Embedding near domains

J.A. Graves and J.J. Malone

A class of near rings which generalizes the class of integral
domains is defined. The definition of near domain is derived
from the desirability of embedding near domains in near fields.
The near domains presented here are shown to contain the D-rings
of Berman and Siverman.

1. Near domains

We take integral domain to mean a commutative ring having no divisors
of zero. Near rings with no zero divisors have been considered by Ligh and
Malone [8], Clay [3] and Ferrero [4]. Ligh and Malone studied the
properties of finite near rings having in one case no zero divisors and in
another a finite number of zero divisors. Their work was not an attempt to
generalize the concept of integral domain. Clay [3] called a near ring
without divisors of zero a near integral domain but as he stated his
definition was for the want of a better name. The main object of his note
was the statement of the conjecture that, except for certain trivial cases,
such a system having finite characteristic must have prime characteristic.
This conjecture was later disproved by Ferrero [4]. Thus the absence of
zero divisors does not enhance the structure of a near ring to the extent
one might expect.

THEOREM 1.1. *Let R be a near ring satisfying the right
cancellation law.*

(i)   For all $a \in R$, $0a = a0 = 0$.

(ii) R has no (proper) zero divisors.

(iii)   The left cancellation law holds in R .

33

Proof. If $a \neq 0$ and $0a = b \neq 0$, then $b = bb$ or $0b = 0bb$ so that $0 = 0b$ and $0 = 0a$. If $ab = 0$, $b \neq 0$, then $ab = 0b$ and $a = 0$. If $ab = ac$, $a \neq 0$, then $a(b-c) = 0$ and $b = c$.   //

Ore has shown in [12] that, given a non-commutative ring having no proper divisors of zero, a necessary and sufficient condition for the ring to have a constructible division ring of quotients is that each pair of non-zero elements has a left (or right) common multiple.

DEFINITION 1.2. A near ring' R is said to satisfy the *left (right) Ore condition* if for all $\dot{a} \neq 0$, $b \neq 0 \in R$, there exist $s \neq 0$, $t \neq 0 \in R$ such that $sa = tb$ $(as = bt)$.

DEFINITION 1.3. A (left) *near domain* is a (left) near ring D satisfying the left Ore condition and the right cancellation law.

The usual proof that a finite integral domain is a field depends, essentially, only on right cancellation and may be used to establish:

THEOREM 1.4. *If R is a finite near ring for which the right cancellation law holds, then R is a near field.*

It can be shown that finite cannot be omitted from the hypothesis of Theorem 1.4. Let G be an additively written free group on two generators x and y and for each non-negative integer n define $T_n : G \to G$ by $h(x, y)T_n = h(nx, ny)$, for all $h(x, y) \in G$. Let R be the near ring generated additively by the T_n . McQuarrie [9] shows that R is a distributively generated near ring with identity and Graves [5] shows that R has the right cancellation property but satisfies neither the right nor the left Ore condition.

By Theorem 1.4, right cancellation implies the left Ore condition (when R is finite). The near ring given as 2.5, 26) in [2] shows that the converse does not hold.

THEOREM 1.5. *A near domain R satisfying the descending chain condition on principal R-subgroups is a near field.*

Proof. Ligh [7] has shown that a near ring satisfying the descending chain condition on principal R-subgroups and having no zero divisors must contain a left identity and moreover, if the left identity is unique, then R is a near field. We have shown right cancellation implies no zero

divisors. Hence an application of Ligh's result yields $e \in R$ such that
$er = r$, for all $r \in R$. If $a \neq 0 \in R$, we have for each $x \in R$,
$xea = xa$ which implies $xe = x$. Hence e is the unique identity and, by
Ligh's result, R is a near field.   //

2. Near rings of quotients

In this section we show that a near domain may be embedded in a near
field and deduce some of the consequences of this embedding. Maxson [11]
has defined near rings of quotients and stated conditions for a near ring
to have a near ring of quotients. We generalize to the case of near rings
of quotients with respect to a multiplicative set.

DEFINITION 2.1. Let R be a near ring and S a multiplicative set
in R . We say R satisfies the *left* Ore *condition with respect to* S if
for each $(s, r) \in S \times R$ there exists $\left(s_1, r_1\right) \in S \times R$ such that
$s_1 r = r_1 s$.

DEFINITION 2.2. Let S be a multiplicative set in a near ring R .
A near ring R_S is called a *near ring of left quotients of R with
respect to S* if

(i) R_S has an identity 1 ,

(ii) there exists an embedding (monomorphism) $\Phi : R \to R_S$,

(iii) for each $s \in S$, $\Phi(s)$ is a unit in R_S ,

(iv) every $q \in R_S$ can be expressed as $q = [\Phi(s)]^{-1}[\Phi(r)]$,
            $(s, r) \in S \times R$.

THEOREM 2.3. *Let $S \neq \emptyset$ be a multiplicative set of (both left and
right) cancellable elements in a (left) near ring R . Then a near ring of
left quotients of R with respect to S exists if and only if R
satisfies the left Ore condition with respect to S .*

Proof. Suppose R_S exists. Then by the definition of R_S there
exists $\Phi : R \to R_S$ embedding R into R_S . For each $(s, r) \in S \times R$,
the product $[\Phi(r)][\Phi(s)]^{-1}$ in R_S is expressible as

$[\Phi(r)][\Phi(s)]^{-1} = [\Phi(s_1)]^{-1}[\Phi(r_1)]$, for some $(s_1, r_1) \in S \times R$. But then $[\Phi(s_1)][\Phi(r)] = [\Phi(r_1)][\Phi(s)]$ implies $s_1 r = r_1 s$ since Φ is a monomorphism.

To show the converse we construct R_S by defining an equivalence relation "$\sim$" on $S \times R$ and binary operations addition and multiplication on the set of equivalence classes $(S \times R)/\sim$. In proving this part of the theorem, there are many details to be verified and not all of these verifications will be given. However, several illustrations of the general techniques of the proof will be presented.

If $(c, x), (d, y) \in S \times R$ we define $(c, x) \sim (d, y)$ if there exists $(s, r) \in S \times R$ such that $sc = rd$ implies $sx = ry$. Note that even though r may not be in S , $sc = rd$ where s, c and d are elements of S implies the product $rd \in S$ and forces r to be cancellable. From the definition, it can be noted that if $(c, x) \sim (d, y)$, then for every $(a_1, a_2) \in R \times R$, $a_1 c = a_2 d$ implies $a_1 x = a_2 y$.

Since the relation $\sim$ is an equivalence relation, we can consider the set of equivalence classes $(S \times R)/\sim = R_S$, where we denote the class containing (c, x) by x/c , and equality of classes is determined by $x/c = y/d$ if and only if $(c, x) \sim (d, y)$.

Define $+$ (addition) on R_S by $x/c + y/d = (sx+ry)/sc = (sx+ry)/rd$ where $(s, r) \in S \times R$ satisfies $sc = rd \in S$.

We observe the definition is independent of the choice of (s, r) by assuming (s_1, r_1) satisfies $s_1 c = r_1 d$ and showing that $(s_1 x + r_1 y)/s_1 c = (sx+ry)/sc$, that is, $(s_1 c, s_1 x + r_1 y) \sim (sc, sx+ry)$. Let (s_2, r_2) be such that $s_2 s_1 c = r_2 sc$. Then $s_2 s_1 = r_2 s$ and since $s_1 c = r_1 d$, $sc = rd$, we have, $s_2 r_1 d = s_2 s_1 c = r_2 sc = r_2 rd$ which implies $s_2 r_1 = r_2 r$. Thus, $s_2 (s_1 x + r_1 y) = s_2 s_1 x + s_2 r_1 y = r_2 sx + r_2 ry = r_2 (sx+ry)$ implying the equivalence required.

Now addition is also well defined in the sense of being independent of

the choice of the representatives for the equivalence classes. For let
$(c, x) \sim (c', x')$ and $(d, y) \sim (d', y')$. We show
$x/c + y/d = x'/c' + y'/d'$ by choosing (s, r) and (s', r') such that
$sc = rd$ and $s'c' = r'd'$ to write $x/c + y/d = (sx+ry)/sc = (sx+ry)/rd$
and $x'/c' + y'/d' = (s'x'+r'y')/s'c' = (s'x'+r'y')/r'd'$. Then find
(s_1, r_1) such that $s_1 sc = r_1 s'c' = s_1 rd = r_1 r'd'$ whence $s_1 sx = r_1 s'x'$
and $s_1 ry = r_1 r'y'$, so that we have $s_1(sx+ry) = r_1(s'x'+r'y')$ which
yields $(sc, sx+ry) \sim (s'c', s'x'+r'y')$.

For any $e \in S$, $0/e$ is an additive right identity for R_S and
$-x/c$ is a right additive inverse for $x/c \in R_S$.

Since addition can be shown to be associative, we now have $(R_S, +)$
is a group and, since addition on R_S is defined in terms of the addition
of R , $(R_S, +)$ is abelian if and only if $(R, +)$ is abelian.

Let $x/c, y/d \in R_S$. Define $\cdot$ (multiplication) on R_S by
$x/c \cdot y/d = ry/sc$ where $(s, r) \in S \times R$ satisfies $sx = rd$.

This multiplication is independent of the choice of (s, r) and is
also independent of the choice of representatives of the equivalence
classes.

For any $e \in S$, $e/e = 1 \in R_S$.

Now multiplication is associative; for let $x/c, y/d, z/e \in R_S$ and
compute $(x/c \cdot y/d) \cdot z/e = ry/sc \cdot z/e = r_1 z/s_1 sc$ where $sx = rd$ and
$s_1 ry = r_1 e$. Also, $x/c \cdot (y/d \cdot z/e) = x/c \cdot r_2 z/s_2 d = r_3 r_2 z/s_3 c$ where
$s_2 y = r_2 e$, and $s_3 x = r_3 s_2 d$. To see that these products are equal we
find (s_4, r_4) such that $s_4 s_1 sc = r_4 s_3 c$, whence $s_4 s_1 s = r_4 r_3$ and,
using the equalities above, $s_4 s_1 rd = s_4 s_1 sx = r_4 s_3 x = r_4 r_3 s_2 d$ which
implies $s_4 s_1 r = r_4 r_3 s_2$. Further, $s_4 r_1 e = s_4 s_1 ry = r_4 r_3 s_2 y = r_4 r_3 r_2 e$
which implies $s_4 r_1 = r_4 r_3 r_2$. So $s_4 r_1 z = r_4 r_3 r_2 z$ which says the desired
equality holds.

The fact that multiplication is left distributive over addition in R_S follows from the left distributive property in R .

At this point we have shown $(R_S, +, \cdot)$ is a left near ring with identity. To show R_S is a near ring of left quotients of R with respect to S we must verify the remaining three defining properties of a near ring of left quotients.

Let $\Phi : R \to R_S$ be defined by $r \to er/e$ where $e \in S$. Then Φ is well defined because $er/e = dr/d$ for all $e, d \in S$. For $r_1, r_2 \in R$,
$\Phi(r_1) + \Phi(r_2) = er_1/e + er_2/e = (er_1 + er_2)/e = e(r_1 + r_2)/e = \Phi(r_1 + r_2)$ and
$\Phi(r_1) \cdot \Phi(r_2) = er_1/e \cdot er_2/e = rer_2/se = ser_1r_2/se = er_1r_2/e = \Phi(r_1r_2)$
where (s, r) satisfies $ser_1 = re$. So Φ is a near ring morphism. Further, kernel $\Phi = 0$, for if $r \in R$, $\Phi(r) = er/e = 0/e = e0/e$ implies $er = e0$ and so $r = 0$. Thus Φ is a monomorphism embedding R into R_S .

For all $s \in S$, $\Phi(s) = es/e$ has an inverse $e/es \in R_S$, for we compute, where (s_1, r_1) satisfies $s_1es = r_1es$, that
$es/e \cdot e/es = r_1e/s_1e = e/e = 1$ since e and s cancellable implies $s_1 = r_1$. Similarly, $e/es \cdot es/e = e/e = 1 \in R_S$.

Every $q = r/s \in R_S$ may be expressed as $[\Phi(s)]^{-1}[\Phi(r)]$ where $(s, r) \in S \times R$. Consider $[\Phi(s)]^{-1}[\Phi(r)] = e/es \cdot er/e = r_1er/s_1es = r/s$ where (s_1, r_1) satisfies $s_1e = r_1e$, an element of S .

Thus R_S is a near ring of left quotients of R with respect to S and the theorem is proved.   //

Several points should be noted about the proof of Theorem 2.3. First, commutativity of addition in R is not assumed. Secondly, the defining properties of a near ring of left quotients and the left Ore condition imply that any two near rings of left quotients for a given near ring R are isomorphic. Thus we speak of *the* near ring of left quotients of R

with respect to S . Thirdly, as pointed out by Maxson [11], a parallel
construction for near rings of right quotients does not hold because the
right distributive law is necessary for addition in R_S to be well
defined, in the sense of being independent of the choice of $(s, r) \in S \times R$
used in the definition of addition.

DEFINITION 2.4. If R is a near ring having at least one
cancellable element and C is the multiplicative set of all cancellable
elements of R , then the near ring of left quotients of R with respect
to C is called the *total near ring of left quotients of R* .

COROLLARY 2.5. *The total near ring of left quotients of a near
domain D is a near field (called the near field of quotients of D).*

Proof. We have shown that in D right cancellation implies there are
no zero divisors and this implies left cancellation. Hence the multiplic-
ative set S of all cancellable elements of D is $D\backslash\{0\}$, so any non-
zero element of the total near ring of left quotients has a multiplicative
inverse.   //

Since a near domain embeds in a near field, we have:

COROLLARY 2.6. *Let R be a near domain, then*

  (i) $(R, +)$ is commutative,

 (ii) for all $a, b \in R$, $a(-b) = -(ab) = (-a)b$,

(iii) if R has characteristic $m < \infty$, m is prime.

We can observe by examples that neither of the defining properties of
a near domain is by itself sufficient hypothesis to assure an embedding in
a near field. The near ring given as 2.5, 26) in [2] is a near ring which
satisfies the left Ore condition but which cannot be a sub near ring of a
near field because it has zero divisors. Malcev [10] has given an example
of a ring which has no divisors of zero but which is not embeddable in a
division ring. Since Malcev's is a ring example, the absence of zero
divisors along with right distributivity show that the right cancellation
law holds. Thus his example provides a near ring satisfying the right
cancellation law that is not embeddable in a near field.

The D-ring of Berman and Silverman [1] is the only embeddable (in a
near field) generalization of integral domain appearing in the near ring

literature. We now show near domains are generalizations of D-rings.

DEFINITION 2.7. A *D-ring* is a near ring D such that

(i) $ab = 0$; $a, b \in D$ implies $a = 0$ or $b = 0$,

(ii) for every $a \in D$ there are non-zero elements $a_e, a_r \in D$
     such that $a_e a$ and $a a_r$ are in the multiplicative center
     of D .

THEOREM 2.8. *If the near ring R is a D-ring, then R is a near domain.*

Proof. First, R satisfies left Ore condition (also right by similar proof). Let $a, b \in R$. Then there exists $a_e \neq 0 \in R$ such that $a_e a$ is in the multiplicative center of R , whence $(a_e a)b = b(a_e a) = (ba_e)a$. Thus there exist $x = a_e a$ and $y = ba_e$ in R such that $xb = ya$, where x and y are non-zero if a and b are non-zero. Also, the right cancellation law holds in R . Let $a, b, c \neq 0 \in R$ satisfy $ac = bc$. There exists $c_r \neq 0 \in R$ such that cc_r is in the multiplicative center and $cc_r \neq 0$ so that $acc_r = bcc_r$, which implies $(cc_r)a = (cc_r)b$, so that $(cc_r)(a-b) = 0$. Hence $a = b$ and R is a near domain.   //

For the ring of integers Z (actually any Ore domain, see [6]), the field of quotients of any non-zero ideal of Z is the same as that of Z , namely the rational numbers. Similarly we have:

THEOREM 2.9. *Let R be a near domain having Q as its near field of quotients. If $A \neq (0)$ is an ideal of R , then Q is also the near field of quotients of A .*

Proof. We first observe that A is in fact a near domain, so A does have a near field of quotients and the statement of the theorem makes sense. Clearly A satisfies the right cancellation law. Given $a \neq 0$, $b \neq 0$ in A , there exist $s \neq 0$, $t \neq 0$ in R such that $sa = tb$. To find similar elements in A , take $x \neq 0 \in A$, whence $xs \neq 0$, $xt \neq 0$ are such that $(xs)a = (xt)b$ with xs and xt in A since A is an ideal. (Note that this actually shows any R-subgroup of a near domain is also a near domain.)

Let $d/c \in Q$ and let $a \neq 0 \in A$. Then there exist $s \neq 0$, $t \neq 0 \in R$ such that $sc = ta \neq 0$, and further there exist $x \neq 0$, $y \in R$ such that $xsd = ya$. So $d/c = sd/sc = sd/ta = xsd/xta = ya/xta$, where $xta \neq 0 \in A$ and $ya \in A$, by A an ideal. Thus Q is contained in the near field of quotients of A and, since $A \subseteq R$, the theorem is proved.   //

References

[1]   Gerald Berman and Robert J. Silverman, "Near-rings", *Amer. Math. Monthly* **66** (1959), 23-34.

[2]   James R. Clay, "The near-rings on groups of low order", *Math. Z.* **104** (1968), 364-371.

[3]   James R. Clay, "A note on integral domains that are not right distributive", *Elem. Math.* **24** (1969), 40-41.

[4]   G. Ferrero, "Stems planari e *BIB*-disegni", *Riv. Mat. Univ. Parma* **11** (1970), 1-16.

[5]   James A. Graves, "Near domains", (Doctoral dissertation, Texas A&M University, College Station, Texas, 1971).

[6]   I.N. Herstein, *Topics in ring theory* (Chicago Lectures in Mathematics; The University of Chicago Press, Chicago, London, 1969).

[7]   Steve Chong Hong Ligh, "On certain classes of near-rings", (Doctoral dissertation, Texas A&M University, College Station, Texas, 1969).

[8]   S. Ligh and J.J. Malone, Jr, "Zero divisors and finite near-rings", *J. Austral. Math. Soc.* **11** (1970), 374-378.

[9]   B.C. McQuarrie, "A non-abelian near ring in which $(-1)r = r$ implies $r = 0$", *Canad. Math. Bull.* (to appear).

[10] A. Malcev, "On the immersion of an algebraic ring into a field", *Math. Ann.* **113** (1937), 686-691.

[11]   Carlton J. Maxson, "On near-rings and near-ring modules", (Doctoral
         dissertation, State University of New York at Buffalo, New York,
         1967).

[12]   Oystein Ore, "Linear equations in non-commutative fields", *Ann. of
         Math.* **32** (1931), 463-477.

El Paso Community College,

El Paso,

Texas,

USA;

Worcester Polytechnic Institute,

Worcester,

Massachussets,

USA.