du} = \sqrt{1-\phi'^2}\left(-\frac{\cot\phi}{H_2}\right) = -\frac{1}{H_2}\frac{\sqrt{1-u^2}}{\sqrt{\exp\left(\frac{1-u^2}{H_2}\right)-1}}.$$

Then

$$t = -\frac{1}{H_2}\int_1^u \frac{\sqrt{1-u^2}}{\sqrt{\exp\left(\frac{1-u^2}{H_2}\right)-1}}du + C, (-1 \le u \le 1), \tag{5.7}$$

where C is a real constant.

We have

$$t_{\min} = t|_{u=1} = C, t_{\max} = t|_{u=-1} = \frac{1}{H_2}\int_{-1}^1 \frac{\sqrt{1-u^2}}{\sqrt{\exp\left(\frac{1-u^2}{H_2}\right)-1}}du + C.$$

$$t_0 = \frac{t_{\min}+t_{\max}}{2} = \frac{1}{H_2}\int_{-1}^0 \frac{\sqrt{1-u^2}}{\sqrt{\exp\left(\frac{1-u^2}{H_2}\right)-1}}du + C.$$

Therefore, we obtain the equations (5.6) and (5.7) of ϕ and t. As u decreases from 1 to -1, t increases from $t_{\min}$ to $t_{\max}$, and ϕ first increases from 0 to $\phi_{\max}$ then decreases from $\phi_{\max}$ to 0. Also Σ is symmetric about $\mathbb{D} \times \{t_0\}$.

From the above analysis, we obtain that

PROPOSITION 5.1. *Let Σ be a complete immersed surface of revolution (about a vertical line $o' \times \mathbb{R}$, $o' \in \mathbb{S}^2$) in $\mathbb{S}^2 \times \mathbb{R}$ with $H_2 = constant > 0$. Then Σ has the following characters:*

(i) Σ *is topologically a sphere and embedded; and*

(ii) Σ *stays in* $\mathbb{D} \times \mathbb{R}$, *where* $\mathbb{D}$ *denotes the open hemisphere of* $\mathbb{S}^2$ *of the center* o'; *and*

(iii) *the generating curve of* Σ *can be denoted by* $\lambda(u) = (\phi(u), t(u))$ *with parameter* $u(-1 \leq u \leq 1)$, *where* t *denotes the* $\mathbb{R}$*-coordinate of* $\mathbb{R}$ *and* ϕ *denotes the polar coordinate (about* o'*) of* $\mathbb{S}^2$, *as follows:*

$$\phi = \cos^{-1} \exp\left(-\frac{1-u^2}{2H_2}\right), \tag{5.8}$$

$$t = -\frac{1}{H_2}\int_1^u \frac{\sqrt{1-u^2}}{\sqrt{\exp\left(\frac{1-u^2}{H_2}\right)-1}}du + C, \tag{5.9}$$

where C *is a real constant.*

The equation of λ *implies that there exists a horizontal level* $\mathbb{D} \times \{t_0\}$, $t_0 \in \mathbb{R}$ *such that* Σ *divides into two (upper and lower) symmetric parts.*

By Theorem 5.1 and Proposition 5.1, we can determine the generating curvature of Σ satisfying the conditions of Th. 5.1 as follows:

COROLLARY 5.1. *Let* $\mathbb{S}^2$ *be the unit sphere in* $\mathbb{R}^3$ *and* $\mathbb{D}$ *be the open upper hemisphere in* $\mathbb{S}^2$. *Let* Σ *be a complete orientable embedded surface in* $\mathbb{S}^2 \times \mathbb{R}$ *with* $H_2 =$ *constant* > 0. *If* $\Sigma \subset \mathbb{D} \times \mathbb{R}$, *then* Σ *has the following properties:*

The 2*-mean curvature* H_2 *satisfies*

$$H_2 > -\frac{1}{2\log\sin\phi_0}, \quad \text{if} \quad \phi_0 \in \left(0, \frac{\pi}{2}\right); \qquad H_2 > 0, \quad \text{if} \quad \phi_0 = 0$$

(where $\phi_0 \in \left[0, \frac{\pi}{2}\right)$ *denotes the polar angle between the north pole of the open upper hemisphere* $\mathbb{D}$ *and* o'*).*

Moreover, the generating curve of the rotational surface Σ *can be denoted by* $\lambda(u) = (\phi(u), t(u))$ *where* $-1 \leq u \leq 1$, *and* ϕ *denotes the polar coordinate (about* o'*) of* $\mathbb{S}^2$, *and:*

$$\phi = \cos^{-1} \exp\left(-\frac{1-u^2}{2H_2}\right), \tag{5.10}$$

$$t = -\frac{1}{H_2}\int_1^u \frac{\sqrt{1-u^2}}{\sqrt{\exp\left(\frac{1-u^2}{H_2}\right)-1}}du + C, \tag{5.11}$$

where C *is a real constant.*

PROOF. We just need to prove $H_2 > -\frac{1}{2\log\sin\phi_0}$. Since $\Sigma \subset \mathbb{D}^2 \times \mathbb{R}$, $\phi_0 + \phi_{\max} < \frac{\pi}{2}$. Here $\phi_{\max}$ denotes the maximum of ϕ. Of course, we have $\phi_{\max} = \phi|_{\phi'=0} = \cos^{-1}\exp\left(-\frac{1}{2H_2}\right)$. By it, $H_2 > -\frac{1}{2\log\sin\phi_0}$. $\square$

REMARK 5.1. We have proved that if Σ is a complete embedded surface in $\mathbb{D}^2 \times \mathbb{R} \subset \mathbb{S}^2 \times \mathbb{R}$ of positive constant 2-mean curvature, Σ is rotationally symmetric about a vertical line. We conjecture this is true for immersed such Σ in $\mathbb{S}^2 \times \mathbb{R}$ (also in $\mathbb{H}^2 \times \mathbb{R}$). This is true if one assumes that the mean curvature is constant instead of positive constant H_2 under the condition that the genus of Σ is zero (Abresch and Rosenberg 2005).

ACKNOWLEDGMENTS

This work was done when the first author was visiting Université Paris 7 (Institut de Mathématiques de Jussieu), supported by a post-doctoral fellowship by Centre Nationale de la Recherche Scientifique (CNRS), France. She would like to thank Université Paris 7 for support and hospitality.

RESUMO

Seja M uma variedade riemanniana de dimensão m, de curvatura seccional limitada de abaixo. Consideramos as hipersuperfícies na variedade produto $M \times \mathbf{R}$ de dimensão $m+1$, com curvatura r-média constante positiva. Obtemos uma estimativa para altura das alguns gráficos verticais em $M \times \mathbf{R}$ com seus fronteiras em $M \times \{0\}$. Aplicamos isto para obter as obstruções topológicas sobre existência das algumas hipersuperfícies. Também discutimos a simetria rotacional das algumas superfícies completas em $\mathbf{S}^2 \times \mathbf{R}$ de curvatura 2-média constante positiva.

Palavras-chave: variedade produto, hipersuperfície, curvatura r-média.

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