

## Emergence of a Small World from Local Interactions: Modeling Acquaintance Networks

Jörn Davidsen, Holger Ebel, and Stefan Bornholdt\*

*Institut für Theoretische Physik, Universität Kiel, Leibnizstrasse 15, D-24098 Kiel, Germany*  
(Received 20 August 2001; revised manuscript received 6 December 2001; published 8 March 2002)

How do we make acquaintances? A simple observation from everyday experience is that often one of our acquaintances introduces us to one of his or her acquaintances. Such a simple triangle interaction may be viewed as the basis of the evolution of many social networks. Here, it is demonstrated that this assumption is sufficient to reproduce major nontrivial features of social networks: short path length, high clustering, and scale-free or exponential link distributions.

DOI: 10.1103/PhysRevLett.88.128701

PACS numbers: 89.75.Hc, 87.23.Ge, 89.75.Da

A remarkable feature of many complex systems is the occurrence of large and stable network structures as, for example, networks on the protein or gene level, ecological webs, communication networks, and social networks [1–3]. Simple models based on disordered networks are quite successful in describing basic properties of such systems. When addressing topological properties, however, neither random networks nor regular lattices provide an adequate framework. A helpful concept along this line is the idea of “small-world networks” introduced by Watts and Strogatz [4,5] which initiated an avalanche of scientific activity in this field [6–11]. Small-world networks interpolate between the two limiting cases of regular lattices with high local clustering and of random graphs with short distances between nodes. High clustering implies that, if node  $A$  is linked to node  $B$ , and  $B$  is linked to node  $C$ , there is an increased probability that  $A$  will also be linked to  $C$ . Another useful measure is the distance between two nodes, defined as the number of edges along the shortest path connecting them. A network is called a “small-world network” if it exhibits the following two characteristic properties [4,12]: (i) high clustering and (ii) a small average shortest path between two random nodes (sometimes called the diameter of the network), scaling logarithmically with the number of nodes. Thus, two randomly chosen nodes in the network are likely to be connected through only a small number of links. The most popular manifestation of a small world is known as “six degrees of separation,” a postulate by the social psychologist Stanley Milgram [13] stating that most pairs of people in the United States can be connected through a path of only about six acquaintances [14].

Let us here focus on social networks, and acquaintance networks in particular, which are typical examples for the small-world property [1,2,12]. What does the concept of small-world networks tell us about such real world systems? In its original definition [4,5] it serves as an elegant toy model demonstrating the consequences of high clustering and short path length. However, since these networks are derived from regular graphs, their applicability to real world systems is very limited. In particular, how a network in a natural system dynamically forms a small-world

topology, often starting from a completely random structure, remains unexplained. The main goal of this Letter is to provide one possible answer to this problem.

A similar problem of dynamical origin is faced (and much progress has been made) in a different, but not completely unrelated field: the dynamics of scale-free networks. Scale-free properties are commonly studied in diverse contexts from the stability of the internet [15] to the spreading of epidemics [16] and are observed in some social networks [1–3,17,18]. The origin of scale-free properties is well understood in terms of interactions that generate this topology dynamically, e.g., on the basis of network growth and preferential linking [2,3,19]. While these models generate scale-free structures they do not, in general, lead to clustering and are therefore of limited use when modeling social networks.

In this Letter, an attempt is made to unify ideas from the two fields of “small-world networks” and “scale-free networks” in order to address the dynamics of social networks and the dynamical emergence of the small-world structure. In particular, a simple dynamical model for the evolution of acquaintance networks is studied. It generates highly clustered networks with small average path lengths that scale logarithmically with network size. Furthermore, for small death-and-birth rates of nodes this model converges towards scale-free degree distributions, in addition to its small-world behavior. Basic ingredients are a local connection rule based on “transitive linking,” and a finite age of nodes.

To be specific, let us formulate a model of an acquaintance network with a fixed number  $N$  of nodes (as persons) and undirected links between those pairs of nodes that represent people who know each other. Acquaintance networks evolve, with new acquaintances forming between individuals, and old ones dying. Let us assume that people are introduced to each other by a common acquaintance and that the network is formed only by people who are still alive. The dynamics is defined as follows:

(i) One randomly chosen person picks two random acquaintances of his and introduces them to each another. If they have not met before, a new link between them is formed. In case the person chosen has less than two

acquaintances, he introduces himself to one other random person.

(ii) With probability  $p$ , one randomly chosen person is removed from the network, including all links connected to this node, and replaced by a new person with one randomly chosen acquaintance.

These steps are then iterated. Note that the number of nodes remains constant, neglecting fluctuations in the number of individuals in acquaintance networks. The finite age implies that the network reaches a stationary state which is an approximation of the behavior of many social networks, and is in contrast to most models based on network growth [2,18,20]. The probability  $p$  determines the separation of the two time scales in the model. In general, the rate at which people make social contacts can be as short as minutes or hours, while the time scale on which people join or leave the network may be as long as years or decades. In the following, we therefore focus on the regime  $p \ll 1$ .

Once the network reaches a statistically stationary state, one of its characteristic quantities is the degree distribution  $P(k)$  of the network. In Fig. 1, the degree distribution is shown for different values of  $p$ . Because of the limited lifetime of persons in the network, the observed numbers of acquaintances of different persons do not grow forever, but rather fall into some finite range. This can be seen in the degree distribution where the cutoff at high  $k$  results from the finite age of nodes. In the regime  $p \ll 1$ , the degree distribution is dominated by the transitive linking process (i), resulting in a power-law range which increases in size with decreasing  $p$ . For larger values of  $p$ , the Poissonian death process (ii) competes with the transitive linking process (i), resulting in a stretched exponential range in the degree distribution until, in the case  $p \approx 1$ , the random linking of (ii) dominates with its Poissonian

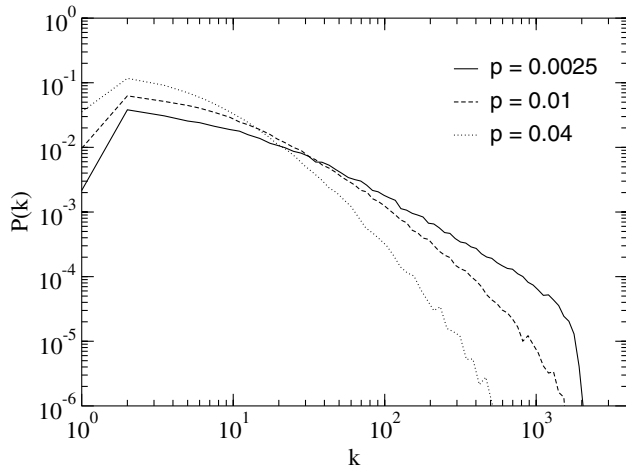


FIG. 1. The degree distribution  $P(k)$  of the model in the statistically stationary state. The distribution exhibits a power-law regime for small  $p$ , with an exponent of 1.35 for  $p = 0.0025$ . Note that the distribution is largely insensitive to system size  $N$ , which here is  $N = 7000$ . The exponential cutoff is a result of the finite age of nodes.

dynamics. Therefore, the above model generates degree distributions spanning scale-free and exponential regimes, both of which are observed in the statistics of social networks. For large enough network size  $N$ , the specific distribution  $P(k)$  only depends on the single free parameter of the model  $p$ . From experimental data, one can estimate  $p$  as well, which is typically very small  $p \ll 1$  such that the two time scales of the network dynamics are well separated.

As noted above, small-world networks are characterized by a high degree of clustering  $C$  and a small average shortest path length  $\ell$  which scales logarithmically with the number of nodes. The degree of clustering is measured by the clustering coefficient defined as follows: For a distinct node  $i$ , the clustering coefficient  $C_i$  is given by the ratio of existing links  $E_i$  between its  $k_i$  neighbors to the possible number of such connections  $\frac{1}{2}k_i(k_i - 1)$ . Then the clustering coefficient  $C$  of the network is defined as the average over all nodes,

$$C = \langle C_i \rangle_i = \left\langle \frac{2E_i}{k_i(k_i - 1)} \right\rangle_i. \quad (1)$$

For the above model,  $C$  can be related to the mean degree  $\langle k \rangle$  and  $p$  because in the stationary state the number of added links on average equals the number of removed links:

$$1 - C = p(\langle k \rangle - 1). \quad (2)$$

In Table I, the clustering coefficient of the above model in the stationary state—calculated according to Eq. (1)—is shown for different values of  $p$ . Within numerical precision, the same values are obtained via Eq. (2) as one can easily deduce from Table I. In comparison to the clustering of a random network with the same size and same mean degree  $C_{\text{rand}}$ , the model coefficient  $C$  is consistently of a much larger size. The clustering coefficient of a random network, i.e., a network with constant probability of linking each pair of nodes  $p_{\text{link}} = \langle k \rangle / (N - 1)$  and, therefore, a Poissonian distribution of the node degree, is just this probability  $C_{\text{rand}} = p_{\text{link}}$ . Obviously,  $C_{\text{rand}}$  is proportional to the mean degree  $\langle k \rangle$  for constant network size. For further comparison, let us derive an estimate  $C'$  for an upper bound of the average clustering coefficient of a network with the same degree distribution, but randomly

TABLE I. Clustering coefficient for different values of  $p$  and a network size of  $N = 7000$ .  $C'$  is an upper bound for the average clustering coefficient of a network with the same degree distribution without transitive linking.  $C_{\text{rand}}$  is the clustering coefficient of a random network of the same size and with the same average degree  $\langle k \rangle$ .

$p$	$\langle k \rangle$	$\langle k^2 \rangle$	$C$	$C'$	$C_{\text{rand}}$
0.04	14.9	912	0.45	0.036	0.0021
0.01	49.1	13 744	0.52	0.29	0.0070
0.0025	149.2	99 436	0.63	0.43	0.021

assigned links. Thus,  $C'$  provides an upper bound on the average clustering which one would expect solely from the degree distribution while neglecting transitive linking of the model. Using the generating function approach for graphs with arbitrary degree distributions [21] and assuming that fluctuations of the average degree of the neighborhood of a node can be neglected, an upper bound  $C'$  can be derived in terms of the first two moments:

$$C' = \frac{1}{\langle k \rangle N} \left( \frac{\langle k^2 \rangle}{\langle k \rangle} - 1 \right)^2. \quad (3)$$

This result holds exactly in the case of the Poissonian degree distribution of a random network ( $C_{\text{rand}} = C'$ ). As Table I shows, the network generated by the model exhibits an even higher average clustering coefficient than a network with links distributed randomly according to the same degree distribution. In particular, the network is much more clustered than a random network with a Poissonian distribution as required for the small-world property. Furthermore, the clustering is not as strongly dependent on the mean degree  $\langle k \rangle$  as  $C'$  of a network with the same degree distribution but no transitive linking. This observation is directly explained by Eq. (2).

The scaling of the average path length with system size is shown in Fig. 2. The data are consistent with a logarithmic behavior and, thus, our model meets the second requirement for a small-world behavior as well. Also, this is what one expects in the framework of a random graph with arbitrary degree distribution [21]. Moreover, we can compare the average path length of our model  $\ell$  with the path length  $\ell'$  of a network with the same degree distribution and randomly assigned links. Applying the generating function approach, we obtain

$$n_2 = \langle k^2 \rangle - \langle k \rangle, \quad (4)$$

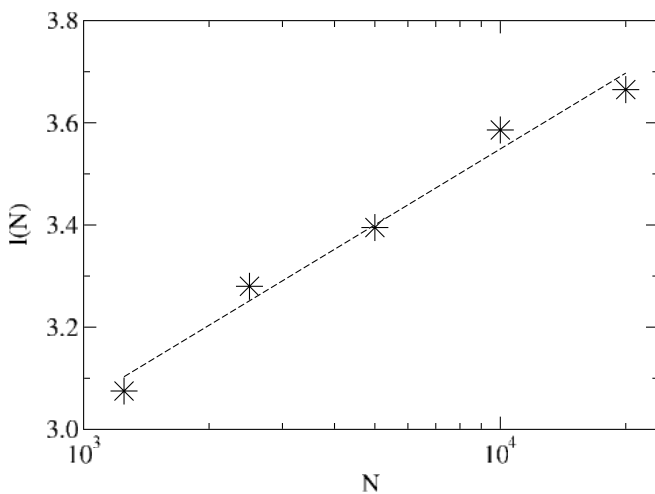


FIG. 2. The average path length  $\ell(N)$  as a function of system size  $N$  on a semilogarithmic scale ( $p = 0.04$ ). The data are in good agreement with a logarithmic fit (straight line).

i.e., the mean number of second neighbors  $n_2$  of a node is the difference between the second and the first moment of the degree distribution. Equation (4), in combination with the results in [21], leads to the following estimate of the average shortest path length:

$$\ell' \approx \frac{\log\left(\frac{N}{\langle k \rangle}\right)}{\log\left(\frac{\langle k^2 \rangle - \langle k \rangle}{\langle k \rangle}\right)} + 1. \quad (5)$$

For the Poissonian distribution of a random network, one obtains

$$\ell_{\text{rand}} \approx \frac{\log N}{\log \langle k \rangle}. \quad (6)$$

Note that only  $N$ ,  $\langle k \rangle$ , and  $\langle k^2 \rangle$  are used to estimate  $\ell$  as also used in the derivation of (3). With the help of Table I, one finds that  $\ell' \approx 1.59$  and  $\ell_{\text{rand}} \approx 1.77$  for  $p = 0.0025$ . Numerical simulations of our model similarly yield a very short path length of  $\ell = 2.38$  which is further evidence that the simple linking rule of our model leads to small-world behavior. Intuitively,  $\ell' < \ell_{\text{rand}}$  follows from the highly connected “hubs” present in the scale-free networks for  $p = 0.0025$ . The fact that  $\ell$  is slightly larger than  $\ell_{\text{rand}}$  is due to the fact that, in step (i) of the model, many links are used to build highly clustered neighborhoods. This price to pay for clustering, however, only slightly affects the small overall mean path length.

One example for an observed small-world effect is the network of coauthorships between physicists in high energy physics [17]. Nodes are researchers who are connected if they have coauthored a paper. In a recent study of the publications in the SPIRES database over the five year period 1995–1999, a graph was reconstructed from the data and analyzed [17,22]. The resulting network consists of 55 627 nodes with a mean degree  $\langle k_S \rangle = 173$ , a mean shortest path length  $\ell_S = 4.0$ , and a very high clustering coefficient  $C_S = 0.726$ . The degree distribution is consistent with a power law of exponent  $-1.2$  [17]. From these data, one can derive  $C'$  and  $\ell'$  for a network with the same degree distribution but random links to  $C' = 0.19$  and  $\ell' = 1.81$ . These numbers show that the real network exhibits clustering which is very much higher than would be expected from the degree distribution alone. Also, the path length is short but still larger than for a randomly linked network of the same degree distribution. Using the logarithmic scaling for  $\ell$ , the data of the example agree with the values of the above model. The basic assumptions made in the model are met by the data set, as the number of researchers in the sample is, to a good approximation, stationary, and as the small rate of researchers entering or leaving the system during the time frame of the sample justifies the regime of small  $p \ll 1$ .

The previous example demonstrates how our model can be applied to a social network in the dynamically stationary state. Moreover, the model studied here is also able to

accommodate other small-world scenarios, e.g., without a scale-free degree distribution [12] as the particular shape of the distribution varies, depending on the turnover rate  $p$ . The question of the origin of small-world behavior in social systems has led to other approaches as well. In an interesting model Mathias and Gopal [23] showed that a small-world topology can arise from the combined optimization of network distance and physical distance. Applications of this principle, however, are more likely to be found in the field of transportation networks rather than acquaintance networks considered here. A more similar approach to the study presented here has been taken by Jin *et al.* [24], who study a model which shares a mechanism similar to transitive linking as defined here. Otherwise, however, it is more complicated than we feel it needs to be, at least for some classes of social networks. Also its upper limit on the degree of a node, motivated by one specific trait of some acquaintance networks, makes the model less suitable to meet experimental data of social networks which exhibit broad degree distributions.

Even though our work has focused on social networks, possible applications also include biological systems. For example, the neural network of the fresh-water polyp *hydra* self-organizes by the continuous addition and removal of neurons maintaining a stationary size [25,26]. It is conceivable that some form of transitive linking may occur when new neurons form connections into an existing dynamical neural network, e.g., when linking between two neurons is guided by the mutual correlation of their activities.

In conclusion, a simple dynamical model for the emergence of small-world network structures has been studied. It is based on a local linking rule which connects nodes who share a common neighbor, as well as on a slow turnover of nodes and links in the system. The network approaches a dynamically stationary state with high clustering and small average path lengths which scale logarithmically with system size. Depending on a single free parameter, the turnover rate of nodes, this model interpolates between networks with a scale-free and an exponential degree distribution, both of which are observed in experimental data of social networks.

\*Email address: bornholdt@theo-physik.uni-kiel.de

- [1] S. H. Strogatz, *Nature (London)* **410**, 268 (2001), and references therein.
- [2] R. Albert and A.-L. Barabási, *Rev. Mod. Phys.* **47**, 74 (2002).
- [3] S. N. Dorogovtsev and J. F. F. Mendes, cond-mat/0106144.
- [4] D. J. Watts and S. H. Strogatz, *Nature (London)* **393**, 440 (1998).
- [5] D. J. Watts, *Small worlds: The Dynamics of Networks between Order and Randomness* (Princeton University Press, Princeton, NJ, 1998).
- [6] M. E. J. Newman, *J. Stat. Phys.* **101**, 819 (2000).
- [7] M. Barthélemy and L. A. N. Amaral, *Phys. Rev. Lett.* **82**, 3180 (1999).
- [8] M. E. J. Newman and D. J. Watts, *Phys. Rev. E* **60**, 7332 (1999).
- [9] R. Monasson, *Eur. Phys. J. B* **12**, 555 (1999).
- [10] A. Barrat and M. Weigt, *Eur. Phys. J. B* **13**, 547 (2000).
- [11] M. E. J. Newman, C. Moore, and D. J. Watts, *Phys. Rev. Lett.* **84**, 3201 (2000).
- [12] L. A. N. Amaral, A. Scala, M. Barthélemy, and H. E. Stanley, *Proc. Natl. Acad. Sci. U.S.A.* **97**, 11 149 (2000), and references therein.
- [13] S. Milgram, *Psych. Today* **2**, 60 (1967).
- [14] *The Small World*, edited by M. Kochen (Ablex, Norwood, 1989).
- [15] R. Albert, H. Jeong, and L.-A. Barabási, *Nature (London)* **406**, 378 (2000).
- [16] R. Pastor-Satorras and A. Vespignani, *Phys. Rev. Lett.* **86**, 3200 (2001).
- [17] M. E. J. Newman, *Phys. Rev. E* **64**, 016131 (2001).
- [18] A.-L. Barabási, H. Jeong, Z. Néda, E. Ravasz, A. Schubert, and T. Vicsek, cond-mat/0104162.
- [19] S. Bornholdt and H. Ebel, *Phys. Rev. E* **64**, 035104(R) (2001).
- [20] K. Klemm and V. Eguíluz, cond-mat/0107607.
- [21] M. E. J. Newman, S. H. Strogatz, and D. J. Watts, *Phys. Rev. E* **64**, 026118 (2001).
- [22] M. E. J. Newman, *Phys. Rev. E* **64**, 016132 (2001).
- [23] N. Mathias and V. Gopal, *Phys. Rev. E* **63**, 021117 (2001).
- [24] E. M. Jin, M. Girvan, and M. E. J. Newman, *Phys. Rev. E* **64**, 046132 (2001).
- [25] U. Technau *et al.*, *Proc. Natl. Acad. Sci. U.S.A.* **97**, 12 127 (2000).
- [26] G. Hager and C. N. David, *Development* **124**, 569 (1997).