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Abstract

A grid is a two-dimensional permutation: an $m \times n$-grid of size mn is an $m \times n$-matrix where the entries run through the elements $\{1, 2, \ldots, mn\}$. We prove that if δ_1 and δ_2 are any two linear orders on $\{1, 2, \ldots, N\}$, then they can be simultaneously embedded (in a well defined sense) into a unique grid having the smallest size.

Keywords: Linear order, biorder, matrix, grid, embedding

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1 Introduction

Let D be a finite set of cardinality n. A *linear order* (or a *permutation*) on D is a sequence $\delta = (x_1, x_2, \ldots, x_n)$ such that each element $x \in D$ occurs exactly once in δ. Also, if δ_1 and δ_2 are two linear orders on a common set $D = \{1, 2, \ldots, N\}$ for some $N \geq 1$, then the pair $\tau = (\delta_1, \delta_2)$ is called a *biorder* (on D).

We shall study the problem of representing biorders in the form of grids, i.e., matrices that have all entries different from each other. We show that each biorder τ has a unique (in a well defined sense) grid of the smallest size representing it.

A biorder (δ_1, δ_2) can be regarded as a partial order $\rho = \delta_1 \cap \delta_2$ of dimension two; see Trotter [5] for the results on the dimensions of partially ordered sets. A theorem of Dushnik and Miller states that a partially ordered set P has dimension at most two if and only if the incomparability graph of P is also a comparability graph. It is shown by Pnueli, Lempel and Even [4] that the partially ordered sets of dimension two are closely related to permutation graphs. Indeed, for a permutation graph $G = (D, E)$ one can find a biorder (δ_1, δ_2) on D such that $ab \in E$ if and only if $(a, b) \in \delta_1 \cap \delta_2$.

Another graph theoretical application of biorders is given in [3], see also [1, 2], where the notion of a *text* is introduced as an ordered triple $\tau = (\lambda, \delta_1, \delta_2)$ consisting of a function $\lambda \colon D \to S$ from the finite domain D into another set S (say, a word semigroup A^*) and of a biorder (δ_1, δ_2) on D. In a sense a text is a structured word.

2 Preliminaries

We denote by $[n, m]$ the interval $\{n, n+1, \ldots, m\}$ of integers. For pairs of integers, $(m, n) < (p, q)$ means that $m \leq p$, $n \leq q$ and $(m, n) \neq (p, q)$.

We shall often identify a singleton set $\{x\}$ with its element x.

Let ρ be a partial order on a finite set D, called the *domain* of ρ and denoted by $\mathrm{dom}(\rho)$. All domains in this paper will be finite, and without loss of generality we shall consider domains consisting of positive integers. The *dual order* of ρ is the partial order $\rho^{-1} = \{(x, y) \mid (y, x) \in \rho\}$.

The structure preserving functions, i.e., embeddings, considered in this paper preserve partial orders. To be more precise, let ρ_1 and ρ_2 be partial orders on the domains D_1 and D_2, respectively. A mapping $\varphi \colon D_1 \to D_2$ is *order preserving*, if $\varphi(\rho_1) \subseteq \rho_2$, where φ maps the relation ρ_1 pointwise, i.e.,

$$\varphi(\rho_1) = \{(\varphi(x), \varphi(y)) \mid (x, y) \in \rho_1\}\,.$$

An injective order preserving function φ is an *order embedding*.

Let ρ_1 and ρ_2 be disjoint partial orders, i.e., $\mathrm{dom}(\rho_1) \cap \mathrm{dom}(\rho_2) = \emptyset$. Then their (*directed*) *sum* is the partial order

$$\rho_1 \oplus \rho_2 = \rho_1 \cup \rho_2 \cup \{(x, y) \mid x \in D_1 \text{ and } y \in D_2\}\,.$$

Also, we adopt the convention that $\rho \oplus \emptyset = \rho = \emptyset \oplus \rho$. Clearly, the operation $\oplus$ is associative on disjoint partial orders, and therefore we can write

$$\sum_{i=1}^{n} \rho_i = \rho_1 \oplus \rho_2 \oplus \cdots \oplus \rho_n$$

for the unique sum $\rho_1 \oplus (\rho_2 \oplus (\cdots \oplus (\rho_{n-1} \oplus \rho_n)))$ of pairwise disjoint partial orders ρ_i, $i = 1, 2, \ldots, n$. Note, however, that the operation $\oplus$ is not commutative.

3 Grid biorders

Let δ_1 and δ_2 be linear orders on a common domain $[1, N]$ for some $N \geq 1$. Recall that the pair $\tau = (\delta_1, \delta_2)$ is a *biorder*. The *domain* of τ is the common domain of its components, $\mathrm{dom}(\tau) = [1, N]$.

If $\tau = (\delta_1, \delta_2)$ is a biorder, and δ'_1 and δ'_2 are two linear orders with $\mathrm{dom}(\delta'_1) = \mathrm{dom}(\delta'_2)$ such that $\delta'_1 \subseteq \delta_1$ and $\delta'_2 \subseteq \delta_2$, then we write $(\delta'_1, \delta'_2) \subseteq \tau$.

Let $\tau = (\delta_1, \delta_2)$ and $\tau' = (\delta'_1, \delta'_2)$ be two biorders (with $\mathrm{dom}(\tau) = [1, N]$ and $\mathrm{dom}(\tau') = [1, N']$). Then τ' is *embeddable* in τ, if there is a mapping $\varphi\colon \mathrm{dom}(\tau') \to \mathrm{dom}(\tau)$ such that φ is an order embedding simultaneously from δ'_1 into δ_1 and from δ'_2 into δ_2.

Example 3.1. Let $\tau = (\delta_1, \delta_2)$ be a biorder such that $\delta_1 = (2, 4, 1, 3)$ and $\delta_2 = (3, 1, 2, 4)$. Also, let $\tau' = (\delta'_1, \delta'_2)$ be a biorder with $\delta'_1 = (3, 1, 2)$ and $\delta'_2 = (2, 1, 3)$. Then τ' is embeddable in τ. Indeed, the mapping $\varphi\colon [1, 3] \to [1, 4]$ given by $\varphi(1) = 1$, $\varphi(2) = 3$, and $\varphi(3) = 4$ is an order embedding from δ'_1 into δ_1 and from δ'_2 into δ_2. $\square$

Let D be a set and $m, n \geq 1$ be integers. The set of all $m \times n$ matrices with entries in D is denoted by $D^{m\times n}$. For a matrix $M \in D^{m\times n}$, its entries are denoted by $M_{(i,j)}$ for $i \in [1, m]$ and $j \in [1, n]$. Let $\mathrm{size}(M) = (m, n)$ denote the size of the matrix M. Also, let $M_i = (M_{(i,1)}, \ldots, M_{(i,n)})$ be the ith row vector and $M^T_j = (M_{(1,j)}, \ldots, M_{(m,j)})$ the jth column vector of M. Here the matrix M^T is the transpose of M and thus the jth column vector of M equals the jth row vector of M^T.

A matrix $M \in D^{m\times n}$ is called an $m \times n$-*grid*, if $D = [1, mn]$ and the entries of M are all distinct, that is, $\{M_{(i,j)} \mid i \in [1, m], j \in [1, n]\} = [1, mn]$. Hence a grid is a generalization of a permutation to two dimensions.

We shall study biorders (δ_1, δ_2) that can be represented by grids in such a way that both linear orders can be read from the representing grid. In the following we choose a basic way of reading a grid to produce a biorder. In order not to loose any biorders, such a way of reading must be carefully chosen.

Let M be an $m \times n$-grid. We note first that the row and column vectors M_i and M^T_i can be interpreted as linear orders in a natural way. Then $(M^T_i)^{-1}$ is the dual order corresponding to the ith column of M, and we denote this also by M_i^{-T}. We define the linear orders $\alpha(M)$ and $\beta(M)$ as follows:

$$\alpha(M) = \sum_{i=1}^{m} M_i \quad \text{and} \quad \beta(M) = \sum_{i=1}^{n} M_i^{-T}\,.$$

The *grid biorder* of M is the biorder $\mathrm{Bi}(M) = (\alpha(M), \beta(M))$; see Fig. 1. It is easy to see that, for each row M_i and for each column M_j^T of M,

$$\alpha(M) \cap M_i = \beta(M) \cap M_i \quad \text{and} \quad \alpha(M) \cap M_j^T = \beta(M)^{-1} \cap M_j^T\,. \tag{1}$$

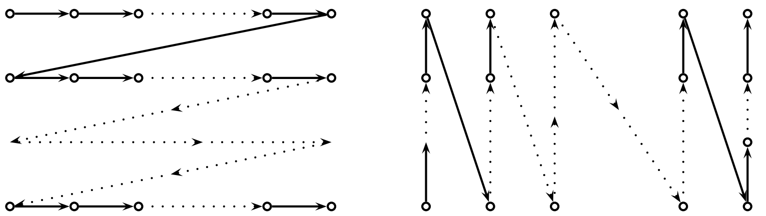

Figure 1: Reading orders of a grid: $\alpha(M)$ and $\beta(M)$.

Example 3.2. (1) Consider the following grid

$$M = \begin{pmatrix} 4 & 7 & 2 & 11 \\ 5 & 8 & 6 & 9 \\ 3 & 10 & 1 & 12 \end{pmatrix}$$

of size $(3, 4)$. Then the grid biorder $\mathrm{Bi}(M) = (\delta_1, \delta_2)$ of M consists of the following linear orders: $\delta_1 = (4, 7, 2, 11, 5, 8, 6, 9, 3, 10, 1, 12)$ and $\delta_2 = (3, 5, 4, 10, 8, 7, 1, 6, 2, 12, 9, 11)$.

(2) The grid biorder of a row vector $M = (x_1, x_2, \ldots, x_n)$ is simply $\mathrm{Bi}(M) = (M, M)$, and for a column vector $M = (x_1, x_2, \ldots, x_n)^T$, we have $\mathrm{Bi}(M) = (M, M^{-1})$. $\square$

4 Embedding biorders into grids

Grids produce rather special biorders $\mathrm{Bi}(M)$ in the sense that the second linear order of $\mathrm{Bi}(M)$ is almost redundant – it is uniquely determined by the first linear order and the size of the grid M. Nevertheless, as we shall see, all biorders can be embedded into grid biorders.

Let $\tau = (\delta_1, \delta_2)$ be a biorder. We say that a sequence $(\sigma_1, \sigma_2, \ldots, \sigma_k)$ of partial orders is a *left partition* of τ (with k components), if $\delta_1 = \sum_{i=1}^{k} \sigma_i$ and $\sigma_i \subseteq \delta_2$ for all $i \in [1, k]$. Similarly, a sequence $(\kappa_1, \kappa_2, \ldots, \kappa_t)$ is a *right partition* of τ (with t components), if $\delta_2 = \sum_{i=1}^{t} \kappa_i$ and $\kappa_i \subseteq \delta_1^{-1}$ for all $i \in [1, t]$. Moreover, a left (right) partition is said to be *maximal* if it has the smallest number of components among the left (right, respectively) partitions of τ.

Example 4.1. Let $\tau = (\delta_1, \delta_2)$ be the biorder where $\delta_1 = (1, 5, 3, 4, 2)$ and $\delta_2 = (3, 2, 1, 4, 5)$, and set $\sigma_1 = (1, 5), \sigma_2 = (3, 4)$, and $\sigma_3 = (2)$. Then $\delta_1 = \sigma_1 \oplus \sigma_2 \oplus \sigma_3$ and $\sigma_i \subseteq \delta_2$ for each i. Therefore $(\sigma_1, \sigma_2, \sigma_3)$ is a left partition of τ. Similarly, $(\kappa_1, \kappa_2, \kappa_3)$ is a right partition of τ, when $\kappa_1 = (3), \kappa_2 = (2, 1), \kappa_3 = (4, 5)$. $\square$

We begin with a lemma concerning pairs of orders contained in grid biorders.

Lemma 4.2. *Let $\tau' = \mathrm{Bi}(M)$ for an $m \times n$-grid M and let τ be a biorder such that $\tau = (\delta_1, \delta_2) \subseteq \tau'$. For each $i \in [1, m]$ and $j \in [1, n]$, let $\sigma_i = \delta_1 \cap M_i$ and $\kappa_j = \delta_2 \cap M_j^{-T}$. Then $(\sigma_1, \ldots, \sigma_m)$ is a left partition and $(\kappa_1, \ldots, \kappa_n)$ is a right partition of τ.*

Proof. It is clear that $\delta_1 = \sum_{i=1}^{m} \sigma_i$ and $\delta_2 = \sum_{j=1}^{n} \kappa_j$. Since $\tau \subseteq \tau'$ and $\sigma_i \subseteq M_i$, it follows by (1) that $\sigma_i \subseteq \delta_2$ for each i. Similarly, $\kappa_j \subseteq \delta_1^{-1}$ for each j. The claim follows from these observations. $\square$

Each biorder $\tau = (\delta_1, \delta_2)$ does have a left and a right partition. Indeed, if $\delta_1 = (a_1, a_2, \ldots, a_n)$ and $\delta_2 = (b_1, b_2, \ldots, b_n)$, then these are trivial left and right partitions: $\delta_1 = \sum_{i=1}^{n} (a_i)$ and $\delta_2 = \sum_{i=1}^{n} (b_j)$ where each (a_i) and (b_j) is a one element linear order. We shall prove in the following that the maximal left and right partitions of a biorder τ are unique.

Lemma 4.3. *Let τ be a biorder and let $(\sigma_1, \ldots, \sigma_k)$ and $(\kappa_1, \ldots, \kappa_t)$ be a left and a right partition of τ, respectively. Then $\mathrm{dom}(\sigma_i \cap \kappa_j)$ has at most one element for each $i \in [1, k]$ and $j \in [1, t]$.*

Proof. Let $\tau = (\delta_1, \delta_2)$. Suppose that there are elements $a, b \in \mathrm{dom}(\tau)$ such that $(a, b) \in \sigma_i$ and $\{a, b\} \subseteq \mathrm{dom}(\kappa_j)$ for some $i \in [1, k]$ and $j \in [1, t]$. Since $\sigma_i \subseteq \delta_1$, also $(a, b) \in \delta_1$. On the other hand, $\sigma_i \subseteq \delta_2$ and $\kappa_j \subseteq \delta_2$, and therefore $(a, b) \in \kappa_j$. However, by definition, $\kappa_j \subseteq \delta_1^{-1}$ and thus $(b, a) \in \delta_1$, which implies that $a = b$ proving the claim. $\square$

Let $\sigma = (\sigma_1, \ldots, \sigma_k)$ and $\kappa = (\kappa_1, \ldots, \kappa_t)$ be left and right partitions of a biorder τ. If $\mathrm{dom}(\sigma_i \cap \kappa_j) \neq \emptyset$, then, by Lemma 4.3, the intersection has exactly one element. In this case, we say that the pair (i, j) is *compatible* in (σ, κ). Trivially, every element $x \in \mathrm{dom}(\tau)$ belongs to exactly one set $\mathrm{dom}(\sigma_i \cap \kappa_j)$. Therefore we can define $\chi_\tau \colon \mathrm{dom}(\tau) \to [1, k] \times [1, t]$ by

$$\chi_\tau(x) = (i, j) \quad \text{if} \quad x = \sigma_i \cap \kappa_j \,.$$

We now give necessary and sufficient conditions for a biorder τ to be embeddable into a given grid biorder.

Theorem 4.4. *Let $\tau = (\delta_1, \delta_2)$ be a biorder and let $\tau' = \mathrm{Bi}(M)$ be a grid biorder for an $m \times n$-grid M. The following two conditions are equivalent for each function $\varphi \colon \mathrm{dom}(\tau) \to \mathrm{dom}(\tau')$.*

(i) φ is an embedding of τ into τ'.

(ii) *There are integers k and t, and left and right partitions $\sigma = (\sigma_1, \ldots, \sigma_k)$ and $\kappa = (\kappa_1, \ldots, \kappa_t)$ of τ, and order embeddings $\pi_1 : [1,k] \to [1,m]$ and $\pi_2 : [1,t] \to [1,n]$ such that for all compatible pairs (i,j) of (σ, κ),*

$$\varphi(\sigma_i \cap \kappa_j) = M_{(\pi_1(i), \pi_2(j))} .$$

Proof. Let $\tau' = (\delta_1', \delta_2')$.

(1) Assume first that φ is an embedding. Let, for each $i \in [1,m]$ and $j \in [1,n]$,

$$\sigma_i' = \varphi(\delta_1) \cap M_i \quad \text{and} \quad \kappa_j' = \varphi(\delta_2) \cap M_j^{-T} .$$

Moreover, let $d_1 < d_2 < \cdots < d_k$ be the increasing sequence of all indices such that $\sigma_{d_1}', \ldots, \sigma_{d_k}' \neq \emptyset$, and let $e_1 < e_2 < \cdots < e_t$ be the increasing sequence of all indices such that $\kappa_{e_1}', \ldots, \kappa_{e_t}' \neq \emptyset$. By Lemma 4.2, $\sigma = (\sigma_{d_1}, \ldots, \sigma_{d_k})$ is a left partition and $\kappa = (\kappa_{e_1}, \ldots, \kappa_{e_t})$ is a right partition of $\varphi(\tau)$. It is immediate that $M_{(d_i, e_j)} = \mathrm{dom}(\sigma_{d_i}' \cap \kappa_{e_j}')$ for all compatible pairs (d_i, e_j) (where $i \in [1,k]$ and $j \in [1,t]$).

Let $\pi_1 : [1,k] \to [1,m]$ and $\pi_2 : [1,t] \to [1,n]$ be defined by $\pi_1(i) = d_i$ and $\pi_2(j) = e_j$, respectively. Obviously, π_1 and π_2 are order embeddings. Let $\sigma_i = \varphi^{-1}(\sigma_{d_i}')$ and $\kappa_j = \varphi^{-1}(\kappa_{e_j}')$ for $i \in [1,k]$ and $j \in [1,t]$. Because φ is an embedding of τ onto $\varphi(\tau)$, $(\sigma_1, \ldots, \sigma_k)$ is a left partition and $(\kappa_1, \ldots, \kappa_t)$ is a right partition of τ. Now, if $x \in \mathrm{dom}(\sigma_i \cap \kappa_j)$, then we have $\varphi(x) \in \mathrm{dom}(\sigma_{\pi_1(i)}' \cap \kappa_{\pi_2(j)}')$. Hence, by the above, $\varphi(x) = M_{(\pi_1(i), \pi_2(j))}$ for each compatible pair (i,j) with $i \in [1,k]$ and $j \in [1,t]$ as required.

(2) Suppose now that (ii) is satisfied. (Note that φ is well defined by Lemma 4.3.) The injectivity of φ follows directly from its definition and from Lemma 4.3. We need only to show that φ is order preserving for both linear orders of τ.

Let $x, y \in \mathrm{dom}(\tau)$, and let $i, p \in [1,k]$ and $j, q \in [1,t]$ be such that $x = \mathrm{dom}(\sigma_i \cap \kappa_j)$ and $y = \mathrm{dom}(\sigma_p \cap \kappa_q)$. If $(x,y) \in \delta_1$ then $i \leq p$, and therefore $\pi_1(i) \leq \pi_1(p)$, since π_1 is order preserving. Moreover, if $i = p$, then also $(x,y) \in \delta_2$, since $\sigma_i \subseteq \delta_2$. Hence in this case, $q \leq j$ and also $\pi_2(q) \leq \pi_2(j)$, since π_2 is order preserving. It follows then that

$$\varphi(x,y) = (\varphi(x), \varphi(y)) = (M_{(\pi_1(i), \pi_2(j))}, M_{(\pi_1(p), \pi_2(q))}) \in \delta_1' .$$

Similarly, if $(x,y) \in \delta_2$, then $j \leq q$ and thus $\pi_2(j) \leq \pi_2(q)$. Moreover, if $q = j$, then $(x,y) \in \delta_1^{-1}$, since $\kappa_j \subseteq \delta_1^{-1}$. In this case, $i \leq p$ and also $\pi_1(i) \leq \pi_1(p)$. As in the above we have now that $\varphi(x,y) \in \delta_2'$. We conclude that φ is an embedding from τ into τ'. $\square$

We note that, in the notations of the previous theorem, the grid biorder $\mathrm{Bi}(M)$ into which the given biorder τ is embeddable, has the size at least (k,t) where k and t are the numbers of the components of the left and right partitions, respectively.

Example 4.5. Consider the grid

$$M = \begin{pmatrix} 3 & 5 & 1 \\ 4 & 2 & 6 \end{pmatrix}$$

for which we have $\mathrm{Bi}(M) = ((3,5,1,4,2,6),(4,3,2,5,6,1))$. Let also $\tau = ((3,2,1),(3,1,2))$. Then τ has a left partition $((3,2),(1))$ and a right partition $((3),(1),(2))$. Now τ can be embedded into $\mathrm{Bi}(M)$ by the embedding φ defined by $\varphi(1) = 2$, $\varphi(2) = 1$ and $\varphi(3) = 3$. Indeed, $\varphi(3,2,1) = (3,1,2)$ is a suborder of $(3,5,1,4,2,6)$ and $\varphi(3,1,2) = (3,2,1)$ is a suborder of $(4,3,2,5,6,1)$. In this case, the order embeddings $\pi_1 \colon [1,2] \to [1,2]$ and $\pi_2 \colon [1,3] \to [1,3]$ of Theorem 4.4 are both identity functions. For instance, we have that $\sigma_1 = (3,2)$ and $\kappa_3 = (2)$, and therefore

$$M_{(1,3)} = 1 = \varphi(2) = \varphi(\sigma_1 \cap \kappa_3) = M_{(\pi_1(1),\pi_2(3))}\,,$$

and hence $\pi_1(1) = 1$ and $\pi_2(3) = 3$.

The grid biorder $\mathrm{Bi}(M)$ is not the smallest one into which τ can be embedded. It is easy to verify that τ can be embedded into $\mathrm{Bi}(M')$ where the grid M' has size $(2,2)$:

$$M' = \begin{pmatrix} 3 & 2 \\ 4 & 1 \end{pmatrix}.$$

Here $\mathrm{Bi}(M') = ((3,2,4,1),(4,3,1,2))$. The embedding φ' is the identity function in this case. □

The following result proves that every left (right) partition of a biorder τ can be extended to a unique maximum left (right, respectively) partition.

Lemma 4.6. *Each biorder τ possesses a unique maximal left partition and a unique maximal right partition.*

Proof. Let $\tau = (\delta_1, \delta_2)$. We prove the claim for left partitions; for right partitions the proof is similar and omitted here. Now there exists at least one left partition for τ; namely the trivial left partition. Let then $(\sigma_1, \ldots, \sigma_k)$ be any left partition of τ. If for some $i \in [1, k-1]$, $\sigma_i \oplus \sigma_{i+1} \subseteq \delta_2$, then $\sigma_i \oplus \sigma_{i+1} \subseteq \delta_1$, and hence also $(\sigma_1, \ldots, (\sigma_i \oplus \sigma_{i+1}), \ldots, \sigma_k)$ is a left partition of τ, and it has $k-1$ components. We may thus assume that in the chosen left partition there are no indices $i \in [1, k-1]$ such that $\sigma_i \oplus \sigma_{i+1} \subseteq \delta_2$. Let $(\sigma'_1, \ldots, \sigma'_p)$ be another left partition of τ. If for some $i \in [1,p]$ and $j \in [1, k-1]$, $\mathrm{dom}(\sigma'_i) \cap \mathrm{dom}(\sigma_j) \neq \emptyset$ and $\mathrm{dom}(\sigma'_i) \cap \mathrm{dom}(\sigma_{j+1}) \neq \emptyset$, then $(a,b) \in \sigma'_i$ for the maximal element a of σ_j and the minimal element b of σ_{j+1}. Since $\sigma_j \subseteq \delta_2$, $\sigma_{j+1} \subseteq \delta_2$ and $(a,b) \in \delta_2$, evidently also $\sigma_j \oplus \sigma_{j+1} \subseteq \delta_2$, contradicting our assumption. Consequently, for each $i \in [1,p]$, we have $\sigma'_i \subseteq \sigma_j$ for some $j \in [1,k]$. Thus $(\sigma_1, \ldots, \sigma_k)$ is a maximal left partition and it is unique as such a partition. □

Let $(\sigma_1, \ldots, \sigma_k)$ and $(\kappa_1, \ldots, \kappa_t)$ be the maximal left and right partitions of a biorder τ, respectively. Then $\mathrm{size}^P(\tau) = (k,t)$ is called the *partitive size* of τ. By Lemma 4.6, this notion is well defined for each biorder τ.

From Theorem 4.4 we deduce the following estimation on the size for the grid biorders into which a given biorder τ can be embedded.

Lemma 4.7. *Let τ be a biorder τ that is embeddable into a grid biorder* $\mathrm{Bi}(M)$ *for a grid M. Then* $\mathrm{size}^P(\tau) \leq \mathrm{size}(M)$.

Proof. The existence of the injective functions π_1 and π_2 in Theorem 4.4 implies that $\mathrm{size}(M) \geq (k,t)$ where k and t equal the number of components in the left and right partitions given by Theorem 4.4. Since the maximal left and right partitions of τ have the least number of components, we have $\mathrm{size}(M) \geq (k,t) \geq \mathrm{size}^P(\tau)$. □

We are going to show now that each biorder can be embedded into a unique grid biorder 'modulo τ'. First we define the meaning of 'modulo τ'.

Let τ be a biorder on $[1,N]$, and let M' and M'' be two $m \times n$-grids. Then $\mathrm{Bi}(M')$ and $\mathrm{Bi}(M'')$ are said to be *congruent modulo* τ, if $\tau \subseteq \mathrm{Bi}(M')$ and $\tau \subseteq \mathrm{Bi}(M'')$, and the elements of $[1,N]$ are in the same places in the grids M' and M'', i.e., if $M'_{(i,j)} = M''_{(i,j)}$ for each $M'_{(i,j)} \in [1,N]$ with $(i,j) \in [1,m] \times [1,n]$.

Example 4.8. Let $\tau' = \mathrm{Bi}(M')$ and $\tau'' = \mathrm{Bi}(M'')$ be grid biorders where

$$M' = \begin{pmatrix} 1 & 2 & 5 \\ 4 & 3 & 6 \end{pmatrix} \quad \text{and} \quad M'' = \begin{pmatrix} 5 & 1 & 2 \\ 4 & 6 & 3 \end{pmatrix}.$$

For the biorder $\tau = ((1,2,3),(1,3,2))$, we have $\tau \subseteq \tau'$ and $\tau \subseteq \tau''$. However, τ' and τ'' are not congruent modulo τ, because $M'_{(1,1)} = 1 \in [1,3]$, but $M'_{(1,1)} \neq M''_{(1,1)}$. (Indeed, we have even that $M''_{(1,1)} \notin [1,3]$.) □

We are now ready to express our main embedding theorem which states that every biorder can be embedded into a unique smallest grid biorder where uniqueness is taken up to congruence of biorders.

Theorem 4.9. *Let τ be a biorder on $D = [1,N]$ with* $\mathrm{size}^P(\tau) = (k,t)$. *There exists a $k \times t$-grid M' such that*

(i) $\tau \subseteq \mathrm{Bi}(M')$, *and*

(ii) if $\tau \subseteq \mathrm{Bi}(M'')$ *for a grid M'', then either* $\mathrm{Bi}(M'')$ *and* $\mathrm{Bi}(M')$ *are equivalent modulo τ or* $\mathrm{size}(M'') > \mathrm{size}^P(\tau)$.

Proof. Let $\tau = (\delta_1, \delta_2)$ and let $(\sigma_1, \dots, \sigma_k)$ and $(\kappa_1, \dots, \kappa_t)$ be the maximal left and right partitions of τ, respectively. Moreover, we let $D' = [N+1, kt]$. For each pair $(i,j) \in [1,k] \times [1,t]$, let $\mu(i,j)$ be the number of pairs (r,s) such that $(r,s) \leq (i,j)$ and $\mathrm{dom}(\sigma_r \cap \kappa_s) = \emptyset$, i.e., for which $\chi_\tau^{-1}(r,s)$ is not defined. By Lemma 4.3, $\chi_\tau^{-1}(r,s)$ is undefined for exactly $|D'| = kt - N$ pairs (r,s).

Define a $k \times t$-grid M' as follows: for each pair $(i,j) \in [1,k] \times [1,t]$,

$$M'_{(i,j)} = \begin{cases} \chi_\tau^{-1}(i,j), & \text{if } \sigma_i \cap \kappa_j \neq \emptyset\,, \\ N + \mu(i,j), & \text{if } \sigma_i \cap \kappa_j = \emptyset\,. \end{cases}$$

Denote $\tau' = \mathrm{Bi}(M')$.

We apply Theorem 4.4 for the case where $\pi_1(i) = i$ and $\pi_2(i) = i$ for each i, and accordingly we define $\varphi(x) = M'_{(\pi_1(i),\pi_2(j))}$ where $(i,j) = \chi_\tau(x)$. By Theorem 4.4, φ is an embedding of τ into τ'. Now, φ is the identity function on $\mathrm{dom}(\tau)$, and thus $\tau \subseteq \tau'$.

Suppose then that τ is contained in another grid biorder $\tau'' = \mathrm{Bi}(M'')$ for a $p \times q$-grid M''. By Lemma 4.7, we have that $(k,t) \leq (p,q)$.

Suppose now that $(k,t) = (p,q)$. By applying Theorem 4.4 to the identity function φ as an embedding of τ into τ'', we have that the left and right partitions in this case are necessarily the maximal left and right partitions of τ, because there are exactly k and t components, respectively. Moreover, necessarily $\pi_1(i) = i$ and $\pi_2(i) = i$ for all i, that is, $M''_{(i,j)} = \chi_\tau^{-1}(i,j)$ for each compatible pair (i,j). Hence $M'_{(i,j)} = M''_{(i,j)}$ for all compatible pairs (i,j), which proves the claim. □

The next example illustrates the construction given in Theorem 4.9.

Example 4.10. Let $\tau = (\delta_1, \delta_2)$ be a biorder with $\delta_1 = (3,5,1,6,2,4)$ and $\delta_2 = (2,1,3,4,6,5)$. Then the maximal left and right partitions of τ are $(\sigma_1, \sigma_2, \sigma_3)$ and (κ_1, κ_2), where $\sigma_1 = (3,5)$, $\sigma_2 = (1,6)$, $\sigma_3 = (2,4)$, and $\kappa_1 = (2,1,3)$, $\kappa_2 = (4,6,5)$. Hence $\mathrm{size}^P(\tau) = (3,2)$ and the entries $M_{(i,j)} = \mathrm{dom}(\sigma_i \cap \kappa_j)$ in the grid obtained in the proof of Theorem 4.9 are: $M_{(1,1)} = 3$, $M_{(1,2)} = 5$, $M_{(2,1)} = 1$, $M_{(2,2)} = 6$, $M_{(3,1)} = 2$, $M_{(3,2)} = 4$. Hence $\tau = \mathrm{Bi}(M)$ for the grid

$$M = \begin{pmatrix} 3 & 5 \\ 1 & 6 \\ 2 & 4 \end{pmatrix}.$$

□

The previous result states that the size of the smallest grid biorder, which contains a given biorder τ, is unique, and the grid biorder itself is 'unique modulo τ'. We shall now 'forget' the elements from the grids that will not be in the domain of τ by introducing a special *free symbol* $\star$. Let S be any set excluding the free symbol $\star$, and let $A \subseteq S$. Define a general purpose function $\Lambda_A \colon S \to S \cup \{\star\}$ by

$$\Lambda_A(s) = \begin{cases} s & \text{if } s \in A\,, \\ \star & \text{if } a \notin A\,. \end{cases}$$

Given a biorder $\tau = (\delta_1, \delta_2)$ and a subset $A \subseteq \mathrm{dom}(\tau)$, a pair $\Lambda_A(\tau) = (\Lambda_A(\delta_1), \Lambda_A(\delta_2))$ is called a *biorder with free symbols* obtained from τ by A. Similarly, for a grid M the matrix $\Lambda_A(M)$ is a *matrix with free symbols.*

Example 4.11. Let $\tau = ((1,2,3,4),(2,4,1,3))$ be a biorder and choose $A = \{1,4\}$. Then $\Lambda_A(\tau) = ((1,\star,\star,4),(\star,4,1,\star))$. The biorder τ is a grid biorder and the corresponding grid M together with the matrix $\Lambda_A(M)$ are given below:

$$M = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \quad \text{and} \quad \Lambda_A(M) = \begin{pmatrix} 1 & \star \\ \star & 4 \end{pmatrix}.$$

□

Using the notation of a free symbol, if two grid biorders τ' and τ'' are congruent modulo a biorder τ with a domain D, then $\Lambda_D(\tau') = \Lambda_D(\tau'')$.

Let $\tau \subseteq \tau'$ for a biorder τ' and let $\mathrm{dom}(\tau) = D$. The biorder $\Lambda_D(\tau')$ with free symbols is called a *cover* of τ. Furthermore, if τ' is a grid biorder, then $\Lambda_D(\tau')$ is said to be a *matrix cover* of τ. A matrix cover of τ is a *minimal matrix cover*, if τ' has minimal size.

Theorem 4.9 can now be restated as follows.

Theorem 4.12. *For each biorder there exists a unique minimal matrix cover.*

Proof. The claim is obvious by Theorem 4.9 and the above definitions. □

Example 4.13. Let $\tau = (\delta_1, \delta_2)$ be a biorder with $\delta_1 = (4, 5, 3, 1, 2, 6)$ and $\delta_2 = (2, 3, 1, 6, 4, 5)$. Then the maximal left and right partitions of τ are $(\sigma_1, \sigma_2, \sigma_3)$ and $(\kappa_1, \kappa_2, \kappa_3, \kappa_4)$, where $\sigma_1 = (4, 5)$, $\sigma_2 = (3, 1)$, $\sigma_3 = (2, 6)$, and $\kappa_1 = (2, 3)$, $\kappa_2 = (1)$, $\kappa_3 = (6, 4)$, $\kappa_4 = (5)$. Now $\mathrm{size}^P(\tau) = (3, 4)$ and the entries in the matrix given by Theorem 4.9 are $M_{(1,3)} = 4$, $M_{(1,4)} = 5$, $M_{(2,1)} = 3$, $M_{(2,2)} = 1$, $M_{(3,1)} = 2$, $M_{(3,3)} = 6$. The rest of the entries are filled with the free symbol $\star$. Hence the minimal matrix cover of τ is the grid biorder $\mathrm{Bi}(M)$ for the grid

$$M = \begin{pmatrix} \star & \star & 4 & 5 \\ 3 & 1 & \star & \star \\ 2 & \star & 6 & \star \end{pmatrix}.$$

□

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