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# EMBEDDING FREE ALGEBRAS IN SKEW FIELDS

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ABSTRACT. This paper constructs a minimal element in the partial order on the set of skew fields generated by a free algebra, and shows that the partial order contains a certain sub partial order. Examples of embedding free algebras in skew fields of heights one and two are also given.

Let $R$ be an integral domain (noncommutative) with identity. We will assume that the integral domain $R$ is always embedded in some skew field (this need not be the case in general [2], [3], [6], [7]). There are many skew fields which contain $R$ but in the commutative case there is only one field which is generated by $R$, namely the field of fractions. In the noncommutative case there may be several distinct skew fields generated by $R$ [4, pp. 277]. For skew fields $D_1$ and $D_2$ generated by $R$, we say $D_1 \geqq D_2$ if there exists a place from $D_1$ to $D_2$ which extends the natural isomorphism between the embeddings of $R$. This paper shows that this is a partial order on the set $\mathcal{P}$ (we identify isomorphic embeddings) of skew fields generated by $R$ and in the case where $R$ is the free algebra on two generators we show that $\mathcal{P}$ contains the subposet $\mathcal{L}$ where $C(x)$ is the unique maximal element [1, Theorem 27] of $\mathcal{P}$ and $K_i$ are distinct elements of $\mathcal{P}$ with $K_2$ minimal in $\mathcal{P}$. We also examine the height of an integral domain and give examples of embeddings of different heights of a free algebra in two skew fields.

## I. The partial order and the chain of domains $Q_i(K, \phi(R))$.

DEFINITION. Let $D$ be a skew field and $\phi$ an isomorphism of $R$ into $D$. $R$ is fully embedded in $D$ if the smallest sub skew field of $D$ containing $\phi(R)$ is $D$ itself. We will denote a full embedding of $(D, \phi(R))$ by $(D, \phi)$.

DEFINITION. If $D_1$ and $D_2$ are two division rings we say $\phi$ is a place from $D_1$ to $D_2$ if $\phi$ is a homomorphism from a local subring $S$ of $D_1$ onto $D_2$ [local means the set of nonunits is an ideal].

DEFINITION. For full embeddings $(D, \alpha)$ and $(K, \gamma)$ we say $(D, \alpha) \geqq (K, \gamma)$ if there is a place $\phi$ from $D$ to $K$ such that $\phi^{-1}(K) \supset \alpha(R)$ and $\phi | \alpha(R) = \gamma\alpha^{-1}$.

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$(D, \alpha) \cong (K, \gamma)$ if there is an isomorphism $\phi$ between $D$ and $K$ such that $\phi | \alpha(R) = \gamma\alpha^{-1}$. If we identify isomorphic embeddings the class of embeddings is a set.

We also make the following notation. Suppose $\phi$ is an isomorphism of $R$ into a skew field $K$.

$Q_0(K, \phi(R)) = \phi(R)$.

$Q_1(K, \phi(R)) =$ subring of $K$ generated by $\{a, b^{-1}$ such that $a, b \in Q_0(K, \phi(R))$ with $b \neq 0\}$.

Inductively we define

$Q_i(K, \phi(R)) =$ subring of $K$ generated by $\{a, b^{-1}$ such that $a, b \in Q_{i-1}(K, \phi(R))$ with $b \neq 0\}$.

When no confusion arises, we will write $Q_i$, $Q_i(K, \phi)$ or $Q_i(\phi(R))$ for $Q_i(K, \phi(R))$.

It is easy to see that $\bigcup_i Q_i(K, \phi(R))$ is the skew subfield of $K$ generated by $R$ and $(\bigcup Q_i, \phi)$ is a full embedding.

Lemma 1. *The relation $\geqq$ is a partial order on the set of full embeddings $\mathcal{P}$ of $R$.*

Proof. We will show that if $(D_1, \alpha) \geqq (D_2, \gamma)$ and $(D_2, \gamma) \geqq (D_1, \alpha)$ that $(D_1, \alpha) \cong (D_2, \gamma)$. The rest is clear. Let $\phi_1$ be a place from $S_1$ onto $D_2$, $D_1 \supset S_1 \supset \alpha(R)$ and $\phi_2$ a place from $S_2$ onto $D_1$, $D_2 \supset S_2 \supset \gamma(R)$. $\phi_1 | \alpha(R) = \gamma\alpha^{-1} = (\alpha\gamma^{-1})^{-1} = \phi_2^{-1} | \alpha(R)$. Hence $r \neq 0 \in Q_0(D_1, \alpha)$ implies $\phi_1(r) \neq 0 \Rightarrow r^{-1} \in S_1$ so that $S_1 \supset Q_1(D_1, \alpha)$. $\phi_1 | Q_1(D_1, \alpha)$ is an isomorphism since $\phi_1(\sum m_i) = 0$, where $m_i$ is a product of elements of $\alpha(R)$ and inverses of elements of $\alpha(R)$, implies $\sum m_i = \phi_2(\sum \phi_2^{-1}(m_i)) = \phi_2(\sum \phi_1(m_i)) = \phi_2[\phi_1(\sum m_i)] = 0$. $\phi_1 = \phi_2^{-1}$ on $Q_1(D_1, \alpha)$. Similarly $\phi_1$ is an isomorphism on all $Q_i$ and hence

$$D_1 = \bigcup Q_i(D_1, \alpha) \overset{\phi_1}{\cong} \bigcup Q_i(D_2, \gamma) = D_2.$$

Lemma 2. *If $(K, \alpha)$ is a full embedding and $Q_i(K, \alpha)$ is simple for each $i \geqq 1$ then $(K, \alpha)$ is minimal in $\mathcal{P}$.*

Proof. Suppose $(K, \alpha) \geqq (D, \gamma)$. Hence $\phi$ is a homomorphism from a local subring $S$ of $K$ onto $D$. $\phi | \alpha(R) = Q_0$ is an isomorphism. Suppose $\phi | Q_j$ is an isomorphism. Hence $Q_{j+1} \subset S$ and $\phi$ maps $Q_{j+1}(K, \alpha)$ onto $Q_{j+1}(D, \gamma)$. Since $Q_{j+1}(K, \alpha)$ is simple $\phi$ is an isomorphism on $Q_{j+1}$. Hence $\phi$ is an isomorphism from $\bigcup Q_i = K$ onto $D$.

Our last general observation is the following. Let $I$ be an ideal of $Q_{i+1}$, and let $i$ be the inclusion of $Q_i$ into $Q_{i+1}$. This induces a monomorphism $\bar{\imath}$ from $Q_i$ into $Q_{i+1}/I$. This is clear since every element of $Q_i$ has an inverse in $Q_{i+1}$ and hence in $Q_{i+1}/I$. $Q_i$ and its inverses still generate $Q_{i+1}/I$.

II. **The $K$-height of a free algebra in two different skew fields.** Suppose $\phi$ is an isomorphism of $R$ into a skew field $K$. The $K$-height of $\phi(R)$ is $h_K(\phi(R)) = \min\{i : Q_i(K, \phi(R))$ is a skew field$\}$ or $\infty$ if $Q_i$ is not a skew field for any $i$.

Denote the free algebra on a set $X$ over the field $F$ by $F[X]$. B. H. Neumann [**8**] gave an example of an embedding of $F[X]$ in a skew field $D$ and conjectured that $h_D(F[X]) = \infty$. Jategaonkar [**5**] gives a different embedding. We show that this has height 2 and construct another embedding of height one.

Let $Q$ be the rationals and let $F = Q(t_1, t_2, \cdots)$ be the commutative field with indeterminates $t_1, t_2, \cdots$. Denote the skew polynomial ring $F[x;\sigma]$ by $S$ where $\sigma$ is the monomorphism of $F$ into $F$ induced by mapping $t_i$ into $t_{i+1}$, and where $\sigma(t_i)x = xt_i$. $S$ is embedded in the division ring $K = \{x^{-n}\sum_{i=0}^{\infty} f_i x^i$ where $f_i \in F\}$. Since $xS + t_1 xS$ is direct, the ring $R = Q[x, t_1 x]$ is a free algebra over $Q$ and embedded in $K$ (Jategaonkar [**5**]).

THEOREM 1. $h_K(R) = 2$.

We will use the following series of lemmas to prove this.

LEMMA 3. *$Q_2(R)$ is a skew field.*

PROOF. The subring $Q_1(R)$ contains $x$ and $x^{-1}$ and hence contains $t_i = x^{i-1}(t_1 x)x^{-i}$. Thus $Q_1(R) \supset Q[t_1, \cdots][x;\sigma]$ which is left Ore with a ring of quotients $K_1$. Thus taking quotients again we get $Q_2(R) \supset K_1 \supset Q_1(R)$. Since $K_1$ is a field, by taking quotients again we have $Q_3(R) \supset K_1 \supset Q_2(R)$ and the lemma is proven.

LEMMA 4. *If $r \in R$ then $r = \sum f_i x^i$ where $f_i = \sum_I \alpha_I t_1^{\epsilon_1} t_2^{\epsilon_2} \cdots t_i^{\epsilon_i}$, $\alpha_I \in Q$, where $I$ ranges over all $i$-tuples $(\epsilon_1, \cdots, \epsilon_i)$ with $\epsilon_j = 1$ or $0$.*

PROOF. Since $R$ is a free algebra it is sufficient to examine a monomial $m$. We claim that if $m$ is of degree $i$ then $m = t_1^{\epsilon_1} t_2^{\epsilon_2} \cdots t_1^{\epsilon_i} x^i$ where $\epsilon_j = 1$ or $0$. If $i = 1$ then this is true, since the only monomials we have are $\alpha x$ and $\beta t_1 x$ where $\alpha, \beta \in Q$. Suppose true for $i \leqq n$. If $m$ is of degree $n+1$, then $m = m' t_1^{\epsilon_{n+1}} x$ where $m'$ has degree $n$, and $\epsilon_{n+1}$ is either 1 or 0. By induction $m' = t_1^{\epsilon_1} t_2^{\epsilon_2} \cdots t_n^{\epsilon_n} x^n$ and $m = t_1^{\epsilon_1} t_2^{\epsilon_2} \cdots t_n^{\epsilon_n} t_{n+1}^{\epsilon_{n+1}} x^{n+1}$. Gathering monomials of same degree we have the lemma.

The following notation will be used in the remaining lemmas. Define $\mathcal{S} = \{\sum_I \alpha_I t_1^{\epsilon_1} t_2^{\epsilon_2} \cdots t_i^{\epsilon_i} \neq 0$ where $I$ ranges over all $i$-tuples $(\epsilon_1, \cdots, \epsilon_i)$ where $\epsilon_j = 1$ or $0$, and $i$ ranges over all integers$\}$. Let $\mathfrak{M}$ be all products of elements of $\mathcal{S}$. It is easy to see that $\sigma(\mathcal{S}) \subset \mathcal{S}$ and thus $\sigma(\mathfrak{M}) \subset \mathfrak{M}$. Since $\mathfrak{M}$ is multiplicatively closed, let $V = Q[t_1, t_2, \cdots]$ localized about $\mathfrak{M}$. $\sigma(V) \subset V$ and from this it is not difficult to see that $U = \{x^{-n}\sum_{i=0}^{\infty} v_i x^i \mid v_i \in V\}$ is a subring of $K$.

LEMMA 5. $Q_1(R) \subset U$.

PROOF. It is sufficient to show that $r \in R$ implies $r^{-1} \in U$. Let $r = \sum_{i=0}^{n} f_i x^{i+i_0}$ where $f_0 \neq 0$ and $f_i \in \mathcal{S}$. $r^{-1} = x^{-i_0}(\sum_{i=0}^{n} f_i x^i)^{-1} = x^{-i_0} \sum_{i=0}^{\infty} b_i x^i$ where $f_0 b_0 = 1$, and $b_n = -f_0^{-1} \sum_{i=1}^{n} f_i \sigma^i(b_{n-i})$. Since $b_0 = f_0^{-1} \in V$ and $\sigma(V) \subset V$ we have by induction that $b_n \in V$. Thus $r^{-1} \in U$.

LEMMA 6. $(t_1 - t_2^2)^{-1}$ *is not contained in* $U$.

PROOF. If $(t_1 - t_2^2)^{-1}$ were in $U$ then it would be in $V$. Thus $(t_1 - t_2^2)^{-1} = f/g$ where $g \in \mathfrak{M}$ and $f \in Q[t_1, \cdots]$. Therefore $g = f(t_1 - t_2^2)$ where $t_1 - t_2^2$ is irreducible and hence divides $g = \pi p_i$ with $p_i \in \mathcal{S}$. This is impossible however since no $p_i$ has terms with square factors.

The theorem now follows since $(t_1 - t_2^2)^{-1} \notin Q_1(R)$ implies $Q_1(R)$ is not a skew field.

We continue with the same notation, and construct an embedding of height one by factoring out a maximal ideal of $Q_1(K, R)$. Let $P$ be the ideal of $Q[t_1, t_2, \cdots]$ generated by $t_1 - t_2^2, t_2 - t_3^2, \cdots$. It is clear that $\sigma(P) \subset P$. It is also not difficult to see that

$$Q[t_1, \cdots]/P \cong Q[\bar{t}_1, \bar{t}_1^{1/2}, \bar{t}_1^{1/4}, \cdots]$$

and hence $P$ is a prime ideal. $\sigma$ extends to an automorphism of $Q[\bar{t}_1, \bar{t}_1^{1/2}, \cdots]$ where $\sigma(\bar{t}_1^{1/2^n}) = \bar{t}_1^{1/2^{n+1}}$. In $Q[t_1, \cdots]/P$ every element can be written uniquely as $\sum_k \sum_I \alpha_{I,k} t^k \bar{t}_2^{\epsilon_1} \cdots \bar{t}_{n+1}^{\epsilon_n}$ so that $\mathfrak{M} \cap P = \varnothing$. Therefore factoring out the ideal of $Q_1(Q[x, t_1 x])$ generated by $P$ we get the free algebra $Q[x, t_1 x]$ embedded in the Ore domain $Q[\bar{t}_1, \bar{t}_1^{1/2}, \bar{t}_1^{1/4}, \cdots][x; \sigma]$ and thus in a skew field, namely the domain's field of quotients $K_2$.

We will write $t$ for $\bar{t}_1$.

THEOREM 2. $Q_1(K_2, Q[x, tx])$ *is a division ring.*

PROOF. Since $x$ and $x^{-1}$ are in $Q_1$, all integral powers of $x$ are in $Q_1$. We first observe that if $p(x) \in Q[t, t^{1/2}, \cdots][x; \sigma]$ then there exists integers $n$ and $m$ such that $x^n p(x) x^m \in Q[x, tx]$. Suppose $p(x) = \sum_{i=0}^{k} a_i x^i$ where

$$a_i = \sum_{j=0}^{k_i} \left( \sum_I \alpha_{I,i,j} t^{\epsilon_1/2} \cdots t^{\epsilon_m/2^m} \right) t^j$$

where $I$ runs over the $m$-tuples $(\epsilon_1, \cdots, \epsilon_m)$ with each $\epsilon_l = 0$ or 1. Pick $n$ to be an integer such that $2^n \geqq \max\{k_i\}$.

Therefore $x^n p(x) = \sum_{i=0}^{k} a_i' x^{i+n}$ where each

$$a_i' = \sum_{j=0}^{k_i} \left( \sum_I \alpha_{I,i,j} t^{\epsilon_1/2^{n+1}} \cdots t^{\epsilon_m/2^{n+m}} \right) t^{j/2^n}.$$

Each $j$ can be written in the form $\gamma_n + \gamma_{n-1}2 + \cdots + \gamma_0 2^n$ with $\gamma_i = 0$, or 1. Hence

$$t^{j/2^n} t^{\epsilon_1/2^{n+1}} \cdots t^{\epsilon_m/2^{n+m}} = t^{\gamma_0} t^{\gamma_1/2} \cdots t^{\gamma_n/2^n} t^{\epsilon_1/2^{n+1}} \cdots t^{\epsilon_m/t^{n+m}}$$

where $(\gamma_0, \gamma_1, \cdots, \gamma_n, \epsilon_1, \cdots, \epsilon_m)$ is an $n+m$ tuple of zeros and ones. Hence $x^n p(x) x^m \in Q[x, tx]$.

Since $R = Q[t, t^{1/2}, \cdots][x; \sigma]$ is Ore and contained in $Q_1$ we may assume that an element of $Q_1$ is written $q(x)^{-1} p(x)$ with $q(x)$ and $p(x) \in R$.

By the above we can choose integers $n$ and $m$ such that both $x^n p(x) x^m$ and $x^n q(x) x^m$ are in $Q[x, tx]$.

Hence $(x^n p(x) x^m)^{-1} \in Q_1$ and

$$1 = (q(x)^{-1} p(x)) [x^m (x^n p(x) x^m)^{-1} (x^n q(x) x^m) x^{-m}] \in Q_1$$

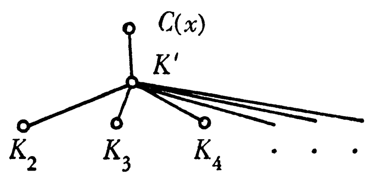
so that $Q_1$ is a division ring.

COROLLARY 1. $h_{K_2}(Q[x, tx]) = 1$.

III. **The poset of skew fields.** Let $K'$ be the field of quotients of $Q(t_1, t_2, \cdots)[x; \sigma]$ where $\sigma(t_i) = t_{i+1}$ and $xf = \sigma(f)x$ for $f \in Q(t_1, t_2, \cdots)$. Let $F_n$ be $Q[t_1, t_2, \cdots]$ localized about the prime ideal $P_n = \langle t_1 - t_2^n, t_2 - t_3^n, \cdots \rangle$ for a positive integer $n > 1$. $\sigma(P_n) \subset P_n$ so we can localize $F_n[x; \sigma]$ about the ideal generated by $P_n$, to obtain $S_n$. $S_n$ is a local ring which maps onto the field of quotients $K_n$ of $Q(s, s^{1/n}, s^{1/n^2}, \cdots)[y; \sigma]$. If $Q[u, v]$ is the free algebra on two generators, the map $\phi(u) = x$ and $\phi(v) = t_1 x$ maps $Q[u, v]$ onto a free algebra generating $K'$. Likewise $\phi_n(u) = y$ and $\phi_n(v) = sy$ maps $Q[u, v]$ onto a free algebra generating $K_n$.

THEOREM 3. *$(K_n, \phi_n) \not\gtreqless (K_m, \phi_m)$ for $n \neq m$.*

PROOF. The quotient field of $Q(s, s^{1/n}, s^{1/n^2}, \cdots)[y; \sigma]$ is $K_n$ where $ys = s^{1/n} y$ and the quotient field of $Q(r, r^{1/m}, r^{1/m^2}, \cdots)[z; \sigma]$ is $K_m$ where $zr = r^{1/m} z$. If $(K_n, \phi_n) \geqq (K_m, \phi_m)$ and $\gamma$ is the place then $\gamma(s) = \gamma(sy\, y^{-1}) = rz\, z^{-1} = r$ but $\gamma(s^{1/n}) = \gamma(y\, sy\, y^{-2}) = zrzz^{-2} = r^{1/m}$. $\gamma[(s^{1/n})^n] = (r^{1/m})^n = r$ which implies $n = m$.



Hence we have $(K', \phi) > (K_n, \phi_n)$ for each $n > 1$. Furthermore, since $Q_1(K_2, \phi_2)$ is a skew field and hence simple, Lemma 2 says $(K_2, \phi_2)$ is minimal. By [1, Theorem 27], $\mathcal{P}$ has a maximal element $C(x)$ and we have the subposet $\mathcal{L}$ embedded in $\mathcal{P}$.

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