EMBEDDING OF A RESTRICTION SEMIGROUP INTO A W-PRODUCT

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Abstract. A necessary and sufficient condition is provided for a (two-sided) restriction semigroup so that it be embeddable into a W-product of a semilattice by a monoid.

Dedicated to the memory of John M. Howie

1. Introduction

Although the term 'restriction semigroup' is fairly recent, these algebraic structures have been studied, under different names, for more than 40 years. For a historical overview, the reader is referred to [4] and [7].

From algebraic point of view, a restriction semigroup is a semigroup equipped with one or two unary operations. According to the number of the unary operations, we consider it a one-sided (left or right) or a two-sided restriction semigroup. In semigroup theory, it is regarded as a non-regular generalization of an inverse semigroup. The well-developed structure theory of inverse semigroups gives strong motivation to investigate the structure of restriction semigroups. Since the late 1970's much effort has been devoted to extending the most important structure theorems of inverse semigroups for restriction semigroups. For a summary of these results, the reader might consult [5]. This paper is a contribution to this area in the context of two-sided restriction semigroups, from now on simply referred to as restriction semigroups.

It is well known ([9], [10], see also [8]) that each inverse semigroup has an E-unitary cover, and the E-unitary inverse semigroups are just the inverse subsemigroups of the semidirect products of semilattices by groups. On the other hand, each inverse semigroup is embeddable into an almost factorizable one, and the almost factorizable inverse semigroups are just the homomorphic images of the semidirect products of semilattices by groups.

A construction analogous to the semidirect product of a semilattice by a group and producing a restriction semigroup is $W(T, Y)$ with Y a semilattice and T a monoid, called a W-product of Y by T.

This construction was introduced in [1] for T being a right cancellative monoid as a construction of a special left restriction semigroup, called left

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ample semigroup. In [3], it was generalized for any monoid T (with no idempotent distinct from 1), and it was noticed that there is a natural unary operation * on $W(T,Y)$, so that it becomes a restriction semigroup (a so-called weakly ample semigroup). Moreover, the main result of [3] (see also [13]) said that the almost left factorizable restriction semigroups are just the homomorphic images of the W-products of semilattices by monoids, thus focusing the attention to this construction. Recently, the author has proved in [12] and [13] that each restriction semigroup has a proper cover embeddable into a W-product of a semilattice by a monoid, and each restriction semigroups is embeddable into an almost left factorizable one.

In this paper we give a necessary and sufficient condition for a restriction semigroup so that it be embeddable into a W-product of a semilattice by a monoid. At the end of the paper, we relate our result to that in [12] and to the conditions obtained in [6] for the analogous question in the one-sided case.

2. Preliminaries

By a *left restriction semigroup* we mean an algebra $S = (S;\cdot,^+)$ of type $(2,1)$ where $(S;\cdot)$ is a semigroup and $^+$ is a unary operation such that the following identities hold:

$$x^+x = x, \quad x^+y^+ = y^+x^+, \quad (x^+y)^+ = x^+y^+, \quad xy^+ = (xy)^+x.$$

The following consequences of them are also frequently applied in the paper:

$$x^+x^+ = x^+, \quad (x^+)^+ = x^+, \quad x^+(xy)^+ = (xy)^+,$$
$$(x^+y^+)^+ = x^+y^+, \quad (xy)^+ = (xy^+)^+.$$

A *right restriction semigroup* $S = (S;\cdot,^*)$ is defined dually, and a *restriction semigroup* $S = (S;\cdot,^+,^*)$ is an algebra of type $(2,1,1)$ where $S = (S;\cdot,^+)$ is a left restriction semigroup, $S = (S;\cdot,^*)$ is a right restriction semigroup, and the identities

$$(x^+)^* = x^+, \quad (x^*)^+ = x^* \tag{2.1}$$

are valid. For restriction semigroups, the notions of a subalgebra, homomorphism, congruence and factor algebra are understood in type $(2,1,1)$. In order to avoid confusion, we call them $(2,1,1)$-subsemigroup, $(2,1,1)$-morphism, $(2,1,1)$-congruence and $(2,1,1)$-factor semigroup, respectively.

If a restriction semigroup S has an identity element with respect to the multiplication, usually denoted by 1, then $1^+ = 1^* = 1$ necessarily holds. Such a restriction semigroup is often called a *restriction monoid*.

The class of restriction semigroups is fairly wide. For example, each inverse semigroup $S_{\mathrm{inv}} = (S;\cdot,^{-1})$ determines a restriction semigroup $S = (S;\cdot,^+,^*)$ where the unary operations are defined by the rules

$$a^+ = aa^{-1} \quad \text{and} \quad a^* = a^{-1}a \quad \text{for every } a \in S.$$

On the other hand, each monoid M becomes a restriction semigroup by defining $a^+ = a^* = 1$ for any $a \in M$. Such a restriction monoid is often called *reduced*. Notice that the congruences and homomorphisms of inverse semigroups and the $(2,1,1)$-congruences and $(2,1,1)$-morphisms, respectively, of the restriction semigroups obtained from them coincide. Monoids and the reduced restriction monoids obtained from them relate to each other in the same way. This allows us to ease our terminology by saying just ‘inverse semigroup’ (in particular, ‘semilattice’) and ‘monoid’ instead of introducing a new name for a restriction semigroup obtained from an inverse semigroup (in particular, from a semilattice) and of saying ‘reduced restriction monoid’, respectively.

Let S be any restriction semigroup. By (2.1), we have $\{x^+ : x \in S\} = \{x^* : x \in S\}$. This set forms a $(2,1,1)$-subsemilattice in S with both unary operations being equal to the identity mapping. Therefore it is a restriction semigroup obtained from a semilattice considered as an inverse semigroup. So we call it, in the above sense, *the semilattice of projections of* S and denote it by $P(S)$.

Given a restriction semigroup S, we consider the following relation σ on S: for any $a, b \in S$, let

$$a \,\sigma\, b \quad \text{if and only if} \quad ea = eb \quad \text{for some} \quad e \in P(S),$$

or, equivalently, if and only if $af = bf$ for some $f \in P(S)$.

This relation is the least congruence on $S = (S;\cdot)$ where $P(S)$ is in a congruence class, and is the least $(2,1,1)$-congruence ρ on $S = (S;\cdot,{}^+,{}^*)$ such that the $(2,1,1)$-factor semigroup S/ρ is a monoid.

A restriction semigroup S is said to be *proper* if the following condition and its dual are fulfilled:

$$a^+ = b^+ \text{ and } a \,\sigma\, b \quad \text{imply} \quad a = b \quad \text{for every } a, b \in S.$$

Note that each $(2,1,1)$-subsemigroup of a proper restriction semigroup is proper.

It is worth mentioning that if a restriction semigroup S is obtained form an inverse semigroup S_{inv} as above then σ is the least group congruence on S_{inv}, and S is proper if and only if S_{inv} is E-unitary. Therefore the relation σ generalizes the least group congruence on an inverse semigroup, and the notion of a proper restriction semigroup generalizes that of an E-unitary inverse semigroup. Moreover, the role played among inverse semigroups by groups is taken over among restriction semigroups by monoids.

Now we introduce a construction producing a restriction semigroup from a semilattice and a monoid which generalizes semidirect products of semilattices by groups.

Let T be a monoid and Y a semilattice. We say that T acts on Y on the right if a monoid homomorphism is given from T into the endomorphism monoid $\operatorname{End} Y$ of Y, or equivalently, if, for any $a \in Y$ and $t \in T$, an element $a^t \in Y$ is given such that

$$(ab)^t = a^t b^t, \quad (a^t)^u = a^{tu}, \quad a^1 = a \tag{2.2}$$

hold for every $a, b \in Y$ and $t, u \in T$. We say that (T, Y) is a *W-pair* if T acts on Y on the right by injective endomorphisms such that the range of each endomorphism corresponding to an element of T forms an order ideal in Y, or equivalently, if additionally to (2.2), conditions

$$a^t = b^t \quad \text{implies} \quad a = b, \tag{2.3}$$

and

$$a \leq b^t \quad \text{implies} \quad a = c^t \quad \text{for some } c \in Y \tag{2.4}$$

are fulfilled for any $a, b \in Y$ and $t \in T$.

Given a W-pair (T, Y), consider the set

$$W(T, Y) = \{(t, a^t) \in T \times Y : a \in Y,\ t \in T\},$$

and define a multiplication and two unary operations on it in the following manner: for any $(t, a^t), (u, b^u) \in W(T, Y)$, let

$$\begin{aligned} (t, a^t)(u, b^u) &= (tu, a^{tu} \cdot b^u), \\ (t, a^t)^+ &= (1, a), \\ (t, a^t)^* &= (1, a^t). \end{aligned}$$

It is straightforward to see that $W(T, Y)$ is a subsemigroup in the reverse semidirect product $T \ltimes Y$. Moreover, this construction has the following basic properties.

Result 2.1. *For any W-pair (T, Y), the following statements hold.*

1. $W(T, Y) = (W(T, Y); \cdot, {}^+, {}^*)$ *is a restriction semigroup, and its semilattice of projections is* $P(W(T, Y)) = \{(1, a) : a \in Y\}$, *which is isomorphic to* Y.
2. *The first projection* $W(T, Y) \to T$ *is a surjective* $(2, 1, 1)$*-morphism whose kernel is* σ. *Consequently,* $W(T, Y)/\sigma$ *is isomorphic to* T.
3. $W(T, Y)$ *is proper.*

The restriction semigroup $W(T, Y)$ is called a *W-product* of the semilattice Y by the monoid T.

3. Embeddability into a W-product

In this section we give a necessary and sufficient condition for a restriction semigroup to be embeddable into a W-product of a semilattice by a monoid.

First, for a restriction semigroup S satisfying certain conditions, we construct a W-product $W(T, \mathcal{Y})$ and an injective $(2, 1, 1)$-morphism from S into $W(T, \mathcal{Y})$.

Let S be a restriction semigroup and, for brevity, put $P = P(S)$ and $T = S/\sigma$. Define a relation $\preceq$ on the set $X = P \times T$ as follows: for any $(e, u), (f, v) \in X$, let

$$(e, u) \preceq (f, v) \quad \text{if} \quad a^+ = e,\ u = a\sigma \cdot v \text{ and } a^* = f \quad \text{for some } a \in S.$$

Lemma 3.1. *The relation $\preceq$ is a preoder.*

Proof. It is clear that $\preceq$ is reflexive. To prove transitivity, assume that $(e, u) \preceq (f, v)$ and $(f, v) \preceq (g, w)$ in X. Then there exist $a, b \in S$ such that $a^+ = e,\ u = a\sigma \cdot v,\ a^* = f$ and $b^+ = f,\ v = b\sigma \cdot w,\ b^* = g$. Therefore we have $(ab)^+ = (ab^+)^+ = (aa^*)^+ = a^+ = e$, $(ab)\sigma \cdot w = a\sigma(b\sigma \cdot w) = a\sigma \cdot v = u$ and $(ab)^* = (a^*b)^* = (b^+b)^* = b^* = g$, whence $(e, u) \preceq (g, w)$. □

In the sequel, we intend to introduce a factor set of X which is partially ordered in a natural way and is acted upon by T in an appropriate way. The construction is motivated by Munn's proof of McAlister's P-theorem ([11]), but it is more complicated, due to the fact that the structure of restriction semigroups is much more complicated than that of inverse semigroups.

First we consider the least equivalence relation ρ on X containing $\preceq$, and having the property that, for every $(e,u),(f,v) \in X$ and $t \in T$, we have $(e,u)\,\rho\,(f,v)$ if and only if $(e,ut)\,\rho\,(f,vt)$. Recursively, we define a sequence of relations η_j $(j \in \mathbb{N}_0)$, and whenever η_j is defined, we denote its transitive closure by ϑ_j. Let $\eta_0 = \preceq \cup \succeq$ and, for every $j \in \mathbb{N}_0$, put

$$\eta_{j+1} = \begin{cases} \{((e,u),(f,v)) \in X \times X : (\exists t \in T)\ (e,ut)\,\vartheta_j\,(f,vt)\} & \text{if } 2 \mid j, \\ \{((e,ut),(f,vt)) \in X \times X : t \in T \text{ and } (e,u)\,\vartheta_j\,(f,v)\} & \text{if } 2 \nmid j. \end{cases}$$

Finally, define $\vartheta = \bigcup_{j=0}^{\infty} \vartheta_j$.

Lemma 3.2. *The relation ϑ is the least equivalence relation ρ on X containing $\preceq$, and having the property that, for every $(e,u),(f,v) \in X$ and $t \in T$, we have $(e,u)\,\rho\,(f,v)$ if and only if $(e,ut)\,\rho\,(f,vt)$.*

Proof. First we check that ϑ is an equivalence on X such that $\preceq \subseteq \vartheta$ and, for every $(e,u),(f,v) \in X$ and $t \in T$, we have $(e,u)\,\vartheta\,(f,v)$ if and only if $(e,ut)\,\vartheta\,(f,vt)$. By definition, it is clear that $\preceq \subseteq \eta_0 \subseteq \vartheta_0 \subseteq \vartheta$ and, for every $j \in \mathbb{N}_0$, the relation η_j is reflexive and symmetric, and so ϑ_j is an equivalence. Moreover, since T is a monoid, we have $\vartheta_j \subseteq \eta_{j+1}$ $(j \in \mathbb{N}_0)$, and so

$$\vartheta_0 \subseteq \vartheta_1 \subseteq \vartheta_2 \subseteq \ldots \subseteq \vartheta_j \subseteq \vartheta_{j+1} \subseteq \ldots. \tag{3.1}$$

This implies that ϑ is an equivalence relation.

If $(e,u)\,\vartheta\,(f,v)$ then, by definition and (3.1), we have $(e,u)\,\vartheta_j\,(f,v)$ for some odd j. Therefore $(e,ut)\,\vartheta_{j+1}\,(f,vt)$ and $(e,ut)\,\vartheta\,(f,vt)$ follow for every $t \in T$. Similarly, we also see that $(e,ut)\,\vartheta\,(f,vt)$ implies $(e,u)\,\vartheta\,(f,v)$ for any $(e,u),(f,v) \in X$ and $t \in T$.

Conversely, let ρ be any equivalence relation on X such that $\preceq \subseteq \rho$ and, for every $(e,u),(f,v) \in X$ and $t \in T$, we have $(e,u)\,\rho\,(f,v)$ if and only if $(e,ut)\,\rho\,(f,vt)$. We intend to show that $\vartheta \subseteq \rho$, that is, $\vartheta_j \subseteq \rho$ for every $j \in \mathbb{N}_0$. We proceed by induction on j. Since ρ is an equivalence, it suffices to verify that $\eta_j \subseteq \rho$ for every $j \in \mathbb{N}_0$.

Clearly, $\eta_0 \subseteq \rho$. Assume that, for some $j \in \mathbb{N}_0$, we have $\vartheta_j \subseteq \rho$. If $2 \mid j$ and $(e,u)\,\eta_{j+1}\,(f,v)$ in X then, by definition, there exists $t \in T$ such that $(e,ut)\,\vartheta_j\,(f,vt)$, and so $(e,ut)\,\rho\,(f,vt)$. By assumption, the latter relation implies $(e,u)\rho(f,v)$, and so $\eta_{j+1} \subseteq \rho$ holds. A similar argument applies in case $2 \nmid j$, completing the proof of the inclusion $\vartheta \subseteq \rho$. □

In the sequel, the following condition will be important:

$$(e,1)\ \vartheta\ (f,1) \quad \text{implies} \quad e = f \quad \text{for every } e,f \in P. \tag{SP}$$

For brevity, denote the set X/ϑ of all ϑ-classes by $\mathcal{X}$, and the ϑ-class containing $(e,u) \in X$ by $[e,u]$. Moreover, put

$$[e,u]^t = [e,ut] \quad \text{for any } [e,u] \in \mathcal{X} \text{ and } t \in T.$$

Lemma 3.2 implies that this rule defines an action of T on the set $\mathcal{X}$ by injective mappings.

Now we make preparations in order to introduce a partial order on $\mathcal{X}$.

Lemma 3.3. (1) *If* $(e,u) \preceq (f,v)$ *in* X *and* $e' \leq e$ *in* P *then there exists* $f' \leq f$ *in* P *such that* $(e',u) \preceq (f',v)$.
(2) *If* $(e,u) \preceq (f,v)$ *in* X *and* $f' \leq f$ *in* P *then there exists* $e' \leq e$ *in* P *such that* $(e',u) \preceq (f',v)$.
(3) *For every* $j \in \mathbb{N}_0$, *if* $(e,u)\,\vartheta_j\,(f,v)$ *in* X *and* $e' \leq e$ *in* P *then there exists* $f' \leq f$ *in* P *such that* $(e',u)\,\vartheta_j\,(f',v)$.
(4) *If* $(e,u)\,\vartheta\,(f,v)$ *in* X *and* $e' \leq e$ *in* P *then there exists* $f' \leq f$ *in* P *such that* $(e',u)\,\vartheta\,(f',v)$.

Proof. Let $(e,u) \preceq (f,v)$ in X. Then, by definition, there exists $a \in S$ such that $a^+ = e$, $u = a\sigma \cdot v$ and $a^* = f$.

(1) Let $e' \in P$ with $e' \leq e$, and define $b = e'a$. Then we easily see that $b^+ = (e'a)^+ = (e'a^+)^+ = e'e = e'$, $b\sigma \cdot v = (e'a)\sigma \cdot v = a\sigma \cdot v = u$ and $b^* = (e'a)^* \leq a^* = f$. Putting $f' = b^*$, we obtain that $(e',u) \preceq (f',v)$.

(2) If $f' \in P$ with $f' \leq f$ then, considering $b = af'$, a similar argument applies.

(3) We proceed by induction on j. Let $(e,u)\,\vartheta_0\,(f,v)$ in X. Then there exists a sequence (e_i,u_i) $(i = 0,1,\ldots,2n)$ of elements in X such that

$$(e,u) = (e_0,u_0) \succeq (e_1,u_1) \preceq (e_2,u_2) \succeq \ldots \preceq (e_{2n},u_{2n}) = (f,v).$$

If $e' \in P$ with $e' \leq e$ then put $e'_0 = e'$, and by applying (1) and (2), we obtain $e'_1, e'_2, \ldots, e'_{2n} \in P$ with $e'_i \leq e_i$ $(i = 1,2,\ldots,2n)$ such that

$$(e',u) = (e'_0,u_0) \succeq (e'_1,u_1) \preceq (e'_2,u_2) \succeq \ldots \preceq (e'_{2n},u_{2n}).$$

Thus, choosing f' to be e'_{2n}, we see that $f' \leq e_{2n} = f$ and $(e',u)\,\vartheta_0\,(f',v)$.

Now suppose that $j \in \mathbb{N}_0$ such that, for every $(e,u),(f,v) \in X$, if $(e,u)\,\vartheta_j\,(f,v)$ and $e' \leq e$ in P then there exists $f' \leq f$ in P such that $(e',u)\,\vartheta_j\,(f',v)$. Furthermore, let $(e,u),(f,v) \in X$ with $(e,u)\,\vartheta_{j+1}\,(f,v)$ and let $e' \in P$ with $e' \leq e$. If $2 \mid j$ then the latter relation means that there exist elements $(e_i,u_i) \in X$ and $t_i \in T$ $(i = 0,1,\ldots,n)$ such that $(e,u) = (e_0,u_0)$, $(f,v) = (e_n,u_n)$, and

$$(e_i,u_it_i)\,\vartheta_j\,(e_{i+1},u_{i+1}t_i) \quad \text{for } i = 0,1,2,\ldots,n-1.$$

Let $e' \in P$ with $e' \leq e$. Then, by the induction hypothesis, there exist $e'_0, e'_1, e'_2, \ldots, e'_n \in P$ with $e'_i \leq e_i$ $(i = 0,1,\ldots,n)$ such that $e'_0 = e'$ and

$$(e'_i,u_it_i)\,\vartheta_j\,(e'_{i+1},u_{i+1}t_i) \quad \text{for } i = 0,1,2,\ldots,n-1.$$

Therefore $f' = e'_n$ has the properties that $f' \leq e_n = f$ and $(e',u)\,\vartheta_{j+1}\,(f',v)$. If $2 \nmid j$ then the argument is similar.

(4) By the definition of ϑ, the statement immediately follows from (3). □

Let us define a relation $\leqslant$ on the set $\mathcal{X}$ by the following rule: for every $[e,u],[f,v] \in \mathcal{X}$, let

$$[e,u] \leqslant [f,v] \quad \text{if} \quad \text{there exists } f' \in P \text{ such that } f' \leq f \text{ and } [e,u] = [f',v].$$

Lemma 3.3(4) ensures that the relation $\leqslant$ is well defined.

Lemma 3.4. *If condition* (SP) *is satisfied by* S *then the relation* $\leqslant$ *is a partial order on* $\mathcal{X}$.

Proof. Reflexivity of $\leqslant$ is obvious by definition. To prove transitivity, let $[e,u] \leqslant [f,v]$ and $[f,v] \leqslant [g,w]$. Then there exist $f' \leq f$ and $g' \leq g$ in P such that $[e,u] = [f',v]$ and $[f,v] = [g',w]$. By Lemma 3.3(4), there exists $g'' \leq g'$ in P such that $[f',v] = [g'',w]$, and so $g'' \leq g$ and $[e,u] = [g'',w]$. This shows that $[e,u] \leqslant [g,w]$.

To check antisymmetry, assume that $[e,u] \leqslant [f,v]$ and $[f,v] \leqslant [e,u]$. By an argument similar to that in the previous paragraph, we can see that there exist $f', e', e'' \in P$ such that $f' \leq f$, $e'' \leq e' \leq e$ and $[e,u] = [f',v]$, $[f,v] = [e',u]$, $[f',v] = [e'',u]$. Thus $[e,u] = [e'',u]$ is implied, and so $(e,1)\,\vartheta\,(e'',1)$ follows by Lemma 3.2. Hence we obtain $e = e''$ by (SP). Thus $e = e'$ also holds, and we see that $[f,v] = [e',u] = [e,u]$. $\square$

From now on, we suppose that S has property (SP).

Next we verify that the action of T on $\mathcal{X}$ is compatible with the partial order $\leqslant$ in a strict sense. In particular, it turns out that T acts on the partially ordered set $\mathcal{X} = (\mathcal{X};\leqslant)$ by injective order preserving mappings.

Lemma 3.5. *For any $[e,u],[f,v] \in \mathcal{X}$ and $t \in T$, we have $[e,u]^t \leqslant [f,v]^t$ if and only if $[e,u] \leqslant [f,v]$. Consequently, if $[e,u],[f,v] \in \mathcal{X}$ and $t \in T$ then $[e,u]^t = [f,v]^t$ implies $[e,u] = [f,v]$.*

Proof. If $[e,u] \leqslant [f,v]$ in $\mathcal{X}$ then there exists $f' \leq f$ in P such that $(e,u)\,\vartheta\,(f',v)$, and so $(e,u)\,\vartheta_j\,(f',v)$ for some $j \in \mathbb{N}_0$. The definition of the ϑ_j's and inclusions (3.1) imply $(e,ut)\,\vartheta_{j'}\,(f',vt)$ for $j' = j+1$ or $j' = j+2$ whence we obtain that $(e,ut)\,\vartheta\,(f',vt)$, and so $[e,ut] \leqslant [f,vt]$. This shows that if $[e,u] \leqslant [f,v]$ then $[e,u]^t \leqslant [f,v]^t$ follows for every $t \in T$. The reverse implication is proved in a similar fashion. Due to the antisymmetry of $\leqslant$, the second statement is clear by the first one. $\square$

By means of $\mathcal{X}$, now we consider a semilattice $\mathcal{Y}$ acted upon by T so that a W-pair be obtained.

Let $\mathcal{Y}$ be the semilattice of order ideals of the partially ordered set $\mathcal{X}$ with respect to $\cap$, the usual intersection of subsets. In particular, $\langle x] = \{z \in \mathcal{X} : z \leqslant x\}$, the principal order ideal generated by x, belongs to $\mathcal{Y}$ for every $x \in \mathcal{X}$. If $x = [e,u]$ then, instead of $\big\langle [e,u]\big]$, we simply write $\langle e,u]$. For any $I \in \mathcal{Y}$ and $t \in T$, define

$$I^t = \{x^t : x \in I\}. \tag{3.2}$$

We present several properties of these subsets.

Lemma 3.6. *If $x \in \mathcal{X}$, $I, J \in \mathcal{Y}$ and $t \in T$ then*

(1) $\langle x]^t = \langle x^t]$,
(2) $I^t \in \mathcal{Y}$,
(3) $I^t \subseteq J^t$ *if and only if* $I \subseteq J$, *and consequently,* $I^t = J^t$ *implies* $I = J$,
(4) $(I \cap J)^t = I^t \cap J^t$,
(5) $J \subseteq I^t$ *implies the existence of* $K \in \mathcal{Y}$ *with* $J = K^t$.

Proof. (1) By Lemma 3.5, if $y \leqslant x$ in $\mathcal{X}$ then $y^t \leqslant x^t$, implying that $\langle x]^t \subseteq \langle x^t]$. To prove the reverse inclusion, assume that $y \in \mathcal{X}$ such that $y \leqslant x^t$. Put $x = [e,u]$ and $y = [f,v]$. Then $[f,v] \leqslant [e,ut]$, that is, by definition, there exists $e' \leq e$ in P such that $[f, v] = [e', ut] = [e', u]^t$. Since $[e', u] \leqslant [e, u] = x$, we see that $[e', u] \in \langle x]$, and so $y \in \langle x]^t$.

(2) Straightforward by (1).

(3) It suffices to verify the first statement since it clearly implies the second one. It is clear by definition that if $I \subseteq J$ then $I^t \subseteq J^t$. Now suppose that $I^t \subseteq J^t$, and let $x \in I$. Since $x^t \in J^t$, we have $y \in J$ with $x^t = y^t$. Hence $x = y \in J$ follows by Lemma 3.5 which verifies the reverse implication.

(4) The inclusion $(I \cap J)^t \subseteq I^t \cap J^t$ is obvious. To show the reverse inclusion, let $z \in I^t \cap J^t$. Then $z = x^t = y^t$ for some $x \in I$ and $y \in J$, and so Lemma 3.5 implies that $x = y \in I \cap J$. Thus the inclusion $(I \cap J)^t \supseteq I^t \cap J^t$ also holds.

(5) If $J \subseteq I^t$ then put $K = \{z \in \mathcal{X} : z^t \in J\}$. By (1), we have $z^t \in J$ if and only if $\langle z]^t \subseteq J$, therefore $K \in \mathcal{Y}$. Furthermore, we clearly have $K^t \subseteq J$. Conversely, if $y \in J$ then $y \in I^t$ which implies that $y = x^t$ for some $x \in I$, and so we obtain that $x \in K$ and $y = x^t \in K^t$. Thus $J = K^t$. □

Statements (2) and (4) of this lemma say that rule (3.2) defines an action of T on the semilattice $\mathcal{Y}$, and statements (3) and (5) imply that conditions (2.3) and (2.4) are fulfilled by this action. This proves the following lemma.

Lemma 3.7. *The pair $(T, \mathcal{Y})$ is a W-pair.*

This allows us to define the W-product $W(T, \mathcal{Y})$ and the mapping

$$\kappa \colon S \to W(T, \mathcal{Y}), \quad a \mapsto \big(a\sigma, \langle a^+, 1]^{a\sigma}\big)\,.$$

Note that, by definition, we have $(a^+, a\sigma) \preceq (a^*, 1)$, and so

$$[a^+, a\sigma] = [a^*, 1] \quad \text{for every } a \in S. \tag{3.3}$$

Lemma 3.8. *If S is proper then the mapping κ is an injective* $(2,1,1)$*-morphism.*

Proof. Let $a, b \in S$. To show that κ is injective, assume that $a\kappa = b\kappa$, that is, $a\sigma = b\sigma$ and $\langle a^+, 1]^{a\sigma} = \langle b^+, 1]^{b\sigma}$. By Lemma 3.7, these equalities imply $\langle a^+, 1] = \langle b^+, 1]$, and so $[a^+, 1] = [b^+, 1]$. By (SP), we deduce that $a^+ = b^+$ and, since S is proper, this equality and $a\sigma = b\sigma$ imply $a = b$.

Applying Lemma 3.6(1) and (3.3), we see that

$$\begin{aligned}
(a\kappa)^+ &= \big(a\sigma, \langle a^+, 1]^{a\sigma}\big)^+ = (1, \langle a^+, 1]) = a^+\kappa,\\
(a\kappa)^* &= \big(a\sigma, \langle a^+, 1]^{a\sigma}\big)^* = \big(1, \langle a^+, 1]^{a\sigma}\big) = (1, \langle a^*, 1]) = a^+\kappa.
\end{aligned}$$

Therefore κ respects both unary operation. Furthermore, we have

$$\begin{aligned}
a\kappa \cdot b\kappa &= \big(a\sigma, \langle a^+, 1]^{a\sigma}\big)\big(b\sigma, \langle b^+, 1]^{b\sigma}\big)\\
&= \big((ab)\sigma, \langle a^+, 1]^{a\sigma \cdot b\sigma} \cap \langle b^+, 1]^{b\sigma}\big),\\
&= \big((ab)\sigma, \big(\langle a^+, 1]^{a\sigma} \cap \langle b^+, 1]\big)^{b\sigma}\big),\\
(ab)\kappa &= \big((ab)\sigma, \langle (ab)^+, 1]^{(ab)\sigma}\big)\\
&= \big((ab)\sigma, \big(\langle (ab)^+, 1]^{a\sigma}\big)^{b\sigma}\big).
\end{aligned}$$

By Lemma 3.6(1), in order to prove that $a\kappa \cdot b\kappa = (ab)\kappa$, it suffices to show that $\langle a^+, a\sigma] \cap \langle b^+, 1] = \langle (ab)^+, a\sigma]$. Since $(ab)^+ \leq a^+$, we see that

$[(ab)^+, a\sigma] \leqslant [a^+, a\sigma]$. Furthermore, since $(ab)^* \leq b^*$, we obtain by (3.3) that $[(ab)^+, a\sigma]^{b\sigma} = [(ab)^+, (ab)\sigma] = [(ab)^*, 1)] \leqslant [b^*, 1] = [b^+, b\sigma] = [b^+, 1]^{b\sigma}$. Hence it follows by Lemma 3.5 that $[(ab)^+, a\sigma] \leqslant [b^+, 1]$. This implies the inclusion $\langle a^+, a\sigma] \cap \langle b^+, 1] \supseteq \langle (ab)^+, a\sigma]$.

To verify the reverse inclusion, let $[e, u] \in \langle a^+, a\sigma] \cap \langle b^+, 1]$. Then $[e, u] \leqslant [a^+, a\sigma] = [a^*, 1]$ and $[e, u] \leqslant [b^+, 1]$, so that there exist $i, j \in P$ such that $i \leq a^*$, $j \leq b^+$ and $[e, u] = [i, 1] = [j, 1]$. By condition (SP), we obtain that $i = j$, and so $i \leq a^* b^+$. Hence $[e, u] \leqslant [a^* b^+, 1]$. Since $(ab)^+ = (ab^+)^+$, $(ab^+)\sigma = a\sigma$ and $(ab^+)^* = a^* b^+$, the relation $((ab)^+, a\sigma) \preceq (a^* b^+, 1)$ follows. Therefore $[e, u] \leqslant [a^* b^+, 1] = [(ab)^+, a\sigma]$, and so $[e, u] \in \langle (ab)^+, a\sigma]$. This shows the reverse inclusion, and so κ respects also the multiplication. □

So far, we have established the 'if' part of our main result:

Theorem 3.9. *A restriction semigroup S is $(2,1,1)$-embeddable into a W-product of a semilattice by a monoid if and only if S is proper and satisfies condition* (SP).

Proof. To verify the 'only if' part, assume that (R, Y) is a W-pair and $\phi\colon S \to W(R, Y)$ is an injective $(2,1,1)$-morphism. Result 2.1(3) immediately implies that S is proper.

Denote by π the first projection $W(R, Y) \to R$, $(r, i^r) \mapsto r$. By Result 2.1(2), $\phi\pi\colon S \to R$ is a $(2,1,1)$-morphism into the monoid R, and so σ is contained in the $(2,1,1)$-congruence of S induced by $\phi\pi$. This implies that there exists a unique monoid homomorphism $\zeta\colon T \to R$ such that $a\phi\pi = (a\sigma)\zeta$ for any $a \in S$.

Moreover, consider the restriction $\phi|_P\colon P \to P(W(R, Y))$ of ϕ to the semilattices of projections. It is clearly an injective homomorphism and, by Result 2.1(1), the second projection $\pi'\colon P(W(R, Y)) \to Y$ is an isomorphism. Hence $\phi|_P\pi'\colon P \to Y$ is an injective homomorphism.

Notice that, for every $a \in S$, we have

$$a\phi = \big(a\phi\pi, (a^+\phi|_P\pi')^{a\phi\pi}\big) = \big(a\phi\pi, a^*\phi|_P\pi'\big). \tag{3.4}$$

We extend $\phi|_P\pi'$ to X as follows: define

$$\xi\colon X \to Y, \quad (e, u)\xi = (e\phi|_p\pi')^{u\zeta}.$$

We show that, for every $(e, u), (f, v) \in X$, if $(e, u)\,\vartheta\,(f, v)$ then $(e, u)\xi = (f, v)\xi$. Denote by ρ the equivalence relation induced by ξ on X. First assume that $(e, u) \preceq (f, v)$. Then we have $a \in S$ with $a^+ = e$, $u = a\sigma \cdot v$ and $a^* = f$. By (3.4), we see that $(e\phi|_P\pi')^{(a\sigma)\zeta} = (a^+\phi|_P\pi')^{(a\sigma)\zeta} = a^*\phi|_P\pi' = f\phi|_P\pi'$, whence $(e, u)\xi = (e\phi|_P\pi')^{u\zeta} = (e\phi|_P\pi')^{(a\sigma)\zeta \cdot v\zeta} = (f\phi|_P\pi')^{v\zeta} = (f, v)\xi$. This shows that $\preceq\, \subseteq \rho$, and since ρ is an equivalence, we obtain that $\eta_0 \subseteq \vartheta_0 \subseteq \rho$. Moreover, since (R, Y) is a W-pair, it is easy to see from the definition of ξ that, for any $(e, u), (f, v) \in X$ and $t \in T$, we have $(e, u)\,\rho\,(f, v)$ if and only if $(e, ut)\,\rho\,(f, vt)$. Thus Lemma 3.2 implies that $\vartheta \subseteq \rho$, whence we see that if $(e, u)\,\vartheta\,(f, v)$ then $(e, u)\xi = (f, v)\xi$.

Finally, notice that property (SP) is a direct consequence of this statement since $\phi_P\pi'$ is injective. This completes the proof of our theorem. □

4. Miscellaneous remarks

In this section we make several remarks on the W-product introduced in the previous section. We simplify the construction, and relate it to constructions introduced in [6] and [12].

First we notice that, instead of the semilattice $\mathcal{Y}$ defined in Section 3, it suffices to consider a subsemilattice of $\mathcal{Y}$. For, by Lemma 3.6(1), the injective $(2,1,1)$-morphism κ assigns, to any element of S, an element of $W(T,\mathcal{Y})$ whose second component is a principal order ideal of $\mathcal{X}$, and the set of all principal order ideals is closed under the action of T. Therefore $\mathcal{Y}_0$ consisting of all finitely generated order ideals of $\mathcal{X}$ (i.e. of those being intersections of finitely many principal order ideals) forms a subsemilattice in $\mathcal{Y}$ which is closed under the action of T, and $S\kappa \subseteq W(T,\mathcal{Y}_0)$. So the definition of κ can be modified by replacing $\mathcal{Y}$ with $\mathcal{Y}_0$.

Proposition 4.1. *If S is a proper restriction semigroup satisfying condition* (SP) *then $(T,\mathcal{Y}_0)$ is a W-pair, and the mapping*

$$\kappa_0 \colon S \to W(T,\mathcal{Y}_0), \quad a \mapsto \big(a\sigma, \langle a^+, 1]^{a\sigma}\big)$$

is an injective $(2,1,1)$-morphism.

As an example, let us consider the free restriction semigroup $F\mathcal{RS}(Z)$ on Z as S, and find κ_0 for it. According to [2], one can obtain a model for $F\mathcal{RS}(Z)$ as a $(2,1,1)$-subsemigroup of the free inverse semigroup $F\mathcal{I}(Z)$ on Z:

$$F\mathcal{RS}(Z) = \{(A,u) \in \mathcal{E} \times F\mathcal{G}(Z) : u \in Z^* \cap A\} \leq F\mathcal{I}(Z)$$

with $F\mathcal{I}(Z)$ being the usual model for the free inverse semigroup on Z where $F\mathcal{G}(Z)$ is the free group on Z, Z^* is the free monoid on Z considered as a submonoid in $F\mathcal{G}(Z)$, and $\mathcal{E}$ is the semilattice (with respect to $\cup$) of all finite connected subgraphs of the Cayley graph of $F\mathcal{G}(Z)$ contaning vertex 1. Note that $F\mathcal{G}(Z)$ acts on the set $\mathcal{C}$ of all finite connected subgraphs of the Cayley graph of $F\mathcal{G}(Z)$ by left multiplication. This action plays crucial role in this construction, and $\mathcal{C} = {}^{F\mathcal{G}(Z)}\mathcal{E}$.

The elements of P can obviously be identified with those of $\mathcal{E}$. It is easy to see that if $(A,u) \preceq (B,v)$ in $X = \mathcal{E} \times Z^*$ then ${}^{u^{-1}}A = {}^{v^{-1}}B$. Based on the fact that each element of $F\mathcal{G}(Z)$ is of the form $w_1 w_2^{-1} \cdots w_{2k-1} w_{2k}^{-1}$ for some $k \in \mathbb{N}_0$ and $w_1, w_2, \ldots, w_{2k-1}, w_{2k} \in Z^*$, an inductive argument can be applied to show that $(A,u)\,\vartheta_0\,(B,v)$ if and only if ${}^{u^{-1}}A = {}^{v^{-1}}B$. Hence it follows immediately that $\vartheta_0 = \vartheta$, and so $\mathcal{X}$ can be identified with the partially ordered set $(\mathcal{Q};\supseteq)$ where

$$\mathcal{Q} = \{{}^{u^{-1}}A \in \mathcal{C} : u \in Z^*,\ A \in \mathcal{E}\}.$$

Since the Cayley graph of $F\mathcal{G}(Z)$ is a tree, this is, actually, a semilattice with respect to the operation $\vee$ of forming the least connected subgraph containing given finite connected subgraphs. This implies that the semilattice $(\mathcal{Q};\vee)$ is isomorphic to $\mathcal{Y}_0$.

However, this is just the semilattice introduced in [12] to define a W-product and give a model for $F\mathcal{RS}(Z)$ as a $(2,1,1)$-subsemigroup in this W-product. Hence we easily obtain the following.

Example 4.2. If $S = F\mathcal{RS}(Z)$, the free restriction semigroup on Z, then $W(T, \mathcal{Y}_0)$ is isomorphic to the W-product $W(Z^*, \mathcal{Q})$ introduced in [12], and so κ_0 is, actually, the mapping

$$\iota\colon F\mathcal{RS}(Z) \to W(Z^*, \mathcal{Q}),\ (A, t) \mapsto (t, A^t)$$

considered in the proof of [12, Theorem 3.3].

Finally, let us compare the condition and construction obtained in Section 3 for restriction semigroups to those obtained in [6] for left restriction semigroups. First of all the necessary and sufficient conditions proved in [6, Theorem 3.3] for a left restriction semigroup to be $(2,1)$-embeddable into a W-product seem to be much simpler to check, especially, if we disregard that the least right cancellative $(2,1)$-congruence ω might be difficult to determine. Similarly to the one-sided case, the least right cancellative $(2,1,1)$-congruence ω exists also on any restriction semigroup S, and if S is $(2,1,1)$-embeddable into a W-product then

$$\tau = \{(a, b) \in S \times S : a^+ = b^+ \text{ and } a\,\omega\,b\}$$

is a projection separating $(2,1,1)$-congruence on S. However, the W-product constructed in the proof of [6, Theorem 3.3] to $(2,1)$-embed $(S; \cdot, ^+)$ into it is far from respecting the operation * in general.

To illustrate this more clearly, as it might be seen from the constructions applied in the one-sided and in the two-sided cases, we can modify our construction of X, ϑ, $\mathcal{X}$ and $\mathcal{Y}$ in Section 3 by replacing σ with ω. Then X becomes the same in the two constructions, but the complicated definition of ϑ cannot be avoided.

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