

Anna B. Romanowska; Anna Zamojska-Dzienio

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EMBEDDING SUMS OF CANCELLATIVE MODES INTO SEMIMODULES

ANNA ROMANOWSKA and ANNA ZAMOJSKA-DZIENIO, Warsaw

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Abstract. A mode (idempotent and entropic algebra) is a Lallement sum of its cancellative submodes over a normal band if it has a congruence with a normal band quotient and cancellative congruence classes. We show that such a sum embeds as a subreduct into a semimodule over a certain ring, and discuss some consequences of this fact. The result generalizes a similar earlier result of the authors proved in the case when the normal band is a semilattice.

Keywords: modes (idempotent and entropic algebras), cancellative modes, sums of algebras, embeddings, semimodules over semirings, idempotent subreducts of semimodules

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1. INTRODUCTION

One of the most efficient ways of describing the structure of an algebra is to embed it into another, usually one with a better known and richer structure. Prototypical examples are given by the embedding of integral domains into fields and commutative cancellative semigroups into commutative groups. Among many other instances of such techniques, let us mention also the embedding of cancellative entropic groupoids into quasigroups [2]. In particular, such methods appear to be quite succesful in investigating the structure of modes, idempotent and entropic algebras, as introduced and investigated in the monographs [7] and [11]. Following a number of previous partial results, Romanowska and Smith [10] showed that each cancellative mode embeds as a subreduct into an affine space, the idempotent reduct of a module over a commutative ring. Using certain advanced categorical techniques, they also

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showed that certain sums of cancellative modes embed as subreducts into the so-called functorial sums of affine spaces. (See also [11, Chapter 7].) However, the categorical proof given there does not provide any direct description of the structure of the algebras under consideration. In [13], Romanowska and Zamojska developed a technique of embedding modes as subreducts into semimodules based on two facts: that cancellative modes embed into appropriate modules, and that a Płonka sum of modules over a certain ring is a semimodule over the same ring. They proved that each so-called semilattice Lallement sum of cancellative modes embeds as a subreduct into a Płonka sum of certain affine spaces, and hence into the Płonka sum of the corresponding modules. Consequently, it embeds into a semimodule. (See also [11, Chapter 7].)

In this paper we extend the above result by proving that Lallement sums of cancellative modes over semigroup modes (i.e. normal bands) may also be embedded as subreducts into semimodules over certain rings. The proof involves some new properties of functorial sums of algebras, and is done by showing that the above mentioned Lallement sums are subalgebras of reducts of Płonka sums of modules. Though the method used may not be universal, it does concern a very general class of modes, and is a good tool for representing modes in certain quasivarieties.

For the convenience of the reader we first recall the basic definitions and results we need later. Section 2 provides a (very) brief survey of what we need about modes. The concept of functorial sums of algebras is recalled in Section 3, where we also provide some new results concerning such sums. Non-functorial sums are recalled in Section 4. The main result is proved in Section 5. The paper concludes with a discussion in Section 6 concerning some implications of the main result and some related open questions.

The terminology and the notation of the paper is basically as in the books [7] and [11]. We refer the reader to these books, and to the surveys [6] and [14], for any otherwise undefined notions and further results.

2. MODES

An algebra (A, Ω) of type $\tau: \Omega \longrightarrow \mathbb{Z}^+$ is called a *mode* if it is *idempotent* and *entropic*, i.e. each singleton in A is a subalgebra and each operation $\omega \in \Omega$ is actually a homomorphism from an appropriate power of the algebra. Both properties can also be expressed by the following identities:

$$(I) \quad \forall \omega \in \Omega, x \dots x\omega = x$$

(E) $\forall \omega, \varphi \in \Omega$, with m -ary ω and n -ary φ ,

$$(x_{11} \dots x_{1m} \omega) \dots (x_{n1} \dots x_{nm} \omega) \varphi = (x_{11} \dots x_{n1} \varphi) \dots (x_{1m} \dots x_{nm} \varphi) \omega,$$

satisfied in the algebra (A, Ω) .

Examples of modes are provided by affine spaces (or affine modules) over commutative rings, their reducts and subreducts (subalgebras of reducts), by normal bands (idempotent and entropic (or mode) semigroups) and many binary (or groupoid) modes appearing in combinatorics and geometry. (See [7] and [11] for more details.) In particular, recall that an affine space can be described as the full idempotent reduct of the corresponding module. More general examples are provided by *affine semimodules*, full idempotent reducts of semimodules over commutative semirings, and their subreducts. The semilattice modes investigated by Kearnes [3] are examples of such modes.

A mode (A, Ω) is *cancellative* if for each $(n$ -ary) ω in Ω , (A, Ω) satisfies the *cancellation law*

$$(a_1 \dots a_{i-1} x_i a_{i+1} \dots a_n \omega = a_1 \dots a_{i-1} y_i a_{i+1} \dots a_n \omega) \longrightarrow (x_i = y_i)$$

for each $i = 1, \dots, n$. Cancellative modes have a property that is basic for our main result.

Theorem 2.1 ([10]). *Each cancellative mode (C, Ω) of a fixed type $\tau: \Omega \rightarrow \mathbb{Z}^+$ embeds as an Ω -subreduct into an affine space $(G, P, \underline{R(M\tau)})$ over the ring $R(M\tau)$.*

The ring $R(M\tau)$ is defined in the following way. Let $\underline{M\tau}$ be the variety of all modes of a given type $\tau: \Omega \rightarrow (\mathbb{N} - \{0, 1\})$. Let S be the integral polynomial ring $\mathbb{Z}[\{X_{\omega i} \mid \omega \in \Omega, 1 \leq i \leq \omega\tau\}]$ over a set $\{X_{\omega i} \mid \omega \in \Omega, 1 \leq i \leq \omega\tau\}$ of $\sum_{\omega \in \Omega} \omega\tau$

commuting indeterminates. Then $R(M\tau)$ is the quotient ring $S / \left(1 - \sum_{i=1}^{\omega\tau} X_{\omega i} \mid \omega \in \Omega \right)$

of S by the ideal obtained by setting each sum $\sum_{i=1}^{\omega\tau} X_{\omega i}$ to be 1.

In the proof of Theorem 2.1, the mode (C, Ω) is first embedded as a subreduct into a certain cancellative commutative monoid equipped with pairwise commuting endomorphisms which is then embedded into an abelian group $(G, +, -, 0)$. This group can also be considered as an affine space and a module over the ring $R(M\tau)$.

Theorem 2.1 was then generalized in [13], where it was shown that certain sums of cancellative modes indexed by semilattices also embed into semimodules. (Cf. Sections 5 and 6.)

3. ALGEBRAIC QUASI-ORDERS AND FUNCTORIAL SUMS OF ALGEBRAS

In this section we recall basic facts about functorial sums, and show that in many situations functorial sums can be reduced to Płonka sums. We also provide an example of a functorial sum that cannot be reduced to a Płonka sum. For more information concerning functorial sums and further references we direct the reader to [12] and [11].

Let (I, Ω) be an algebra of a type $\tau: \Omega \rightarrow \mathbb{N}$. The *algebraic quasi-order* of the algebra (I, Ω) is the quasi-order \preceq defined on the set I as follows:

$$\begin{aligned} \preceq := \{ (i, j) \mid & \exists x_1 \dots x_n t \in X\Omega \text{ and} \\ & \exists i_1, \dots, i_{k-1}, i_{k+1}, \dots, i_n \in I \text{ such that} \\ & j = i_1 \dots i_{k-1} i i_{k+1} \dots i_n t \}. \end{aligned}$$

(See e.g. [12] and [11].) If additionally the algebra (I, Ω) satisfies the condition

$$\text{if } a_i \preceq b_i, \text{ then } a_1 \dots a_{\omega\tau} \omega \preceq b_1 \dots b_{\omega\tau} \omega$$

for all ω in Ω and $a_1, \dots, a_{\omega\tau}, b_1, \dots, b_{\omega\tau}$ in I , then we say that it is *naturally quasi-ordered*. If $\preceq = I \times I$, then the algebra (I, Ω) has a *full* algebraic quasi-order. Note that algebras in strongly irregular varieties always have a full algebraic quasi-order. Algebras in the regularization of a strongly irregular variety are naturally quasi-ordered.

Let (I, Ω) and (A_i, Ω) , for $i \in I$, be algebras of a (plural) type $\tau: \Omega \rightarrow \mathbb{N}$. Let \preceq be the algebraic quasi-order of (I, Ω) . Consider (I, Ω) as a category (I) with the elements of I as objects and a unique morphism $i \rightarrow j$ if and only if $i \preceq j$. Consider also the category (Ω) with Ω -algebras as objects and Ω -homomorphisms as morphisms. For a covariant functor $F: (I) \rightarrow (\Omega)$, the *functorial sum* (or *Agassiz sum*) of the functor F , or of the algebras $iF = (A_i, \Omega)$ for $i \in I$, is the disjoint union $IF = \bigcup (iF \mid i \in I)$ of the underlying sets of iF with the Ω -algebra structure given by

$$\omega: i_1 F \times \dots \times i_n F \rightarrow iF; \quad (a_{i_1}, \dots, a_{i_n}) \mapsto a_{i_1} \varphi_{i_1, i} \dots a_{i_n} \varphi_{i_n, i} \omega$$

for each n -ary ω in Ω and $i = i_1 \dots i_n \omega$, where for $k = 1, \dots, n$, the mapping $\varphi_{i_k, i}: i_k F \rightarrow iF$ is the Ω -homomorphism $(i_k \rightarrow i)F$.

Note that all the iF are subalgebras of the sum IF , and that there is an Ω -homomorphism

$$\pi: (IF, \Omega) \rightarrow (I, \Omega); \quad a_i \mapsto i,$$

called a *projection*.

If there is no danger of confusion, an algebra (A, Ω) will be sometimes denoted by the symbol A of its underlying set. Similarly, if (A, Ω) is a functorial sum of (A_i, Ω) over (I, Ω) , then we will write briefly $(A, \Omega) = \sum_{i \in I} (A_i, \Omega)$, or simply $A = \sum_{i \in I} A_i$ or $A = \sum (A_i \mid i \in I)$.

If the indexing algebra (I, Ω) is an Ω -semilattice, then the functorial sum is called a *Plonka sum*. Note that a direct product of an Ω -algebra and an idempotent Ω -algebra can always be considered as a functorial sum. If I has a full algebraic quasi-order, then we obtain another important special case.

Proposition 3.1 ([4]). *Let F be a functor defining a functorial sum, as above. If the algebraic quasi-order \preceq of I is full, then for any two i, j in I , the algebras iF and jF are isomorphic. Moreover, the sum IF is isomorphic to the direct product of the algebras iF and I , i.e.*

$$(IF, \Omega) = \sum_{i \in I} (iF, \Omega) \cong (iF, \Omega) \times (I, \Omega).$$

For an algebra which is a functorial sum over another algebra which is itself a functorial sum, we obtain the following result.

Theorem 3.2. *Let (E, Ω) be a functorial sum of algebras (E_n, Ω) over an algebra (N, Ω) . Let (N, Ω) be a functorial sum of algebras (N_s, Ω) over an algebra (S, Ω) . Then (E, Ω) is a functorial sum of algebras (B_s, Ω) over the algebra (S, Ω) , where (B_s, Ω) is a functorial sum of (E_n, Ω) over (N_s, Ω) . Briefly:*

$$\sum \left(E_n \mid n \in \sum_{s \in S} N_s \right) = \sum_{s \in S} \left(\sum (E_n \mid n \in N_s) \right).$$

Proof. Let E be a functorial sum of E_n over N . In particular, for all $m, n \in N$ with $m \preceq n$, there exist homomorphisms $\varphi_{m,n}: E_m \rightarrow E_n$ satisfying the functoriality condition. On the other hand, the algebra N is a functorial sum of algebras N_s over the algebra S . So we also have homomorphisms $\psi_{s,t}: N_s \rightarrow N_t$ for $s, t \in S$ with $s \preceq t$. Note that for each $s \in S$ one has $N_s \leq N$. It follows that $B_s := \sum_{n \in N_s} E_n$ is also a functorial sum, and $B_s \leq E$.

We will show that E is a functorial sum of algebras B_s over the algebra S . For all $s, t \in S$ with $s \preceq t$, define the mappings

$$h_{s,t}: B_s \rightarrow B_t; \quad x \mapsto x\varphi_{n_s, n_s\psi_{s,t}},$$

where $x \in E_{n_s}$ and $n_s \in N_s$. We first check that each $h_{s,t}$ is a homomorphism. Let $1_s, \dots, p_s \in N_s$, let $x_n \in E_{n_s}$ for $n = 1, \dots, p$, and let $\omega \in \Omega$ be a p -ary operation. Since B_s is a functorial sum of E_{n_s} over N_s , it is clear that

$$x_1 \dots x_p \omega \in E_{1_s \dots p_s \omega}.$$

Then

$$\begin{aligned} (x_1 \dots x_p \omega) h_{s,t} &= (x_1 \dots x_p \omega) \varphi_{1_s \dots p_s \omega, 1_s \dots p_s \omega \psi_{s,t}} \\ &= (x_1 \varphi_{1_s, 1_s \dots p_s \omega} \dots x_p \varphi_{p_s, 1_s \dots p_s \omega} \omega) \varphi_{1_s \dots p_s \omega, 1_s \dots p_s \omega \psi_{s,t}} \\ &= (x_1 \varphi_{1_s, 1_s \dots p_s \omega} \varphi_{1_s \dots p_s \omega, 1_s \dots p_s \omega \psi_{s,t}} \dots x_p \varphi_{p_s, 1_s \dots p_s \omega} \varphi_{1_s \dots p_s \omega, 1_s \dots p_s \omega \psi_{s,t}}) \omega \\ &= (x_1 \varphi_{1_s, 1_s \dots p_s \omega \psi_{s,t}} \dots x_p \varphi_{p_s, 1_s \dots p_s \omega \psi_{s,t}}) \omega. \end{aligned}$$

On the other hand,

$$(x_1 h_{s,t} \dots x_p h_{s,t}) \omega = (x_1 \varphi_{1_s, 1_s \psi_{s,t}} \dots x_p \varphi_{p_s, p_s \psi_{s,t}}) \omega.$$

Now $x_n \varphi_{n_s, n_s \psi_{s,t}} \in E_{n_s \psi_{s,t}}$, and since N is a functorial sum of N_s over S and B_s is a functorial sum of E_{n_s} over N_s , we obtain

$$\begin{aligned} (x_1 \varphi_{1_s, 1_s \psi_{s,t}} \dots x_p \varphi_{p_s, p_s \psi_{s,t}}) \omega &= ((x_1 \varphi_{1_s, 1_s \psi_{s,t}} \varphi_{1_s \psi_{s,t}, 1_s \psi_{s,t} \dots p_s \psi_{s,t} \omega} \dots (x_p \varphi_{p_s, p_s \psi_{s,t}} \varphi_{p_s \psi_{s,t}, 1_s \psi_{s,t} \dots p_s \psi_{s,t} \omega})) \omega \\ &= (x_1 \varphi_{1_s, 1_s \psi_{s,t} \dots p_s \psi_{s,t} \omega} \dots x_p \varphi_{p_s, 1_s \psi_{s,t} \dots p_s \psi_{s,t} \omega} \omega) \omega \\ &= (x_1 \varphi_{1_s, 1_s \dots p_s \omega \psi_{s,t}} \dots x_p \varphi_{p_s, 1_s \dots p_s \omega \psi_{s,t}}) \omega. \end{aligned}$$

Hence all $h_{s,t}$ are indeed homomorphisms.

Now we will show that the homomorphisms $h_{s,t}$ satisfy the functoriality condition

$$h_{s,t} h_{t,u} = h_{s,u}$$

for all $s \preceq t \preceq u$ in S . Let $x \in E_{n_s}$ and $n_s \in N_s$. Then indeed

$$\begin{aligned} x h_{s,t} h_{t,u} &= x \varphi_{n_s, n_s \psi_{s,t}} \varphi_{n_s \psi_{s,t}, n_s \psi_{s,t} \psi_{t,u}} \\ &= x \varphi_{n_s, n_s \psi_{s,t}} \varphi_{n_s \psi_{s,t}, n_s \psi_{s,u}} \\ &= x \varphi_{n_s, n_s \psi_{s,u}} = x h_{s,u}. \end{aligned}$$

The last detail to check is whether the operations induced by the definition of the functorial sum $\sum(B_s \mid s \in S)$ and the original operations of the algebra (E, Ω)

coincide. Let $x_{s_i} \in E_{n_{s_i}}$ for $n_{s_i} \in N_{s_i}$. Since the algebra E is a functorial sum of E_n over N and N is a functorial sum over S , it follows that

$$\begin{aligned} x_{s_1} \dots x_{s_p} \omega &= (x_{s_1} \varphi_{n_{s_1}, n_{s_1} \dots n_{s_p} \omega} \dots x_{s_p} \varphi_{n_{s_p}, n_{s_1} \dots n_{s_p} \omega}) \omega \\ &= x_{s_1} \varphi_{n_{s_1}, (n_{s_1} \psi_{s_1, s_1 \dots s_p \omega} \dots n_{s_p} \psi_{s_p, s_1 \dots s_p \omega}) \omega} \\ &\quad \dots x_{s_p} \varphi_{n_{s_p}, (n_{s_1} \psi_{s_1, s_1 \dots s_p \omega} \dots n_{s_p} \psi_{s_p, s_1 \dots s_p \omega}) \omega} \omega. \end{aligned}$$

On the other hand, in the sum $\sum (B_s \mid s \in S)$ we have

$$\begin{aligned} x_{s_1} \dots x_{s_p} \omega &= x_{s_1} h_{s_1, s_1 \dots s_p \omega} \dots x_{s_p} h_{s_p, s_1 \dots s_p \omega} \omega \\ &= x_{s_1} \varphi_{n_{s_1}, n_{s_1} \psi_{s_1, s_1 \dots s_p \omega}} \dots x_{s_p} \varphi_{n_{s_p}, n_{s_p} \psi_{s_p, s_1 \dots s_p \omega}} \omega \\ &= x_{s_1} \varphi_{n_{s_1}, n_{s_1} \psi_{s_1, s_1 \dots s_p \omega}} \varphi_{n_{s_1} \psi_{s_1, s_1 \dots s_p \omega}, n_{s_1} \psi_{s_1, s_1 \dots s_p \omega} \dots n_{s_p} \psi_{s_p, s_1 \dots s_p \omega}} \\ &\quad \dots x_{s_p} \varphi_{n_{s_p}, n_{s_p} \psi_{s_p, s_1 \dots s_p \omega}} \varphi_{n_{s_p} \psi_{s_p, s_1 \dots s_p \omega}, n_{s_1} \psi_{s_1, s_1 \dots s_p \omega} \dots n_{s_p} \psi_{s_p, s_1 \dots s_p \omega}} \omega \omega. \end{aligned}$$

□

The functorial sums in Theorem 3.2 have better properties in the case where the indexing algebra (N, Ω) is idempotent and naturally quasi-ordered. To discuss these, recall the following. On the set N define a relation α as

$$(x, y) \in \alpha : \Leftrightarrow x \preceq y \quad \text{and} \quad y \preceq x.$$

It is well known that α is an equivalence relation, and that the relation $x^\alpha \leq y^\alpha$ iff $x \preceq y$ is an ordering relation. Moreover, the following holds.

Proposition 3.3 ([12], [11, Chapter 4]). *An idempotent algebra (N, Ω) is naturally quasi-ordered iff the relation α is a congruence on (N, Ω) and (N^α, Ω) is an Ω -semilattice.*

Note that in general a naturally quasi-ordered algebra N does not need to be a Płonka sum of the algebras n^α over the semilattice N^α .

Theorem 3.4 ([12], [11, Chapter 4]). *Let an algebra (E, Ω) be a functorial sum of algebras (E_n, Ω) over an idempotent naturally quasi-ordered algebra (N, Ω) with a natural homomorphism nata onto the Ω -semilattice (N^α, Ω) . Then the homomorphism πnata provides the decomposition of (E, Ω) as a disjoint union of subalgebras each isomorphic to the direct product $(E_n, \Omega) \times (n^\alpha, \Omega)$ over the Ω -semilattice (N^α, Ω) . Briefly:*

$$E = \bigcup (E_n \times n^\alpha \mid n^\alpha \in N^\alpha).$$

In some quite common but still very general special cases, the decomposition given in Theorem 3.4 is in fact a decomposition into a Płonka sum.

Corollary 3.5. *Let an algebra (E, Ω) be a functorial sum of algebras (E_n, Ω) over an idempotent naturally quasi-ordered algebra (N, Ω) . Then the following hold.*

- (a) *If $|N^\alpha| = 1$, then the algebra (E, Ω) is just the direct product $(E_n, \Omega) \times (N, \Omega)$.*
- (b) *If $|n^\alpha| = 1$ for each $n \in N$, then (E, Ω) is a Płonka sum of (E_n, Ω) over the Ω -semilattice (N, Ω) .*
- (c) *If (N, Ω) is a Płonka sum of (n^α, Ω) over (N^α, Ω) , then (E, Ω) is a Płonka sum of subalgebras each isomorphic to the direct product $(E_n, \Omega) \times (n^\alpha, \Omega)$ over the Ω -semilattice (N^α, Ω) . Briefly:*

$$E = \sum_{n \in N} E_n = \sum (E_n \times n^\alpha \mid n^\alpha \in N^\alpha).$$

Obviously (a) and (b) are special cases of (c). One can briefly say that a functorial sum over a Płonka sum is itself a Płonka sum. In particular, if the algebra N is a Płonka sum of algebras n^α that are members of an irregular variety, and the indexing algebra N^α is the semilattice replica of N , then N is naturally quasi-ordered, and the algebra E has a representation as given in (c).

As an example, consider a functorial sum $C = \sum_{n \in N} C_n$ of convex sets C_n over a Płonka sum $\sum_{s \in S} N_s$ of convex sets N_s . Convex sets are considered here as barycentric algebras. (See [11] for the definition and basic properties.) Note that each barycentric algebra embeds into a Płonka sum of convex sets ([11]). It is known (see [12]) that barycentric algebras are naturally quasi-ordered. It follows that C is the Płonka sum of convex sets $C_n \times N_s$, where n is an arbitrarily chosen element of N_s .

Another example is given by functorial sums of Ω -algebras over normal Ω -bands. For a given (plural) type of Ω -algebras define normal Ω -bands as follows. (See e.g. [4].) For each normal band (i.e. semigroup mode) $(B, +)$ and each n -ary operator $\omega \in \Omega$, define the operation ω in B by

$$x_1 \dots x_n \omega := x_1 + \dots + x_n.$$

Then the algebra (B, Ω) obtained in this way is called a normal Ω -band. Obviously each normal Ω -band is a band. And similarly, as each normal band is a Płonka sum of rectangular bands, each normal Ω -band is a Płonka sum of rectangular Ω -bands. All these algebras are naturally quasi-ordered.

Corollary 3.6. *Let (E, Ω) be a functorial sum of algebras (E_n, Ω) over a normal Ω -band (N, Ω) , and let (N, Ω) be a Płonka sum of rectangular Ω -bands (n^α, Ω) over its semilattice replica (N^α, Ω) . Then (E, Ω) is the Płonka sum of the subalgebras $(E_n, \Omega) \times (n^\alpha, \Omega)$ over (N^α, Ω) .*

For a related result see [12, Example 2.7].

Note that in general the disjoint union obtained in Theorem 3.4 is not necessary a Płonka sum. This is illustrated by the following example.

Example 3.7. Consider the half-closed unit real interval $N = [0, 1[$ as a barycentric algebra. For each $n \in N$, let E_n be a copy of the open unit interval $]0, 1[$ considered also as a barycentric algebra. The semilattice replica S of N has two elements, say $0 < 1$, with corresponding fibres $N_0 = \{0\}$ and $N_1 =]0, 1[$. Denote the elements of E_n by i_n , and let a be a fixed element of $]0, 1[$. For $m, n \in]0, 1[$, define the sum homomorphisms by

$$\varphi_{0,m}: E_0 \rightarrow E_m; \quad i_0 \mapsto a_m \quad \text{and} \quad \varphi_{m,n}: E_m \rightarrow E_n; \quad i_m \mapsto a_n.$$

With these homomorphisms, the disjoint union E of E_n becomes a functorial sum of the algebras E_n over the algebra N . Note that the open interval $]0, 1[$ has a full algebraic quasi-order, and that the algebra N cannot be reconstructed as a Płonka sum of the fibres N_0 and N_1 . (See [11, Example 4.5.1].) Note that $B_0 \cong]0, 1[\times \{0\} \cong]0, 1[$ and $B_1 \cong]0, 1[\times]0, 1[$. Moreover, the algebra E is a disjoint union of B_0 and B_1 . However, the algebra E cannot be reconstructed as a Płonka sum of B_0 and B_1 .

4. NON-FUNCTORIAL SUMS OF ALGEBRAS

The conditions on a functorial sum are very strong, and not satisfied in general. The most general method of constructing algebras with a homomorphism onto an idempotent naturally quasi-ordered algebra is given by the construction of a *generalized coherent Lallement sum of algebras* or briefly just a *Lallement sum* as introduced in [7], [9], [11], [12]. The general context of the definition of a Lallement sum is the same as in the case of a functorial sum. We have a naturally quasi-ordered indexing algebra (I, Ω) with algebraic quasi-order \preceq , and for each i in I , an algebra (A_i, Ω) . The algebras (A_i, Ω) come together with certain extensions (E_i, Ω) , and for $i \preceq j$ in (I, \preceq) , there are Ω -homomorphisms $\varphi_{i,j}: (A_i, \Omega) \rightarrow (E_j, \Omega)$ with identity mappings $\varphi_{i,i}$ and with the functoriality condition replaced by the following ones

(L1) For each (n -ary) ω in Ω and for i_1, \dots, i_n in I with $i_1 \dots i_n \omega = i$,

$$(A_{i_1} \varphi_{i_1, i}) \dots (A_{i_n} \varphi_{i_n, i}) \omega \subseteq A_i;$$

(L2) For each $i_1 \dots i_n \omega = i \preceq j$ in (I, \preceq)

$$a_{i_1} \varphi_{i_1, i} \dots a_{i_n} \varphi_{i_n, i} \omega \varphi_{i, j} = a_{i_1} \varphi_{i_1, j} \dots a_{i_n} \varphi_{i_n, j} \omega,$$

where $a_{i_k} \in A_{i_k}$ for $k = 1, \dots, n$.

Moreover,

$$(L3) \ E_i = \{a_j \varphi_{j,i} \mid j \preceq i\}.$$

Then the Lallement sum $\mathcal{L}_{i \in I}(A_i, \Omega)$ or simply $\mathcal{L}_{i \in I} A_i$ of A_i over I is the disjoint union $A = \bigcup (A_i \mid i \in I)$ equipped with Ω -operations defined in the same way as in the case of functorial sums. As proved in [12], (see also [11, Theorem 4.5.3]), an algebra (A, Ω) with a homomorphism onto an idempotent, naturally quasi-ordered algebra (I, Ω) is a Lallement sum of the corresponding fibres over (I, Ω) . The extensions (E_i, Ω) are built in a certain canonical way as the so-called envelopes of the fibres. (Each preserves the fibre subalgebra.)

In the final section we will need the following result.

Theorem 4.1 ([12], [11, Chapter 4]). *Let (A, Ω) be a Lallement sum of fibres (A_i, Ω) over a naturally quasi-ordered algebra (I, Ω) . Then (A, Ω) is a semi-lattice Lallement sum of the preimages (B_{i^α}, Ω) of i^α over the Ω -semilattice (I^α, Ω) . Each fibre (B_{i^α}, Ω) is a Lallement sum of (A_i, Ω) over (i^α, Ω) , and each (i^α, Ω) has a full algebraic quasi-order,*

$$A = \mathcal{L}(A_i \mid i \in I) = \mathcal{L}(B_{i^\alpha} \mid i^\alpha \in I^\alpha) = \mathcal{L}_{i^\alpha \in I^\alpha} (\mathcal{L}_{i \in i^\alpha} A_i).$$

We are especially interested in Lallement sums embeddable into functorial sums. As was shown in [9], [12] and [11, Theorem 4.5.18], a Lallement sum $\mathcal{L}_{i \in I} A_i$ is a subalgebra of a functorial sum precisely if its sum homomorphisms $\varphi_{i,j}$ satisfy the following condition for all i, j, k, l in I with $i, j \preceq k \preceq l$ and all $a_i \in A_i, b_j \in B_j$:

$$(4.1) \quad a_i \varphi_{i,k} = b_j \varphi_{j,k} \implies a_i \varphi_{i,l} = b_j \varphi_{j,l}.$$

The embedding obtained in the proof of Theorem 4.5.18 of [11] is constructed in such a way that $A = \mathcal{L}_{i \in I} A_i$ is a subalgebra of the functorial sum E of the canonical envelopes E_i of A_i over the same algebra I . Now in each subalgebra $\mathcal{L}_{i \in i^\alpha} A_i$ of A , the envelopes E_i of A_i are isomorphic, and $\mathcal{L}_{i \in i^\alpha} A_i$ is a subalgebra of the direct product $E_i \times i^\alpha$. It follows by Theorem 3.4 that the Lallement sum A is a subalgebra of the disjoint union of direct products $E_i \times i^\alpha$ over the semilattice I^α .

5. EMBEDDING A SUM OF CANCELLATIVE Ω -MODES OVER A NORMAL Ω -BAND INTO A SEMIMODULE

In this section we will consider Ω -modes in the Mal'cev product $\underline{Cl} \circ \underline{Nb}$ of the quasivarieties \underline{Cl} of cancellative Ω -modes and \underline{Nb} of normal Ω -bands. Each such mode has a congruence with a normal Ω -band quotient, the corresponding congruence classes being cancellative submodes (cf. [5]). By the results mentioned in the previous sections, it is clear that such modes are Lallement sums of cancellative submodes over a normal Ω -band. In [12] the condition (4.1) given in the previous section was used to show that such modes embed into functorial sums, yielding the following result.

Theorem 5.1 ([12], [9], [11, Chapter 4]). *Let a mode (A, Ω) be a Lallement sum of cancellative modes (A_i, Ω) over a naturally quasi-ordered mode (I, Ω) . Then (A, Ω) is a subalgebra of a functorial sum (E, Ω) of canonical cancellative envelopes (E_i, Ω) of (A_i, Ω) over (I, Ω) . Briefly:*

$$A = \mathcal{L}_{i \in I} A_i \leq \sum_{i \in I} E_i.$$

In particular, Theorem 5.1 holds if the indexing algebra I is a normal Ω -band. In this case we will improve Theorem 5.1 by showing that the Lallement sum A embeds as a subreduct into a semimodule. To prove this result we will need the following facts.

Lemma 5.2. *Let M_1, M_2, N_1, N_2 be modules over a ring R , and let $h_1: M_1 \rightarrow N_1, h_2: M_2 \rightarrow N_2$ be module homomorphisms. Then the mapping*

$$h: M_1 \times M_2 \rightarrow N_1 \times N_2; \quad (a, b) \mapsto (ah_1, bh_2)$$

is also a module homomorphism.

Proposition 5.3. *Each normal band embeds as a subreduct into a semimodule.*

Proof. Each normal band N is a Płonka sum of rectangular bands, say N_s , over its semilattice replica S . We will show that this sum embeds as a subreduct into a Płonka sum of modules. Since Płonka sums of modules are semimodules (see e.g. [13]), this gives the required embedding.

Each rectangular band N_s is a direct product $L_s \times R_s$ of a left-zero semigroup L_s (a binary mode defined by the identity $x \cdot y = x$) and a right-zero semigroup R_s (a binary mode defined by the identity $x \cdot y = y$). The left-zero semigroup L_s can be considered

as the set of free generators of a free module $L_s \text{Mod}_{R(\text{Lz})}$ over the ring $R(\text{Lz}) = \mathbb{Z}[X]/\langle X \rangle$. Similarly, the right-zero semigroup R_s can be considered as the set of free generators of a free module $R_s \text{Mod}_{R(\text{Rz})}$ over the ring $R(\text{Rz}) = \mathbb{Z}[X]/\langle 1 - X \rangle$. (This follows by the affinization process described in [11, Chapter 7].) Both these modules can be considered as modules over the ring $R(\text{Re}) = \mathbb{Z}[X]/\langle X(1 - X) \rangle$. We will denote them by $M(L_s)$ and $M(R_s)$, respectively. In this way we obtain an embedding of N_s into the $R(\text{Re})$ -module $M(N_s) := M(L_s) \times M(R_s)$.

Now each sum homomorphism $\psi_{s,t}: N_s \rightarrow N_t$, for $s \leq t$, is uniquely determined by a pair of functions $(f_{s,t}, g_{s,t})$, where $f_{s,t}: L_s \rightarrow L_t$ and $g_{s,t}: R_s \rightarrow R_t$. Then the universal mapping property for free modules implies that for each mapping $f_{s,t}$ there exists a uniquely defined module homomorphism $\bar{f}_{s,t}: M(L_s) \rightarrow M(L_t)$ such that $\bar{f}_{s,t}|_{L_s} = f_{s,t}$. In particular, the following diagram is commutative:

$$\begin{array}{ccc} L_s & \xrightarrow{\iota} & M(L_s) \\ f_{s,t} \downarrow & & \downarrow \bar{f}_{s,t} \\ L_t & \xrightarrow{\iota} & M(L_t) \end{array}$$

In a similar way one extends each mapping $g_{s,t}$ to a module homomorphism $\bar{g}_{s,t}: M(R_s) \rightarrow M(R_t)$.

Note that under the operation \underline{X} both $M(L_s)$ and $M(R_s)$ are also rectangular bands. (In fact, the former is a left-zero band and the latter a right-zero band.) Moreover, the rectangular band $N_s = L_s \times R_s$ embeds into the rectangular band $M(N_s) = M(L_s) \times M(R_s)$. Now each sum homomorphism $\psi_{s,t}$, for $s \leq t$, is uniquely determined by the pair $(f_{s,t}, g_{s,t})$ that can be uniquely extended to the pair $(\bar{f}_{s,t}, \bar{g}_{s,t})$ of module homomorphisms. By Lemma 5.2, there exists a unique module homomorphism $\bar{\psi}_{s,t}: M(L_s) \times M(R_s) \rightarrow M(L_t) \times M(R_t)$ determined by the pair $(\bar{f}_{s,t}, \bar{g}_{s,t})$. The homomorphisms $\bar{\psi}_{s,t}$ are functorial since the mappings $f_{s,t}$ and $g_{s,t}$ defined on the generators are. They determine a Płonka sum structure on the disjoint union of the modules $M(N_s) = M(L_s) \times M(R_s)$. Now the Płonka sum of these modules is a semimodule over the same ring $R(\text{Re})$. \square

Now we can present the main result of this paper:

Theorem 5.4. *Let τ be a given type of Ω -modes. Let (A, Ω) be a Lallement sum of cancellative modes (A_n, Ω) over a normal Ω -band (N, Ω) . Then (A, Ω) embeds as a subreduct into a semimodule over the ring $R(M\tau)$.*

Proof. First note that for the case where the normal band N is a semilattice, the theorem was already proved in [13] (see also [11, Chapter 7]). So in what follows we assume that N is not a semilattice.

By Theorem 5.1, the algebra A is a subalgebra of a functorial sum E of cancellative envelopes E_n of A_n over N with sum homomorphisms $\varphi_{m,n}$. Let $N = \sum_{s \in S} N_s$ with sum homomorphisms $\psi_{s,t}$. The semilattice S can be identified with N^α , and N_s with $(\text{nat}\alpha)^{-1}(s)$. By Corollary 3.6, the algebra E is the Płonka sum of its subalgebras $B_s = E_n \times N_s$, where $n \in N_s$, over the semilattice S with sum homomorphisms $h_{s,t}$. By Theorem 2.1, each algebra E_n embeds as a subreduct into an $R(M\tau)$ -module G_n . And since Proposition 5.3 extends in an obvious way to normal Ω -bands, each rectangular Ω -band N_s embeds as a subreduct into an $R(\text{Re})$ -module $M(N_s)$. However, the latter module can also be considered as an $R(M\tau)$ -module. In this way one obtains an embedding of the algebra B_s as a subreduct of the $R(M\tau)$ -module $\overline{B}_s = G_n \times M(N_s)$. The Płonka sum E of algebras B_s extends to the Płonka sum of the modules $\overline{B}_s = G_n \times M(N_s)$ in the way described below.

Similarly to the proof of Proposition 3.3 of [13], one proves that each homomorphism $\varphi_{m,n}: E_m \rightarrow E_n$ extends to a module homomorphism $\overline{\varphi}_{m,n}: G_m \rightarrow G_n$. And each homomorphism $\psi_{s,t}: N_s \rightarrow N_t$ extends to a module homomorphism $\overline{\psi}_{s,t}: M(N_s) \rightarrow M(N_t)$. Obviously, the homomorphisms $\overline{\varphi}_{m,n}$ and $\overline{\psi}_{s,t}$ satisfy the functoriality condition. We want to extend the homomorphisms $h_{s,t}$ to functorial module homomorphisms $\overline{h}_{s,t}: \overline{B}_s \rightarrow \overline{B}_t$. To do this, first note that each \overline{B}_s , as a direct product $G_n \times M(N_s)$, can also be considered as a functorial sum $\sum_{m_s \in M(N_s)} G_{m_s}$ of pairwise isomorphic G_{m_s} . For each $m_s \in M(N_s)$ and $x \in G_{m_s}$, define the mapping $\overline{h}_{s,t}$ as follows:

$$x\overline{h}_{s,t} := x\overline{\varphi}_{m_s, m_s}\overline{\psi}_{s,t}.$$

We have to check that the mappings $\overline{h}_{s,t}$ are functorial module homomorphisms.

First we show that they are homomorphisms. Let $x \in G_{m_s}$ and $y \in G_{n_s}$ with $m_s, n_s \in M(N_s)$. Then $x + y \in G_{m_s + n_s}$, and the following holds:

$$\begin{aligned} (x + y)\overline{h}_{s,t} &= (x + y)\overline{\varphi}_{m_s + n_s, (m_s + n_s)}\overline{\psi}_{s,t} \\ &= (x\overline{\varphi}_{m_s, m_s} + y\overline{\varphi}_{n_s, m_s + n_s})\overline{\varphi}_{m_s + n_s, (m_s + n_s)}\overline{\psi}_{s,t} \\ &= x\overline{\varphi}_{m_s, m_s}\overline{\psi}_{s,t} + n_s\overline{\psi}_{s,t} + y\overline{\varphi}_{n_s, m_s}\overline{\psi}_{s,t} + n_s\overline{\psi}_{s,t}. \end{aligned}$$

On the other hand,

$$\begin{aligned} x\overline{h}_{s,t} + y\overline{h}_{s,t} &= x\overline{\varphi}_{m_s, m_s}\overline{\psi}_{s,t} + y\overline{\varphi}_{n_s, n_s}\overline{\psi}_{s,t} \\ &= x\overline{\varphi}_{m_s, m_s}\overline{\psi}_{s,t} + n_s\overline{\psi}_{s,t} + y\overline{\varphi}_{n_s, m_s}\overline{\psi}_{s,t} + n_s\overline{\psi}_{s,t}. \end{aligned}$$

Now let $x \in G_{m_s}$ and $r \in R(M\tau)$. Then

$$(xr)\overline{h}_{s,t} = (xr)\overline{\varphi}_{m_s, m_s}\overline{\psi}_{s,t} = x\overline{\varphi}_{m_s, m_s}\overline{\psi}_{s,t} r = x\overline{h}_{s,t}r.$$

Next we check that the module homomorphisms $\overline{h}_{s,t}$ satisfy the functoriality condition. Let $x \in G_{m_s}$ with $m_s \in M(N_s)$, and let $s, t, u \in S$ with $s \leq t \leq u$. Then

$$x\overline{h}_{s,t}\overline{h}_{t,u} = x\overline{\varphi}_{m_s, m_s}\overline{\psi}_{s,t}\overline{\varphi}_{m_s}\overline{\psi}_{s,t, m_s}\overline{\psi}_{s,t}\overline{\psi}_{t,u} = x\overline{\varphi}_{m_s, m_s}\overline{\psi}_{s,u} = x\overline{h}_{s,u}.$$

The homomorphisms $\overline{h}_{s,t}$ define the Plonka sum of $R(M\tau)$ -modules \overline{B}_s over the semilattice S . As mentioned before, this Plonka sum is a semimodule over the ring $R(M\tau)$. It is obvious that the algebra E is a subreduct of this semimodule. \square

6. CONCLUDING REMARKS

In this final section we will summarize what is known about embedding Lallement sums of modes into semimodules over commutative semirings, and consider some implications. Note that such modes are always idempotent subreducts of semimodules, so that we can always consider them also as subreducts of affine semimodules.

First recall the two basic results, Theorem 2.1 and Proposition 5.3, which show that each cancellative mode and each rectangular band embed as subreducts into a module. It follows also that direct products of cancellative modes and rectangular bands have this property. However, differential groupoids provide examples of idempotent subreducts of affine spaces that are not of this type. (See [11, Chapters 5, 7].)

The next step deals with embeddings of Lallement sums of cancellative and other modes into semimodules. First note that each semilattice can be considered as a semimodule over the semiring of its endomorphisms. Then Proposition 5.3 tells us that each normal band is a subreduct of a semimodule over a certain ring. More generally, Theorem 5.4 shows that each Lallement sum of cancellative modes over a normal band is a subreduct of a semimodule over a certain ring. This theorem has two important special cases. If the indexing normal band N is a rectangular band, then by the results of Section 4 and Theorem 5.1, the algebra A is a subalgebra of the direct product E of a cancellative envelope and the band N , and the algebra E is a subreduct of a module, the direct product of two $R(M\tau)$ -modules. On the other hand, if the indexing normal band N is a semilattice, then the algebra A is a semilattice Lallement sum. In this case Theorem 5.4 is just the result proved earlier in [13] and [11, Chapter 7] about embedding semilattice Lallement sums into semimodules over a ring. Note as well that by Corollary 3.6, a functorial sum E of cancellative modes E_n over a normal band N is a Plonka sum of direct products of E_n and n^α over the semilattice N^α , and hence by the previous result embeds into a semimodule.

The next question concerns the embeddability into semimodules of Lallement sums of cancellative modes over Lallement sums that are known to be embeddable into

semimodules. At present we have some partial results. We will consider a Lallement sum D of cancellative modes D_a over a naturally quasi-ordered mode A that is itself a Lallement sum of cancellative modes A_n over a normal Ω -band N . Here the question arises as to whether a Lallement sum of cancellative modes over a normal band is necessarily naturally quasi-ordered.

Theorem 6.1. *Let (A, Ω) be a naturally quasi-ordered mode which is a functorial sum $\sum_{n \in N} A_n$ of cancellative modes (A_n, Ω) , each with a full algebraic quasi-order over a normal Ω -band $N = \sum_{s \in S} N_s$. Let (D, Ω) be a Lallement sum of cancellative modes (D_a, Ω) over the mode (A, Ω) . Then the mode (D, Ω) embeds as a subreduct into a semimodule.*

Proof. First note that by Theorem 5.1, the algebra D is a subalgebra of the sum

$$\sum (F_a \mid a \in A) = \sum \left(F_a \mid a \in \sum (A_n \mid n \in N) \right),$$

where F_a are canonical envelopes of D_a . By Theorem 3.2, the latter sum can also be written as

$$\sum_{n \in N} \sum (F_a \mid a \in A_n).$$

By Proposition 3.1,

$$\sum (F_a \mid a \in A_n) \cong F_a \times A_n,$$

and as a product of two cancellative modes, it is also cancellative. It follows that the mode D is a subalgebra of the functorial sum $\sum_{n \in N} (F_a \times A_n)$ of cancellative modes over the normal Ω -band N . By Theorem 5.4, it embeds as a subreduct into a semimodule. \square

It is still an open question how far the assumptions of Theorem 6.1 can be relaxed. In particular, is this theorem still true if the functorial sum of A_n is replaced by a Lallement sum of A_n ? We know that this can be done in the case presented in the following theorem.

Theorem 6.2. *Let (I, Ω) be a naturally quasi-ordered mode with a semilattice quotient (I^α, Ω) , such that each (i^α, Ω) , for $i \in I$, is cancellative. Then a Lallement sum (A, Ω) of cancellative modes (A_i, Ω) over (I, Ω) embeds as a subreduct into a semimodule.*

Proof. Let $A = \mathcal{L}(A_i \mid i \in I)$. By Theorem 4.1,

$$A = \mathcal{L}(B_{i^\alpha} \mid i^\alpha \in I^\alpha)$$

and

$$B_{i^\alpha} = \mathcal{L}(A_i \mid i \in i^\alpha).$$

By Theorem 5.1 and Proposition 3.1,

$$B_{i^\alpha} \leq \sum (E_i \mid i \in i^\alpha) = E_i \times i^\alpha,$$

and clearly it is a cancellative mode since each E_i is. It follows that the mode A is a semilattice Lalletment sum of cancellative modes B_{i^α} over the semilattice I . Hence by Theorem 5.4, it embeds into a semimodule. \square

Example 6.3. It is well known ([8] and [11, Chapter 7]) that each barycentric algebra I is a semilattice Lalletment sum of open convex sets i^α over its semilattice replica I^α . By Theorems 5.1 and 6.2, a Lalletment sum A of convex sets A_i over a barycentric algebra I is itself a barycentric algebra, and hence embeds into a semimodule.

Example 6.4. Similarly to the case of barycentric algebras, each commutative binary mode I (a groupoid mode with commutative multiplication) is a semilattice Lalletment sum of cancellative submodes i^α over its semilattice replica I^α . (See [9] and [11, Chapter 7].) Again Theorems 5.1 and 6.2 imply that a Lalletment sum A of cancellative binary modes over a commutative binary mode I is itself a commutative binary mode, and hence embeds into a semimodule.

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Author's address: Faculty of Mathematics and Information Sciences, Warsaw University of Technology, 00-661 Warsaw, Poland, e-mails: aroman@alpha.mini.pw.edu.pl, azamojsk@elka.pw.edu.pl.