EMBEDDINGS IN DIVISION RINGS

BY

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Abstract. A method for embedding a certain class of integral domains in division rings is devised. Integral domains A are constructed with a generalized valuation into a (noncommutative) totally ordered semigroup that need not be discrete. Then the multiplicative semigroup $A\setminus\{0\}$ is expressed as an inverse limit of semigroups each of which is embeddable in a group. Thus $A\setminus\{0\}$ can be embedded in a group G. The main problem is to introduce addition on G in order that G becomes a division ring by the use of eventually commuting maps of inverse limits.

Introduction. Suppose that the multiplicative semigroup A^* of a noncommutative integral domain A can be embedded in a group G. Some recent surprising discoveries show that there exist rings A for which an embedding $A^* \subset G$ is possible, but such that for any embedding of A^* into any group G whatever, addition cannot be extended to all of $G \cup \{0\}$ in order to obtain an embedding of A into a division ring ([1] and [6]). Under certain appropriate additional hypotheses on an integral domain with a valuation into the integers, P. M. Cohn embeds $A \subset G \cup \{0\}$, introduces a group topology on G, then defines addition on the subset A^*A^{*-1} which happens to be dense in G, and then finally extends addition to all of G [2]. Recent interest in the subject ([1], [6], and [3]), as well as the fact that treatises on ring theory find it necessary to quote this result [5, p. 257] but do not prove it because existing proofs are too complicated, are two reasons why a purely algebraic proof which avoids topology altogether is needed.

The first objective of this note is to introduce a much wider class of rings than simply integral domains with an integer valued valuation. The second aim is to prove that any ring in this wider class can be embedded in a division ring. One of the main factors contributing to the length and complexity of the present proof is that in place of the integers, the range of the valuation is in a not necessarily commutative semigroup Γ. One of the appealing features of the present development is that if Γ is specialized to be commutative, our proof simplifies considerably. Furthermore, it involves no topological considerations of any kind whatsoever.

1. **Rings with a noncommutative valuation.** In an ordered ring, the valuation

should be an order preserving map. For this reason unfortunately, the order relation in the subsequent definition is just the reverse of that commonly accepted by most of the previous authors.

1.1 DEFINITION. Consider a *totally ordered semigroup* Γ, i.e. Γ is a totally ordered set, where if $\alpha \leq \beta$, $\bar{\alpha} \leq \bar{\beta}$, then $\alpha\bar{\alpha} \leq \beta\bar{\beta}$. Suppose Γ is cancellative ($\alpha y = \beta y$ or $y\alpha = y\beta$ implies $\alpha = \beta$). Let $\Gamma \cup \{0\}$ be the ordered semigroup where $0\gamma = \gamma 0 < \Gamma$ for all $\gamma \in \Gamma$. Here, a *ring with a valuation* is a triple A, v, Γ where $v: A \to \Gamma \cup \{0\}$ is a function such that for all $a, b \in A$, the following are satisfied

  (i) $v(a) = 0$ if and only if $a = 0$,

  (ii) $v(a - b) \leq \max [v(a), v(b)] = v(a) \vee v(b)$,

  (iii) $v(ab) = v(a)v(b)$.

For any semigroup Γ, as usual $\Gamma^1 = \Gamma$ if Γ has an identity, or $\Gamma^1 = \Gamma \cup \{1\}$ is Γ with an identity adjoined otherwise. Let $P \subseteq \Gamma$ be the strict two-sided increasers, i.e. $P = \{y \in \Gamma \mid y < \gamma y, y < y\gamma \text{ for all } y \in \Gamma\}$. If P is nonvoid, then it is a subsemigroup.

1.2. The following identities valid for all $a, b \in A^* = A \backslash \{0\}$ are proved just the same as in the case when Γ are the integers under addition.

  (i) $v(-a) = v(a)$,

  (ii) $v(b) < v(a) \Rightarrow v(a - b) = v(a)$,

  (iii) $v(a) \neq v(b) \Rightarrow v(a - b) = v(a) \vee v(b)$,

  (iv) $v(a) > v(a - b) \Rightarrow v(a) = v(b)$,

  (v) A is an integral domain.

1.3 DEFINITION. For $\gamma \in P$, let $q(\gamma)$ denote all $(a, b) \in A^* \times A^*$ such that for any $s \in \Gamma^1$, the following two equations hold

  (i) $\gamma s v(a - b) \leq s v(a)$,

  (ii) $v(a - b)s\gamma \leq v(a)s$.

To recapitulate, a product of γ, s, and $v(a - b)$ is less than $sv(a)$ or $v(a)s$, where s precedes $v(a)$ if and only if it also precedes $v(a - b)$. In subsequent proofs it will automatically be assumed that $s \in \Gamma^1$ is an arbitrary element without quantifying s each time.

1.4 REMARK. The following observations will not be used in subsequent proofs. If $L = \{y \in \Gamma \mid y < \gamma y, \text{ all } y \in \Gamma\}$ and R are the subsemigroups of strict left and right increasers of Γ, then define

for $\gamma \in L$, $\qquad q(\gamma, L) = \{(a, b) \in A^* \times A^* \mid \gamma v(a - b) \leq v(a)\}$;

for $\gamma \in R$, $\qquad q(\gamma, R) = \{(a, b) \in A^* \times A^* \mid v(a - b)\gamma \leq v(a)\}$;

for $\gamma \in P = L \cap R$, $\quad q(\gamma, P) = q(\gamma, L) \cap q(\gamma, R)$.

All three are equivalence relations. For an arbitrary $c \in A^*$, the following implications hold

$$(a, b) \in q(\gamma, L) \Rightarrow (ac, bc) \in q(\gamma, L),$$

$$(a, b) \in q(\gamma, R) \Rightarrow (ca, cb) \in q(\gamma, R).$$

Unfortunately, there is no reason why (a, b), $(c, d) \in q(\gamma, P)$ should imply that

also $(ac, bd) \in q(\gamma, P)$. Nevertheless, the latter does hold if Γ is commutative. In that case we may take $q(\gamma, P) = q(\gamma, L) = q(\gamma, R)$ in place of $q(\gamma)$ in Definition 1.3 and all the subsequent proofs become much simpler. However, one of our main objectives is to embed A in a division ring when Γ is not commutative.

1.5 LEMMA. *If A, v, Γ is a valuated ring, then $q(\gamma)$ is an equivalence relation for each $\gamma \in P$.*

Proof. Symmetric. If $a = b$, then the left side of 1.3 (i) and (ii) contains $0 = v(a-a)$, hence equals 0, and thus is (strictly) less than the right side.

Reflexive. If $(a, b) \in q(\gamma)$, then $v(a-b) < \gamma v(a-b) \leq v(a)$. By 1.2 (iv), $v(a) = v(b)$. Replacement of $v(a)$ by $v(b)$ in 1.3 (i), (ii) shows that also $(b, a) \in q(\gamma)$.

Transitive. Given (a, b) and $(b, c) \in q(\gamma)$, then as above $v(a) = v(b) = v(c)$. Thus 1.3 (i) and (ii) become

(i) $\gamma s v(a-c) \leq \gamma s v(a-b) \vee \gamma s v(c-b) \leq s v(a)$,

(ii) $v(a-c) s\gamma \leq v(a-b) s\gamma \vee v(c-b) s\gamma \leq v(a)s$.

Hence also $(a, c) \in q(\gamma)$.

1.6. For each $\gamma \in P$, $q(\gamma)$ is a semigroup congruence.

Proof. For $(a, b), (c, d) \in q(\gamma)$, perform the indicated operations, where $s, \bar{s} \in \Gamma^1$ are arbitrary:

(i) $\gamma s v(a-b) \leq s v(a)$, $s = \bar{s} v(c) \Rightarrow \gamma \bar{s} v(ca-cb) \leq \bar{s} v(ca)$;

(ii) $v(a-b) s\gamma \leq v(a)s \Rightarrow v(ca-cb) \bar{s}\gamma \leq v(ca)\bar{s}$.

Thus $(ca, cb) \in q(\gamma)$, and similarly

(i) $\gamma s v(c-d) \leq s v(c) \Rightarrow \gamma \bar{s} v(cb-db) \leq \bar{s} v(cb)$;

(ii) $v(c-d) s\gamma \leq v(c)s$, $s = v(b)\bar{s} \Rightarrow v(cb-db) \bar{s}\gamma \leq v(cb)\bar{s}$.

Hence $(cb, db) \in q(\gamma)$. Finally, by transitivity, also $(ca, db) \in q(\gamma)$.

1.7. For $\gamma \in P$, let $S_\gamma = A^*/q(\gamma)$. Let $q(\gamma) : A^* \to S_\gamma$ be the projection given by the equivalence relation $q(\gamma)$, where $aq(\gamma) = \{b \in A^* \mid (a, b) \in q(\gamma)\}$. For $(a, b) \in q(\gamma)$, $aq(\gamma) = bq(\gamma) \in S_\gamma$ and abbreviate this by $a \sim b \mod (\gamma)$. If $\alpha < \beta$ with $\alpha \in P$, then $q(\beta) \subseteq q(\alpha)$ and thus there is an induced map $\varphi_\alpha^\beta : S_\beta \to S_\alpha$, where $aq(\beta)\varphi_\alpha^\beta = aq(\alpha)$. Thus $\{\varphi_\alpha^\beta : S_\beta \to S_\alpha; \alpha < \beta; \alpha, \beta \in P\}$ is an inverse system of semigroups with inverse limit S and maps $S \to S_\alpha$ for each $\alpha \in P$ (for definition of S and $S \to S_\alpha$, see §2).

1.8. At this point an additional hypothesis on Γ and one on A have to be invoked.

(i) In addition to Γ being totally ordered and cancellative, assume that for any $\lambda_1, \lambda_2 \in \Gamma$ with $\lambda_1 \leq \lambda_2$, there is a $\gamma \in P$ such that $\lambda_2 < \gamma \lambda_1$.

(ii) In addition to A being a valuated integral domain, assume that for any $\gamma \in P$ and any $a, b \in A^*$, there exist $r, \bar{r} \in A^*$ such that $(ar, b\bar{r}) \in q(\gamma)$. Furthermore, assume that the cancellativity condition that for $c \in A^*$, if either $(ac, bc) \in q(\gamma)$ or $(ca, cb) \in q(\gamma)$, that then also $(a, b) \in q(\gamma)$.

1.9 REMARKS (1) Note that condition 1.8 (ii) implies that S_γ may be embedded in a group, i.e. that S_γ is cancellative on both sides and it satisfies the so-called right Ore condition that for any $\gamma \in P$ and any $x, y \in S_\gamma$, $xS_\gamma \cap yS_\gamma \neq \varnothing$.

(2) If Γ is commutative, then the cancellativity requirement in 1.8 (ii) is super-fluous, since then it follows immediately from 1.3 and the fact that Γ is cancellative.

(3) The whole theory could equally well be developed by imposing analogous conditions in both (i) and (ii) on the other (right) side.

1.10. If $(a, b) \in \cap \{q(\gamma) \mid \gamma \in P\}$ then in particular $\gamma v(a-b) \leq v(a)$ holds for all $\gamma \in P$. But for $a \neq b$, if $\lambda_1 = v(a-b) \neq 0$, $\lambda_2 = v(a)$ and γ is as in 1.8 (i), then $v(a) < \gamma v(a-b)$. Thus $\cap \{q(\gamma) \mid \gamma \in P\} = \{(a, a) \mid a \in A^*\}$, and the semigroup homo-morphisms $q(\beta): A^* \to S_\beta$ which commute with the φ_α^β induce an isomorphism $q: A^* \to S$. By cancellativity and the right Ore condition 1.8 (ii), each S_β may be embedded in its group of quotients G_β. That is, for $x, y \in S_\beta$, let

$$x/y = \{(s, t) \in S_\beta \times S_\beta \mid xr = s\bar{r}, \ yr = t\bar{r} \text{ for some } r, \bar{r} \in S_\beta\}$$

and define a group operation on the set $G_\beta = \{x/y \mid x, y \in S_\beta\}$ by $(x/y)(u/w) = x\rho/w\bar{\rho}$, where $y\rho = u\bar{\rho} \in yS_\beta \cap uS_\beta$. If $e \in S_\beta$ is arbitrary, then the embedding $S_\beta \cong \{se/e \mid s \in S_\beta\} \subset G_\beta$ is independent of e. Any homomorphism $\varphi_\alpha^\beta: S_\beta \to S_\alpha$ into any other cancellative semigroup S_α also with a right Ore condition, extends uniquely to a homomorphism $\pi_\alpha^\beta: G_\beta \to G_\alpha$ of their respective groups of quotients, where $(x/y)\pi_\alpha^\beta = x\varphi_\alpha^\beta/y\varphi_\alpha^\beta$. Then $\{\pi_\alpha^\beta: G_\beta \to G_\alpha; \ \alpha < \beta; \ \alpha, \beta \in P\}$ is an inverse system of groups with an inverse limit G and the usual maps $G \to G_\alpha$ for $\alpha \in P$. Furthermore, a monomorphism $S_\beta \to G_\beta$, $\beta \in P$, of inverse systems always induces also a semi-group monomorphism $S \to G$. Abbreviate the element $aq(\beta)/bq(\beta)$ of G_β simply as $a/b\bar{q}(\beta)$, and the equality $a/b\bar{q}(\beta) = c/d\bar{q}(\beta)$ by $a/b \equiv c/d \bmod (\beta)$. Thus equality in G_β will be denoted by "$\equiv$" and in S_β by "$\sim$". There are the usual commutative diagrams for $\alpha < \beta \in \Gamma$.
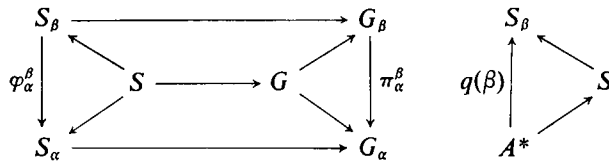
Note that in the next definition if Γ is commutative, $\delta \lhd \alpha$ means simply that $\delta < \alpha$.

1.11 DEFINITION. For δ, $\alpha \in P$ call δ much less than α and write $\delta \lhd \alpha$ provided the following two conditions hold for any λ_1, λ_2, $\mu \in \Gamma$. Any one of the three in-equalities on the left imply that all three on the right hold both in (i) and (ii).

(i) $\alpha\lambda_1\lambda_2 \leq \mu$, or $\lambda_1\alpha\lambda_2 \leq \mu$, or $\lambda_1\lambda_2\alpha \leq \mu \Rightarrow \delta\lambda_1\lambda_2 < \mu$, $\lambda_1\delta\lambda_2 < \mu$, $\lambda_1\lambda_2\delta < \mu$;

(ii) $\mu \leq \delta\lambda_1\lambda_2$, or $\mu \leq \lambda_1\delta\lambda_2$, or $\mu \leq \lambda_1\lambda_2\delta \Rightarrow \mu < \alpha\lambda_1\lambda_2$, $\mu < \lambda_1\alpha\lambda_2$, $\mu < \lambda_1\lambda_2\alpha$.

The main objective now is to introduce subtraction in G compatible with the embedding $A \subset G \cup \{0\}$.

1.12. Suppose δ, α, β are any three elements of P satisfying $\delta \lhd \alpha < \alpha^2 \leq \beta$.

Define $Y \subset G_\beta$ as the set of all $x \in G_\beta$ such that no matter which representatives a, $b \in A^*$ are chosen which give $x = a/b\bar{q}(\beta)$, they satisfy $v(a) \vee v(b) \leq \delta v(a \cdot - b)$. Define $\psi_\alpha^\beta \colon Y \to G_\alpha$ by $x\psi_\alpha^\beta = (b-a)/b\bar{q}(\alpha)$. It has to be shown that ψ_α^β is well defined. Suppose that x is also represented as $x = a_1/b_1\bar{q}(\beta)$. This implies that $ar \sim a_1\bar{r}$ and $br \sim b_1\bar{r}$ mod (β) for $r, \bar{r} \in A^*$. It suffices to show that $(b-a)r \sim (b_1-a_1)\bar{r}$ mod (α). But this is a consequence of the next lemma (with $k = br$, $m = ar$, $k_1 = b_1\bar{r}$, and $m_1 = a_1\bar{r}$).

1.13 LEMMA. *Consider* $\delta \lhd \alpha < \alpha^2 \leq \beta$, *where* $\delta \in P$ *and* k, k_1, m, $m_1 \in A^*$ *with* $k \sim k_1$ *and* $m \sim m_1$ mod (β), *and* $v(k) \vee v(m) \leq \delta v(k-m)$. *Then* $k - m \sim k_1 - m_1$ mod (α)

Proof. For any $s \in \Gamma^1$, straightforward estimates show that

(i)
$$\beta sv(k - m - k_1 + m_1) \leq \beta sv(k - k_1) \vee \beta sv(m - m_1)$$
$$\leq sv(k) \vee sv(m) = s[v(k) \vee v(m)]$$
$$\leq s\delta v(k - m) < \alpha sv(k - m).$$

Use of $\alpha^2 \leq \beta$ and cancellation of α gives 1.3 (i) while an entirely parallel argument establishes 1.3 (ii). Thus $k - m \sim k_1 - m_1$ mod (α).

1.14 REMARK. In the context of 1.12, define Z as the set $Y \subseteq Z \subset G_\beta$ consisting of all $x \in G_\beta$ such that there exist some representatives a, $b \in A^*$ with $x = a/b\bar{q}(\beta)$ such that $v(a) \vee v(b) \leq \delta v(a - b)$. Suppose also $x = a_1/b_1\bar{q}(\beta)$, where it is not assumed that a_1, b_1 satisfy the last inequality. As before let $ar \sim a_1\bar{r}$ and $br \sim b_1\bar{r}$ mod (β) for r, $\bar{r} \in A^*$. Then the above arguments actually show that $(b-a)r \sim (b_1-a_1)\bar{r}$ mod (α); hence $(b-a)/b \equiv (b_1-a_1)/b_1$ mod (α). Thus the domain of the map ψ_α^β could be enlarged to Z.

Some final restrictions in addition to 1.8 (i) and (ii) have to be imposed on Γ.

1.15. Assume that there is a cofinal subset P_1 in $P_1 \subseteq P$ and two order preserving maps $\theta \colon P_1 \to P$, and $\sigma \colon P_1 \to P$ satisfying:

(i) $\sigma(\beta) \lhd \theta(\beta) < \theta(\beta)^2 \leq \beta$ for all $\beta \in P_1$.

(ii) For any $\eta \in \Gamma$, there exist α_1, $\alpha_2 \in P_1$ with $\eta < \sigma(\alpha_1) \lhd \theta(\alpha_1) \leq \alpha_1 < \sigma(\alpha_2) \lhd \theta(\alpha_2) \leq \alpha_2$.

1.16 REMARKS. 1. Note that 1.15 (ii) implies that both $\theta(P_1)$ and $\sigma(P_1)$ are cofinal in P, that P is cofinal in Γ, and that σ and θ are order decreasing maps.

2. Actually 1.15 (ii) is equivalent to (ii') below, which at first glance appears to be a weaker assumption.

(ii') For any $\eta \in \Gamma$, there exists $\alpha \in P_1$ with $\eta < \sigma(\alpha)$.

3. In the examples of totally ordered semigroups Γ given at the end, $P \neq \varnothing$ and each $\alpha \in P$ satisfies $\alpha \lhd \alpha^2$. In this case there is a natural choice of P_1, θ, and σ satisfying the conditions in 1.15. Set $P_1 = \{\alpha^4 \mid \alpha \in P\}$, $\theta(\alpha^4) = \alpha^2$, and $\sigma(\alpha^4) = \theta(\theta(\alpha^4)) = \alpha$. Then 1.15 (i) and (ii) clearly hold.

The next definition combines 1.12 with 1.15. If $\eta > \delta \in P$, then $\eta \in P$. Thus if $P \neq \varnothing$, P is cofinal.

1.17 DEFINITION. For Γ as in 1.8 and 1.15 and for any $\beta \in P_1$, define $Y_\beta \subseteq G_\beta$ as all $x \in G_\beta$ such that no matter which representatives a, $b \in A^*$ of $x = a/b\bar{q}(\beta)$ are chosen, they satisfy the inequality $v(a) \vee v(b) \leq \sigma(\beta)v(a-b)$. Let $\psi^\beta \colon Y_\beta \to G_{\theta(\beta)}$ be the map ψ_α^β of 1.12 with $\delta = \sigma(\beta) \lhd \alpha = \theta(\beta) < \theta(\beta)^2 \leq \beta$.

1.18 LEMMA. *For any* $1 \neq x = \{x_\alpha\} \in G$, *there is an* $\eta = \eta(x) \in P$ *such that* $x_\beta \in Y_\beta$ *for all* $\beta > \eta$.

Proof. Let $1 \neq x_\mu \in G_\mu$. By 1.15 (ii), choose an element η such that $\delta = \sigma(\eta) \in P$ with $\mu \lhd \delta$. Suppose an arbitrary $\beta > \eta$ is given. Then $\sigma(\beta) > \sigma(\eta)$. Take any a, $b \in A^*$ with $a/b\bar{q}(\beta) = x_\beta$. Then $a/b\bar{q}(\mu) \neq 1$ implies $aq(\mu) \neq bq(\mu)$, and thus at least one of the two equations for the equality of a and b modulo $q(\mu)$ in S_μ is violated, i.e. for some $s \in \Gamma^1$ (s depending on a and b) either
  (i) $sv(a) < \mu sv(a-b) < s\delta v(a-b)$, or
  (ii) $v(a)s < v(a-b)s\mu < \delta v(a-b)s$.
Thus in either case (i) or (ii), it follows that $v(a) < \delta v(a-b)$. Suppose $\delta v(a-b) < v(b)$ so that $v(a-b) < v(b)$. Then $v(a) = v(b)$ by 1.2 (iv), which contradicts $v(a) < \delta v(a-b) < v(b)$. Thus $v(a) \vee v(b) \leq \delta v(a-b)$ and $x_\beta \in Y_\beta$ for all $\beta > \eta$.

2. Inverse limits and eventually commuting maps.

The maps ψ^β for $\beta \in P_1$ of the previous section will induce a function $\psi \colon G\backslash\{1\} \to G\backslash\{1\}$ satisfying the four requirements (see 3.1) for defining subtraction on $G \cup \{0\}$. In order not to obscure the fact that the method for constructing ψ is very generally applicable, and not at all dependent on the various specialized properties of Γ, A and the G_α, this section is developed for an inverse limit of arbitrary sets X_α in place of the G_α and any arbitrary partially ordered upper directed set P.

2.1 NOTATION. Consider any inverse system of sets $\{\pi_\alpha^\beta \colon X_\beta \to X_\alpha; \alpha < \beta; \alpha, \beta \in P\}$, where P is any upper directed partially ordered set, and where for $\alpha < \beta < \gamma$, the usual identities $\pi_\beta^\gamma \pi_\alpha^\beta = \pi_\alpha^\gamma$ and $\pi_\alpha^\alpha = 1$ are satisfied. Form the inverse limit

$$X = \lim X_\alpha \subset \prod \{X_\alpha \mid \alpha \in P\}, \qquad x = \{x_\alpha\} \in X$$

if and only if for any α, $\beta \in P$ with $\alpha < \beta$, $x_\alpha = x_\beta \pi_\alpha^\beta$.

2.2 DEFINITION. Suppose $P_1 \subseteq P$ is a subset, and $\theta \colon P_1 \to P$ a given function. Also suppose that for each $\beta \in P_1$, there is a subset $Y_\beta \subseteq X_\beta$ and a function $\psi^\beta \colon Y_\beta \to X_{\theta(\beta)}$. These $\{\psi^\beta \mid \beta \in P_1\}$ are said to *commute eventually* provided the next three conditions are satisfied.

  (i) For any $x = \{x_\alpha \mid \alpha \in P\} \in X$, there is an $\eta(x) \in P$ such that for any $\gamma \in P$ with $\gamma > \eta(x)$, $x_\gamma \in Y_\gamma$.

  (ii) $\theta \colon P_1 \to P$ is an order preserving and order decreasing map (i.e. $\alpha < \beta \Rightarrow \theta(\alpha) \leq \theta(\beta)$, and $\theta(\alpha) \leq \alpha$). For any $\eta \in \Gamma$, there exist α_1, $\alpha_2 \in P_1$ with $\eta < \theta(\alpha_1) \leq \alpha_1 < \theta(\alpha_2) \leq \alpha_2$.

  (Thus both P_1 and $\theta(P_1)$ are cofinal in P.)

(iii) For any $x \in X$ and any δ, $\lambda \in P_1$ with $\eta(x) < \delta < \lambda$ (and hence also $\theta(\delta) \leq \theta(\lambda)$), the identity holds

$$x_\lambda \pi_\delta^\lambda \psi^\delta = x_\lambda \psi^\lambda \pi_{\theta(\delta)}^{\theta(\lambda)}.$$

2.3 REMARK. The conditions in 2.2 say that the diagram

$$
\begin{array}{ccc}
X_\lambda & \xrightarrow{\;\psi\;} & X_{\theta(\lambda)} \\
\pi \downarrow & & \downarrow \pi \\
X_\delta & \xrightarrow{\;\psi\;} & X_{\theta(\delta)}
\end{array}
$$

eventually commutes in the sense that for a fixed $x = \{x_\alpha\} \in X$ and for all sufficiently large λ with $\delta < \lambda$, the above maps commute pointwise at x_λ.

2.4. Let $\alpha \in P$ and $x = \{x_\alpha\} \in X$ be arbitrary. Define $\psi: X \to X$ by $x\psi = \{x_\alpha\}\psi = \{x_\alpha'\}$, where x_α' is defined by selecting any β with $\beta > \eta(x)$ such that $\theta(\beta) > \alpha$, and then setting $x_\alpha' = x_\beta \psi^\beta \pi_\alpha^{\theta(\beta)}$. Suppose another element γ also satisfying $\gamma > \eta(x)$ with $\theta(\gamma) > \alpha$ had been chosen in place of β. Set $\tilde{x}_\alpha = x_\gamma \psi^\gamma \pi_\alpha^{\theta(\gamma)}$. In order to show that $x_\alpha' = \tilde{x}_\alpha$, pick any λ with $\lambda > \beta, \gamma$, use 2.2(iii) with $\delta = \beta, \gamma$, and then multiply the equation 2.2 (iii) respectively by $\pi_\alpha^{\theta(\beta)}$, $\pi_\alpha^{\theta(\gamma)}$ to get

$$\delta = \beta: x_\alpha' = (x_\lambda \pi_\beta^\lambda \psi^\beta)\pi_\alpha^{\theta(\beta)} = (x_\lambda \psi^\lambda \pi_{\theta(\beta)}^{\theta(\lambda)})\pi_\alpha^{\theta(\beta)},$$

$$\delta = \gamma: \tilde{x}_\alpha = (x_\lambda \pi_\gamma^\lambda \psi^\gamma)\pi_\alpha^{\theta(\gamma)} = (x_\lambda \psi^\lambda \pi_{\theta(\gamma)}^{\theta(\lambda)})\pi_\alpha^{\theta(\gamma)}.$$

Thus $x_\alpha' = \tilde{x}_\alpha$ and ψ is well defined. For later purposes it will be useful to observe that for any $\beta > \eta(x)$, $(x\psi)_{\theta(\beta)} = x_{\theta(\beta)}' = x_\beta \psi^\beta$.

Algebraic properties of eventually commuting maps, such as the idempotence of the map $a/b \to (b-a)/b$ encountered in the last section, carries over to their inverse limit.

2.5. Assume that the maps ψ^α are idempotent in the sense that

$$x_\beta \psi^\beta \pi_\gamma^{\theta(\beta)} \psi^\gamma = x_{\theta(\gamma)}$$

holds for any γ, β with $\eta(x) < \gamma < \theta(\beta) \leq \beta$ (and $\beta, \gamma \in P_1$). Then $\psi^2 = \psi$.

Proof. Define $x' = x\psi$, $x'' = x'\psi$. Now for any $\alpha \in P$ and any $\gamma, \beta \in P_1$ sufficiently large, i.e. $\eta(x) \vee \eta(x') \vee \alpha < \theta(\gamma) \leq \gamma < \theta(\beta) \leq \beta$, we have

$$x_\alpha'' = x_\gamma' \psi^\gamma \pi_\alpha^{\theta(\gamma)} \quad \text{and} \quad x_\gamma' = x_\beta \psi^\beta \pi_\gamma^{\theta(\beta)}.$$

Thus use of the hypothesis shows that $x_\alpha'' = x_{\theta(\gamma)} \pi_\alpha^{\theta(\gamma)} = x_\alpha$. Thus $(x\psi)\psi = x$.

2.6 REMARK. Since P_1 is cofinal in P, the reader may wish to take the direct limit with respect to P_1 in place of P in the above considerations.

3. **Additive structure on** $G \cup \{0\}$. The hypotheses and notation of §1 are resumed and these are now combined with the results of §2. That is, A is an integral

domain with a valuation $v: A\backslash\{0\} \to \Gamma$ into a totally ordered cancellative semi-group Γ such that 1.8 (i), (ii) and 1.15 (i), (ii) hold.

The following well-known lemma ([2, p. 514] or [5, p. 258]) will be used to introduce subtraction on $G \cup \{0\}$.

3.1 LEMMA. *Let G be a group and G_1 the subset $G_1 = \{x \in G \mid x \neq 1\}$. A necessary and sufficient condition for addition to be definable on $G \cup \{0\}$ so that $G \cup \{0\}$ becomes a division ring with the original group operation of G as multiplication is that there exists an element $e \in G$ and a function $\psi: G_1 \to G$ which for any $x, y \in G_1$ satisfy the following:*

I. $x\psi\psi = x$,

II. $x^{-1}\psi = e(x\psi)x^{-1}$,

III. $(yxy^{-1})\psi = y(x\psi)y^{-1}$,

IV. $(xy^{-1})\psi(y^{-1}\psi)^{-1} = [(x\psi)(y\psi)^{-1}]\psi$ for $x \neq y$.

If there do exist such an e and ψ, then $e = -1$, $-x = ex$, $x\psi = 1-x$, and $x - y = [(x\psi)(y\psi)^{-1}]\psi(y\psi) = (xy^{-1})\psi ey$.

3.2. By 2.5, the condition I in 3.1 holds. In II, even if $1 \notin A$, take any $0 \neq c \in A$, and then define $e = \{(-c)/c\bar{q}(\alpha) \mid \alpha \in P\} \in G$. The proof of II is omitted. Verification of any identity, such as II, is usually a routine computation with direct limits, provided the right Ore condition need not be used. However, III and IV do require the Ore condition. In subsequent computations, it will be sometimes convenient to write simply $a/b \in G_\beta$ for $a, b \in A^*$ in place of the longer expression $a/b\bar{q}(\beta)$.

3.3. In order to prove 3.1 III that $(yxy^{-1})\psi = y(x\psi)y^{-1}$, first, it is readily seen that it suffices to show that for $x, y \in G\backslash\{1\}$ and for all sufficiently large β in P_1 (for $\eta(y)$, $\eta(y^{-1}) < \beta$) with $x_\beta = a/b$, $y_\beta = c/d \in G_\beta$, the following holds

(1) $\qquad cd^{-1}(ab^{-1}\psi^\beta)dc^{-1} \equiv (cd^{-1}ab^{-1}dc^{-1})\psi^\beta \mod (\theta(\beta))$.

For any $c, k, n \in A^*$,

$$c(kn^{-1}\psi^\beta)c^{-1} \equiv c\frac{n-k}{n}c^{-1} \equiv \frac{cn-ck}{cn} \equiv \frac{ck}{cn}\psi^\beta \equiv (ckn^{-1}c^{-1})\psi^\beta \mod (\theta(\beta)).$$

That is, (1) holds rather trivially if $d = 1$ while (1) with $c = 1$ is not so immediate on account of the nonsymmetry in the Ore condition, i.e. $d^{-1}(k/n)d \neq (kd)/(nd)$. It suffices to prove equation (1) with $c = 1$, i.e. that

(2) $\qquad d^{-1}(ab^{-1}\psi^\beta)d \equiv (d^{-1}ab^{-1}d)\psi^\beta \mod (\theta(\beta))$.

Assuming (2), we complete the proof by showing (1), and then at the end prove (2). First, since $d^{-1}ab^{-1}d\bar{q}(\beta) \in G_\beta$, there are k and $n \in A^*$ such that $d^{-1}ab^{-1}d \equiv kn^{-1} \mod (\beta)$. It now follows from, first, (2) and then (1) that

$$c[d^{-1}(ab^{-1}\psi^\beta)d]c^{-1} \equiv c[(d^{-1}ab^{-1}d)\psi^\beta]c^{-1}$$

$$\equiv c[(kn^{-1})\psi^\beta]c^{-1} \equiv (ckn^{-1}c^{-1})\psi^\beta$$

$$\equiv (cd^{-1}ab^{-1}dc^{-1})\psi^\beta \mod (\theta(\beta)).$$

3.4. To prove (2) above it will be more convenient to show that

(3) $$d[(d^{-1}ab^{-1}d)\psi^\beta]d^{-1} \equiv (ab^{-1})\psi^\beta \bmod (\theta(\beta)).$$

Write $br \sim d\bar{r}$, $d\rho \sim ar\bar{\rho} \bmod (\beta)$ for $r, \bar{r}, \rho, \bar{\rho} \in A^*$. Then

$$d^{-1}ab^{-1}d \equiv \frac{1}{d}\frac{a}{b}d \equiv \frac{1}{d}\frac{ar}{br}\frac{d\bar{r}}{\bar{r}} \equiv \frac{\rho}{d\rho}\frac{ar\bar{\rho}}{\bar{r}\bar{\rho}} \equiv \frac{\rho}{\bar{r}\bar{\rho}} \bmod (\theta(\beta)).$$

Use of the latter shows that

$$d[(d^{-1}ab^{-1}d)\psi^\beta]d^{-1} \equiv d\frac{\bar{r}\bar{\rho}-\rho}{\bar{r}\bar{\rho}}\frac{1}{d} \equiv \frac{d\bar{r}\bar{\rho}-d\rho}{d\bar{r}\bar{\rho}} \bmod (\theta(\beta)).$$

Set $k = br\bar{\rho} \sim k_1 = d\bar{r}\bar{\rho}$, and $m = ar\bar{\rho} \sim m_1 = d\rho \bmod (\beta)$. From the definition of equality in G_β, it follows that $x_\beta = a/b\bar{q}(\beta) = m/k\bar{q}(\beta)$. The assumption that $\beta > \eta(x)$ and hence that $x_\beta \in Y_\beta$ means that for any representatives of x_β, and hence in particular for m and k, the inequality $v(k) \vee v(m) \leq \sigma(\beta)v(k-m)$ holds. Lemma 1.13 with $\alpha = \theta(\beta)$ shows that $k - m \sim k_1 - m_1 \bmod (\theta(\beta))$. Since also $k \sim k_1 \bmod (\beta)$, it follows that $(k-m)/k \equiv (k_1 - m_1)/k_1 \bmod (\theta(\beta))$, or equivalently that the following substitutions are legal in

$$\frac{d\bar{r}\bar{\rho}-d\rho}{d\bar{r}\bar{\rho}} \equiv \frac{br\bar{\rho}-ar\bar{\rho}}{br\bar{\rho}} \equiv \frac{b-a}{b} \bmod (\theta(\beta)).$$

Thus equation (3) above or equivalently conclusion 3.1 III that $(xyx^{-1})\psi = x(y\psi)x^{-1}$ now follows.

3.5. It only now remains to show that for any $x \neq y \in G$ and any $\alpha \in \Gamma$, the αth components of both sides of IV. $[(x\psi)(y\psi)^{-1}]\psi = (xy^{-1})\psi(y^{-1}\psi)^{-1}$ are equal. Set $z = (x\psi)(y\psi)^{-1}$. Then $z \neq 1$, for if $x\psi = y\psi$, then it follows from 2.5 that $x = x\psi^2 = y\psi^2 = y$. In order to guarantee that various maps ψ^β, ψ^γ can be applied, pick any γ, β sufficiently large, that is to be precise

$$\eta(x), \eta(y), \eta(xy^{-1}), \eta(y^{-1}), \eta((x\psi)(y\psi)^{-1}) < \alpha < \theta(\beta) \leq \beta < \theta(\gamma) \leq \gamma.$$

In order to show the equality of the αth component, it merely suffices to show that any higher components, such as $\theta(\beta)$, are equal. Then $(z\psi)_{\theta(\beta)} = z_\beta\psi^\beta$, where $z_\beta = (x_\gamma\psi^\gamma)(y_\gamma\psi^\gamma)^{-1}\pi_\beta^{\theta(\gamma)}$. Thus it suffices to show that

$$[(xy^{-1})\psi(y^{-1}\psi)^{-1}]_{\theta(\beta)} = (x_\gamma y_\lambda^{-1})\psi^\gamma(y_\gamma^{-1}\psi^\gamma)^{-1}\pi_{\theta(\beta)}^{\theta(\gamma)} = z_\beta\psi^\beta$$
$$= (x_\gamma\psi^\gamma)(y_\gamma\psi^\gamma)^{-1}\pi_\beta^{\theta(\gamma)}\psi^\beta.$$

Let $x = x_1/x_2$, $y = y_1/y_2 \in Y_\gamma$ with $x_1, x_2, y_1, y_2 \in A^*$. For any $\zeta \geq \gamma$, elements r, $\bar{r} \in A^*$ such that $x_2 r \sim y_2\bar{r} \bmod (\zeta)$ may be chosen and later a judicious choice of ζ will have to be made. Thus $(xy^{-1})_\gamma \equiv x_1 r/y_1\bar{r} \in Y_\gamma$ also. A straightforward computation shows that the left side of the above equation is

$$((xy^{-1})\psi^\beta)(y^{-1}\psi^\beta)^{-1} \equiv \frac{x_1 r - y_1\bar{r}}{(y_2 - y_1)\bar{r}} \bmod (\theta(\beta)),$$

while the right side becomes

$$(x_\gamma\psi^\gamma)(y_\gamma\psi^\gamma)^{-1}\pi_\beta^{\theta(\gamma)}\psi^\beta \equiv \frac{(x_1r - y_1\bar{r}) - (x_2r - y_2\bar{r})}{(y_2 - y_1)\bar{r}} \bmod (\theta(\beta)).$$

Since $x_2r \sim y_2\bar{r} \bmod (\gamma)$, on first glance the reader might simply cancel $x_2r - y_2\bar{r}$. However, a moments reflection will show that even $v(x_2r - y_2\bar{r})$ might very well dominate the value of any other term in the expression.

3.6. It has now to be shown that for any $s \in \Gamma^1$, the usual two inequalities 1.3 (i) and (ii) with $a = x_1r - y_1\bar{r}$ and $b = a - (x_2r - y_2\bar{r})$ hold:

   (i) $\theta(\beta)sv(x_2r - y_2\bar{r}) \leqq sv(x_1r - y_1\bar{r})$,

   (ii) $v(x_2r - y_2\bar{r})s\theta(\beta) \leqq v(x_1r - y_1\bar{r})s$.

By 1.8 (i) choose $\bar{\lambda} \in P$ such that $v(x_2) < \bar{\lambda}v(x_1)$. Secondly, by 1.15 (ii) choose $\lambda \in P$ such that $\bar{\lambda}\sigma(\gamma) \lhd \lambda$; and, thirdly, choose any ζ satisfying $\theta(\beta)\lambda \vee \lambda\theta(\beta) \leqq \zeta$. Then $x_2r \sim y_2\bar{r} \bmod (\zeta)$ means that

   (1) $v(x_2r - y_2\bar{r})s\theta(\beta)\lambda < v(x_2r)s < \bar{\lambda}v(x_1r)s$,

   (2) $\lambda\theta(\beta)sv(x_2r - y_2\bar{r}) < sv(x_2r) < s\bar{\lambda}v(x_1r)$.

Since $xy^{-1} \in Y_\gamma$ and $xy^{-1} \equiv x_1r/y_1\bar{r} \bmod (\gamma)$, we have

$$v(x_1r) \vee v(y_1\bar{r}) \leqq \sigma(\gamma)v(x_1r - y_1\bar{r}).$$

Use of the latter in (1) and (2) gives us that

   (3) $\bar{\lambda}v(x_1r)s \leqq [\bar{\lambda}\sigma(\gamma)]v(x_1r - y_1\bar{r})s < v(x_1r - y_1\bar{r})s\lambda$,

   (4) $s\bar{\lambda}v(x_1r) \leqq s[\bar{\lambda}\sigma(\gamma)]v(x_1r - y_1\bar{r}) < \lambda sv(x_1r - y_1\bar{r})$.

Substitution of (3) and (4) in (1) and (2) and cancellation of λ gives (i) and (ii) above. Thus 3.1, I–IV hold.

The results of the previous sections are collected and summarized in the next theorem and the first corollary which simplifies condition (c).

3.7 THEOREM. *Consider an integral domain A with a valuation v: $A\backslash\{0\} = A^* \to \Gamma$ (see 1.1) into a cancellative totally ordered semigroup Γ. (It is not assumed that either A or Γ is commutative or that $1 \in A$.) Let $P \subseteq \Gamma$ be the subsemigroup of strict two-sided increasers. Suppose the following hold.*

   (a) *For any $\lambda_1, \lambda_2 \in \Gamma$ there is a $\gamma \in P$ such that $\lambda_2 < \gamma\lambda_1$.*

   (b) *For $\gamma \in P$, the semigroups $S_\gamma = A^*/q(\gamma)$ (see 1.7) satisfies the right Ore condition that $xS_\gamma \cap yS_\gamma \neq \varnothing$ for each $x, y \in S_\gamma$ and also that S_γ is cancellative on both sides.*

   If (a) and (b) hold then A^* can be embedded in a group G. Now in addition to (a) and (b) assume that

   (c) (i) *there is a subset $P_1 \subseteq P$ and functions θ: $P_1 \to P$, σ: $P_1 \to P$ such that*

$$\sigma(\beta) \lhd \theta(\beta) < \theta(\beta)^2 \leqq \beta$$

for all $\beta \in P_1$ (for the definition of "$\lhd$" see 1.11).

   (ii) *For any $\eta \in \Gamma$, there exists an $\alpha \in P$ with $\eta < \sigma(\alpha)$.*

   Then A can be embedded in the division ring $A \subset G \cup \{0\}$.

3.8 COROLLARY I. *Assume* (a), (b), *and* (c'):

(c') $P \neq \varnothing$ *and each* $\alpha \in P$ *satisfies* $\alpha \lhd \alpha^2$ (*see* 1.11). *Then* $P_1 = \{\alpha^4 \mid \alpha \in P\}$, $\theta(\alpha^4) = \alpha^2$, $\sigma(\alpha^4) = \alpha$ *satisfy* (c). *Consequently* $A \subset G \cup \{0\}$ *is an embedding of* A *in a division ring.*

3.9 COROLLARY II. *Under the assumptions* (a), (b), *and* (c) *of the theorem, suppose that* Γ *also satisfies the right Ore condition that* $\alpha\Gamma \cap \beta\Gamma \neq \varnothing$ *for any* α, $\beta \in \Gamma$. *Then* v *has a unique, natural extension to a valuation* $\bar{v} \colon G \cup \{0\} \to \bar{\Gamma}$, *where* $\bar{\Gamma}$ *is the group of quotients of* Γ.

Proof. If α/β, $\gamma/\delta \in \bar{\Gamma}$ with α, β, γ, and $\delta \in \Gamma$, then find ρ, $\bar{\rho} \in \Gamma$ with $\beta\rho = \delta\bar{\rho} \in \beta\Gamma \cap \delta\Gamma$. Then the definition

$$\alpha/\beta < \gamma/\delta \text{ if and only if } \alpha\rho < \gamma\bar{\rho} \text{ in } \Gamma$$

makes $\bar{\Gamma}$ into a totally ordered group. For $x = \{x_\alpha\} \in G$, pick any $\beta \in \Gamma$, any x_1, $x_2 \in A^*$ with $x_\beta = x_1/x_2\bar{q}(\beta)$. Then the definition $\bar{v}(x) = v(x_1)/v(x_2)$ is independent of both β and x_1, x_2. In order to verify 1.1 (i)–(iii), take y_1, $y_2 \in A^*$ with $y_\beta = y_1/y_2\bar{q}(\beta)$. If c, $\bar{c} \in A^*$ with $x_2c \sim y_1\bar{c}$ mod (β), then

(iii)
$$\bar{v}(xy) = \bar{v}(x_1c/y_2\bar{c}) = v(x_1)v(c)/v(y_2)v(\bar{c})$$
$$= [v(x_1)/v(x_2)][v(y_1)/v(y_2)] = \bar{v}(x)\bar{v}(y).$$

By 3.1, $\bar{v}(x - y) = \bar{v}[(xy^{-1})\psi(-y)]$. Take $\beta > \eta(xy^{-1})$. There exist r, $\bar{r} \in A^*$ with $x_2r \sim y_2\bar{r}$ mod (β) so that $(xy^{-1})_\beta \equiv x_1r/y_1\bar{r}$ mod (β) and

$$\bar{v}[(xy^{-1})\psi(-y)] = \bar{v}[(xy^{-1})\psi]\bar{v}(y) = v(y_1\bar{r} - x_1r)/v(y_2\bar{r}).$$

If $v(y_1\bar{r} - x_1r) \leq v(y_1\bar{r})$, then $\bar{v}(x - y) \leq \bar{v}(y)$. If $v(y_1\bar{r} - x_1r) \leq v(x_1r)$, then since $v(y_2\bar{r}) = v(x_2r)$, it follows that $\bar{v}(x - y) \leq \bar{v}(x)$. Thus $\bar{v}$ satisfies 1.1 (i)–(iii).

The next definition and corollary give the main source of examples of rings A which can be embedded in division rings.

3.10 DEFINITION. Consider a totally ordered cancellative semigroup Γ and any commutative field R (not necessarily of characteristic zero). If $b \colon \Gamma \to R$ is any function, then the *support of* b is the set supp $(b) = \{s \in \Gamma \mid b(s) \neq 0\}$. The additive group of all functions b such that supp (b) satisfies the ascending chain condition will be denoted by $V = V(\Gamma, R)$. In particular, let $v(b)$ denote the maximal element of supp (b). If $a, b \in V$ then $ab \in V$ is defined at $s \in \Gamma$ by

$$(ab)(s) = \sum a(t)b(u)$$
$$(t, u), \ s = tu.$$

It is known that the above sum for each s is finite and also that supp (ab) satisfies the ascending chain condition. The subset $P(\Gamma, R)$ consisting of all $b \in V$ with supp (b) finite is just the ordinary semigroup ring over Γ. Then v is a valuation in the sense of 1.1 for both V and $P(\Gamma, R)$.

Note that any commutative totally ordered group satisfies (i)–(iii) in the next corollary.

3.11 COROLLARY III. *With* $V = V(\Gamma, R)$ *as above and for* $P \subseteq \Gamma$ *the subsemigroup of strict increasers, assume the following.*

(i) *For any* $\lambda_1, \lambda_2 \in \Gamma$, *there is a* $\gamma \in P$ *with* $\lambda_2 < \gamma \lambda_1$.

(ii) *For each* $\gamma \in P$, $\gamma \lhd \gamma^2$ *(see 1.11).*

(iii) *For any* $\gamma, \lambda \in \Gamma$, *there exists* $t \in \Gamma$ *such that* $\gamma st \leqq s\lambda$ *and* $ts\gamma \leqq \lambda s$ *holds for all* $s \in \Gamma^1$.

(iv) *Suppose* $A \subseteq V(\Gamma, R)$ *is a subring having the property that for arbitrary* $a, b \in A^*$ *there exist* $a_1, b_1 \in A^*$ *with* $v(a_1), v(b_1) \in P$ *such that* $(aa_1)(s) = (bb_1)(s)$ *holds for all* $s \geqq t$ *for any given* $t \in \Gamma$.

(v) *If* $\alpha, \gamma, \beta, \bar{\alpha}, \bar{\beta} \in \Gamma^1$ *satisfy* $\alpha\gamma\beta \leqq \bar{\alpha}\gamma\bar{\beta}$, *then* $\alpha\beta \leqq \bar{\alpha}\bar{\beta}$.

Then A *can be embedded in a division ring.*

Proof. Given $a, b \in A^*$ and $\gamma \in P$, it has to be shown that in our previous notation $(aa_1, bb_1) \in q(\gamma)$ for some $a_1, b_1 \in A^*$. Take $\lambda = v(a)$ and let t be as given by (iii). Then $aa_1 \sim bb_1 \bmod (\gamma)$, since for any $s \in \Gamma^1$, it follows from (iv) that

(i) $\gamma sv(aa_1 - bb_1) \leqq \gamma st \leqq sv(a) < sv(aa_1)$,

(ii) $v(aa_1 - bb_1)s\gamma \leqq ts\gamma \leqq v(a)s < v(aa_1)s$.

Thus 3.8(a), (b), (c′) hold and A can be embedded in a division ring.

3.12. Suppose A is a ring with a valuation into the integers, i.e. $v: A \to \Gamma = Z$. Define $A_n = \{a \in A \mid v(a) = n\}$ and B_n as the additive quotient group $B_n = A_n - A_{n-1}$. Form the associated graded algebra $G(A) = \bigoplus \{B_n \mid n = 0, \pm 1, \ldots\}$ (see [5, p. 257]). Then Cohn has shown [2, p. 523, Theorem 4.2] that in this particular case the assumption 3.7(b), i.e. that for each $a, b \in A^*$, the function $f(x, y) = v(ax - by) - v(ax)$, with $x, y \in A^*$, is not bounded below is equivalent to requiring that the ring $G(A)$ satisfies the right Ore condition. If in the last corollary $\Gamma = Z$, then A is a subring of the ordinary power series ring $V(\Gamma, R)$, while $G(A) = P(\Gamma, R)$, and condition 3.11 (iv) specializes to requiring that $P(\Gamma, R)$ satisfies the Ore condition.

In conclusion, some totally ordered semigroups are described which could be used for a valuation and which are also useful in illustrating the relation "$\lhd$" (see 1.11). Various subsemigroups of wreath products of totally ordered groups are too well known and will not be mentioned.

3.13 EXAMPLE. Let $\cdots < x_2^{-1} < x_1^{-1} < e = 1 < x_1 < x_2 < \cdots$ be any finite or infinite totally ordered set indexed by ordinals. Let Γ be the group generated by this set, i.e. all expressions $z = z_1 z_2 \cdots z_m$, where m is an integer and where each $z_i = x_j$ or x_j^{-1} and $x_j x_j^{-1} = x_j^{-1} x_j = e$. If $p(z)$ denotes the total number of z_i with positive and $n(z)$ the number with negative exponents, set $\partial(z) = p(z) - n(z)$. Note that $\partial(e) = p(e) = n(e) = 0$ and $\partial(x_j^k) = k$ for any positive or negative integer k. If $u = u_1 u_2 \cdots u_r \in \Gamma$ is another element (where r is an integer and $u_i = x_j$ or x_j^{-1}), define $z < u$ if $\partial z < \partial u$; or $\partial z = \partial u$, but $z_t < u_t$ for the first t with $z_t \neq u_t$. Then Γ is a totally ordered group, $P = \{z \in \Gamma \mid 1 \leqq \partial z\}$, and $z \lhd u$ if and only if $\partial z < \partial u$.

3.14 EXAMPLE. Consider again a totally ordered set $1 = e < x_1 < x_2 < \cdots$ indexed by ordinals. Let Γ consist of e and the set of all finite or infinite formal expressions z of the form $z = z_1 z_2 \cdots z_\alpha$, where each z_i equals some x_j, and where the z_i are indexed by ordinals in increasing order, and where α is the biggest index. Then Γ is a semigroup with juxtaposition as multiplication. If $u = u_1 u_2 \cdots u_\beta \in \Gamma$, where u_i equals some x_j, then $zu = z_1 z_2 \cdots z_\alpha t_{\alpha+1} \cdots t_{\alpha+\beta}$ where $t_{\alpha+1} = u_1$ and $t_{\alpha+\gamma} = u_\gamma$. The degree ∂z of z is $\partial z = \alpha$. Define $z < u$ if either $\partial z < \partial u$, or if $\partial z = \partial u$, but $z_t < u_t$ for the first ordinal t such that $z_t \neq u_t$. Then Γ is a totally ordered cancellative semigroup where $z \triangleleft u$ if and only if $\partial z < \partial u$.

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