EMBEDDINGS OF REDUCED FREE PRODUCTS OF OPERATOR ALGEBRAS

Etienne F. Blanchard and Kenneth J. Dykema

EMBEDDINGS OF REDUCED FREE PRODUCTS OF OPERATOR ALGEBRAS

Etienne F. Blanchard and Kenneth J. Dykema

Given reduced amalgamated free products of C*-algebras $(A, \phi) = \underset{\iota \in I}{*}(A_\iota, \phi_\iota)$ and $(D, \psi) = \underset{\iota \in I}{*}(D_\iota, \psi_\iota)$, an embedding $A \hookrightarrow D$ is shown to exist assuming there are conditional–expectation–preserving embeddings $A_\iota \hookrightarrow D_\iota$. This result is extended to show the existence of the reduced amalgamated free product of certain classes of unital completely positive maps. Finally, analogues of the above mentioned results are proved for amagamated free products of von Neumann algebras.

Introduction.

The reduced free product construction, and more generally the reduced amalgamated free product construction for C*-algebras, introduced independently by Voiculescu [**21**] and (somewhat less generally) Avitzour [**1**], has received much recent attention.

It is natural to ask: To what extent does the reduced free product construction satisfy a universal property, analogous to those for the free product of groups or the full free product of C*-algebras? Since the reduced free product of C*-algebras frequently gives rise to simple C*-algebras, (see [**1**], [**15**], [**10**], [**11**] and [**8**]), it is clear that any universal property for the reduced free product should be quite a bit more restrictive in character than for the full free product; however, at first glance the following question still seems reasonable.

Question 1. If

$$(A, \phi) = \underset{\iota \in I}{*}(A_\iota, \phi_\iota)$$

is a reduced free product of C*-algebras, where the ϕ_ι are states on the unital C*-algebras A_ι having faithful GNS representations, and if D is a unital C*-algebra with a state ψ and with unital $*$-homomorphisms $\kappa_\iota : A_\iota \to D$ such that

  (i) $\psi \circ \kappa_\iota = \phi_\iota$ for every $\iota \in I$,
  (ii) the family $\big(\kappa_\iota(A_\iota)\big)_{\iota \in I}$ is free with respect to ψ,

does it follow that there is a ∗-homomorphism $\kappa : A \to D$ such that, denoting by $\alpha_\iota : A_\iota \to A$ the injective ∗-homomorphisms arising from the free product construction, $\kappa \circ \alpha_\iota = \kappa_\iota$ for every $\iota \in I$? (Note that κ would necessarily be injective.)

Note that the homomorphism κ exists if and only if the GNS representation $\pi_\psi : D \to \mathcal{L}(L^2(D, \psi))$ of ψ is faithful when restricted to the subalgebra of D generated by $\bigcup_{\iota \in I} \kappa_\iota(A_\iota)$. As observed in [**16**, 1.3], the answer to Question 1 is "yes" if the state ψ on D is assumed to be faithful, (and a similar result holds in the amalgamated case). However, in general the answer is "no", as was shown by the elementary example [**16**, 1.4], (see also the erratum to [**16**]).

The main result of this paper is an embedding result (Theorem 1.3) implying that the ∗-homomorphism π in Question 1 does exist provided that the free subalgebras $\pi_\iota(A_\iota)$ lie in free subalgebras of D that taken together generate D. Namely, we have the following property.

Property 2. Let I be a set and for every $\iota \in I$ let $A_\iota \subseteq D_\iota$ be a unital inclusion of unital C∗-algebras. Suppose ϕ_ι is a state on D_ι such that the GNS representations of ϕ_ι and of the restriction $\phi_\iota\!\restriction_{A_\iota}$ are faithful. Consider the reduced free products of C∗-algebras,

$$(D, \phi) = \underset{\iota \in I}{*}(D_\iota, \phi_\iota)$$
$$(A, \psi) = \underset{\iota \in I}{*}(A_\iota, \phi_\iota\!\restriction_{A_\iota}).$$

Then there is a ∗-homomorphism $\pi : A \to D$ such that for every $\iota \in I$ the diagram

$$\begin{array}{ccc} D_\iota & \hookrightarrow & D \\ \cup & & \uparrow \pi \\ A_\iota & \hookrightarrow & A \end{array}$$

commutes, where the horizontal arrows are the inclusions arising from the free product construction.

This property was previously known under the additional assumption that every ψ_ι is faithful, which by [**9**] implies that ψ is faithful on D; as noted after Question 1, this in turn implies the existence of π. Theorem 1.3 actually proves more generally a version of Property 2 for reduced amalgamated free products of C∗-algebras. Such an embedding result is frequently useful for understanding reduced free product C∗-algebras; it has been used in [**13**] and several times in [**12**].

We should point out that M. Choda has in [**7**] stated a theorem about reduced free products of completely positive maps which is more general than Property 2. However, her proof is incomplete, as it implicitly uses the full generality of Property 2 without justifying its validity.

In §1, the main theorem about embeddings of reduced amalgamated free products of C*-algebras is proved. In §2, Choda's argument proving the existence of reduced free products of state–preserving completely positive maps is generalized to prove existence of reduced amalgamated free products of certain sorts of completely positive maps. In §3, we consider the reduced free product with amalgamation of von Neumann algebras and prove analogues of the results in §1 and §2 for von Neumann algebras.

1. Embeddings.

In this section we prove the main embedding result. We use the same notation as in [**12**] for the reduced amalgamated free product construction.

In the following lemma, with the reduced amalgamated free product of C*-algebras $(A, \phi) = \underset{\iota \in I}{*}(A_\iota, \phi_\iota)$ we view each A_ι as a C*-subalgebra of A via the canonical embedding arising from the free product construction.

Lemma 1.1. *Let B be a unital C*-algebra, let I be a set and for every $\iota \in I$ let A_ι be a unital C*-algebra containing a copy of B as a unital C*-subalgebra and having a conditional expectation $\phi_\iota : A_\iota \to B$ whose GNS representation is faithful. Let*

$$(A, \phi) = \underset{\iota \in I}{*}(A_\iota, \phi_\iota)$$

be the reduced amalgamated free product. Then for every $\iota_0 \in I$ there is a conditional expectation $\Phi_{\iota_0} : A \to A_{\iota_0}$ such that $\Phi_{\iota_0}\restriction_{A_\iota} = \phi_\iota$ for every $\iota \in I\backslash\{\iota_0\}$ and $\Phi_{\iota_0}(a_1 a_2 \cdots a_n) = 0$ whenever $n \geq 2$ and $a_j \in A_{\iota_j} \cap \ker\phi_{\iota_j}$ with $\iota_1 \neq \iota_2, \dots, \iota_{n-1} \neq \iota_n$.

Proof. We let $E_\iota = L^2(A_\iota, \phi_\iota)$, $\xi_\iota = \widehat{1_{A_\iota}} \in E_\iota$, $E_\iota = \xi_\iota B \oplus E_\iota^{\mathrm{o}}$. Then A acts (by definition) on the Hilbert B-module

$$E = \xi B \oplus \bigoplus_{\substack{n \geq 1 \\ \iota_1, \dots, \iota_n \in I \\ \iota_1 \neq \iota_2, \iota_2 \neq \iota_3, \dots, \iota_{n-1} \neq \iota_n}} E_{\iota_1}^{\mathrm{o}} \otimes_B E_{\iota_2}^{\mathrm{o}} \otimes_B \cdots \otimes_B E_{\iota_n}^{\mathrm{o}}.$$

Identify the submodule $\xi B \oplus E_{\iota_0}^{\mathrm{o}}$ of E with the Hilbert B-module E_{ι_0} and let $Q_{\iota_0} : E \to E_{\iota_0}$ be the projection. Then $\Phi_{\iota_0}(x) = Q_{\iota_0} x Q_{\iota_0}$ has the desired properties. $\square$

Explication 1.2. Consider the GNS representation $\left(\sigma, L^2(A, \Phi_{\iota_0}), \eta\right) = \mathrm{GNS}(A, \Phi_{\iota_0})$ associated with the conditional expectation $\Phi_{\iota_0} : A \to A_{\iota_0}$

found in Lemma 1.1. Since A is the closed linear span of B and the set of reduced words of the form $a_1 a_2 \cdots a_n$ where $a_j \in A_{\iota_j} \cap \ker \phi_{\iota_j}$ and $\iota_j \neq \iota_{j+1}$, we see that the Hilbert A_{ι_0}-module in the GNS representation is

$$
(1) \qquad L^2(A, \Phi_{\iota_0}) = A_{\iota_0} \oplus \bigoplus_{\substack{n \geq 1 \\ \iota_1, \ldots, \iota_n \in I \\ \iota_1 \neq \iota_2, \ldots, \iota_{n-1} \neq \iota_n \\ \iota_n \neq \iota_0}} E^{\mathrm{o}}_{\iota_1} \otimes_B \cdots \otimes_B E^{\mathrm{o}}_{\iota_n} \otimes_B A_{\iota_0}.
$$

Moreover, the action σ of A on $L^2(A, \Phi_{\iota_0})$ is determined by its restrictions $\sigma\restriction_{A_\iota}$, which are easily described.

Let $\rho : A_{\iota_0} \to \mathcal{L}(\mathcal{V})$ be a unital $*$-homomorphism, for some Hilbert space $\mathcal{V}$. Then $\sigma \otimes 1 : A \to \mathcal{L}\big(L^2(A, \Phi_{\iota_0}) \otimes_\rho \mathcal{V}\big)$ is a $*$-homomorphism; it is the representation induced, in the sense of Rieffel [19], from ρ up to A, with respect to the conditional expectation Φ_{ι_0}, and we will denote this induced representation by $\rho\restriction^A$. We have the following explicit description of $\rho\restriction^A$, obtained by tensoring (1) with $\otimes_\rho \mathcal{V}$ on the right. Writing $\mathcal{H} = L^2(A, \Phi_{\iota_0}) \otimes_\rho \mathcal{V}$ we have

$$
(2) \qquad \mathcal{H} = \mathcal{V} \oplus \bigoplus_{\substack{n \geq 1 \\ \iota_1, \ldots, \iota_n \in I \\ \iota_1 \neq \iota_2, \ldots, \iota_{n-1} \neq \iota_n \\ \iota_n \neq \iota_0}} E^{\mathrm{o}}_{\iota_1} \otimes_B \cdots \otimes_B E^{\mathrm{o}}_{\iota_n} \otimes_\rho \mathcal{V}.
$$

Moreover, the $*$-homomorphism $\sigma \otimes 1$ is determined by its restrictions

$$
\sigma_\iota \overset{\mathrm{def}}{=} (\sigma \otimes 1)\restriction_{A_\iota} : A_\iota \to \mathcal{L}(\mathcal{H}),
$$

given as follows. Consider the Hilbert spaces

$$
\mathcal{H}(\iota) = \begin{cases} (\eta_\iota B \otimes_{\rho\restriction_B} \mathcal{V}) \oplus \displaystyle\bigoplus_{\substack{n \geq 1 \\ \iota_1, \ldots, \iota_n \in I \\ \iota_1 \neq \iota_2, \ldots, \iota_{n-1} \neq \iota_n \\ \iota_n \neq \iota_0, \iota_1 \neq \iota}} E^{\mathrm{o}}_{\iota_1} \otimes_B \cdots \otimes_B E^{\mathrm{o}}_{\iota_n} \otimes_\rho \mathcal{V} & \text{if } \iota \neq \iota_0 \\[2em] \displaystyle\bigoplus_{\substack{n \geq 1 \\ \iota_1, \ldots, \iota_n \in I \\ \iota_1 \neq \iota_2, \ldots, \iota_{n-1} \neq \iota_n \\ \iota_n \neq \iota_0, \iota_1 \neq \iota_0}} E^{\mathrm{o}}_{\iota_1} \otimes_B \cdots \otimes_B E^{\mathrm{o}}_{\iota_n} \otimes_\rho \mathcal{V} & \text{if } \iota = \iota_0, \end{cases}
$$

where $\eta_\iota B$ is just the Hilbert B-module B with identity element denoted by η_ι. If $\iota \in I \backslash \{\iota_0\}$ let

$$
(3) \qquad\qquad\qquad W_\iota : E_\iota \otimes_B \mathcal{H}(\iota) \to \mathcal{H}
$$

be the unitary defined, using the symbol $\ddot{\otimes}$ to denote the tensor product in (3), by

$$W_\iota : \xi_\iota \ddot{\otimes} (\eta_\iota \otimes v) \mapsto v$$
$$\zeta \ddot{\otimes} (\eta_\iota \otimes v) \mapsto \zeta \otimes v$$
$$\xi_\iota \ddot{\otimes} (\zeta_1 \otimes \cdots \otimes \zeta_n \otimes v) \mapsto \zeta_1 \otimes \cdots \otimes \zeta_n \otimes v$$
$$\zeta \ddot{\otimes} (\zeta_1 \otimes \cdots \otimes \zeta_n \otimes v) \mapsto \zeta \otimes \zeta_1 \otimes \cdots \otimes \zeta_n \otimes v$$

whenever $v \in \mathcal{V}$, $\zeta \in E_\iota^{\mathrm{o}}$, $\zeta_j \in E_{\iota_j}^{\mathrm{o}}$ and $\iota \neq \iota_1$, $\iota_1 \neq \iota_2$, $\ldots$, $\iota_{n-1} \neq \iota_n$, $\iota_n \neq \iota_0$. Then for every $\iota \in I \backslash \{\iota_0\}$ and $a \in A_\iota$, we have

$$\sigma_\iota(a) = W_\iota(a \otimes 1_{\mathcal{H}(\iota)})W_\iota^*.$$

Similarly, define the unitary

$$W_{\iota_0} : \mathcal{V} \oplus \big(E_\iota \otimes_B \mathcal{H}(\iota_0)\big) \to \mathcal{H}$$

by

$$W_{\iota_0} : v \oplus 0 \mapsto v$$
$$0 \oplus \big(\xi_{\iota_0} \ddot{\otimes} (\zeta_1 \otimes \cdots \otimes \zeta_n \otimes v)\big) \mapsto \zeta_1 \otimes \cdots \otimes \zeta_n \otimes v$$
$$0 \oplus \big(\zeta \ddot{\otimes} (\zeta_1 \otimes \cdots \otimes \zeta_n \otimes v)\big) \mapsto \zeta \otimes \zeta_1 \otimes \cdots \otimes \zeta_n \otimes v.$$

Then

$$\sigma_{\iota_0}(a) = W_{\iota_0}\big(\rho(a) \oplus (a \otimes 1_{\mathcal{H}(\iota_0)})\big)W_{\iota_0}^*.$$

Note that the above description is related to the construction of the conditionally free product, due to Bożejko and Speicher [5], (see also [4]).

Theorem 1.3. *Let $B \subseteq \widetilde{B}$ be a (not necessarily unital) inclusion of unital C^*-algebras. Let I be a set and for each $\iota \in I$ suppose*

$$\begin{array}{ccccc} 1_{\widetilde{A}_\iota} \in & \widetilde{B} & \subseteq & \widetilde{A}_\iota \\ & \cup & & \cup \\ 1_{A_\iota} \in & B & \subseteq & A_\iota \end{array}$$

are inclusions of C^*-algebras. Suppose that $\tilde{\phi}_\iota : \widetilde{A}_\iota \to \widetilde{B}$ is a conditional expectation such that $\tilde{\phi}_\iota(A_\iota) \subseteq B$ and assume that $\tilde{\phi}_\iota$ and the restriction $\tilde{\phi}_\iota \restriction_{A_\iota}$ have faithful GNS representations, for all $\iota \in I$. Let

$$(\widetilde{A}, \tilde{\phi}) = \underset{\iota \in I}{*}(\widetilde{A}_\iota, \tilde{\phi}_\iota)$$
$$(A, \phi) = \underset{\iota \in I}{*}(A_\iota, \tilde{\phi}_\iota \restriction_{A_\iota})$$

be the reduced amalgamated free products of C^*-algebras. Then there is a unique $*$-homomorphism $\kappa : A \to \widetilde{A}$ such that for every $\iota \in I$ the diagram

(4)
$$\begin{array}{ccc} \widetilde{A}_\iota & \hookrightarrow & \widetilde{A} \\ \cup & & \uparrow \kappa \\ A_\iota & \hookrightarrow & A \end{array}$$

commutes, where the horizontal arrows are the inclusions arising from the free product construction. Moreover, κ is necessarily injective.

Proof. Since A is generated by $\bigcup_{\iota \in I} A_\iota$, it is clear that κ will be unique if it exists. Let 1 denote the identity element of $\widetilde{B}$ and let p be the identity element of B. If $p \neq 1$ then we may replace B by $B + \mathbf{C}(1-p)$ and each A_ι by $A_\iota + \mathbf{C}(1-p)$; hence we may without loss of generality assume that B is a unital C*-subalgebra of $\widetilde{B}$ and thus each A_ι is a unital C*-subalgebra of $\widetilde{A}_\iota$. Let

$$(\widetilde{\pi}_\iota, \widetilde{E}_\iota, \widetilde{\xi}_\iota) = \mathrm{GNS}(\widetilde{A}_\iota, \widetilde{\phi}_\iota),$$
$$(\pi_\iota, E_\iota, \xi_\iota) = \mathrm{GNS}(A_\iota, \phi_\iota)$$

and

$$(\widetilde{E}, \widetilde{\xi}) = \underset{\iota \in I}{*}(\widetilde{E}_\iota, \widetilde{\xi}_\iota),$$
$$(E, \xi) = \underset{\iota \in I}{*}(E_\iota, \xi_\iota).$$

The inclusion $A_\iota \hookrightarrow \widetilde{A}_\iota$ gives an inner–product–preserving isometry of Banach spaces $E_\iota \hookrightarrow \widetilde{E}_\iota$ sending ξ_ι to $\widetilde{\xi}_\iota$, and we identify E_ι with this subspace of $\widetilde{E}_\iota$ and thereby E_ι^o with the subspace of $\widetilde{E}_\iota^\mathrm{o}$. This allows canonical identification of the tensor product module

$$E_{\iota_1}^\mathrm{o} \otimes_B \cdots \otimes_B E_{\iota_{p-1}}^\mathrm{o} \otimes_B \widetilde{E}_{\iota_p}^\mathrm{o} \otimes_{\widetilde{B}} \cdots \otimes_{\widetilde{B}} \widetilde{E}_{\iota_n}^\mathrm{o}$$

with a closed subspace of $\widetilde{E}_{\iota_1}^\mathrm{o} \otimes_{\widetilde{B}} \cdots \otimes_{\widetilde{B}} \widetilde{E}_{\iota_n}^\mathrm{o}$. Hence, we may and do identify E with the subspace

$$\widetilde{\xi}B \oplus \bigoplus_{\substack{n \geq 1 \\ \iota_1,\ldots,\iota_n \in I \\ \iota_1 \neq \iota_2,\ldots,\iota_{n-1} \neq \iota_n}} E_{\iota_1}^\mathrm{o} \otimes_B \cdots \otimes_B E_{\iota_n}^\mathrm{o}$$

of $\widetilde{E}$. Let $\mathfrak{A} = (*_B)_{\iota \in I} A_\iota$ be the universal algebraic free product with amalgamation over B. Let $\sigma : \mathfrak{A} \to \mathcal{L}(E)$, respectively $\widetilde{\sigma} : \mathfrak{A} \to \mathcal{L}(\widetilde{E})$, be the homomorphism extending the homomorphisms $\pi_\iota : A_\iota \to \mathcal{L}(E)$, respectively $\widetilde{\pi}_\iota\!\restriction_{A_\iota} : A_\iota \to \mathcal{L}(\widetilde{E})$, ($\iota \in I$). In particular, we have $\overline{\sigma(\mathfrak{A})} = A$. In order to show that κ exists, it will suffice to show that

$$\forall x \in \mathfrak{A} \qquad \|\widetilde{\sigma}(x)\| \leq \|\sigma(x)\|.$$

Note that the subspace E of $\widetilde{E}$ is invariant under $\widetilde{\sigma}(\mathfrak{A})$ and that the restriction of $\widetilde{\sigma}(\cdot)$ to E gives σ. This implies

$$\forall x \in \mathfrak{A} \qquad \|\widetilde{\sigma}(x)\| \geq \|\sigma(x)\|,$$

which will in turn imply that κ is injective, once it is known to exist. Let τ be a faithful representation of $\widetilde{B}$ on a Hilbert space $\mathcal{W}$. Consider the Hilbert

space $\widetilde{E} \otimes_\tau \mathcal{W}$ and let $\widetilde{\lambda} : \mathcal{L}(\widetilde{E}) \to \mathcal{L}(\widetilde{E} \otimes_\tau \mathcal{W})$ be the $*$-homomorphism given by $\widetilde{\lambda}(x) = x \otimes 1_\mathcal{W}$. Then $\widetilde{\lambda}$ is faithful, and hence it will suffice to show that

$$(5) \qquad \forall x \in \mathfrak{A} \qquad \|\widetilde{\lambda} \circ \widetilde{\sigma}(x)\| \le \|\sigma(x)\|.$$

Our strategy will be to show that $\widetilde{\lambda} \circ \widetilde{\sigma}$ decomposes as a direct sum of subrepresentations, each of which is of the form $(\nu \restriction^A) \circ \sigma$, where $\nu \restriction^A$ is the $*$-representation of A induced from a representation ν of some A_ι.

Given $n \ge 1$ and $\iota_1, \ldots, \iota_n$ with $\iota_1 \ne \iota_2, \ldots, \iota_{n-1} \ne \iota_n$, and given $p \in \{1, 2, \ldots, n\}$, consider the Hilbert space

$$E_{\iota_1}^o \otimes_B \cdots \otimes_B E_{\iota_{p-1}}^o \otimes_B K_{\iota_p} \otimes_{\widetilde{B}} \widetilde{E}_{\iota_{p+1}}^o \otimes_{\widetilde{B}} \cdots \otimes_{\widetilde{B}} \widetilde{E}_{\iota_n}^o \otimes_\tau \mathcal{W} =$$

$$\stackrel{\text{def}}{=} \left(\begin{array}{l} E_{\iota_1}^o \otimes_B \cdots \otimes_B E_{\iota_{p-1}}^o \otimes_B \widetilde{E}_{\iota_p}^o \otimes_{\widetilde{B}} \widetilde{E}_{\iota_{p+1}}^o \otimes_{\widetilde{B}} \cdots \otimes_{\widetilde{B}} \widetilde{E}_{\iota_n}^o \otimes_\tau \mathcal{W} \\ \ominus \;\; E_{\iota_1}^o \otimes_B \cdots \otimes_B E_{\iota_{p-1}}^o \otimes_B E_{\iota_p}^o \otimes_B \widetilde{E}_{\iota_{p+1}}^o \otimes_{\widetilde{B}} \cdots \otimes_{\widetilde{B}} \widetilde{E}_{\iota_n}^o \otimes_\tau \mathcal{W} \end{array} \right).$$

Heuristically, K_ι takes the place of $\widetilde{E}_\iota \ominus E_\iota$, even when the latter does not make sense. Then

$$\widetilde{E} \otimes_\tau \mathcal{W} = (E \otimes_{\tau \restriction_B} \mathcal{W}) \oplus$$

$$\oplus \bigoplus_{\substack{n \ge 1 \\ \iota_1, \ldots, \iota_n \in I \\ \iota_1 \ne \iota_2, \ldots, \iota_{n-1} \ne \iota_n \\ p \in \{1, 2, \ldots, n\}}} E_{\iota_1}^o \otimes_B \cdots \otimes_B E_{\iota_{p-1}}^o \otimes_B$$
$$\otimes_B K_{\iota_p} \otimes_{\widetilde{B}} \widetilde{E}_{\iota_{p+1}}^o \otimes_{\widetilde{B}} \cdots \otimes_{\widetilde{B}} \widetilde{E}_{\iota_n}^o \otimes_\tau \mathcal{W}.$$

As mentioned earlier, $\widetilde{\sigma}(\mathfrak{A})E \subseteq E$ and $\widetilde{\sigma}(\cdot)\restriction_E = \sigma(\cdot)$, so $E \otimes_{\tau \restriction_B} \mathcal{W}$ is invariant under $\widetilde{\lambda} \circ \widetilde{\sigma}(\mathfrak{A})$, and

$$\forall x \in \mathfrak{A} \qquad \|\widetilde{\lambda} \circ \widetilde{\sigma}(x)\restriction_{E \otimes_\tau \mathcal{W}}\| = \|\sigma(x)\|.$$

Since $\widetilde{\pi}_\iota(A_\iota)E_\iota \subseteq E_\iota$, it is not difficult to check that for every $n \ge 1$ and for every $\iota_1, \ldots, \iota_n \in I$ with $\iota_1 \ne \iota_2, \ldots, \iota_{n-1} \ne \iota_n$,

$$\widetilde{\mathcal{W}}(\iota_1, \ldots, \iota_n)$$
$$\stackrel{\text{def}}{=} \overline{\widetilde{\lambda} \circ \widetilde{\sigma}(\mathfrak{A})(K_{\iota_1} \otimes_{\widetilde{B}} \widetilde{E}_{\iota_2}^o \otimes_{\widetilde{B}} \cdots \otimes_{\widetilde{B}} \widetilde{E}_{\iota_n}^o \otimes_\tau \mathcal{W})}$$
$$= (K_{\iota_1} \otimes_{\widetilde{B}} \widetilde{E}_{\iota_2}^o \otimes_{\widetilde{B}} \cdots \otimes_{\widetilde{B}} \widetilde{E}_{\iota_n}^o \otimes_\tau \mathcal{W}) \quad \oplus$$
$$\oplus \bigoplus_{\substack{q \ge 1 \\ \iota_1', \ldots, \iota_q' \in I \\ \iota_1' \ne \iota_2', \ldots, \iota_{q-1}' \ne \iota_q' \\ \iota_q' \ne \iota_1}} E_{\iota_1'}^o \otimes_B \cdots \otimes_B E_{\iota_q'}^o \otimes_B K_{\iota_1} \otimes_{\widetilde{B}} \widetilde{E}_{\iota_2}^o \otimes_{\widetilde{B}} \cdots \otimes_{\widetilde{B}} \widetilde{E}_{\iota_n}^o \otimes_\tau \mathcal{W}.$$

Thus

$$\widetilde{E} \otimes_\tau \mathcal{W} = (E \otimes_{\tau \restriction_B} \mathcal{W}) \oplus \bigoplus_{\substack{n \ge 1 \\ \iota_1, \ldots, \iota_n \in I \\ \iota_1 \ne \iota_2, \ldots, \iota_{n-1} \ne \iota_n}} \widetilde{\mathcal{W}}(\iota_1, \ldots, \iota_n);$$

hence in order to prove the theorem it will suffice to show that for every choice of $\iota_1, \ldots \iota_n$,

$$(6) \qquad \forall x \in \mathfrak{A} \qquad \|\widetilde{\lambda} \circ \widetilde{\sigma}(x)\restriction_{\widetilde{\mathcal{W}}(\iota_1,\ldots,\iota_n)}\| \leq \|\sigma(x)\|.$$

But letting $\mathcal{V} = K_{\iota_1} \otimes_{\widetilde{B}} \widetilde{E}^{\mathrm{o}}_{\iota_2} \otimes_{\widetilde{B}} \cdots \otimes_{\widetilde{B}} \widetilde{E}^{\mathrm{o}}_{\iota_n} \otimes_\tau \mathcal{W}$, letting $\nu : A_{\iota_1} \to \mathcal{L}(\mathcal{V})$ be the $*$-homomorphism $\nu(a) = (\widetilde{\pi}_{\iota_1}(a) \otimes 1_{\widetilde{E}^{\mathrm{o}}_{\iota_2} \otimes_B \cdots \otimes_B \widetilde{E}^{\mathrm{o}}_{\iota_n} \otimes_\tau \mathcal{W}})\restriction_{\mathcal{V}}$, and appealing to Explication 1.2, it is straightforward to check that

$$\widetilde{\lambda} \circ \widetilde{\sigma}(\cdot)\restriction_{\widetilde{\mathcal{W}}(\iota_1,\ldots,\iota_n)} = (\nu \restriction^A) \circ \sigma,$$

where $\nu \restriction^A$ is the representation of A induced from ν with respect to the conditional expectation $\Phi_{\iota_1} : A \to A_{\iota_1}$ found in Lemma 1.1; this in turn implies (6). $\qquad \square$

Remark 1.4. Let us consider for a moment Theorem 1.3 when the sub-algebra B over which we amalgamate is the scalars, $\mathbf{C}$. When taking the reduced free product $(A, \phi) = \underset{\iota \in I}{*}(A_\iota, \phi_\iota)$ of C*-algebras, one usually requires the states ϕ_ι to have faithful GNS representation. However, one could extend the construction to the case of completely general states ϕ_ι; one then obtains

$$\underset{\iota \in I}{*}(A_\iota, \phi_\iota) = \underset{\iota \in I}{*}((A_\iota / \ker \pi_\iota), \dot{\phi}_\iota),$$

where π_ι is the GNS representation of ϕ_ι and where $\dot{\phi}_\iota$ is the state induced on the quotient $A_\iota / \ker \pi_\iota$ by ϕ_ι. Thus the canonical $*$-homomorphism $A_\iota \to A$ has the same kernel as π_ι.

As a caveat, we would like to point out that with this relaxed definition of reduced free product, (allowing ϕ_ι with nonfaithful GNS representation), the statement of Theorem 1.3 does not in general hold. Indeed, if for some $\iota \in I$ $A_\iota = \mathbf{C} \oplus \mathbf{C}$ with ϕ_ι non-faithful, if $\widetilde{A}_\iota = M_2(\mathbf{C})$ with a unital embedding $A_\iota \hookrightarrow \widetilde{A}_\iota$ and if $\widetilde{\phi}_\iota$ is a state on $M_2(\mathbf{C})$ such that $\widetilde{\phi}\restriction_{A_\iota} = \phi_\iota$, then $A_\iota \to A$ is not injective, while $\widetilde{A}_\iota \hookrightarrow \widetilde{A}$ is injective. This shows that there can be no $*$-homomorphism $\kappa : A \to \widetilde{A}$ making the diagram (4) commute. However, there is no problem allowing the $\widetilde{\phi}_\iota$ to have nonfaithful GNS representations, as long as the restrictions ϕ_ι are taken with faithful GNS representations.

2. Completely positive maps.

M. Choda [7] gave an argument which, when combined with an embedding result like Property 2, proves that if $\theta_\iota : A_\iota \to D_\iota$ is a unital completely positive map between unital C*-algebras for every $\iota \in I$, if ϕ_ι and ψ_ι are states on A_ι and respectively D_ι, each having faithful GNS representation,

and if $\psi_\iota \circ \theta_\iota = \phi_\iota$ then letting

$$(A, \phi) = \underset{\iota \in I}{*}(A_\iota, \phi_\iota)$$
$$(D, \psi) = \underset{\iota \in I}{*}(D_\iota, \psi_\iota)$$

be the reduced free products of C*-algebras, there is a unital completely positive map $\theta : A \to D$ such that $\theta\restriction_{A_\iota} = \theta_\iota$ for every $\iota \in I$, and such that $\theta(a_1 a_2 \cdots a_n) = \theta(a_1)\theta(a_2)\cdots\theta(a_n)$ whenever $a_j \in A_{\iota_j} \cap \ker \phi_{\iota_j}$ for some $\iota_j \in I$ with $\iota_1 \neq \iota_2, \ldots, \iota_{n-1} \neq \iota_n$.

In this section, we generalize this argument of Choda's to the case of reduced amalgamated free products of C*-algebras. The generalization consists of, in essence, replacing Stinespring's dilation theorem for completely positive maps into bounded operators on a Hilbert space by Kasparov's generalization [**17**] to the case of completely positive maps into the algebra of bounded adjointable operators on a Hilbert B-module (see alternatively the book [**18**]). We would like to point out that Theorem 2.2 is quite similar in appearance to analogous results of F. Boca [**2**], [**3**] about completely positive maps on universal amalgamated free products of C*-algebras. However, the universal and reduced free products of C*-algebras are quite different in character, and we do not believe that Boca's results can be used directly to prove Theorem 2.2.

Lemma 2.1. *Let A and B be C*-algebras, let E and $\widetilde{E}$ be Hilbert A-modules, let F and $\tilde{F}$ be Hilbert B-modules and let $v \in \mathcal{L}(E, \widetilde{E})$, $w \in \mathcal{L}(F, \tilde{F})$. Suppose $\pi : A \to \mathcal{L}(F)$ and $\widetilde{\pi} : A \to \mathcal{L}(\tilde{F})$ are $*$-homomorphisms and suppose that*

(7) $$\forall a \in A \quad \forall \xi \in F \qquad w(\pi(a)\xi) = \widetilde{\pi}(a)w(\xi).$$

Let $E \otimes_\pi F$ and $\widetilde{E} \otimes_{\widetilde{\pi}} \tilde{F}$ be the interior tensor products. Then there is an element $v \otimes w \in \mathcal{L}(E \otimes_\pi F, \widetilde{E} \otimes_{\widetilde{\pi}} \tilde{F})$ such that

$$\forall \zeta \in E \quad \forall \xi \in F \qquad (v \otimes w)(\zeta \otimes \xi) = (v\zeta) \otimes (w\xi).$$

If, moreover, $\langle v(\zeta), v(\zeta) \rangle = \langle \zeta, \zeta \rangle$ for every $\zeta \in E$ and $\langle w(\xi), w(\xi) \rangle = \langle \xi, \xi \rangle$ for every $\xi \in F$ then $\langle v \otimes w(\eta), v \otimes w(\eta) \rangle = \langle \eta, \eta \rangle$ for every $\eta \in E \otimes_\pi F$.

Proof. That $v \otimes w$ is bounded is a standard argument (compare p. 42 of [**18**]). Then one sees $(v \otimes w)^* = v^* \otimes w^*$. The final statement follows using the polarization identity. $\qquad\square$

Theorem 2.2. *Let B be a unital C*-algebra, let I be a set and for every $\iota \in I$ let A_ι and D_ι be unital C*-algebras containing copies of B as unital C*-subalgebras and having conditional expectations $\phi_\iota : A_\iota \to B$, respectively $\psi_\iota : D_\iota \to B$, whose GNS representations are faithful. Suppose that for each $\iota \in I$ there is a unital completely positive map $\theta_\iota : A_\iota \to D_\iota$ that is also a*

B–B bimodule map and satisfies $\psi_\iota \circ \theta_\iota = \phi_\iota$. Let

$$(A, \phi) = \mathop{*}_{\iota \in I} (A_\iota, \phi_\iota)$$
$$(D, \phi) = \mathop{*}_{\iota \in I} (D_\iota, \psi_\iota)$$

be the reduced amalgamated free products of C^*-algebras. Then there is a unital completely positive map $\theta : A \to D$ such that for all $\iota \in I$ the diagram

(8)

commutes, where the vertical inclusions are those arising from the free product construction, and satisfying

(9) $$\theta(a_1 a_2 \cdots a_n) = \theta(a_1)\theta(a_2) \cdots \theta(a_n)$$

whenever $a_j \in A_{\iota_j} \cap \ker \phi_{\iota_j}$ and $\iota_1 \neq \iota_2$, $\iota_2 \neq \iota_3$, $\ldots$, $\iota_{n-1} \neq \iota_n$.

Proof. Note first that the assumptions imply that each θ_ι is the identity map on B. Let

$$(\pi_\iota, E_\iota, \xi_\iota) = \mathrm{GNS}(D_\iota, \psi_\iota), \qquad (E, \xi) = \mathop{*}_{\iota \in I}(E_\iota, \xi_\iota).$$

(We will usually write simply $d\zeta$ instead of $\pi_\iota(d)\zeta$, when $d \in D_\iota$ and $\zeta \in E_\iota$.) Recall that then $E_\iota = \xi_\iota B \oplus E_\iota^\mathrm{o}$, that the action $\pi_\iota {\restriction}_B$ leaves E_ι^o globally invariant, and that

$$E = \xi B \oplus \bigoplus_{\substack{n \geq 1 \\ \iota_1,\ldots,\iota_n \in I \\ \iota_1 \neq \iota_2,\ldots,\iota_{n-1} \neq \iota_n}} E_{\iota_1}^\mathrm{o} \otimes_B \cdots \otimes_B E_{\iota_n}^\mathrm{o}.$$

Consider the Hilbert B-module $F_\iota = A_\iota \otimes_{\pi_\iota \circ \theta_\iota} E_\iota$ and the specified element $\eta_\iota = 1 \otimes \xi_\iota \in F_\iota$. Since θ_ι restricts to the identity map on B, in F_ι we have $b \otimes \zeta = 1 \otimes (b\zeta)$ for every $b \in B$ and $\zeta \in E$. Consider the unital $*$-homomorphism $\sigma_\iota : A_\iota \to \mathcal{L}(F_\iota)$ given by

$$\forall a', a \in A_\iota \quad \forall \zeta \in E_\iota \qquad \sigma_\iota(a')(a \otimes \zeta) = (a'a) \otimes \zeta,$$

(cf. page 48 of [**18**]). Consider the map $\rho_\iota : \mathcal{L}(F_\iota) \to B$ given by $\rho_\iota(x) = \langle \eta_\iota, x\eta_\iota \rangle$. If $x \in \mathcal{L}(F_\iota)$ and if $b_1, b_2 \in B$ then $\rho_\iota\big(\sigma_\iota(b_1)x\sigma_\iota(b_2)\big) = b_1\rho_\iota(x)b_2$. If we use σ_ι to identify B with $\sigma_\iota(B) \subseteq \mathcal{L}(F_\iota)$ then we have that $\rho_\iota : \mathcal{L}(F_\iota) \to B$ is a conditional expectation. Clearly $L^2(\mathcal{L}(F_\iota), \rho_\iota) \cong F_\iota$ and the GNS representation of ρ_ι is faithful on $\mathcal{L}(F_\iota)$. We have that $\rho_\iota \circ \sigma_\iota = \phi_\iota$ since

(10) $$\rho_\iota \circ \sigma_\iota(a) = \langle 1 \otimes \xi_\iota, a \otimes \xi_\iota \rangle = \langle \xi_\iota, \theta_\iota(a)\xi_\iota \rangle = \psi_\iota \circ \theta_\iota(a) = \phi_\iota(a).$$

Let
$$(\mathcal{M}, \rho) = \underset{\iota \in I}{*}(\mathcal{L}(F_\iota), \rho_\iota)$$
be the reduced amalgamated free product of C*-algebras. Note that $\mathcal{M} \subseteq \mathcal{L}(F)$ where
$$(F, \eta) = \underset{\iota \in I}{*}(F_\iota, \eta_\iota).$$
By Theorem 1.3 there is a $*$-homomorphism $\sigma : A \to \mathcal{M}$ such that $\sigma\!\restriction_{A_\iota} = \sigma_\iota$, $(\iota \in I)$.

Consider the operator $v_\iota : E_\iota \to F_\iota$ given by $\zeta \to 1 \otimes \zeta$, and note that $\langle v_\iota \zeta, v_\iota \zeta \rangle = \langle \zeta, \zeta \rangle$ for every $\zeta \in E_\iota$, hence $v_\iota(E_\iota^\mathrm{o}) \subseteq F_\iota^\mathrm{o}$. A calculation using e.g. Lemma 5.4 of [18] shows that there is a bounded operator $F_\iota \to E_\iota$ sending $a \otimes \zeta$ to $\theta_\iota(a)\zeta$, which is then the adjoint of v_ι. Hence $v_\iota \in \mathcal{L}(E_\iota, F_\iota)$, and clearly $v_\iota^* v_\iota = 1$. Since θ_ι is a left B-module map, we have for every $b \in B$ and $\zeta \in E_\iota$ that $v_\iota(b\zeta) = 1 \otimes (b\zeta) = b \otimes \zeta = b(v_\iota(\zeta))$. Therefore, taking direct sums of operators $v_{\iota_1} \otimes \cdots \otimes v_{\iota_n}$ given by Lemma 2.1, we get $v \in \mathcal{L}(E, F)$ such that $\langle v\zeta, v\zeta \rangle = \langle \zeta, \zeta \rangle$ for every $\zeta \in E$, $v\xi = \eta$ and
$$v(\zeta_1 \otimes \zeta_2 \otimes \cdots \otimes \zeta_n) = (v_{\iota_1}\zeta_1) \otimes (v_{\iota_2}\zeta_2) \otimes \cdots \otimes (v_{\iota_n}\zeta_n)$$
whenever $\zeta_j \in E_{\iota_j}^\mathrm{o}$, $\iota_1, \ldots, \iota_n \in I$ and $\iota_j \neq \iota_{j+1}$. Let $\theta : A \to \mathcal{L}(E)$ be the unital completely positive map $\theta(x) = v^*\sigma(x)v$.

We will show that the diagram (8) commutes and and that (9) holds, which will furthermore imply that $\theta(A) \subseteq D$. In order to show (8) commutes, let $w_\iota : E \to E_\iota \otimes_B E(\iota)$ and $y_\iota : F \to F_\iota \otimes_B F(\iota)$ be the unitaries used in the free product constructions to define the inclusions $A_\iota \hookrightarrow A$ and, respectively, $\mathcal{L}(F_\iota) \hookrightarrow \mathcal{M}$. Note that $v_\iota(E(\iota)) \subseteq F(\iota)$ and that $y_\iota v = (v_\iota \otimes v\!\restriction_{E(\iota)})w_\iota$. Furthermore, observe that for $a \in A_\iota$ and $\zeta \in E_\iota$,
$$(v_\iota^*\sigma_\iota(a)v_\iota)\zeta = v_\iota^*(a \otimes \zeta) = \theta_\iota(a)\zeta.$$
Hence for $a \in A_\iota$,
$$\theta(a) = v^*\sigma(a)v = v^*\sigma_\iota(a)v = v^*y_\iota^*\big(\sigma_\iota(a) \otimes 1_{F(\iota)}\big)y_\iota v =$$
$$= w_\iota^*\big(v_\iota^*\sigma_\iota(a)v_\iota \otimes (v\!\restriction_{E(\iota)})^* v\!\restriction_{E(\iota)}\big)w_\iota = w_\iota^*(\theta_\iota(a) \otimes 1_{E(\iota)})w_\iota = \theta_\iota(a),$$
and thus (8) commutes. Now to show that (9) holds, consider $a_j \in A_{\iota_j} \cap \ker \phi_{\iota_j}$ for some $\iota_j \in I$ $(1 \leq j \leq n)$ with $\iota_j \neq \iota_{j+1}$. It is easy to see that

$$(11) \quad \theta_{\iota_1}(a_1) \cdots \theta_{\iota_n}(a_n)\xi = \widehat{\theta_{\iota_1}(a_1)} \otimes \cdots \otimes \widehat{\theta_{\iota_n}(a_n)}$$
$$= v^*\big((a_1 \otimes \xi_{\iota_1}) \otimes \cdots \otimes (a_n \otimes \xi_{\iota_n})\big) = \theta(a_1 \cdots a_n)\xi.$$

Now consider an element $\zeta_1 \otimes \cdots \otimes \zeta_p \in E$, where $\zeta_j \in E_{k_j}^\mathrm{o}$ for some $k_j \in I$ with $k_j \neq k_{j+1}$.

Let $P_0 : E \to \xi B$ be the projection and for $\ell \in \mathbf{N}$ let
$$P_\ell : E \to \bigoplus_{\substack{\iota_1, \ldots, \iota_\ell \in I \\ \iota_1 \neq \iota_2, \ldots, \iota_{\ell-1} \neq \iota_\ell}} E_{\iota_1}^\mathrm{o} \otimes \cdots \otimes E_{\iota_\ell}$$

be the projection. Taking adjoints and using (11), we see that

$$P_0\theta_{\iota_1}(a_1)\cdots\theta_{\iota_n}(a_n)(\zeta_1\otimes\cdots\otimes\zeta_p) = P_0\theta(a_1\cdots a_n)(\zeta_1\otimes\cdots\otimes\zeta_p).$$

Now letting $\ell \in \mathbf{N}$ we will use standard techniques (see, for example, [14] and [12]) to show that

$$(12) \qquad P_\ell\theta_{\iota_1}(a_1)\cdots\theta_{\iota_n}(a_n)(\zeta_1\otimes\cdots\otimes\zeta_p) = P_\ell\theta(a_1\cdots a_n)(\zeta_1\otimes\cdots\otimes\zeta_p).$$

If $\ell > n + p$ or $\ell < |n - p|$ then it is clear that both sides of (12) are zero. If $\ell = n + p$ then both sides of (12) are zero unless $\iota_n \neq k_1$, in which case a calculation similar to (11) shows that (12) holds. Let $Q_\iota^{\mathrm{o}} : E_\iota \to E_\iota^{\mathrm{o}}$ and $R_\iota^{\mathrm{o}} : F_\iota \to F_\iota^{\mathrm{o}}$ be the projections, and note that $R_\iota^{\mathrm{o}}v_\iota = v_\iota Q_\iota^{\mathrm{o}}$. Consider when $n + p - \ell = 1$. Then both sides of (12) are zero unless $\iota_n = k_1$, in which case

$$P_\ell\theta_{\iota_1}(a_1)\cdots\theta_{\iota_n}(a_n)(\zeta_1\otimes\cdots\otimes\zeta_p)$$
$$= \widehat{\theta_{\iota_1}(a_1)}\otimes\cdots\otimes\widehat{\theta_{\iota_{n-1}}(a_{n-1})}\otimes Q_{\iota_n}^{\mathrm{o}}(\theta_{\iota_n}(a_n)\zeta_1)\otimes\zeta_2\otimes\cdots\otimes\zeta_p$$
$$= v_\iota^*\big((a_1\otimes\xi_{\iota_1})\otimes\cdots\otimes(a_{n-1}\otimes\xi_{\iota_{n-1}})\otimes R_{\iota_n}^{\mathrm{o}}(a_n\otimes\zeta_1)$$
$$\otimes(1\otimes\zeta_2)\otimes\cdots\otimes(1\otimes\zeta_p)\big)$$
$$= P_\ell\theta(a_1\cdots a_n)(\zeta_1\otimes\cdots\otimes\zeta_p).$$

If $p + n - \ell = 2r + 1$ for $r \in \{1, 2, \ldots, \min(p, n) - 2\}$ then both sides of (12) are zero unless $\iota_n = k_1$, $\iota_{n-1} = k_2, \ldots, \iota_{n-r+1} = k_r$, in which case

$$P_\ell\theta_{\iota_1}(a_1)\cdots\theta_{\iota_n}(a_n)(\zeta_1\otimes\cdots\otimes\zeta_p)$$

$$= \widehat{\theta_{\iota_1}(a_1)}\otimes\cdots\otimes\widehat{\theta_{\iota_{n-r-1}}(a_{n-r-1})}\otimes$$
$$\otimes Q_{\iota_{n-r}}^{\mathrm{o}}\big(\theta_{\iota_{n-r}}(a_{n-r})\langle\xi, \theta_{\iota_{n-r+1}}(a_{n-r+1})\cdots\theta_{\iota_n}(a_n)\zeta_1\otimes\cdots\otimes\zeta_r\rangle\zeta_{r+1}\big)\otimes$$
$$\otimes\zeta_{r+2}\otimes\cdots\otimes\zeta_p$$

$$= \widehat{\theta_{\iota_1}(a_1)}\otimes\cdots\otimes\widehat{\theta_{\iota_{n-r-1}}(a_{n-r-1})}\otimes$$
$$\otimes Q_{\iota_{n-r}}^{\mathrm{o}}\big(\theta_{\iota_{n-r}}(a_{n-r})\langle\widehat{\theta_{\iota_n}(a_n^*)}\otimes\cdots\otimes\widehat{\theta_{\iota_{n-r+1}}(a_{n-r+1}^*)}, \zeta_1\otimes\cdots\otimes\zeta_r\rangle\zeta_{r+1}\big)$$
$$\otimes\zeta_{r+2}\otimes\cdots\otimes\zeta_p$$

$$= v^*\Big((a_1\otimes\xi_{\iota_1})\otimes\cdots\otimes(a_{n-r-1}\otimes\xi_{\iota_{n-r-1}})\otimes$$
$$\otimes R_{\iota_{n-r}}^{\mathrm{o}}\Big(\theta_{\iota_{n-r}}(a_{n-r})\,\cdot$$
$$\cdot\langle\sigma_{\iota_n}(a_n^*)\cdots\sigma_{\iota_{n-r+1}}(a_{n-r+1}^*)\eta, (1\otimes\zeta_1)\otimes\cdots\otimes(1\otimes\zeta_r)\rangle(1\otimes\zeta_{r+1})\Big)\otimes$$
$$\otimes(1\otimes\zeta_{r+2})\otimes\cdots\otimes(1\otimes\zeta_p)\Big)$$

$$
\begin{aligned}
= v^* \Big(& (a_1 \otimes \xi_{\iota_1}) \otimes \cdots \otimes (a_{n-r-1} \otimes \xi_{\iota_{n-r-1}}) \otimes \\
& \otimes R^{\mathrm{o}}_{\iota_{n-r}} \Big(\theta_{\iota_{n-r}}(a_{n-r}) \cdot \\
& \cdot \big\langle \eta, \, \sigma_{\iota_{n-r+1}}(a_{n-r+1}) \cdots \sigma_{\iota_n}(a_n)(1 \otimes \zeta_1) \otimes \cdots \otimes (1 \otimes \zeta_r) \big\rangle (1 \otimes \zeta_{r+1}) \Big) \otimes \\
& \otimes (1 \otimes \zeta_{r+2}) \otimes \cdots \otimes (1 \otimes \zeta_p) \Big)
\end{aligned}
$$

$$
= P_\ell \theta(a_1 \cdots a_n)(\zeta_1 \otimes \cdots \otimes \zeta_p).
$$

A similar calculation shows that (12) holds also when $n+p-\ell = 2\min(p,n) - 1$.

If $n + p - \ell = 2r$ is even for $r \in \{1, 2, \ldots, \min(p,n) - 1\}$ then both sides of (12) are zero unless $\iota_n = k_1$, $\iota_{n-1} = k_2, \ldots, \iota_{n-r+1} = k_r$ and $\iota_{n-r} \neq k_{r+1}$, in which case

$$
P_\ell \theta_{\iota_1}(a_1) \cdots \theta_{\iota_n}(a_n)(\zeta_1 \otimes \cdots \otimes \zeta_p)
$$

$$
\begin{aligned}
= \, & \widehat{\theta_{\iota_1}(a_1)} \otimes \cdots \otimes \widehat{\theta_{\iota_{n-r}}(a_{n-r})} \otimes \\
& \otimes \big\langle \xi, \, \theta_{\iota_{n-r+1}}(a_{n-r+1}) \cdots \theta_{\iota_n}(a_n)\zeta_1 \otimes \cdots \otimes \zeta_r \big\rangle \zeta_{r+1} \otimes \\
& \otimes \zeta_{r+2} \otimes \cdots \otimes \zeta_p
\end{aligned}
$$

$$
\begin{aligned}
= \, & \widehat{\theta_{\iota_1}(a_1)} \otimes \cdots \otimes \widehat{\theta_{\iota_{n-r}}(a_{n-r})} \otimes \\
& \otimes \big\langle \widehat{\theta_{\iota_n}(a_n^*)} \otimes \cdots \otimes \widehat{\theta_{\iota_{n-r+1}}(a_{n-r+1}^*)}, \, \zeta_1 \otimes \cdots \otimes \zeta_r \big\rangle \zeta_{r+1} \otimes \\
& \otimes \zeta_{r+2} \otimes \cdots \otimes \zeta_p
\end{aligned}
$$

$$
\begin{aligned}
= v^* \Big(& (a_1 \otimes \xi_{\iota_1}) \otimes \cdots \otimes (a_{n-r} \otimes \xi_{\iota_{n-r}}) \otimes \\
& \otimes \big\langle \eta, \, \sigma_{\iota_{n-r+1}}(a_{n-r+1}) \cdots \sigma_{\iota_n}(a_n) \big((1 \otimes \zeta_1) \otimes \cdots \otimes (1 \otimes \zeta_r) \big) \big\rangle (1 \otimes \zeta_{r+1}) \otimes \\
& \otimes (1 \otimes \zeta_{r+2}) \otimes \cdots \otimes (1 \otimes \zeta_p) \Big)
\end{aligned}
$$

$$
= P_\ell \theta(a_1 \cdots a_n)(\zeta_1 \otimes \cdots \otimes \zeta_p).
$$

Similar calculations show that (12) holds also when $p + n - \ell = 2\min(p,n)$. This finishes the proof of (9), and of the theorem. $\qquad \square$

3. Amalgamated free products of von Neumann algebras.

This section contains results for amalgamated free products of von Neumann algebras that are analogous to those for C*-algebras found in §1 and §2. The free product of von Neumann algebras with respect to given normal states was defined by Voiculescu in [21] and has been much studied. See also Ching's paper [6], where the free product of von Neumann algebras with respect to normal faithful tracial states from a certain class was first

defined. We begin this section by describing the "folklore" construction of amalgamated free products of von Neumann algebras (with respect to normal conditional expectations onto a von Neumann subalgebra). We are grateful to the referee for pointing out this simplification of the construction we originally gave.

Construction 3.1. Let B be a von Neumann algebra contained as a unital von Neumann subalgebra of von Neumann algebras A_ι ($\iota \in I$). Suppose there are normal conditional expectations $\phi_\iota : A_\iota \to B$. The amalgamated free product of von Neumann algebras, which we will denote

$$(A, \phi) = \mathop{\overline{\ast}}_{\iota \in I} (A_\iota, \phi_\iota),$$

is constructed as follows. Let

$$(\mathcal{A}, \varphi) = \mathop{\ast}_{\iota \in I} (A_\iota, \phi_\iota)$$

be the C*-algebra reduced amalgamated free product. Let ψ be a normal state on B with faithful GNS representation and consider the state $\psi \circ \varphi$ on $\mathcal{A}$. Let $\pi_{\psi \circ \varphi}$ be the GNS representation of $\mathcal{A}$ associated to $\psi \circ \varphi$, let $A = \pi_{\psi \circ \varphi}(\mathcal{A})'' \subseteq \mathcal{L}(L^2(\mathcal{A}, \psi \circ \varphi))$ and thereby regard $\mathcal{A}$ as a weakly dense subalgebra of A. The Hilbert space projection $L^2(\mathcal{A}, \psi \circ \varphi) \to L^2(B, \psi)$ gives rise to a normal conditional expectation $\phi : A \to B$ whose restriction to $\mathcal{A}$ is φ. It remains to see that the pair (A, ϕ) is independent of the choice of ψ. If ψ' is any normal state on B then the state $\psi' \circ \varphi$ of $\mathcal{A}$ extends to the normal state $\psi' \circ \phi$ of A. Hence $\pi_{\psi' \circ \varphi}(\mathcal{A})'' = \pi_{\psi' \circ \phi}(A)$ is a quotient of A. Thus (A, ϕ) is independent of the choice of ψ.

Remark 3.2. In the above construction, we have $\mathcal{A} \subseteq \mathcal{L}(E)$ for the Hilbert B, B-bimodule $E = L^2(\mathcal{A}, \varphi)$. The GNS Hilbert space $L^2(\mathcal{A}, \psi \circ \varphi)$ is canonically isomorphic to the internal tensor product $E \otimes_{\pi_\psi} L^2(B, \psi)$, where $\pi_\psi : B \to \mathcal{L}(L^2(B, \psi))$ is the GNS representation of ψ, and the representation $\pi_{\psi \circ \varphi}$ is given by $\pi_{\psi \circ \varphi}(a) = a \otimes 1 \in \mathcal{L}(E \otimes_{\pi_\psi} L^2(B, \psi))$. We will later use this picture in proofs.

Following Rieffel [20, 1.5], if A and B are von Neumann algebras, if E is a Hilbert B-module and if $\theta : A \to \mathcal{L}(E)$ is a completely positive map, we say that θ is *normal* if for every $\zeta_1, \zeta_2 \in E$, the map $A \ni a \mapsto \langle \zeta_1, \theta(a)\zeta_2 \rangle \in B$ is normal. This coincides with the usual notion of normality when $B = \mathbf{C}$ (in which case E is a Hilbert space). It is clear that if B is a von Neumann subalgebra of a von Neumann algebra A having a normal conditional expectation $\phi : A \to B$ then the GNS representation of A as bounded adjointable operators on the Hilbert B-module $L^2(A, \phi)$ is normal.

Part (i) of the following straightforward lemma was proved in the case of a ∗-homomorphism by Rieffel as part of [20, 5.2].

Lemma 3.3. *Let A and B be von Neumann algebras, let E be a Hilbert B-module and suppose that $\theta : A \to \mathcal{L}(E)$ is completely positive map. Let $\mathcal{H}$ be a Hilbert space and let $\tau : B \to \mathcal{L}(\mathcal{H})$ be a normal $*$-representation. Let $\theta \otimes 1$ denote the completely positive map $A \ni a \mapsto \pi(a) \otimes 1 \in \mathcal{L}(E \otimes_\tau \mathcal{H})$ of A into bounded operators on the Hilbert space $E \otimes_\tau \mathcal{H}$. We have:*

  (i) *if θ is normal then $\theta \otimes 1$ is normal;*
  (ii) *if $\theta \otimes 1$ is normal and if τ is faithful then θ is normal.*

Proof. If $\zeta_1, \zeta_2 \in E$ and $v_1, v_2 \in \mathcal{H}$ then

$$(13) \qquad \langle \zeta_1 \otimes v_1, (\theta \otimes 1)(x)(\zeta_2 \otimes v_2) \rangle = \langle v_1, \tau(\langle \zeta_1, \theta(x)\zeta_2 \rangle)v_2 \rangle.$$

If θ is normal then (13) shows that $\theta \otimes 1$ is continuous from the $\sigma(A, A_*)$ topology on A to the weak–operator topology on $\mathcal{L}(E \otimes_\tau \mathcal{H})$, which implies $\theta \otimes 1$ is normal and proves (i).

If $\theta \otimes 1$ is normal then (13) shows that $x \mapsto \tau(\langle \zeta_1, \theta(x)\zeta_2 \rangle)$ is normal. Assuming also τ is faithful, it follows that θ is normal. $\square$

The following application of Lemma 3.3(i) is in [**20**, 5.2].

Lemma 3.4. *Let B be a unital von Neumann subalgebra of a von Neumann algebra A with a normal conditional expectation $\Phi : A \to B$. Let τ be a normal $*$-representation of B on a Hilbert space $\mathcal{H}$. Then the induced representation, $\tau \upharpoonright^A$, of τ to A with respect to the conditional expectation Φ is normal.*

Proof. By definition, and in the notation of Lemma 3.3, $\tau \upharpoonright^A = \pi \otimes 1 : A \to \mathcal{L}(L^2(A, \Phi) \otimes_\tau \mathcal{H})$, where π is the GNS representation of A on $L^2(A, \Phi)$. $\square$

Lemma 3.5. *Let A, B_1 and B_2 be von Neumann algebras and let E_j be a Hilbert B_j-module $(j = 1, 2)$. If $\theta : A \to \mathcal{L}(E_1)$ and $\sigma : B_1 \to \mathcal{L}(E_2)$ are normal completely positive maps, then the completely positive map $\theta \otimes 1 : A \to \mathcal{L}(E_1 \otimes_\sigma E_2)$ is normal.*

Proof. Let $\tau : B_2 \to \mathcal{L}(\mathcal{H})$ be a faithful normal $*$-representation of B_2 on a Hilbert space $\mathcal{H}$. Applying Lemma 3.3 in succession we find that $\sigma \otimes 1 : B_1 \to \mathcal{L}(E_2 \otimes_\tau \mathcal{H})$ is normal, $\theta \otimes 1 \otimes 1 : A \to \mathcal{L}(E_1 \otimes_\sigma E_2 \otimes_\tau \mathcal{H})$ is normal, and thus $\theta \otimes 1 : A \to \mathcal{L}(E_1 \otimes_\sigma E_2)$ is normal. $\square$

Lemma 3.6. *Let B be a unital von Neumann algebra, let I be a set and for every $\iota \in I$ let A_ι be a unital von Neumann algebra containing a copy of B as a unital von Neumann subalgebra and having a normal conditional expectation $\phi_\iota : A_\iota \to B$ whose GNS representation is faithful. Let*

$$(A, \phi) = \underset{\iota \in I}{\overline{\ast}} (A_\iota, \phi_\iota)$$

be the reduced amalgamated free product of von Neumann algebras. Then for every $\iota_0 \in I$, there is a normal conditional expectation $\Phi_{\iota_0} : A \to A_{\iota_0}$ such

that $\Phi_{\iota_0}\!\restriction_{A_{\iota}} = \phi_\iota$ *for every* $\iota \in I\backslash\{\iota_0\}$ *and* $\Phi_{\iota_0}(a_1 a_2 \cdots a_n) = 0$ *whenever* $n \geq 2$ *and* $a_j \in A_{\iota_j} \cap \ker \phi_{\iota_j}$ *with* $\iota_1 \neq \iota_2, \ldots, \iota_{n-1} \neq \iota_n$.

Proof. Let ψ be a normal faithful state on B and let $\tau = \pi_\psi : B \to \mathcal{L}(\mathcal{H})$ be the associated GNS representation. The construction of A can be realized on the Hilbert space $E \otimes_\tau \mathcal{H}$. The projection $Q_{\iota_0} : E \to E_{\iota_0}$ from the proof of Lemma 1.1 gives rise to the projection $Q_{\iota_0} \otimes 1_\mathcal{H} : E \otimes_\tau \mathcal{H} \to E_{\iota_0} \otimes_\tau \mathcal{H}$, compression with which provides a normal positive linear map $\Theta_{\iota_0} : A \to \mathcal{L}(E_{\iota_0} \otimes_\tau \mathcal{H})$. Let $\lambda_{\iota_0} : A_{\iota_0} \to \mathcal{L}(E_{\iota_0} \otimes_\tau \mathcal{H})$ be the GNS representation $A_{\iota_0} \hookrightarrow \mathcal{L}(E_{\iota_0})$ followed by the inclusion $\mathcal{L}(E_{\iota_0}) \ni x \mapsto x \otimes 1_\mathcal{H} \in \mathcal{L}(E_{\iota_0} \otimes_\tau \mathcal{H})$. Then Θ_{ι_0} maps a weakly dense $*$-subalgebra of A into the image of λ_{ι_0}, hence maps all of A there. Let $\Phi_{\iota_0} = \lambda_{\iota_0}^{-1} \circ \Theta_{\iota_0}$. The desired properties of Φ_{ι_0} are easily verified. $\qquad\qquad\square$

Here is an embedding result, analogous to Theorem 1.3, for amalgamated free products of von Neumann algebras.

Theorem 3.7. *Let* $B \subseteq \widetilde{B}$ *be a* (*not necessarily unital*) *inclusion of von Neumann algebras. Let* I *be a set and for each* $\iota \in I$ *suppose*

$$1_{\widetilde{A}_\iota} \in \quad \widetilde{B} \quad \subseteq \quad \widetilde{A}_\iota$$
$$\cup \qquad\qquad \cup$$
$$1_{A_\iota} \in \quad B \quad \subseteq \quad A_\iota$$

are inclusions of von Neumann algebras. Suppose that $\tilde{\phi}_\iota : \widetilde{A}_\iota \to \widetilde{B}$ *is a normal conditional expectation such that* $\tilde{\phi}_\iota(A_\iota) \subseteq B$ *and assume that* $\tilde{\phi}_\iota$ *and the restriction* $\tilde{\phi}_\iota\!\restriction_{A_\iota}$ *have faithful GNS representations, for all* $\iota \in I$. *Let*

$$(\widetilde{A}, \tilde{\phi}) = \underset{\iota \in I}{\overset{*}{\ast}}(\widetilde{A}_\iota, \tilde{\phi}_\iota)$$

$$(A, \phi) = \underset{\iota \in I}{\overset{*}{\ast}}(A_\iota, \tilde{\phi}_\iota\!\restriction_{A_\iota})$$

be the amalgamated free products of von Neumann algebras. Then there is a unique normal $*$-*homomorphism* $\kappa : A \to \widetilde{A}$ *such that for every* $\iota \in I$ *the diagram*

(14)
$$\widetilde{A}_\iota \quad \hookrightarrow \quad \widetilde{A}$$
$$\cup \qquad\qquad \uparrow \kappa$$
$$A_\iota \quad \hookrightarrow \quad A$$

commutes, where the horizontal arrows are the inclusions arising from the free product construction. Moreover, κ *is necessarily injective.*

Proof. This is very much like the proof of Theorem 1.3, to which we refer in detail. Assume without loss of generality that B is a unital subalgebra of $\widetilde{B}$. We now insist that τ be a normal faithful representation of $\widetilde{B}$; we must show that the algebra homomorphism $\widetilde{\lambda} \circ \widetilde{\sigma} : \mathfrak{A} \to \mathcal{L}(\widetilde{E} \otimes_\tau \mathcal{W})$ extends to a

normal representation of the von Neumann algebra A. But $\widetilde{E} \otimes_\tau \mathcal{W}$ is the direct sum of $E \otimes_{\tau \upharpoonright_B} \mathcal{W}$ and the various $\widetilde{\mathcal{W}}(\iota_1, \ldots, \iota_n)$. The homomorphism $\widetilde{\lambda} \circ \widetilde{\sigma}$ restricted to $E \otimes_{\tau \upharpoonright_B} \mathcal{W}$ extends to the defining representation of A. Let $n \geq 1$ and let $\iota_1, \ldots, \iota_n \in I$ be such that $\iota_j \neq \iota_{j+1}$; we have the normal $*$-representation, μ, of A_{ι_1} on the Hilbert space $K_{\iota_1} \otimes_{\widetilde{B}} E_{\iota_2} \otimes_{\widetilde{B}} \cdots \otimes_{\widetilde{B}} E_{\iota_n} \otimes_\tau \mathcal{W}$, obtained from the normal representation of A_{ι_1} in $\mathcal{L}(\widetilde{E}_{\iota_1})$; let $\mu \!\upharpoonright^A$ be the representation of the von Neumann algebra A on $\widetilde{\mathcal{W}}(\iota_1, \ldots, \iota_n)$ induced from μ with respect to the normal conditional expectation $\Phi_{\iota_1} : A \to A_{\iota_1}$ found in Lemma 3.6; then $\widetilde{\lambda} \circ \widetilde{\sigma}$ restricted to $\widetilde{\mathcal{W}}(\iota_1, \ldots, \iota_n)$ extends to the $*$-homomorphism $\mu \!\upharpoonright^A : A \to \mathcal{L}(\widetilde{\mathcal{W}}(\iota_1, \ldots, \iota_n))$, which by Lemma 3.4 is normal. $\qquad\square$

Here is the construction, analogous to Theorem 2.2, of free products of certain completely positive maps in the von Neumann algebra setting.

Theorem 3.8. *Let B be a von Neumann algebra, let I be a set and for every $\iota \in I$ let A_ι and D_ι be von Neumann algebras containing copies of B as unital von Neumann subalgebras and having normal conditional expectations $\phi_\iota : A_\iota \to B$, respectively $\psi_\iota : D_\iota \to B$, whose GNS representations are faithful. Suppose that for each $\iota \in I$ there is a normal unital completely positive map $\theta_\iota : A_\iota \to D_\iota$ that is also a B–B bimodule map and satisfies $\psi_\iota \circ \theta_\iota = \phi_\iota$. Let*

$$(15) \qquad (A, \phi) = \overline{\underset{\iota \in I}{*}} (A_\iota, \phi_\iota)$$

$$(D, \phi) = \overline{\underset{\iota \in I}{*}} (D_\iota, \psi_\iota)$$

be the reduced amalgamated free products of von Neumann algebras. Then there is a normal unital completely positive map $\theta : A \to D$ satisfying

$$(16) \qquad \forall \iota \in I \qquad \theta \!\upharpoonright_{A_\iota} = \theta_\iota$$

and

$$(17) \qquad \theta(a_1 a_2 \cdots a_n) = \theta(a_1)\theta(a_2)\cdots\theta(a_n)$$

whenever $a_j \in A_{\iota_j} \cap \ker \phi_{\iota_j}$ and $\iota_1 \neq \iota_2,\ \iota_2 \neq \iota_3,\ \ldots,\ \iota_{n-1} \neq \iota_n$.

Proof. Let

$$(\mathcal{A}, \varphi) = \underset{\iota \in I}{*}(A_\iota, \phi_\iota)$$

$$(\mathcal{D}, \rho) = \underset{\iota \in I}{*}(D_\iota, \psi_\iota)$$

be the C*-algebra reduced amalgamated free products. Thus A and D are the closures in strong–operator topology of $\mathcal{A}$ and respectively $\mathcal{D}$ in the appropriate representations. We need only show that the unital completely positive map $\theta : \mathcal{A} \to \mathcal{D}$ found in Theorem 2.2 extends to a normal completely positive map $\overline{\theta} : A \to D$.

Consider, from the proof of Theorem 2.2, the Hilbert B-modules $(E, \xi) = \ast_{\iota \in I}(E_\iota, \xi_\iota)$, and $(F, \eta) = \ast_{\iota \in I}(F_\iota, \eta_\iota)$, the $\ast$-homomorphism $\sigma : \mathcal{A} \to \mathcal{L}(F)$ and the bounded operator $v \in \mathcal{L}(E, F)$; denote by $i_\mathcal{A}$ the GNS representation of $\mathcal{A}$ on $L^2(\mathcal{A}, \varphi)$. Recall that σ is a free product of embeddings $\sigma_\iota : A_\iota \to \mathcal{L}(F_\iota)$. From the proof of Theorem 1.3, and letting $\tau = \pi_\mu : B \to \mathcal{L}(\mathcal{V})$ be the GNS representation of a normal faithful state μ of B, we see that the representation $\sigma \otimes 1$ of $\mathcal{A}$ on $F \otimes_\tau \mathcal{V}$ given by $a \mapsto \sigma(a) \otimes 1$ splits as a direct sum, $\sigma \otimes 1 = \bigoplus_{\lambda \in \Lambda}(\sigma \otimes 1)\!\restriction_{\mathcal{W}_\lambda}$, where each summand $(\sigma \otimes 1)\!\restriction_{\mathcal{W}_\lambda}$ is either a copy of $i_\mathcal{A} \otimes 1 : \mathcal{A} \to \mathcal{L}(L^2(\mathcal{A}, \varphi) \otimes_\tau \mathcal{V})$ or is the induced representation $\nu \!\restriction^\mathcal{A}$ of a representation ν of some A_ι on a Hilbert space, where ν is the restriction to an invariant subspace of the representation $\sigma_\iota \otimes 1 : A_\iota \to \mathcal{L}(F_\iota \otimes_\tau \mathcal{V})$. The representation $i_\mathcal{A} \otimes 1$ extends to a normal $\ast$-representation of A as seen in Construction 3.1. Using Lemma 3.3 we see that $\sigma_\iota \otimes 1$ is normal; hence ν is normal and by Lemma 3.4, $\nu \!\restriction^\mathcal{A}$ extends to a normal $\ast$-representation of A. Hence $\sigma \otimes 1$ extends to a normal $\ast$-representation of A, which we will denote by $\overline{\sigma} : A \to \mathcal{L}(F \otimes_\tau \mathcal{V})$.

The isometry $v \in \mathcal{L}(E, F)$ gives rise to an isometry $v \otimes 1 : E \otimes_\tau \mathcal{V} \to F \otimes_\tau \mathcal{V}$. Letting $i_D : \mathcal{D} \to \mathcal{L}(E)$ be the defining representation, by Remark 3.2 the weak closure of the image of $i_D \otimes 1 : \mathcal{D} \to \mathcal{L}(E \otimes_\tau \mathcal{V})$ is the von Neumann algebra D. Consider the normal unital completely positive map $\overline{\theta} : A \to \mathcal{L}(E \otimes_\tau \mathcal{V})$ given by $\overline{\theta}(x) = (v \otimes 1)^*\overline{\sigma}(x)(v \otimes 1)$. If $a \in \mathcal{A}$ then $\overline{\theta}(a) = i_D(\theta(a)) \otimes 1$. So $\overline{\theta}$ extends the map $\theta : \mathcal{A} \to \mathcal{D}$; hence $\overline{\theta}(A) \subseteq D$. $\qquad \square$

References

[1] D. Avitzour, *Free products of C^*-algebras*, Trans. Amer. Math. Soc., **271** (1982), 423-465, MR 83h:46070, Zbl 513.46037.

[2] F. Boca, *Free products of completely positive maps and spectral sets*, J. Funct. Anal., **97** (1991), 251-263, MR 92f:46064, Zbl 741.46024.

[3] ———, *Completely positive maps on amalgamated free product C^*-algebras*, Math. Scand., **72** (1992), 212-222, MR 94k:46110.

[4] M. Bożejko, M. Leinert and R. Speicher, *Convolution and limit theorems for conditionally free random variables*, Pacific J. Math., **175** (1996), 357-388, MR 98j:46069, Zbl 874.60010.

[5] M. Bożejko and R. Speicher, *ψ-independent and symmetrized white noises* in 'Quantum Probability & Related Topics, QP–PQ VI', 219-236, World Sci. Publishing, 1991, CMP 1 149 828.

[6] W.-M. Ching, *Free products of von Neumann algebras*, Trans. Amer. Math. Soc., **178** (1973), 147-163, MR 48 #4749.

[7] M. Choda, *Reduced free products of completely positive maps and entropy for free product of automorphisms*, Publ. Res. Inst. Math. Sci., **32** (1996), 371-382, MR 97d:46077, Zbl 857.46040.

[8] M. Choda and K.J. Dykema, *Purely infinite simple C^*-algebras arising from free product constructions*, III, Proc. Amer. Math. Soc., **128**(11) (2000), 3269-3273, MR 2001b:46095.

[9] K.J. Dykema, *Faithfulness of free product states*, J. Funct. Anal., **154** (1998), 323-329, MR 99e:46066, Zbl 927.46034.

[10] _____ , *Simplicity and the stable rank of some free product C*-algebras*, Trans. Amer. Math. Soc., **351** (1999), 1-40, MR 99g:46078.

[11] _____ , *Purely infinite simple C*-algebras arising from free product constructions*, II, Math. Scand., to appear.

[12] _____ , *Exactness of reduced amalgamated free product C*-algebras*, preprint, 1999.

[13] _____ , *Topological entropy of some automorphisms of reduced amalgamated free product C*-algebras*, Ergodic Theory Dynam. Systems, to appear.

[14] K.J. Dykema, U. Haagerup and M. Rørdam, *The stable rank of some free product C*-algebras*, Duke Math. J., **90** (1997), 95-121, MR 99g:46077a; *Correction*, idem., **94** (1998), 213, Zbl 943.46034.

[15] K.J. Dykema and M. Rørdam, *Purely infinite simple C*-algebras arising from free product constructions*, Canad. J. Math., **50** (1998), 323-341, MR 99d:46074, Zbl 919.46040.

[16] _____ , *Projections in free product C*-algebras*, Geom. Funct. Anal., **8** (1998), 1-16, MR 99d:46075, Zbl 907.46045; *Erratum*, idem., **10**(**4**) (2000), 975, CMP 1 791 146.

[17] G.G. Kasparov, *Hilbert C*-modules: Theorems of Stinespring and Voiculescu*, J. Operator Theory, **4** (1980), 133-150, MR 82b:46074, Zbl 456.46059.

[18] E.C. Lance, *Hilbert C*-modules, a Toolkit for Operator Algebraists*, London Math. Soc. Lecture Note Series, **210**, Cambridge University Press, 1995, MR 96k:46100, Zbl 822.46080.

[19] M.A. Rieffel, *Induced representations of C*-algebras*, Adv. Math., **13** (1974), 176-257, MR 50 #5489, Zbl 284.46040.

[20] _____ , *Morita equivalence for C*-algebras and W*-algebras*, J. Pure Appl. Algebra, **5** (1974), 51-96, MR 51 #3912, Zbl 295.46099.

[21] D. Voiculescu, *Symmetries of some reduced free product C*-algebras*, Operator Algebras and Their Connections with Topology and Ergodic Theory, Lecture Notes in Mathematics, **1132**, Springer-Verlag, (1985), 556-588, MR 87d:46075, Zbl 618.46048.

Institut de Mathématiques de Luminy
163, avenue de Luminy, case 907
F-13288 Marseille
France
E-mail address: blanch@iml.univ-mrs.fr

Dept. of Mathematics Texas A&M University
College Station TX 77843-3368
E-mail address: Ken.Dykema@math.tamu.edu

The Mathematical Sciences Research Institute
1000 Centennial Drive, #5070
Berkeley CA 94720-5070