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EMBEDDING OF URN SCHEMES INTO CONTINUOUS TIME MARKOV BRANCHING PROCESSES AND RELATED LIMIT THEOREMS[1]

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1. Introduction. In this paper we present a technique of embedding certain urn schemes into continuous time Markov branching processes. Typically these urn schemes could be represented by a discrete parameter Markov chain $\{Y_n; n = 0, 1, 2, \cdots\}$ where the state space is the nonnegative integer lattice in p dimensions for some integer p. We shall establish the existence of certain continuous time Markov branching processes with p-types $\{X(t) = (X_1(t), \cdots, X_p(t)); t \geqq 0\}$ such that for an appropriate sequence $\tau_n, n = 0, 1, 2, \cdots$, of increasing stopping times, the stochastic process $\{X(\tau_n); n = 0, 1, 2, \cdots\}$ is equivalent to $\{Y_n, n = 0, 1, 2, \cdots\}$. Thus from limit theorems for $X(t)$ as $t \to \infty$ we can deduce results on the limit behavior of the random variables $\{Y_n\}$ as $n \to \infty$. It turns out that this technique yields many classical and some new results on urn schemes in a relatively simple and more transparent manner.

In this paper we shall be mainly concerned with B. Friedman's scheme (see [7], [8] and Section 2 for a definition). Although we describe the technique in detail only in this case the fundamental idea can easily be adapted to more general situations.

It is interesting to note that urn models have been basic in the study of the spread of contagious diseases and certain ecological and branching processes (see [6], [11]). In this work we proceed in the reverse direction by exploiting properties of branching processes with view to investigate the fluctuation behavior of urn schemes.

An outline of the paper follows. Section 2 introduces the background material concerning multitype continuous time branching processes and reviews some relevant limit theorems for these processes. Section 3 describes the structure of the Friedman and Pólya urn schemes and highlights their connections to branching processes. The principal theorem on the relation of certain results pertaining to multitype continuous time branching processes to those on the embedded Markov chain is contained in Section 4. The applications of this fundamental limit theorem to the case of the Friedman urn are summarized earlier in Section 3.

2. Some results on multitype continuous time Markov branching processes. To make the paper reasonably self contained we devote this section to summarizing results on multitype continuous time Markov branching processes needed later on; for details see ([1], [2], [3]).

Let $\{X(t) = (X_1(t), \cdots, X_p(t)); t \geqq 0\}$ be a p-type continuous time Markov

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branching process defined on a probability space $(\Omega, \mathbf{F}, P)$ and let the associated infinitesimal generating functions be

$$u_i(s) = a_i[h_i(s) - s_i] \tag{1}$$

where for $i = 1, 2, \cdots, p$,

$$0 < a_i < \infty, \qquad s = (s_1, s_2, \cdots, s_p), \qquad 0 \leqq s_i \leqq 1$$

and $h_i(s)$ is a probability generating function on the p dimensional nonnegative integer lattice.

We make the assumption

$$\partial h_i(s)/\partial s_j \,|_{s=(1,1,\cdots,1)} < \infty, \qquad i, j = 1, 2, \cdots, p. \tag{2}$$

It is known that a conservative Markov branching process fulfilling the above specifications exists (see [3], [10]). If $\sigma(D)$ denotes the sub σ-field of $\mathbf{F}$ induced by a collection D of random variables on $(\Omega, \mathbf{F}, P)$, then let $\mathbf{F}_t = \sigma(\{X(s, \omega); s \leqq t\})$. Without loss of generality we may stipulate that $\{X(t); t \geqq 0\}$ is strong Markov with respect to the family $\mathbf{F}_t$ and that the sample paths are right continuous and possess left limits in t with probability one.

Set $\tau_0(\omega) \equiv 0$ and let $\tau_n(\omega)$ for $n = 1, 2, 3, \cdots$ denote the nth discontinuity point of the sample path $X(t, \omega)$. It can be shown that for every n, $\tau_n(\omega)$ is a stopping time with respect to the family $\mathbf{F}_t$. Since the process is strong Markov and the sequence τ_n is increasing we conclude that the family $\{X(\tau_n), \mathfrak{F}_n; n = 0, 1, 2, \cdots\}$ is a discrete parameter Markov chain, where $\mathfrak{F}_n$ is the σ-field associated with the stopping time τ_n.

It aids intuition to interpret $X(t) = (X_1(t), \cdots, X_p(t))$ as a vector denoting the population sizes at time t of a system comprised of p types of particles evolving in the following manner:

(α) a type i particle lives an exponentially distributed length of time with mean a_i^{-1} and on death creates particles of all types following the distribution law whose probability generating function is $h_i(s)$,

(β) all particles engender independent lines of descent,

(γ) the initial population consists of $X_i(0)$ particles of type i, $i = 1, 2, \cdots, p$.

From (2) we can deduce that $m_{ij}(t) \equiv E(X_j(t) \mid X_r(0) = \delta_{ir}$ for $r = 1, 2, \cdots, p)$ is finite for all i, j and t. (E denotes the expectation operator.)

Moreover, $M(t) = \|m_{ij}(t)\| = \exp(At)$ where $A = \|a_{ij}\|$, $a_{ij} = a_i b_{ij}$, $b_{ij} = \partial h_i/\partial s_j \,|_{s=(1,1,\cdots,1)} - \delta_{ij}$.

We postulate the existence of $t_0 > 0$ such that

$$m_{ij}(t_0) > 0, \qquad i, j = 1, 2, \cdots, p. \tag{3}$$

This implies

(i) the eigenvalues λ_i of A can be arranged so that

$$\lambda_1 > \operatorname{Re} \lambda_2 \geqq \operatorname{Re} \lambda_3 \geqq \cdots \geqq \operatorname{Re} \lambda_p ; \tag{4}$$

(ii) there exist strictly positive eigenvectors u and v such that

(5) $$Av = \lambda_1 v, \quad u^* A = \lambda_1 u^*, \quad \text{and} \quad u^* v = 1,$$

where * denotes the transpose operation;

(iii) the right and left eigenspaces of A corresponding to λ_1 are one dimensional.

To avoid trivial degeneracies we henceforth exclude the singular case where

$$h_i(s) = \sum_{j=1}^{p} p_{ij} s_j \quad \text{for} \quad i = 1, 2, \cdots, p.$$

We now state a few results to be used in the sequel. For proofs see [1], [2] and [3].

PROPOSITION 1. *For any right eigenvector ξ of A with eigenvalue λ the family $\{\xi \cdot X(t)e^{-\lambda t}; \mathbf{F}_t; t \geqq 0\}$ is a martingale (possibly complex valued) where*

(6) $$\mathbf{F}_t = \sigma(\{X(s, \omega); s \leqq t\})$$

as specified earlier. (The notation $\xi \cdot \eta$ denotes the inner product of the given vectors.)

COROLLARY 1. *For v in (5) $\{v \cdot X(t)e^{-\lambda_1 t}; \mathbf{F}_t; t \geqq 0\}$ is a nonnegative martingale and therefore*

(7) $$\lim_{t\to\infty} v \cdot X(t, \omega)e^{-\lambda_1 t} = W(\omega)$$

exists almost surely and

$$E(W) \leqq v_i \quad \text{if} \quad X_i(0) = 1 \quad \text{and} \quad X_j(0) = 0 \quad \text{for} \quad j \neq i.$$

PROPOSITION 2. *Subject to (2) and (3)*

(8) $$\lim_{t\to\infty} X(t, \omega)e^{-\lambda_1 t} = W(\omega)u \quad \text{a.s.}$$

Now we impose the following additional assumptions: (consult (2) and (3))

(9) $$\lambda_1 > 0$$

corresponding to the supercritical case and

(10) $$\partial^2 h_i / \partial s_l \partial s_m |_{s=(1,1,\cdots,1)} < \infty \quad \text{for} \quad i, l, m = 1, 2, \cdots, p.$$

Let ξ be a right eigenvector of A with eigenvalue λ. We have the following results (see [1], [2]).

CASE 1. $2 \operatorname{Re} \lambda > \lambda_1$. The martingale

$$\{Y(t) = \xi \cdot X(t)e^{-\lambda t}; \mathbf{F}_t; t \geqq 0\}$$

satisfies $\sup_{t>0} E(|Y(t)|^2) < \infty$ and hence there exists a random variable Y such that

(11) $$\lim_{t\to\infty} E|Y(t) - Y|^2 = 0 \quad \text{and} \quad P\{\lim_{t\to\infty} Y(t, \omega) = Y(\omega)\} = 1.$$

CASE 2. $2 \operatorname{Re} \lambda = \lambda_1$. Let $Y(t) = (l_1 \operatorname{Re} \xi \cdot X(t) + l_2 \operatorname{Im} \xi \cdot X(t))$ where l_1

and l_2 are real numbers not both zero. Then

$$(12)\quad \lim_{t\to\infty} P\{0 < x_1 \leqq W \leqq x_2 < \infty, Y(t)[v\cdot X(t) \log v\cdot X(t)]^{-\frac{1}{2}} \leqq x\} = P\{0 < x_1 \leqq W \leqq x_2 < \infty\}\Phi(x\sigma^{-1})$$

where

$$\sigma^2 = \sum_{i=1}^{p} u_i\sigma_i^2,$$

$$\sigma_i^2 = \lim_{t\to\infty} e^{-\lambda_1 t}t^{-1} \operatorname{Var}(Y(t) \mid X(0) = e_i),$$

$$e_i = (\delta_{i1}, \delta_{i2}, \cdots, \delta_{ip}),$$

and $\Phi(x)$ is the standard cumulative normal distribution function.

CASE 3. $2 \operatorname{Re} \lambda < \lambda_1$. Then with $Y(t)$ being the same as in Case 2,

$$(13)\quad \lim_{t\to\infty} P\{0 < x_1 \leqq W \leqq x_2 < \infty, Y(t)[v\cdot X(t)]^{-\frac{1}{2}} \leqq x\} = P\{0 < x_1 \leqq W \leqq x_2 < \infty\}\Phi(x\sigma^{-1})$$

where

$$\sigma^2 = \sum_{i=1}^{p} u_i\sigma_i^2, \qquad \sigma_i^2 = \lim_{t\to\infty} e^{-\lambda_1 t} \operatorname{Var}(Y(t) \mid X(0) = e_i).$$

We next state the generalization of the above to the case of an arbitrary vector η (not necessarily an eigenvector). Using the spectral decomposition of $M(t)$ one can define for any η, a real number $a = a(\eta)$, an integer $\gamma = \gamma(\eta)$ and an index set $I(\eta)$ contained in $\{1, 2, \cdots, p\}$ such that:

CASE 1′. When $2a > \lambda_1$. There exist random variables Z_j, $j \in I(\eta)$ such that

$$(14)\qquad Z(t) = [\gamma!\, t^{-\gamma}e^{-at}\eta\cdot X(t) - \sum_{j\in I(\eta)} Z_j e^{ib_j t}]$$

converges to zero in mean square and almost surely. Here $b_j = \operatorname{Im}(\lambda_j)$.

CASES 2′ AND 3′. When $2a = \lambda_1$ and $2a < \lambda_1$: the results are the same as in the eigen-vector case.

3. Embedding.

3.1. *B. Friedman's urn.* In 1949 B. Friedman [8] proposed the following generalization of the classical Pólya's urn scheme. Start with W_0 white and B_0 black balls ($W_0 + B_0 > 0$). A *draw* is effected as follows: (i) Choose a ball at random from the urn; (ii) observe its color, return the ball to the urn, and add α balls of the same color and β balls of the opposite color. Let (W_n, B_n) denote the composition of the urn after n successive draws. The stochastic process $\{(W_n, B_n); n = 0, 1, 2, \cdots\}$ is called *B. Friedman's Urn scheme.* When $\beta = 0$ the process reduces to Pólya's urn scheme [6].

3.2. *The embedding.* We will now construct the Friedman's urn process as the standard embedded Markov chain of a continuous time two type Markov branching process. Let $\{X(t) = (X_1(t), X_2(t)); t \geqq 0\}$ be a two type process as defined in Section 2 with the associated parameters

$$
\begin{aligned}
a_1 &= a_2 = 1, \\
h_1(s) &= s_1^{\alpha+1} s_2^{\beta}, \\
h_2(s) &= s_1^{\beta} s_2^{\alpha+1}, \\
X_1(0) &= W_0, X_2(0) = B_0.
\end{aligned} \tag{15}
$$

THEOREM 1. *The stochastic processes* $\{Y_n = (W_n, B_n); n = 0, 1, 2, \cdots\}$ *and* $\{X(\tau_n) = (X_1(\tau_n), X_2(\tau_n)); n = 0, 1, 2, \cdots\}$ *are equivalent (see Section 2 regarding notation).*

PROOF. Since the case $\alpha + \beta = 0$ is degenerate we will assume $\alpha + \beta > 0$. Hence $\tau_n(\omega)$ coincides with the nth split time in the realization corresponding to ω. Consider the situation at $\tau_n + 0$. There are $X_1(\tau_n)$ particles of type 1 and $X_2(\tau_n)$ of type 2 respectively. (Recall the paths are assumed right continuous.) Since $a_1 = a_2 = 1$ all particles irrespective of type have independent exponential lifetime distribution with mean one. Therefore, the ensuing split will involve a type 1 particle with probability $X_1(\tau_n)/[X_1(\tau_n) + X_2(\tau_n)]$ and a type 2 particle with probability $X_2(\tau_n)/[X_1(\tau_n) + X_2(\tau_n)]$. The split particle is lost but $(\alpha + 1)$ new particles of its own kind and β of the opposite kind are created resulting in a net addition of α particles of the type that split and β of the opposite type. This implies that the stochastic mechanism yielding the movement from $X(\tau_n)$ to $X(\tau_{n+1})$ is the same as that from Y_n to Y_{n+1}. Furthermore, the two processes are Markov. This completes the proof.

3.3. *A remark.* From the proof of Theorem 1 it should be clear that the embedding prevails under quite general conditions. For example, consider the following extension of B. Friedman's urn scheme.

An urn has balls of p different colors. We start with Y_{0i} balls of color i ($i = 1, 2, \cdots, p$). A *draw* consists of the following operations: (i) Select a ball at random from the urn, (ii) notice its color C and return the ball to the urn, and (iii) if $C = i$, add a random number R_{ij} of balls of color j ($j = 1, 2, \cdots, p$) where the vector $R_i = (R_{i1}, \cdots, R_{ip})$ has the probability generating function $h_i(s)$. Let $Y_n = (Y_{n1}, Y_{n2}, \cdots, Y_{np})$ denote the composition of the urn after n successive draws. The stochastic process $\{Y_n; n = 0, 1, 2, \cdots\}$ on the p dimensional integer lattice is called a *Generalized Friedman's Urn Process* (GFP).

Consider a p dimensional branching process $\{X(t); t \geqq 0\}$ of the type described in Section 2 with the additional specification $a_i = 1$ and $X_i(0) = Y_{0i}$ for all i. Then the two stochastic processes $\{X(\tau_n); n = 0, 1, 2, \cdots\}$ and $\{Y_n; n = 0, 1, 2, \cdots\}$ are equivalent.

3.4. *Some results on Friedman's urn.* D. Freedman [7] derived the following results about B. Friedman's urn process $\{(W_n, B_n); n = 0, 1, 2, \cdots\}$ with parameters W_0, B_0, α and β. We assume $\beta \neq 0$. Let $\rho = (\alpha - \beta)/(\alpha + \beta)$.

(1) If $\rho > \frac{1}{2}$ then

$$
\lim_{n\to\infty} (W_n - B_n) n^{-\rho} = T'' \quad \text{exists a.s.} \tag{16}
$$

(2) If $\rho < \frac{1}{2}$ then as $n \to \infty$

$$(W_n - B_n)n^{-\frac{1}{2}} \to_d N(0, (\alpha - \beta)^2/(1 - 2\rho)). \tag{17}$$

(3) If $\rho = \frac{1}{2}$ then as $n \to \infty$

$$(W_n - B_n)(n \log n)^{-\frac{1}{2}} \to_d N(0, (\alpha - \beta)^2) \tag{18}$$

where $\to_d$ stands for convergence in law and $N(0, \sigma^2)$ is a Gaussian random variable with mean 0 and variance σ^2.

Now we will show how (16) follows easily from a martingale theorem for the continuous time process $\{X(t); t \geqq 0\}$ in which the Friedman's urn is embedded as described in 3.2. Later we will establish (17) and (18) in a considerably more general context. Our methods differ substantially from those of D. Freedman. He employed mainly moment methods. Furthermore, D. Freedman was puzzled by the nature of the factor ρ. From our analysis the significance of ρ will become clear.

In the notation of Section 2, for the $\{X(t); t \geqq 0\}$ process the A matrix becomes

$$A = \begin{pmatrix} \alpha & \beta \\ \beta & \alpha \end{pmatrix}; \qquad \lambda_1 = (\alpha + \beta); \qquad u = (2^{-\frac{1}{2}}, 2^{-\frac{1}{2}}) = v. \tag{19}$$

The second eigenvalue is $\lambda_2 = (\alpha - \beta)$ and the associated right eigenvector is $\xi = (1, -1)$. Since the condition $\rho > \frac{1}{2}$ is the same as $2\lambda_2 > \lambda_1$, by appealing to (7) and (11) we get, if $\rho > \frac{1}{2}$,

$$\begin{aligned} &\lim_{t\to\infty} (X_1(t) + X_2(t))e^{-\lambda_1 t} = W\, 2^{\frac{1}{2}}, \\ &\lim_{t\to\infty} (X_1(t) - X_2(t))e^{-\lambda_2 t} = V \qquad \text{exist a.s.} \end{aligned} \tag{20}$$

Also one can show that $P\{W > 0\} = 1$ (see [3], [12]). Hence using (20) we conclude that

$$\lim_{t\to\infty} (X_1(t) - X_2(t))e^{-\lambda_2 t}[(X_1(t) + X_2(t))e^{-\lambda_1 t}]^{-\rho} = T \quad \text{exists a.s.} \tag{21}$$

where $T = V/(W2^{\frac{1}{2}})^\rho$ is a bonafide random variable.

Because $P\{\lim_{n\to\infty} \tau_n = \infty\} = 1$, (21) implies

$$\lim_{n\to\infty} [X_1(\tau_n) - X_2(\tau_n)][X_1(\tau_n) + X_2(\tau_n)]^{-\rho} = T \qquad \text{a.s.} \tag{22}$$

By virtue of the embedding, we have

$$\lim_{n\to\infty} (W_n - B_n)(W_n + B_n)^{-\rho} = T' \qquad \text{exists a.s.} \tag{23}$$

where T' is a random variable with the same distribution as T.

Clearly (23) implies (16) with $T'' = T'(\alpha + \beta)^\rho$ since $W_n + B_n = W_0 + B_0 + n(\alpha + \beta)$.

Now let us examine the assertion of (17). This assertion can be rephrased in the form

$$(W_n - B_n)(W_n + B_n)^{-\frac{1}{2}} \to_d N(0, C) \tag{24}$$

where C is a suitable positive constant. Expressed in terms of the embedding this means that

$$[X_1(\tau_n) - X_2(\tau_n)][X_1(\tau_n) + X_2(\tau_n)]^{-\frac{1}{2}} \to_d N(0, C). \tag{25}$$

This suggests the conjecture that

$$[X_1(t) - X_2(t)][X_1(t) + X_2(t)]^{-\frac{1}{2}} \to_d N(0, C) \qquad \text{as} \quad t \to \infty \tag{26}$$

or what is the same

$$\xi \cdot X(t)[v \cdot X(t)]^{-\frac{1}{2}} \to_d N(0, C) \tag{27}$$

where $\xi = (1, -1)$, $v = (2^{-\frac{1}{2}}, 2^{-\frac{1}{2}})$ satisfy respectively $A\xi = \lambda_2 \xi$, $Av = \lambda_1 v$.

Clearly (27) follows from (13). But to go from (27) to

$$\xi \cdot X(\tau_n)[v \cdot X(\tau_n)]^{-\frac{1}{2}} \to_d N(0, C) \tag{28}$$

turns out to be nontrivial and is a special case of the principal result of the next section. Similar remarks apply with respect to the assertion (18).

4. Limit theorems for the embedded process $\{X(\tau_n); n = 0, 1, 2, \cdots\}$. In this section we shall develop several general limit theorems for the sequence $X(\tau_n)$, $n = 0, 1, 2, \cdots$.

4.1. *Split times.* Consider a branching process $\{X(t); t \geqq 0\}$ of the set up in Section 2. We assume

$$\partial h_i(s)/\partial s_i |_{s=(0,0,\cdots 0)} = 0 \qquad \text{for all} \quad i \tag{29}$$

so that when a particle splits, it never creates exactly one particle of the same type i. This assumption entails no loss of generality since the process is Markov and now the discontinuities of the sample path $\{X(t, \omega)$ for $t \geqq 0\}$ coincide with the splitting times. The discontinuities can be ordered and we designate them by $\tau_n(\omega)$ for $n = 1, 2, \cdots, N(\omega)$ where $N(\omega)$ is the total number of discontinuities of the sample point ω. Clearly the event $A_1 \equiv \{\omega: N(\omega) < \infty\}$ is contained in the set $A_2 = \{\omega: X(t, \omega) = 0$ for some $t\}$ a.s. To avoid unimportant technical complications, we make the assumption $P(A_2) = 0$, so that extinction cannot occur from any nontrivial initial state. Formally we assume

$$h_i(0) = 0 \qquad \text{for all} \quad i. \tag{30}$$

It then follows

$$P\{\lim_{n\to\infty} \tau_n = \infty\} = 1. \tag{31}$$

For later purposes it is convenient to record two facts proved in [4]:

$$\lim_{n\to\infty} n e^{-\lambda_1 \tau_n} = CW \qquad \text{a.s.} \tag{32a}$$

(Here C is a positive constant.)

$$\lim_{n\to\infty} (\tau_n - \lambda_1^{-1} \log n) = W' \qquad \text{exists a.s.} \tag{32b}$$

4.2. *Almost sure properties.* An immediate consequence of (31) is to the effect that if some limit relation holds a.s. as $t \to \infty$ for the process $\{X(t); t \geqq 0\}$ the same limit relation holds a.s. when $n \to \infty$ for the process $\{X(\tau_n); n \geqq 0\}$. Thus from (7), (11) and (31) we get

$$P\{\lim_{n\to\infty} X(\tau_n)e^{-\lambda_1\tau_n} = Wu\} = 1 \tag{33}$$

and provided $2 \operatorname{Re} \lambda > \lambda_1$

$$P\{\lim_{n\to\infty} Y(\tau_n) = Y\} = 1.$$

As pointed out in the previous section the almost sure limit relations for the urn processes are immediate consequences of results like (33). For example, we establish using (14) the following generalization of (16).

THEOREM 2. *Let $\{Y_n; n = 0, 1, 2, \cdots\}$ be a* GFP *(see Section* 3.3 *for the definition). Let η be an arbitrary vector in p dimensions. Assume that the process $\{X(t); t \geqq 0\}$ in which $\{Y_n; n = 0, 1, 2, \cdots\}$ is embedded conforms to the setup of Section 2. Define $a(\eta)$, $\gamma(\eta)$ and $I(\eta)$ as before. Let $2a > \lambda_1$. Then there exist random variables Z_j' and constants b_j' for $j \in I(\eta)$*

$$P\{\lim_{n\to\infty} |\eta\cdot Y_n/n^\rho(\log n)^\gamma - \textstyle\sum_{j\in I(\eta)} Z_j' \exp(ib_j' \log n)| = 0\} = 1, \tag{34}$$

where $\rho = a/\lambda_1$.

PROOF. Use (14), (31) and (32). q.e.d.

It turns out where $2 \operatorname{Re} \lambda_j > \lambda_1$ that if η is an eigenvector corresponding to a real eigenvalue λ_j then $a(\eta) = \lambda_j$, $I(\eta) = \{j\}$, $b_j' = 0$ and $\gamma(\eta) = 0$. In this case (34) resembles (16).

We next investigate the limit behavior of $\eta\cdot Y_n$ in the case $2a(\eta) \leqq \lambda_1$. It is anticipated that the analogues of (12) and (13) should prevail for the embedded process $\{X(\tau_n); n = 0, 1, 2, \cdots\}$. However, it is quite nontrivial to demonstrate that if a limit relation holds in law for the process $\{X(t); t \geqq 0\}$ as $t \to \infty$ then the same holds for the process $\{X(\tau_n); n = 0, 1, 2, \cdots\}$ as $n \to \infty$. This raises the following general problem concerning Markov processes and associated embedded chains.

Let $\{Z(t); t \geqq 0\}$ be a continuous time Markov process with a discrete state space and let $\{\tau_n(\omega)\}$ denote the sequence of discontinuity times of the sample path $Z(t, \omega)$. Suppose $P\{\tau_n \to \infty\} = 1$ and $Y(t) = f(Z(t))$ is such that the distribution functions $F(t, x) \equiv P\{Y(t, \omega) \leqq x\}$ converge as $t \to \infty$ in distribution to a distribution function $F(x)$. (Caution: The $Z(t)$ and $Y(t)$ used here differ from those in the statement and proof of Theorem 3.) Now since $Y(t, \omega) = Y(\tau_n, \omega)$ for $\tau_n \leqq t < \tau_{n+1}$ for $n = 1, 2, \cdots$ (assuming $Y(t, \omega)$ to be right continuous in t) it is tempting to conjecture

$$F_n(x) = P\{Y(\tau_n, \omega) \leqq x\} \to_d F(x). \tag{$*$}$$

Of course, $(*)$ cannot be valid without the minimum requirement of aperiodicity on the embedded Markov chain $X(\tau_n)$. The random telegraph signal process with $f(x) \equiv x$ provides a simple counterexample. Indeed consider a continuous time homogeneous Markov chain $\{X(t); t \geqq 0\}$ whose state values alternate between $+1$ and -1 and where the sojourn times at each of these values is exponentially distributed with unit mean. If τ_n denote the successive discontinuity times of $X(t)$, then

$$\lim_{t\to\infty} P\{X(t) = 1 \mid X(0) = 1\} = \tfrac{1}{2},$$

$$\lim_{n\to\infty} P\{X(\tau_{2n}) = 1 \mid X(0) = 1\} = 1,$$

$$\lim_{n\to\infty} P\{X(\tau_{2n+1}) = 1 \mid X(0) = 1\} = 0.$$

However, if the embedded Markov chain $\{X(\tau_n); n = 0, 1, 2, \cdots\}$ is aperiodic and ergodic then both $X(t)$ and $X(\tau_n)$ have the same limiting distributions as $t \to \infty$ and $n \to \infty$ respectively. For a non Markov process, $\{X(t), t \geqq 0\}$ and the embedded process $\{X(\tau_n); n = 0, 1, 2, \cdots\}$ may exhibit diverse limit behavior. For example, such situations occur frequently in queueing theory.

The above discussion focused on the case $f(x) \equiv x$. The case of general f is more interesting and presents a formidable problem.

If we restrict attention to special subsequences of τ_n then $(*)$ need not prevail. To wit, consider a compound Poisson process $\{X(t); t > 0\}$ with mean zero and variance σt. Examine the process only at the special times when $X(t) = 0$. These comprise an infinite subsequence τ_{n_j} of discontinuities of $\{X(t); t \geqq 0\}$ and obviously

$$X(\tau_{n_j})(\sigma\tau_{n_j})^{-\frac{1}{2}} = 0 \quad \text{but} \quad X(t)(\sigma t)^{-\frac{1}{2}} \to_d N(0, 1).$$

It is easy to see that $(*)$ certainly applies in the case $f(x) = x$ and undoubtedly for a large class of f's when considering the entire sequence of split times.

A further helpful remark is the following. Our proof of the main theorem uses a step slightly resembling Rényi's generalization of the central limit theorem (see [14]) to the effect that if N_n is a sequence of nonnegative integer valued random variables and S_n is a sequence of partial sums of independent identically distributed random variables ξ_i with $E\xi_i = 0$, $E\xi_i^2 = 1$, then

$$S_{N_n} N_n^{-\frac{1}{2}} \to_d N(0, 1) \qquad \text{as} \quad n \to \infty$$

provided $N_n/n \to W$ a.s. where W is a positive random variable. The technique employed by Rényi [14] in the simplest case where $W \equiv c$, a positive constant, involves decomposing $S_{N_n} N_n^{-\frac{1}{2}}$ in the form

$$S_{N_n} N_n^{-\frac{1}{2}} = (S_{N_n} - S_{[nc]}) N_n^{-\frac{1}{2}} + S_{[nc]}[nc]^{-\frac{1}{2}}([nc]/N_n)^{\frac{1}{2}}.$$

With the aid of Kolmogorov's inequality it is shown that

$$(S_{N_n} - S_{[nc]})[nc]^{-\frac{1}{2}} \to 0$$

in probability and Slutsky's theorem combined with the classical central limit theorem is used on the second term.

In our case we know that the functional $Y(t)$ defined in (13) is such that on $\{\omega: 0 < x_1 \leqq W \leqq x_2 < \infty\}$, $Y(t)/(e^{\lambda_1 t})^{\frac{1}{2}} \to_d N(0, \sigma^2)$. We decompose $Y(\tau_{n+k})/(e^{\lambda_1 \tau_{n+k}})^{\frac{1}{2}}$ in a way somewhat similar to Rényi's (see (36) to (39)).

We further use the martingale analog of Kolmogorov's inequality. The complete analysis is more delicate and difficult.

We shall prove in this section, relying on (13), the following theorem.

THEOREM 3. *Conforming to the set up of Section 2 if* ξ *is an eigenvector of* A *with eigenvalue* λ *and* $2 \operatorname{Re} \lambda < \lambda_1$ *then*

$$(35) \quad \lim_{n\to\infty} P\{0 < x_1 \leqq W \leqq x_2 < \infty,\ Y(\tau_n)[v\cdot X(\tau_n)]^{-\frac{1}{2}} \leqq x\} = P\{0 < x_1 \leqq W \leqq x_2 < \infty\}\Phi(x/\sigma)$$

where $Y(t)$ is $(l_1 \operatorname{Re}(\xi\cdot X(t)) + l_2 \operatorname{Im}(\xi\cdot X(t)))$ with l_1 and l_2 arbitrary but fixed real numbers not both zero (see (13)).

REMARK. Clearly Theorem 3 implies that

$$(\operatorname{Re} \xi\cdot X(\tau_n)/[v\cdot X(\tau_n)]^{\frac{1}{2}},\ \operatorname{Im} \xi\cdot X(\tau_n)/[v\cdot X(\tau_n)]^{\frac{1}{2}})$$

jointly converge in distribution as $n \to \infty$ to a bivariate normal law. Also by virtue of Theorem 3, the result in (17) emerges as a very special case. A similar theorem prevails in the log case, i.e., when $2 \operatorname{Re} \lambda = \lambda_1$. We omit the details. For ease of exposition, we discuss only the eigenvector case of Theorem 3. However the arguments can be adapted to treat the case of a general vector η. Actually, the only place we use the fact that ξ is an eigenvector is by appeal to martingale and semi-martingale inequalities contained in the discussion of Lemma 3. In the general case the corresponding inequalities can be established with the aid of stopping time arguments.

SOME PRELIMINARIES TO THE PROOF OF THEOREM 3. Let

$$(36) \qquad F_n(x) = P\{\omega\colon W \varepsilon [x_1, x_2],\ Y(\tau_n)/[v\cdot X(\tau_n)]^{\frac{1}{2}} \leqq x\}$$

where we have suppressed "the conditioning value of $X(0)$". Observe that

$$(37a) \quad F_{k+n}(x) \leqq P\{\omega\colon W \varepsilon [x_1, x_2],\ Y(\tau_k + \mu_{n,k})/[v\cdot X(\tau_k + \mu_{n,k})]^{\frac{1}{2}} \leqq x + \epsilon\} + P\{\omega\colon W \varepsilon [x_1, x_2],\ |B_1(n, k) + B_2(n, k)| > \epsilon\}$$

and

$$(37b) \quad F_{k+n}(x) \geqq P\{\omega\colon W \varepsilon [x_1, x_2],\ Y(\tau_k + \mu_{n,k})/[v\cdot X(\tau_k + \mu_{n,k})]^{\frac{1}{2}} \leqq x - \epsilon\} - P\{\omega\colon W \varepsilon [x_1, x_2],\ |B_1(n, k) + B_2(n, k)| > \epsilon\}$$

where

$$(37c) \quad B_1(n, k) = [Y(\tau_{n+k}) - Y(\tau_k + \mu_{n,k})]/[v\cdot X(\tau_{n+k})]^{\frac{1}{2}},$$

$$(37d) \quad B_2(n, k) = Y(\tau_k + \mu_{n,k})[v\cdot X(\tau_k + \mu_{n,k})]^{-\frac{1}{2}} \cdot ([v\cdot X(\tau_k + \mu_{n,k})/v\cdot X(\tau_{n+k})]^{\frac{1}{2}} - 1)$$

on the set $\{\omega\colon W \varepsilon [x_1, x_2]\}$ and 0 otherwise and where

$$(37e) \qquad \mu_{n,k} = \lambda_1^{-1} \log((n + k)/k).$$

Let $\epsilon_1 > 0$ be arbitrary. Determine $\epsilon > 0$ such that

$$|\Phi((x + \epsilon)/\sigma) - \Phi((x - \epsilon)/\sigma)| < \epsilon_1 . \tag{38}$$

Next let

$$A = \{\omega\colon W(\omega) \in [x_1, x_2]\} \tag{39a}$$

and

$$A_k = \{\omega\colon (x_1 - \eta)u_j < X_j(\tau_k)e^{-\lambda_1\tau_k} < (x_2 + \eta)u_j \quad \text{for} \quad j = 1, 2, \cdots, p\} \tag{39b}$$

where η is chosen to satisfy

$$P\{\omega\colon W \notin [x_1, x_2];\, W \in [x_1 - \eta, x_2 + \eta]\} < \epsilon_1 . \tag{39c}$$

This is possible since the distribution of W is continuous on $(0, \infty)$. Now using (37), (38) and (39) we conclude that

$$\begin{aligned}
&\limsup_{N\to\infty} F_N(x) \\
&\leqq \limsup_{n\to\infty} P\{\omega\colon \omega \in A_k,\, Y(\tau_k + \mu_{n,k})/[v\cdot X(\tau_k + \mu_{n,k})]^{\frac{1}{2}} \leqq x + \epsilon\} \\
&\quad + \limsup_{n\to\infty} P\{\omega\colon \omega \in A_k,\, |B_1(n, k) + B_2(n, k)| > \epsilon\} \\
&\quad + 2P\{A_k \triangle A\}
\end{aligned} \tag{40a}$$

and

$$\begin{aligned}
&\liminf_{N\to\infty} F_N(x) \\
&\geqq \liminf_{n\to\infty} P\{\omega\colon \omega \in A_k,\, Y(\tau_k + \mu_{n,k})/[v\cdot X(\tau_k + \mu_{n,k})]^{\frac{1}{2}} \leqq x - \epsilon\} \\
&\quad - \liminf_{n\to\infty} P\{\omega\colon \omega \in A_k,\, |B_1(n, k) + B_2(n, k)| > \epsilon\} \\
&\quad - 2P\{A_k \triangle A\}
\end{aligned} \tag{40b}$$

where

$$A_k \triangle A = \{\omega\colon \omega \in A_k \cup A,\, \omega \notin A_k \cap A\}. \tag{40c}$$

Now we state three key lemmas which provide the estimates from which the theorem quickly follows. The proofs of the lemmas will be presented subsequently.

LEMMA 1.

$$\limsup_{\epsilon_1 \downarrow 0} \limsup_{k\to\infty} P\{A_k \triangle A\} = 0. \tag{41}$$

LEMMA 2. *For every* k

$$\lim_{n\to\infty} P\{\omega\colon \omega \in A_k,\, Y(\tau_k + \mu_{n,k})/[v\cdot X(\tau_k + \mu_{n,k})]^{\frac{1}{2}} \leqq x\} = P\{A_k\}\Phi(x/\sigma). \tag{42}$$

LEMMA 3.

$$\limsup_{\epsilon_1\to 0} \limsup_{k\to\infty} \limsup_{n\to\infty} P\{\omega\colon \omega \in A_k,\, |B_1(n, k) + B_2(n, k)| > \epsilon\} = 0. \tag{43}$$

PROOF OF THEOREM 3. Assuming the validity of these lemmas it follows from (40) that

$$(44)\quad P\{A\}\Phi((x-\epsilon)/\sigma) + h_1(\epsilon_1) \leqq \liminf_{N\to\infty} F_N(x) \leqq \limsup_{N\to\infty} F_N(x) \leqq P\{A\}\Phi((x+\epsilon)/\sigma) + h_2(\epsilon_1)$$

where $h_1(t)$ and $h_2(t) \to 0$ as $t \to 0$. But by our choice of ϵ in (38) and since $\epsilon_1 > 0$ is arbitrarily prescribed (44) implies (35).

We turn now to the proofs of the lemmas.

PROOF OF LEMMA 1. Clearly

$$P\{A_k \triangle A\} = P\{\omega:\omega \in A_k, \omega \notin A\} + P\{\omega: \omega \notin A_k, \omega \in A\}.$$

Now

$$P\{\omega: \omega \in A_k, \omega \notin A\} \leqq P\{\omega: v\cdot X(\tau_k)e^{-\lambda_1\tau_k} \in [x_1-\eta, x_2+\eta], W \notin [x_1, x_2]\}$$

and

$$P\{\omega:\omega \notin A_k ; \omega \in A\} \leqq \sum_{j=1}^{p} P\{\omega: X_j(\tau_k)e^{-\lambda_1\tau_k} \notin [u_j(x_1-\eta), u_j(x_2+\eta)], W \in [x_1, x_2]\}.$$

But since $X(\tau_k)e^{-\lambda_1\tau_k}$ converges a.s. to Wu as $k \to \infty$ we conclude that

$$\limsup_{k\to\infty} P\{\omega:\omega \in A_k ; \omega \notin A\} \leqq P\{\omega: W \in [x_1-\eta, x_2+\eta], W \notin [x_1, x_2]\}$$

and

$$\limsup_{k\to\infty} P\{\omega:\omega \notin A_k ; \omega \in A\} = 0.$$

Our choice of η in (39c) now implies (41).

PROOF OF LEMMA 2. Recall that the process $\{X(t); t \geqq 0\}$ is assumed to be strong Markov and adapted to the family of fields $\mathbf{F}_t = \sigma\{X(s); s \leqq t\}$. Let $\mathfrak{F}_k$ be the field associated with the stopping time τ_k. By the strong Markov property

$$(45)\quad P\{\omega:\omega \in A_k, Y(\tau_k+\mu_{n,k})/[v\cdot X(\tau_k+\mu_{n,k})]^{\frac{1}{2}} \leqq x \mid \mathfrak{F}_k\} = \chi_{A_k}(\omega)P_{X(\tau_k)}(Y(\mu_{n,k})/[v\cdot X(\mu_{n,k})]^{\frac{1}{2}} \leqq x) \quad \text{a.s.}$$

where P_y denotes the probability measure for the branching process $\{X(t); t \geqq 0\}$ with $X(0) = y$ and $\chi_{A_k}(\omega)$ is the indicator function of the set A_k.

Now for a fixed k, $\mu_{n,k} \to \infty$ as $n \to \infty$. Thus appealing to (13), we conclude that

$$\lim_{n\to\infty} P_{X(\tau_k)}(Y(\mu_{n,k})/[v\cdot X(\mu_{n,k})]^{\frac{1}{2}} \leqq x) = \Phi(x/\sigma) \quad \text{a.s.}$$

provided $X(\tau_k) \neq 0$. The result of the lemma follows from (45). q.e.d.

We turn next to the proof of Lemma 3. We shall break it up into two further lemmas.

LEMMA 3a.

$$(46)\quad \limsup_{\epsilon_1\downarrow 0} \limsup_{k\to\infty} \limsup_{n\to\infty} P\{\omega:\omega \in A_k, |B_1(n,k)| > \tfrac{1}{2}\epsilon\} = 0.$$

LEMMA 3b.

$$\lim_{\epsilon_1 \downarrow 0} \limsup_{k\to\infty} \limsup_{n\to\infty} P\{\omega : \omega \in A_k, |B_2(n, k)| > \tfrac{1}{2}\epsilon\} = 0.$$

PROOF OF LEMMA 3a. We first observe using (32b) and the definition of $\mu_{n,k}$ that

$$\lim_{n\to\infty} v\cdot X(\tau_{n+k}) \exp[-\lambda_1(\tau_k + \mu_{n,k})] = W(\omega) \exp[-\lambda_1[\tau_k - \lambda_1^{-1} \log k - W']]$$

almost surely on the set $\{W > 0\}$ where $W' = \lim_{n\to\infty} (\tau_n - (1/\lambda_1) \log n)$. Thus it suffices to show

$$\text{(47)} \quad \limsup_{k\to\infty} \limsup_{n\to\infty} P\{\omega : \omega \in A_k, |Y(\tau_{n+k}) - Y(\tau_k + \mu_{n,k})| > \tfrac{1}{2}y\epsilon \exp[\tfrac{1}{2}\lambda_1(\tau_k + \mu_{n,k})]\} \to 0$$

as $\epsilon_1 \downarrow 0$ where y satisfies $0 < y < (x_1 - \eta)$. Consider

$$\pi = P\{\omega : \omega \in A_k, |Y(\tau_{n+k}) - Y(\tau_k + \mu_{n,k})| > \tfrac{1}{2}y\epsilon \exp[\tfrac{1}{2}\lambda_1(\tau_k + \mu_{n,k})]\}.$$

Let

$$Z(t) = \xi\cdot X(t)e^{-\lambda t}, \qquad \eta_{n,k} = \tau_{n+k} - \tau_k - \mu_{n,k} \quad \text{and} \quad a = \operatorname{Re} \lambda.$$

We observe since $Y(t) = [l_1 \operatorname{Re}(\xi\cdot X(t)) + l_2 \operatorname{Im}(\xi\cdot X(t))]$ that

$$e^{-a(\tau_k+\mu_{n,k})} |Y(\tau_{n+k}) - Y(\tau_k + \mu_{n,k})|$$

$$\text{(48a)} \qquad \leqq (l_1^2 + l_2^2)^{\frac{1}{2}} |\{\xi\cdot X(\tau_{n+k}) - \xi\cdot X(\tau_k + \mu_{n,k})\}e^{-\lambda(\tau_k+\mu_{n,k})}|$$

$$\text{(48b)} \qquad \leqq (l_1^2 + l_2^2)^{\frac{1}{2}}\{|Z(\tau_{n+k}) - Z(\tau_k + \mu_{n,k})| + |Z(\tau_{n+k})|\,|e^{\lambda\eta_{n,k}} - 1|\}.$$

It follows, setting $c = (l_1^2 + l_2^2)^{\frac{1}{2}}$, that

$$\text{(49a)} \qquad \pi \leqq p_1 + p_2$$

where

$$\text{(49b)} \qquad p_1 = P\{E_{n,k}\},$$

$$\text{(49c)} \qquad p_2 = P\{F_{n,k}\},$$

$$\text{(49d)} \qquad E_{n,k} = \{\omega : \omega \in A_k, |Z(\tau_{n+k}) - Z(\tau_k + \mu_{n,k})| > \tfrac{1}{4}\epsilon y c^{-1} \exp[(\tfrac{1}{2}\lambda_1 - a)(\tau_k + \mu_{n,k})]\},$$

$$\text{(49e)} \qquad F_{n,k} = P\{\omega : \omega \in A_k, |Z(\tau_{n+k})|\,|e^{\lambda\eta_{n,k}} - 1| > \tfrac{1}{4}\epsilon y c^{-1} \exp[(\tfrac{1}{2}\lambda_1 - a)(\tau_k + \mu_{n,k})]\}.$$

If we show $\limsup_{\epsilon_1 \downarrow 0} \limsup_{k\to\infty} \limsup_{n\to\infty} p_i = 0$ for $i = 1, 2$ we are done.

STEP 1. First consider p_1. Clearly

$$\text{(50a)} \qquad p_1 \leqq p_{11} + p_{12}$$

where

$$\text{(50b)} \qquad p_{11} = P(\omega; \omega \in E_{n,k}, |\eta_{n,k}| \leqq \delta),$$

$$\text{(50c)} \qquad p_{12} = P(\omega; \omega \in E_{n,k}, |\eta_{n,k}| > \delta).$$

But $p_{12} \leqq P\{|\eta_{n,k}| > \delta\}$. Now consulting (32b) we infer for any $\delta > 0$ that

$$\limsup_{k\to\infty} \limsup_{n\to\infty} p_{12} = 0. \tag{51}$$

Obviously,

$$p_{11} \leqq p_{111} + p_{112} \tag{52a}$$

where

$$p_{111} = P(\omega;\ \omega \in E_{n,k},\ 0 \leqq \eta_{n,k} \leqq \delta), \tag{52b}$$

$$p_{112} = P(\omega;\ \omega \in E_{n,k},\ -\delta \leqq \eta_{n,k} \leqq 0). \tag{52c}$$

In order to establish $\limsup_{\epsilon_1 \downarrow 0} \limsup_{k\to\infty} \limsup_{n\to\infty} p_{111} = 0$, we develop some additional estimates. Note that the family,

$$\{Z(\tau_k + \mu_{n,k} + u) - Z(\tau_k + \mu_{n,k}), \mathfrak{F}_{\tau_k+\mu_{n,k}+u}\ ;\ u \geqq 0\}$$

constitutes a martingale owing to the facts that τ_k is a stopping time, the process $\{Z(t), t \geqq 0\}$, is strong Markov and Doob's optional sampling theorem is applicable (see [13]).

Manifestly

$$\begin{aligned} p_{111} \leqq P\{\omega:\ \omega \in A_k\ ;\ \sup_{0 \leqq u \leqq \delta} |Z(\tau_k + \mu_{n,k} + u) - Z(\tau_k + \mu_{n,k})| \\ > \tfrac{1}{4}\epsilon y c^{-1} \exp(\tfrac{1}{2}\lambda_1 - a)(\tau_k + \mu_{n,k})]\}. \end{aligned} \tag{53a}$$

Now using Kolmogorov's inequality for submartingales (see [5]) and conditioning on the values of $X(t)$ at $\tau_k + \mu_{n,k}$ we deduce that the right hand side of (53a) is bounded above by

$$\begin{aligned} &\leqq 16c^2\epsilon^{-2}y^{-2}E(|Z(\tau_k + \mu_{n,k} + \delta) - Z(\tau_k + \mu_{n,k})|^2 \\ &\qquad \cdot\{\exp[(\lambda_1 - 2a)(\tau_k + \mu_{n,k})]\}^{-1}; A_k) \\ &\leqq 16c^2\epsilon^{-2}y^{-2}E(\textstyle\sum_{i=1}^{p} e^{-\lambda_1(\tau_k+\mu_{n,k})} X_i(\tau_k + \mu_{n,k})V_i(\delta);\ A_k) \end{aligned} \tag{53b}$$

where

$$V_i(t) = E(|Z(t) - EZ(t)|^2 \,|\, X(0) = e_i).$$

Since

$$E(X_i(\tau_k + \mu_{n,k}) \,|\, \mathfrak{F}_{\tau_k}) = \textstyle\sum_{j=1}^{p} X_j(\tau_k) m_{ji}(\mu_{n,k})$$

where

$$m_{ji}(t) = E(X_i(t) \,|\, X(0) = e_j),$$

we have

$$p_{111} \leqq 16c^2\epsilon^{-2}y^{-2}E\{\textstyle\sum_{j=1}^{p} e^{-\lambda_1\tau_k} X_j(\tau_k)(\sum_{i=1}^{p} V_i(\delta) m_{ji}(\mu_{n,k}) e^{-\lambda_1\mu_{n,k}});\ A_k\}.$$

We know on the basis of the Frobenius theory of positive matrices (see [11], Appendix) that there exists a finite positive constant C_1 such that

$$\sup_{j,i,n,k} m_{ji}(\mu_{n,k}) e^{-\lambda_1\mu_{n,k}} \leqq C_1 < \infty.$$

Also on A_k

$$\sum_{j=1}^{p} X_j(\tau_k)e^{-\lambda_1\tau_k} \leqq C_2$$

where C_2 depends only on x_1, x_2 and η.

Now $\lim_{t\downarrow 0} V_i(t) = 0$ and so determine δ small enough such that

$$\sup_{1\leqq i\leqq p} V_i(\delta) < \epsilon_1\epsilon^2.$$

Combining these estimates, we may conclude that

$$p_{111} \leqq (16c^2y^{-2}C_2C_1p)\epsilon_1. \tag{54}$$

It follows that

$$\limsup_{\epsilon_1\downarrow 0} \limsup_{k\to\infty} \limsup_{n\to\infty} p_{111} = 0 \tag{55a}$$

as desired to be proved.

A similar computation yields the result

$$\limsup_{\epsilon_1\downarrow 0} \limsup_{k\to\infty} \limsup_{n\to\infty} p_{112} = 0. \tag{55b}$$

The relations (50a), (51), (52a), (55a), and (55b) in conjunction embrace the arguments of Step 1 to the effect that $\limsup_{\epsilon_1\downarrow 0} \limsup_{k\to\infty} \limsup_{n\to\infty} p_1 = 0$.

Now we turn to

STEP 2. To show $\limsup_{\epsilon_1\downarrow 0} \limsup_{k\to\infty} \limsup_{n\to\infty} p_2 = 0$, we recall that

$$p_2 = P\{\omega: \omega \varepsilon A_k, |Z(\tau_{n+k})| \, |e^{\lambda\eta_{n,k}} - 1| > \tfrac{1}{4}\epsilon y c^{-1} \exp [(\tfrac{1}{2}\lambda_1 - a)(\tau_k + \mu_{n,k})]\}. \tag{56}$$

Thus $p_2 \leqq p_2' + p_2''$ where

$$p_2' = P\{\omega: \omega \varepsilon A_k ; |Z(\tau_{n+k}) - Z(\tau_k)| \, |e^{\lambda\eta_{n,k}} - 1| > \tfrac{1}{8}\epsilon y c^{-1} \exp [(\tfrac{1}{2}\lambda_1 - a)(\tau_k + \mu_{n,k})]\} \tag{57a}$$

and

$$p_2'' = P\{\omega: \omega \varepsilon A_k ; |Z(\tau_k)| \, |e^{\lambda\eta_{n,k}} - 1| \geqq \tfrac{1}{8}\epsilon y c^{-1} \exp [(\tfrac{1}{2}\lambda_1 - a)(\tau_k + \mu_{n,k})]\}. \tag{57b}$$

However

$$p_2'' \leqq P\{\omega: \omega \varepsilon A_k ; |Z(\tau_k)| > \tfrac{1}{8}\epsilon y c^{-1}(e^{a\delta} + 1)^{-1} \exp [(\tfrac{1}{2}\lambda_1 - a)(\tau_k + \mu_{n,k})], |\eta_{n,k}| \leqq \delta\} + P\{\omega: |\eta_{n,k}| > \delta\}$$

where $\delta > 0$ is arbitrary but fixed. Since $\mu_{n,k} \to \infty$ as $n \to \infty$ for fixed k and $|Z(\tau_k)|$ is a finite valued random variable

$$\limsup_{n\to\infty} p_2'' \leqq 0 + \limsup_{n\to\infty} P\{\omega: |\eta_{n,k}| > \delta\}$$

and thus

$$\limsup_{k\to\infty} \limsup_{n\to\infty} p_2'' \leqq \limsup_{k\to\infty} \limsup_{n\to\infty} P\{\omega: |\eta_{n,k}| > \delta\} = 0. \tag{58}$$

It remains only to show $\limsup_{k\to\infty} \limsup_{n\to\infty} p_2' = O(\epsilon_1)$. Again we decompose p_2' in the form $p_2' = p_{21} + p_{22}$ where

$$(59a) \quad p_{21} = P\{\omega\colon \omega \varepsilon A_k, |Z(\tau_{n+k}) - Z(\tau_k)| |e^{\lambda\eta_{n,k}} - 1| > \tfrac{1}{8}\epsilon y c^{-1} \exp[(\tfrac{1}{2}\lambda_1 - a)(\tau_k + \mu_{n,k})], |\eta_{n,k}| \leqq \delta\}$$

and

$$(59b) \qquad p_{22} = p_2' - p_{21}.$$

As before $p_{22} \leqq P\{|\eta_{n,k}| > \delta\}$ and hence for any δ

$$(60) \qquad \limsup_{k\to\infty} \limsup_{n\to\infty} p_{22} = 0.$$

Thus we need only show

$$(61) \qquad \limsup_{k\to\infty} \limsup_{n\to\infty} p_{21} = 0.$$

But

$$p_{21} = P\{\omega\colon \omega \varepsilon A_k, |Z(\tau_k + \mu_{n,k} + \eta_{n,k}) - Z(\tau_k)| |e^{\lambda\eta_{n,k}} - 1| > \tfrac{1}{8}\epsilon y c^{-1} \exp[(\tfrac{1}{2}\lambda_1 - a)(\tau_k + \mu_{n,k})], |\eta_{n,k}| \leqq \delta\}.$$

Choose δ small enough such that

$$|e^{\lambda u} - 1| \leqq \epsilon_1\epsilon \qquad \text{for} \qquad |u| \leqq \delta.$$

For a fixed k let n_0 be such that $\mu_{n_0,k} > \delta$. Then for $n \geqq n_0$

$$p_{21} \leqq P\{\omega\colon \omega \varepsilon A_k; \sup_{-\delta\leqq u\leqq\delta} |Z(\tau_k + \mu_{n,k} + u) - Z(\tau_k)| > \tfrac{1}{8} y \epsilon_1^{-1} c^{-1} \exp[(\tfrac{1}{2}\lambda_1 - a)(\tau_k + \mu_{n,k})]\}.$$

Now the family $\{Z(\tau_k + \mu_{n,k} + u) - Z(\tau_k); \mathfrak{F}_{\tau_k+\mu_{n,k}+u} - \delta \leqq u \leqq \delta\}$ is a martingale as before. Employing Kolomogorov's inequality for submartingales and paraphrasing the analysis as in the case of Step 1, we infer that

$$(62) \quad p_{21} \leqq (\epsilon_1 8c/y)^2 E\{e^{\lambda_1\delta} \textstyle\sum_{i=1}^{p} X_i(\tau_k) V_i(\mu_{n,k} + \delta) e^{-\lambda_1(\tau_k+\mu_{n,k}+\delta)}; A_k\}$$

where we recall that

$$V_i(t) = E\{|Z(t) - EZ(t)|^2 \mid X(0) = e_i\}.$$

Now we shall use the following easily proved fact (see [3]). There exists, if $2 \operatorname{Re} \lambda < \lambda_1$, a finite positive constant $C < \infty$ such that $\sup_i \sup_t V_i(t) e^{-\lambda_1 t} \leqq C < \infty$. Further on A_k, $\sup_{1\leqq i\leqq p} X_i(\tau_k) e^{-\lambda_1\tau_k} \leqq C_2$ for some C_2. It follows that

$$(63) \qquad p_{21} \leqq (\epsilon_1 8c/y)^2 e^{\lambda_1\delta} C_2 C.$$

Thus $\limsup_{\epsilon_1\downarrow 0} \limsup_{k\to\infty} \limsup_{n\to\infty} p_{21} = 0$ establishing (61). Now combining (58), (60) and (61) finishes the essentials of Step 2 to show $\limsup_{\epsilon_1\downarrow 0} \limsup_{k\to\infty} \limsup_{n\to\infty} p_2 = 0$.

The proof of Lemma 3a is complete.

PROOF OF LEMMA 3b. Recall that

$$(64)\quad B_2(n, k) = Y(\tau_k + \mu_{n,k})[v\cdot X(\tau_k + \mu_{n,k})]^{-\frac{1}{2}} \cdot([v\cdot X(\tau_k + \mu_{n,k})/[v\cdot X(\tau_{n+k})]^{\frac{1}{2}} - 1).$$

On the set $A = \{\omega\colon W \varepsilon [x_1, x_2]\}$ we see from Lemma 2 that

$$(65)\quad Y(\tau_k + \mu_{n,k})/[v\cdot X(\tau_k + \mu_{n,k})]^{\frac{1}{2}} \to_d N(0, \sigma^2) \qquad \text{as} \quad n \to \infty.$$

Also a.s. on A, $v\cdot X(t)e^{-\lambda_1 t} \to W > 0$ as $t \to \infty$. This implies that almost surely on A we have

$$(66)\quad \lim_{n\to\infty} v\cdot X(\tau_k + \mu_{n,k})/v\cdot X(\tau_{n+k}) = \exp[\lambda_1(\tau_k - \lambda_1^{-1} \log k - W')].$$

Now using (32b) we see that as $k \to \infty$, $\exp[\lambda_1(\tau_k - (1/\lambda_1) \log k - W')] \to 1$ a.s. Invoking Slutsky's theorem, the lemma follows.

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