Emergent Universe from Energy-Momentum Squared Gravity

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(Dated: March 28, 2022)

In order to bypass the big bang singularity, we develop an emergent universe scenario within a covariant extension of General Relativity known as *"Energy-Momentum Squared Gravity"*. The extra terms of the model emerge in the high energy regime. Considering dynamics in a Friedmann-Lemaître-Robertson-Walker background, critical points, representing stable Einstein static states of the phase space, result as solutions. It then turns out that as the equation of state parameter ω gradually declines from a constant value as $t \to -\infty$, eventually some of the static past eternal solutions find the chance to naturally enter into thermal history through a graceful exit mechanism. In this way, the successful realization of the emergent universe allows an expanding thermal history without the big bang singularity for the spatially flat universe free of cosmological constant.

PACS numbers: 98.80.-k, 04.50.Kd

I. INTRODUCTION

Although inflationary paradigm seems to be sufficiently supported by data so that one can take it as the default approach to describe the early universe, the initial singularity issue (the so-called Friedmann-Lemaître-Robertson-Walker (FLRW) big bang) is far from being clear and definitely solved in the framework of this paradigm. On the basis of some theorems [1], such singularities are generic and unavoidable, meaning that the classical spacetime has a threshold point beyond which the standard general relativity (GR) is not applicable [2, 3]. The singularities are commonly diagnosed by divergences of the scalar invariants of curvature or torsion tensors or the collapse of geodesics at some given points. Therefore, it seems that the initial singularity issue will potentially shed light on the answer to the question of whether our universe has a beginning or eternally existed. The lack of a solution for this fundamental issue in the context of inflationary cosmology[1] motivated many authors to construct some pre-inflationary scenarios such as: emergent universe (EU) [7, 8], cyclic/ekpyriotic scenarios [9–12] which are commonly non-singular or past eternal. Motivations by the widespread belief that including a quantum gravity (QG) effects at very short scales leads to the natural disappearance of singularities has led to some other singularity free cosmological models [13–16]. Those models have been merely derived from semi-classical corrections to QG, see [17–19], or considering non-local corrections [20, 21] as well.

The so-called EU scenario is one of the popular candidates that has been highly considered by several authors. This scenario, which is equivalent to *"a creation in absence of beginning of time"*, includes striking traits: no initial singularity (no beginning of time i.e infinite past), no horizon problem and no QG era. More exactly, Ellis et al [7, 8] proposed a scenario to overcome the initial singularity within

[1] It is worth recalling two points. First of all, the inflationary scenario is in conflict with singularity theorems by Penrose and Hawking, due to explicit violation of strong energy condition. Secondly, inflation cannot be eternal in the past since it suffers from geodesic incompleteness issue. For a detailed discussion about the Pros and Cons of inflationary cosmology, see [4, 5] and also [6].

the framework of GR. In this framework a closed[2] FLRW cosmology with a positive spatial curvature is preceded by an initially static state known as the Einstein Static Universe (ESU)[3] in the eternal past (instead of a big bang singularity). Finally this closed FLRW is superseded by the inflationary era. According to this scenario, the closed universe has existed eternally, but eventually, at some point, it begins inflating [26]. As a result, there are two required conditions for the scenario to provide a successful description for fixing the initial singularity: a self-consistent exit from the ESU i.e. stable as well as a graceful exit from the ESU to the inflation. Note that the former and the latter are respectively necessary and sufficient conditions for the singularity circumvention so that the failure in satisfying any of the two conditions will cease the EU scenario. Failure to meet the former condition caused this scenario to face a challenge in the first place. The original EU scenario failed to solve the big bang singularity issue successfully since Barrow et al [27] discovered that the ESU in GR is not stable, meaning that the universe in such an initial static state is not able to survive for a long time against the existence of dominating perturbations. However, the extreme physical conditions expected in the early universe (such as those arising from quantization of gravity, or corrections to GR) may turn the stability situation of initial state in favor of EU scenario. In other words, due to the failure of the EU scenario in the context of GR, modified theories of gravity may have the potential to improve the situation. This idea has led to several studies on the natural extensions of the original EU setup into modified gravitational theories with the aim to derive some promising outputs in comparison with GR, [28]-[54].

Recently, a modification of the matter Lagrangian (instead of gravitational Lagrangian) in a nonlinear way using a term proportional to $T_{\mu\nu}T^{\mu\nu}$ has been proposed as a new covariant generalization of GR [55, 56]. This theory is known as *"energy-momentum squared gravity"* (EMSG) and induces quadratic contributions to gravity from matter side (without the appearance of novel forms of fluid stresses such as the scalar field and so on [57]) so that it affects the cosmological dynamics considerably at high energy phases. In other words, the self-coupling of matter, instead of geometry, is assumed having interesting cosmological consequences, in particular, at early epochs. It is worth noticing that the so-called EMSG actually is a special case of theories with the general form of the Lagrangian as $f(R, T_{\mu\nu}T^{\mu\nu})$ which were first investigated in [58]. Naturally, the expected deviations from GR in early universe can lead to non-trivial consequences for some key issues in modern cosmology such as: initial singularity, inflation and big bang nucleosynthesis. Actually due to the lack of a final theory of QG, one of the motivations behind considering modified gravities, such as EMSG, is trying to remove the big bang singularity. A previous work [55] has shown that this theory is singularity free due to the bounce at early epochs. In other words, EMSG, by predicting a minimum length and a finite maximum energy density as common features of most QG approaches, cancels out the singularity issue of standard GR scenario of early universe. It is remarkable that it is the incentive to work with the EMSG model and also its generalized versions as it is not restricted to discarding the initial singularity, but, depending on the underlying EMSG model, it may result in some interesting modifications to the whole cosmic history, too. Other motivation for using EMSG come from trying to improve the usual paradigm in ΛCDM-based cosmology. Indeed, despite the ΛCDM is successful in fitting a wide range of the observational data, it not able to give a self-consistent description of the cosmic acceleration [59]. Tensions in measurements of the universe acceleration between early time and late time, as the so-called "Hubble tension", may be a robust confirmation in the direction of modifying the ΛCDM picture as recently reported in [60]. In other words, one may imagine the EMSG and its generalized models as phenomenological extensions of the ΛCDM, such that, despite the existence of a cosmological constant, the nonlinear matter Lagrangian, embedded in these models, leads to additional terms in Einstein's equations, which finally can be constrained via cosmological observations. The cosmological applications of this novel modified theory of gravity along with its generalized types has attracted much attention in recent years (see some recent studies as [61]-[75]). In [57, 62], the viability of EMSG has been studied via contrasting the relevant free parameter in light of observations. In [57],

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[2] Despite that most data suggest the universe is effectively spatially flat, meaning it has no curvature similar to a sheet of paper, recent Planck measurements do not exclude that it could be closed [22, 23].

[3] It may be interesting to know that the idea of ESU as a vital component in EU scenario actually was originated in the seminal papers by Eddington and Lemaître, respectively [24, 25] .

authors used the existing observational constraints on the masses and radii of neutron stars and derived some constraints on the EMSG free parameter. In [62], by adopting recent observational data such as *i*) cosmic chronometer and SNe Type-Ia Riess (292) $H(z) - z$ data-sets, *ii*) baryon acoustic oscillation (BAO) peak parameter and *iii*) cosmic microwave background (CMB) peak parameter, bounds on the model parameter α have been derived. It is worth mentioning that, for low redshift data, some constraints on the relevant model parameter of energy-momentum-powered gravity models have been achieved [67]. Although that constraints obtained on EMSG free parameter are tight, still this theory is at the play.

The above descriptions motivated us to look for another cosmological implication of EMSG model i.e. the full realization of the EU scenario. More exactly, we seek for the possibility whether EMSG model allows the EU scenario to be a viable solution of the early universe singularity. It is worth noticing that the study of the EU within the context of EMSG is well-motivated, because it changes the GR-based picture at early times, exactly when the EU scenario comes into play.

This manuscript is structured as follows. In Sec. II, we give a general review of the action as well as dynamical field equations in the framework of EMSG model. In Sec. III ,we extract ESU solutions related to the underlying gravity model and then investigate their stability using first order dynamical system analysis. In Sec. IV, we look for a realistic EU scenario via providing a graceful exit mechanism for the stable ESU solutions derived in Sec. III. Finally, we end the paper with a discussion summarized in Sec. V. Across this manuscript, for the signature of the spacetime metric, we will set the $(-, +, +, +)$ convention.

II. ENERGY-MOMENTUM SQUARED GRAVITY AND BACKGROUND FIELD EQUATIONS

In EMSG model, the action can be written as [55, 56]

$$S_{EMSG} = \frac{1}{2\kappa} \int \sqrt{-g}(R - 2\Lambda + \alpha(T_{\mu\nu}T^{\mu\nu})) \, d^4x + \int L_m \sqrt{-g} \, d^4x, \quad \kappa \equiv 8\pi G \tag{1}$$

in which the Einstein-Hilbert action with a cosmological constant Λ is extended via adding a self contracting term of energy-momentum tensor (EMT), $T_{\mu\nu}T^{\mu\nu}$. Here, R, L_m and α, respectively refer to the Ricci scalar, the matter Lagrangian density and a real constant which addresses the gravitational coupling strength of the underlying modification. The mass dimension of the EMSG parameter α is $[M]^{-6}$ and potentially can be any non-zero real value. So it is expected that, in early universe with high energy density, the EMSG is different from the standard GR so that, by going to lower energies, this deviation has to disappear. The action (1) can be re-express as follows

$$S_{EMSG} = \int \sqrt{-g}\left(\frac{R - 2\Lambda}{2\kappa} + L_{m,eff}\right) \, d^4x, \quad L_{m,eff} = L_m + \frac{\alpha}{2\kappa}T_{\mu\nu}T^{\mu\nu} \tag{2}$$

Varying the above action with respect to the inverse metric, we acquire the following modified Einstein's field equations

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} + \Lambda g_{\mu\nu} = \kappa T_{\mu\nu}^{\text{eff}} , \tag{3}$$

where we have defined

$$
\begin{aligned}
T_{\mu\nu}^{\text{eff}} &= -\frac{2}{\sqrt{-g}}\frac{\delta(\sqrt{-g}L_{m,eff})}{\delta g^{\mu\nu}} \\
&= T_{\mu\nu} + \frac{\alpha}{\kappa}\left(\frac{1}{2}T_{\alpha\beta}T^{\alpha\beta}g_{\mu\nu} + 2L_m(T_{\mu\nu} - \frac{1}{2}g_{\mu\nu}T) + TT_{\mu\nu} - 2T_\mu^\alpha T_{\nu\alpha} + 4T^{\alpha\beta}\frac{\partial^2 L_m}{\partial g^{\mu\nu}\partial g^{\alpha\beta}}\right) .
\end{aligned}
\tag{4}
$$

which clearly indicate that further degrees of freedom related to EMSG can be formally dealt under the standard of perfect fluids. See for example [93]. The above equations allow us to show that $\nabla^\mu T_{\mu\nu}^{\text{eff}} = 0$ not

$\nabla^\mu T_{\mu\nu} = 0$ [4]. It is assumed here that L_m is free of derivatives of the metric components. By considering the perfect fluid form for the EMT, we have

$$T_{\mu\nu} = (\rho + p)U_\mu U_\nu + pg_{\mu\nu}, \qquad T_{\mu\nu}T^{\mu\nu} = \rho^2 + 3p^2 \tag{5}$$

where ρ, p and U_μ respectively denote the energy density, the pressure and the co-moving four-velocity satisfying the conditions, $U_\mu U^\mu = -1$, $\nabla_\nu U^\mu U_\mu = 0$. Now by taking the Lagrangian $L_m = p$ [5], the last term in the right hand side of Eq.(4) cancels so that after inserting Eq.(5) into it, the modified Einstein's field equations finally read as

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} + \Lambda g_{\mu\nu} = \kappa((\rho + p)U_\mu U_\nu + pg_{\mu\nu}) + \alpha\left(\frac{1}{2}(\rho^2 + 3p^2)g_{\mu\nu} + (\rho + p)(\rho + 3p)U_\mu U_\nu\right). \tag{6}$$

The right-hand side of the above equation indicates that we no longer deal with a standard perfect fluid but with an effective fluid. More precisely, the whole budget of EMT does not come from standard matter fields only but there are other contributions coming from the non-Einsteinian part of the gravitational interaction [90–92]. The cosmological applications of such effective fluids, arising from generalized theories of gravity, have been considered in recent years, see e.g. [93–95]. The paradigm is that contributions coming from geometric degrees of freedom or further fields are brought to the r.h.s. of field equations and then the r.h.s. is globally considered as a source satisfying the Bianchi Identity. We want to investigate the implications of these field equations in cosmology. Therefore, we consider the FLRW cosmological models. Using the FLRW metric

$$ds^2 = -dt^2 + a^2(t)\left(\frac{dr^2}{1 - kr^2} + r^2\left(d\theta^2 + \sin^2\theta d\phi^2\right)\right), \tag{7}$$

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[4] Since EMT is a delicate issue of GR, as well as of any modified theory of gravity, it is worthy of further discussion. First of all, it is necessary to point out that the equation $\nabla^\mu T_{\mu\nu} = 0$, in curved spacetime, does not has the same interpretation of conservation law as $\partial^\mu T_{\mu\nu} = 0$ in absence of gravity. Actually, equation $\nabla^\mu T_{\mu\nu} = 0$ is a consequence of the contracted Bianchi Identity $\nabla^\mu G_{\mu\nu} = 0$ and expresses not only the conservation of energy-momentum tensor but also the exchange of energy and momentum of matter with gravitational field [76]. Thanks to the Equivalence Principle, one can locally adopt an *Inertial Frame* and then $\nabla^\mu T_{\mu\nu} = 0$ can be interpreted as a *local conservation law*. As a result, the meaning of energy conservation, in the context of GR and modified gravities as well, is a *local concept* and, in principle, it is not a general conserved quantity in presence of farther degrees of freedom and fields. Generally speaking, energy localization is one of the challenging issues in GR. Its investigation, in the context of alternative theories of gravity, attracted a lot of attention in recent years leading to generalizations of the gravitational stress-energy tensor, i.e. the so-called energy-momentum complex [77–79]. However, it is worth stressing that any consistent theory of gravity, in particular any extension of GR, must respect the Bianchi Identity and, subsequently, the local conservation equation. However, due to the appearance of corrections in modified theories of gravity (e.g. higher-order curvature terms, further scalar fields or $(T^{\mu\nu}T_{\mu\nu})^n$ as in the present case), we can deal with generalized local conservation laws as $\nabla^\mu T_{\mu\nu}^{\text{eff}} = 0$. In other words, these corrections can be dealt as effective fluids whose interpretation differs from the conventional matter fluids commonly considered as sources of the Einstein field equations [80, 81]. In the context of modified gravity, such as EMSG considered here, imposing the divergence-free condition in the r.h.s. of (3), the equation $\nabla^\mu T_{\mu\nu} = 0$ unlike the common imagine [82] is no longer a criterion to measure the healthiness of theory but, instead, its generalized counterpart $\nabla^\mu T_{\mu\nu}^{\text{eff}} = 0$ has to be considered. However, there is a noteworthy point related to the generalization of energy conditions in modified gravity. See [80, 81] for a detailed discussion. In this regard, it is helpful to refer also to [83] where conservation conditions are discussed for a wide class of models with matter non-minimally coupled to gravity. There extended conservation laws are obtained. In particular, the conservation issue can be probed from the perspective of particle creation. In [84], non-conservation issue related to non-minimal coupling between the matter and curvature is discussed. These results are criticized in [85]. The key argument of the criticism in [85] is based on the claim that non-minimal coupling, in essence, induces a change in the particle-momentum at a cosmological timescale, which is not relevant for the particle creation process. So, in the framework of particle creation phenomena, the EMSG as a theory with non-minimally coupling between curvature and matter, still can preserve the conservation properties.

[5] As it is well-known $L_m = p$ is not the unique choice for the matter Lagrangian representing the EMT perfect fluid. In fact, one can consider other cases like, for instance, $L_m = -\rho$. According to some results [86, 87] this is not problematic in the context of GR, but in some modified gravity models, such as EMSG, it is expected that the choice of L_m affects the dynamics, see [88, 89]. However, it is worth noticing that, in case of minimal coupling of the fluid with gravity, the two matter Lagrangian $L_m = p$ and $L_m = -\rho$ are equivalent. While there are no definitive criterion for the choice of these two matter Lagrangians, the former seems more natural. Hence, we choose it to conform with the EMSG literature [69].

characterized by the expansion scale factor $a(t)$ and the constant curvature parameter $k = -1, 0, 1$ (corresponding to spatially open, flat and closed universes respectively), we obtain the following modified dynamical equations

$$\frac{\dot{a}^2}{a^2} = -\frac{k}{a^2} + \frac{\Lambda}{3} + \kappa\frac{\rho}{3} + \frac{\alpha}{6}\left(\rho^2 + 3p^2 + 8\rho p\right), \tag{8}$$

and

$$\frac{\ddot{a}}{a} = -\kappa\frac{\rho + 3p}{6} + \frac{\Lambda}{3} - \frac{\alpha}{3}\left(\rho^2 + 3p^2 + 2\rho p\right), \tag{9}$$

for an isotropic and homogeneous universe where dot means derivative with respect to the cosmic time. Assuming that in a spacetime with the FLRW background metric, the matter field obeys a barotropic equation of state (EoS), $p/\rho = w = Const$, then we can extend the dynamical equations expressed in the following final forms

$$\frac{\dot{a}^2}{a^2} = -\frac{k}{a^2} + \frac{\Lambda}{3} + \kappa\frac{\rho}{3} + \frac{\alpha A_1(w)}{3}\rho^2, \tag{10}$$

and

$$\frac{\ddot{a}}{a} = -\kappa\frac{1 + 3w}{6}\rho + \frac{\Lambda}{3} - \frac{\alpha A_2(w)}{3}\rho^2, \tag{11}$$

with constants $A_{1,2}(w)$ designated by

$$A_1(w) \equiv \frac{3}{2}w^2 + 4w + \frac{1}{2}, \quad A_2(w) \equiv 3w^2 + 2w + 1 . \tag{12}$$

Merging the above dynamical equations, we obtain

$$\dot{H} = -\frac{1 + w}{2}\rho - \frac{\alpha}{3}(A_1 + A_2)\rho^2 + \frac{k}{a^2}, \tag{13}$$

where $H = \frac{\dot{a}}{a}$ and $\dot{H} = \frac{\ddot{a}}{a} - H^2$. Differentiating the Friedman equation (10), we see that the GR-based conservation equation also is modified as

$$\dot{\rho} + 3\frac{\dot{a}}{a}(1 + w)\rho\left(\frac{\kappa + \alpha\rho(1 + 3w)}{\kappa + 2\alpha\rho A_1(w)}\right) = 0. \tag{14}$$

The above modified conservation equation is the cosmological counterpart of $\nabla^\mu T_{\mu\nu}^{\text{eff}} = 0$. It means that we do not have to demand conservation of the ordinary $T_{\mu\nu}$, but the whole right-hand side of Einstein equation (3), that is conservation of $T_{\mu\nu}^{\text{eff}}$. As a consistency check, one can see immediately that if $\alpha = 0$, then dynamical equations (10), (11) as well as the modified conservation equation (14), reproduce exactly GR behaviors. Due to the fact that the source of the modification in the gravity model under consideration comes from the extension of EMT, it is quite natural that the modified terms in dynamical equations (10) and (11) appear as $\mathcal{O}(\rho^2)$. Eqs. (10) and (11), together with (13), form a complete dynamical set to investigate the existence and stability status of static solutions, as a prerequisite for the realization of EU. Hereinafter, for simplicity we will set $\kappa = 1$ in our calculations. However, the other values of κ can be considered under the same standard.

III. FIRST ORDER COSMOLOGICAL DYNAMICS AND STABILITY ANALYSIS OF EINSTEIN STATIC SOLUTIONS

The system of Eqs. (10)-(13) admit static solutions characterized by $\dot{a} = \ddot{a} = \dot{\rho} = 0$ (i.e. $a_{ES} = \rho_{ES} = $const) which address the ESU phase. Let us start with a flat FLRW metric. Considering the

cosmological dynamical system with $k = 0$ and applying this condition in Eqs. (10) and (11), we obtain the critical points (CPs) as follows

$$\text{CP: A} \quad \rho_{ES} = \frac{-1 \pm \sqrt{1 - 4\Lambda A_1 \alpha}}{2A_1 \alpha}, \tag{15}$$

with any positive arbitrary value of $\frac{1}{a_{ES}^2}$. However, for the above CPs to be physically meaningful, we have to check the non-negative energy density condition $\rho_{ES} \geq 0$. Thus, the CPs $A_\pm$, are physically meaningful if

$$
\begin{aligned}
&\Lambda = 0, \quad \alpha \neq 0, \quad \left(w < -\tfrac{4+\sqrt{13}}{3}, \quad \text{or} \quad -\tfrac{4+\sqrt{13}}{3} < w < \tfrac{-4+\sqrt{13}}{3} \quad \text{or} \quad w > \tfrac{-4+\sqrt{13}}{3} \right) \\
&\Lambda < 0, \quad \alpha < 0, \quad \left(-\tfrac{4+\sqrt{13}}{3} < w < \tfrac{-4+\sqrt{13}}{3} \quad \text{or} \quad \tfrac{-4+\sqrt{13}}{3} < w \leq -\tfrac{4}{3} + \tfrac{\sqrt{26+\frac{3}{\alpha\Lambda}}}{3\sqrt{2}} \right) \\
&\Lambda < 0, \quad 0 < \alpha < -\tfrac{3}{26\Lambda}, \quad \left(-\tfrac{4+\sqrt{13}}{3} < w < \tfrac{-4+\sqrt{13}}{3}, \quad \text{or} \quad w > \tfrac{-4+\sqrt{13}}{3} \right) \\
&\Lambda < 0, \quad \alpha > -\tfrac{3}{26\Lambda}, \quad \left(-\tfrac{4+\sqrt{13}}{3} < w \leq -\tfrac{4}{3} - \tfrac{\sqrt{26+\frac{3}{\alpha\Lambda}}}{3\sqrt{2}}, \quad \text{or} \quad -\tfrac{4}{3} + \tfrac{\sqrt{26+\frac{3}{\alpha\Lambda}}}{3\sqrt{2}} \leq w < \tfrac{-4+\sqrt{13}}{3}, \quad \text{or} \quad w > \tfrac{-4+\sqrt{13}}{3} \right)
\end{aligned} \tag{16}
$$

and

$$
\begin{aligned}
&\Lambda \geq 0, \quad \alpha < 0, \quad w > \tfrac{-4+\sqrt{13}}{3} \\
&\Lambda \geq 0, \quad \alpha > 0, \quad -\tfrac{4+\sqrt{13}}{3} < w < \tfrac{-4+\sqrt{13}}{3} \\
&\Lambda < 0, \quad 0 < \alpha < -\tfrac{3}{26\Lambda}, \quad -\tfrac{4+\sqrt{13}}{3} < w < \tfrac{-4+\sqrt{13}}{3}, \\
&\Lambda < 0, \quad \alpha > -\tfrac{3}{26\Lambda}, \quad \left(-\tfrac{4+\sqrt{13}}{3} < w \leq -\tfrac{4}{3} - \tfrac{\sqrt{26+\frac{3}{\alpha\Lambda}}}{3\sqrt{2}}, \quad \text{or} \quad -\tfrac{4}{3} + \tfrac{\sqrt{26+\frac{3}{\alpha\Lambda}}}{3\sqrt{2}} \leq w < \tfrac{-4+\sqrt{13}}{3}, \quad \text{or} \quad w > \tfrac{-4+\sqrt{13}}{3} \right)
\end{aligned} \tag{17}
$$

respectively. So in case of satisfying any of conditions released in Eqs. (16) and (17), the physical meaning of the CPs $A_\pm$ can be guaranteed, respectively.

The dynamical systems with $k = \pm 1$ can be obtained by applying the condition related to ESU in Eqs. (10) and (11). We obtain the following CPs

$$
\begin{aligned}
\text{CP: C} \quad \frac{1}{a_{ES}^2} &= \frac{(1+3w)A_1\left(-1\pm\sqrt{1-\frac{16\Lambda A_2\alpha}{(1+3w)^2}}\right)^2}{48\alpha A_2^2} + \frac{(1+3w)\left(-1\pm\sqrt{1-\frac{16\Lambda A_2\alpha}{(1+3w)^2}}\right)}{12\alpha A_2} + \frac{\Lambda}{3}, \\
\text{CP: D} \quad \frac{1}{a_{ES}^2} &= -\frac{(1+3w)A_1\left(-1\pm\sqrt{1-\frac{16\Lambda A_2\alpha}{(1+3w)^2}}\right)^2}{48\alpha A_2^2} - \frac{(1+3w)\left(-1\pm\sqrt{1-\frac{16\Lambda\alpha A_2}{(1+3w)^2}}\right)}{12\alpha A_2} - \frac{\Lambda}{3},
\end{aligned} \tag{18}
$$

with the energy density

$$\rho_{ES} = \frac{(1+3w)\left(-1\pm\sqrt{1-\frac{16\Lambda A_2\alpha}{(1+3w)^2}}\right)}{4A_2\alpha} \tag{19}$$

for both CPs. Note that the CPs $C_\pm$ and $D_\pm$ come from the closed and open universes, respectively. Here because of the added conditions $\frac{1}{a_{ES}^2} > 0$ and $\rho_{ES} \geq 0$, the extraction of the limits on parameters (w, α, Λ) to ensure the physical meaning of the aforementioned CPs is very complicated. However, we will do that for CPs $C_\pm$ and $D_\pm$, via the parameter space plots in terms of $w - \alpha$ with some positive, negative and zero fixed values of Λ. For similar analysis in extended gravity theories, see e.g. [90–92].

Now, in order to investigate the stability of these CPs, we follow the strategy that the second order dynamics of the cosmological model after introducing two variables $X_1 = a, \quad X_2 = \dot{a}$ can be reduced to the first order dynamics, as proposed for instance in Refs. [34, 35, 39, 43]. Subsequently, we have a system of coupled equations as follows

$$
\begin{aligned}
\dot{X}_1 &= X_2 = \mathcal{G}_1(X_1, X_2) , \\
\dot{X}_2 &= \frac{X_2^2}{X_1} - \left(\frac{1+w}{2}\rho + \frac{\alpha}{3}(A_1 + A_2)\rho^2 \right) X_1 + \frac{k}{X_1^2} = \mathcal{G}_2(X_1, X_2) .
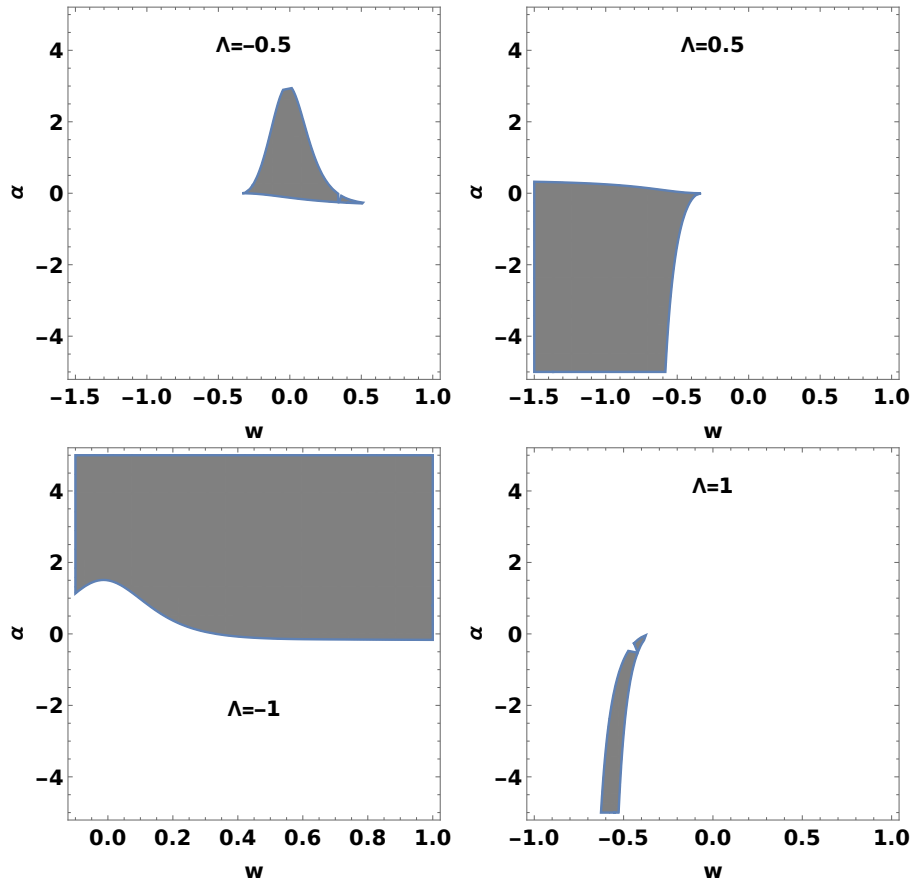\end{aligned} \tag{20}
$$

FIG. 1: Regions of existence in the (w, α) parameter space for the stable CPs C_+ (up row) and D_+ (bottom row). C_+ and D_+ refer to the FLRW with $k = 1$ and $k = -1$, respectively.

The behavior of the coupled equations near the critical points can be studied by the Jacobian matrix of the coupled system. More precisely, the eigenvalues of the Jacobian matrix are related to the stability or instability of the system. Moreover, we extract the eigenvalues squared, λ^2, via the following Jacobian matrix

$$J\left(\mathcal{G}_1(X_1, X_2), \mathcal{G}_2(X_1, X_2)\right) = \begin{pmatrix} \frac{\partial \mathcal{G}_1}{\partial X_1} & \frac{\partial \mathcal{G}_1}{\partial X_2} \\ \frac{\partial \mathcal{G}_2}{\partial X_1} & \frac{\partial \mathcal{G}_2}{\partial X_2} \end{pmatrix} \quad \Rightarrow \quad \lambda^2 = \frac{\partial \mathcal{G}_2}{\partial X_1}|_{(a_{ES}, \rho_{ES})} =$$

$$-\left(\frac{3}{2}\alpha w^2 + 2\alpha w + \frac{\alpha}{2}\right)\rho_{ES}^2 - \frac{1+w}{2}\rho_{ES} - \frac{k}{a_{ES}^2} \ . \tag{21}$$

When $\lambda^2 < 0$, we can consider the relevant physical CP as a stable ESU. Physically, $\lambda^2 < 0$ means that the relevant CP is a stable concentric center and if it undergoes some small perturbations, it does not collapse. In contrast, when $\lambda^2 > 0$, we expect to encounter an unstable saddle point [6].

Let us start our analysis with CPs $A_\pm$ which come from the flat FLRW case ($k = 0$). By imposing the

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[6] It is important to mention that $\lambda^2 > 0$ does not always address an unstable saddle point since conditions $\frac{\partial \mathcal{G}_1}{\partial X_1} \neq \frac{\partial \mathcal{G}_2}{\partial X_1}$ and $\frac{\partial \mathcal{G}_1}{\partial X_2} = 0 = \frac{\partial \mathcal{G}_2}{\partial X_2}$, in the Jacobian matrix (21) can lead to $\lambda^2 > 0$ which represents a stable node.
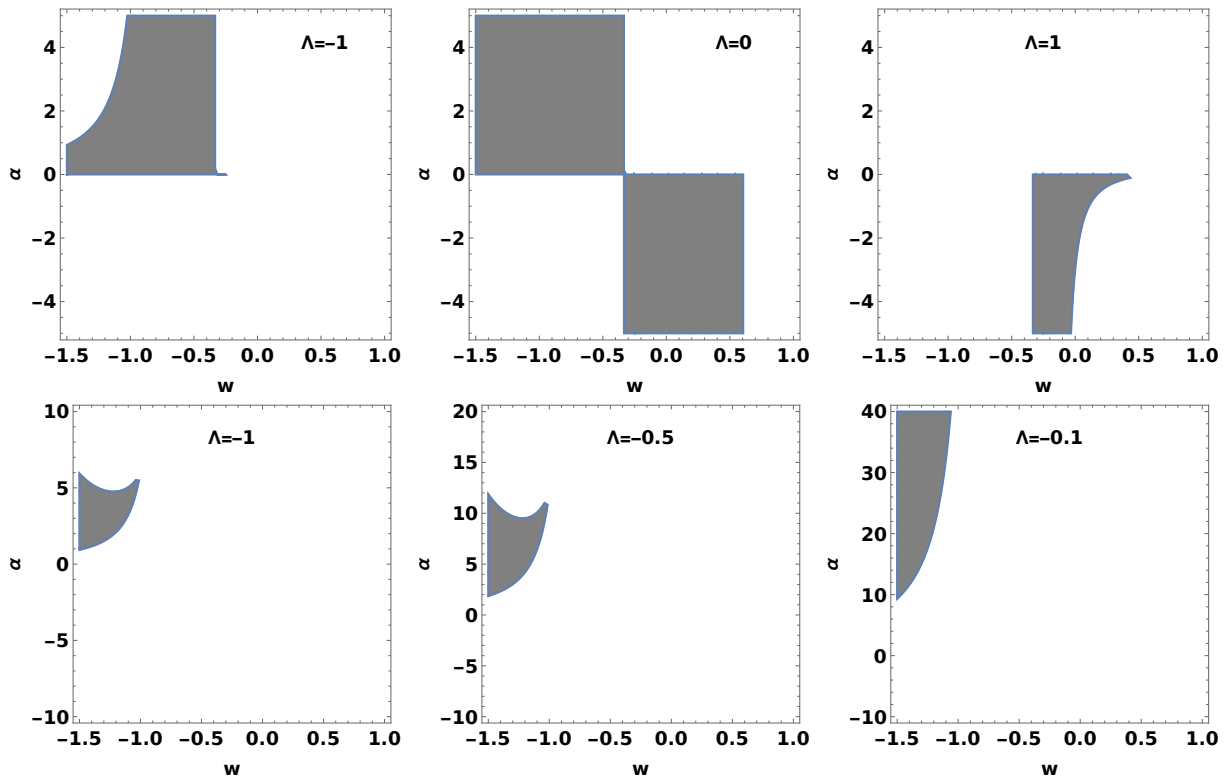
8

FIG. 2: Regions of existence in the (w, α) parameter space for the stable CPs C_- (up row) and D_- (bottom row). C_- and D_- refer to the FLRW with $k = 1$ and $k = -1$, respectively.
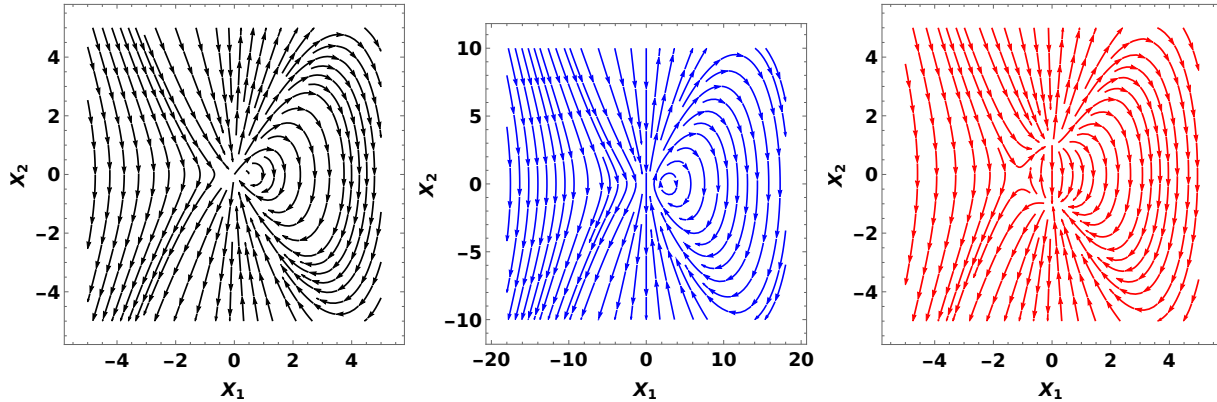

FIG. 3: The phase diagram in $(a, \dot{a})$ or (X_1, X_2) space for spatially flat, closed and open cosmology with the CPs A_-, C_+ and D_+, respectively from left to right panels. We use numerical values $(0, 2, 0)$, $(-0.8, -4, 0.5)$ and $(0, 5, -1)$ for the set parameters (w, α, Λ), respectively.

conditions $\lambda^2 < 0$ as well as $\rho_{ES} \geq 0$ simultaneously, we get the following constraints on the set (w, α, Λ)

$$\Lambda < 0, \quad \alpha \leq \tfrac{3-\sqrt{13}}{4\Lambda}, \quad -1 < w < \sqrt{\tfrac{4}{54\alpha\Lambda+9}} - \tfrac{1}{3}$$

$$\Lambda < 0, \quad \tfrac{3-\sqrt{13}}{4\Lambda} < \alpha \leq \tfrac{1}{2\Lambda}, \quad \left(-1 < w < \tfrac{\sqrt{13}-4}{3} \text{ or } \tfrac{\sqrt{13}-4}{3} < w < \sqrt{\tfrac{4}{54\alpha\Lambda+9}} - \tfrac{1}{3} \right)$$

$$\Lambda < 0, \quad \tfrac{1}{2\Lambda} < \alpha < 0, \quad \left(-1 < w < \tfrac{\sqrt{13}-4}{3}, \text{ or } \tfrac{\sqrt{13}-4}{3} < w \leq \sqrt{\tfrac{26\alpha\Lambda+3}{18\alpha\Lambda}} - \tfrac{4}{3} \right)$$

$$\Lambda < 0, \quad 0 < \alpha \leq -\tfrac{1}{8\Lambda}, \quad \left(-1 < w < \tfrac{\sqrt{13}-4}{3} \text{ or } w > \tfrac{\sqrt{13}-4}{3} \right)$$

$$\Lambda < 0, \quad -\tfrac{1}{8\Lambda} < \alpha \leq \tfrac{3}{4\Lambda} + \tfrac{\sqrt{13}}{4|\Lambda|}, \quad \left(-\sqrt{\tfrac{4}{54\alpha\Lambda+9}} - \tfrac{1}{3} < w \leq -\sqrt{\tfrac{26\alpha\Lambda+3}{18\alpha\Lambda}} - \tfrac{4}{3} \text{ or} \right.$$

$$\left. \sqrt{\tfrac{26\alpha\Lambda+3}{18\alpha\Lambda}} - \tfrac{4}{3} \leq w < \tfrac{\sqrt{13}-4}{3} \text{ or } w > \tfrac{\sqrt{13}-4}{3} \right) \tag{22}$$

$$\Lambda < 0, \quad \tfrac{3}{4\Lambda} + \tfrac{\sqrt{13}}{4|\Lambda|} < \alpha < -\tfrac{1}{6\Lambda}, \quad \left(-\sqrt{\tfrac{4}{54\alpha\Lambda+9}} - \tfrac{1}{3} < w < -\tfrac{4+\sqrt{13}}{3} \text{ or } -\tfrac{4+\sqrt{13}}{3} < w \leq -\sqrt{\tfrac{26\alpha\Lambda+3}{18\alpha\Lambda}} - \tfrac{4}{3} \text{ or} \right.$$

$$\left. \sqrt{\tfrac{26\alpha\Lambda+3}{18\alpha\Lambda}} - \tfrac{4}{3} \leq w < \tfrac{\sqrt{13}-4}{3} \text{ or } w > \tfrac{\sqrt{13}-4}{3} \right)$$

$$\Lambda < 0, \quad \alpha \geq -\tfrac{1}{6\Lambda}, \quad \left(w < -\tfrac{4+\sqrt{13}}{3} \text{ or } -\tfrac{4+\sqrt{13}}{3} < w \leq -\sqrt{\tfrac{26\alpha\Lambda+3}{18\alpha\Lambda}} - \tfrac{4}{3} \text{ or } \sqrt{\tfrac{26\alpha\Lambda+3}{18\alpha\Lambda}} - \tfrac{4}{3} \leq w < \tfrac{\sqrt{13}-4}{3} \text{ or} \right.$$

$$\left. w > \tfrac{\sqrt{13}-4}{3} \right)$$

and

$$\Lambda = 0, \quad \alpha < 0, \quad w > \tfrac{1}{3}$$

$$\Lambda = 0, \quad \alpha > 0, \quad \left(-\tfrac{4+\sqrt{13}}{3} < w < -1 \text{ or } -1 < w < \tfrac{\sqrt{13}-4}{3} \right)$$

$$\Lambda > 0, \quad -\tfrac{1}{6\Lambda} < \alpha < 0, \quad w > \sqrt{\tfrac{4}{54\alpha\Lambda+9}} - \tfrac{1}{3}$$

$$\Lambda > 0, \quad \alpha > 0, \quad \left(-\tfrac{4+\sqrt{13}}{3} < w < -1 \text{ or } -\sqrt{\tfrac{4}{54\alpha\Lambda+9}} - \tfrac{1}{3} < w < \tfrac{\sqrt{13}-4}{3} \right) \tag{23}$$

$$\Lambda < 0, \quad \tfrac{1}{2\Lambda} < \alpha < 0, \quad \sqrt{\tfrac{4}{54\alpha\Lambda+9}} - \tfrac{1}{3} < w \leq \sqrt{\tfrac{26\alpha\Lambda+3}{18\alpha\Lambda}} - \tfrac{4}{3}$$

$$\Lambda < 0, \quad 0 < \alpha \leq -\tfrac{1}{8\Lambda}, \quad \left(-\tfrac{4+\sqrt{13}}{3} < w < -\sqrt{\tfrac{4}{54\alpha\Lambda+9}} - \tfrac{1}{3} \text{ or } -1 < w < \tfrac{\sqrt{13}-4}{3} \right)$$

respectively corresponding to the CPs A_+, A_-. As a result, CPs $A_\pm$ can address the physical stable ESUs in the context of EMSG model with a flat spacetime, if the above conditions are satisfied.

Let's go to CPs $C_\pm$ and $D_\pm$ arising from the spatially non-flat cases. Here, by imposing the conditions $\rho_{ES} \geq 0$ and $\frac{1}{a_{ES}^2} > 0$ along with the stability condition $\lambda^2 < 0$, it is not possible to list constraints such as (22) and (23). Instead we illustrate the parameter space plots in terms of $w - \alpha$ which address the allowed regions for existence of real and stable ESUs, see Figs. 1 and 2. As it is clear, the only CP which admits three values for the cosmological constant is C_-. The CPs C_+ and D_+ admit values $\Lambda < 0$ and $\Lambda > 0$ while for the CP D_- just the value $\Lambda < 0$ is allowed. As an additional check, in the Fig. 3 optionally for three CPs A_-, C_+ and D_+, we present the phase-space behavior for various set parameters (w, α, Λ).

An important issue has to be clarified at this point. The EMSG scenario modifies GR in the early phase of the universe. According to this statement, the CPs seem no longer applicable in classical evolution, because, in the EMSG phase, the universe undergoes a quantum regime. In this situation, the classical evolution and expansion seem forbidden. To address this concern, one has to note the following discussion. In [55], it is shown that, in the EMSG picture, by controlling the values of free parameter α, the universe no longer enter a quantum era during its evolution. In other words, to guarantee that the CPs are free from quantum effects and allow the classical evolution, one has to impose additional conditions as $\rho_{ES} < \rho_p$ and $a_{ES} < \ell_p$ where ρ_p and ℓ_p are the Planck energy density and the Planck length, respectively. More precisely, in our set up, the above mentioned constraints read as $\frac{1}{a_{ES}^2} < 8\pi$ and $\rho_{ES} < 64\pi$ since c, $\hbar$, $\kappa = 1$ and $G = \frac{1}{8\pi}$. All numerical values taken into account for the EMSG parameter α throughout this paper satisfy both constraints, indicating that our analysis is safely classic and there is no need for any quantum considerations on the CPs. According to this result, the classical evolution is allowed.

A.   Non-Singular oscillations around stable ESU(s)

To ensure that the previous results do show stable points, we perform analysis of small perturbations around the ESU phases. In what follows, we will show that if we impose some small perturbations around the above mentioned stable CPs (in particular spatially non-flat cases), the perturbations will undergo an infinite series of non-singular oscillations around the inception state. For this purpose, we linearly perturb Eqs. (10) and (11) around the ESUs. We define the perturbations in the scale factor and matter density as follows

$$
\begin{aligned}
a(t) &\to a_{ES}(1 + \delta a(t)), \\
\rho(t) &\to \rho_{ES}(1 + \delta\rho(t)) ,
\end{aligned}
\tag{24}
$$

and also we have the linear expansions

$$
\begin{aligned}
(1 + \delta a(t))^n &\simeq 1 + n\delta a(t), \\
(1 + \delta\rho(t))^n &\simeq 1 + n\delta\rho(t).
\end{aligned}
\tag{25}
$$

After discarding the cross differential and non-linear terms, Eq. (10) yields

$$
\frac{\delta\rho}{\delta a} = -\frac{6k}{a_{ES}^2(\rho_{ES} + 2\alpha A_1)} .
\tag{26}
$$

In the same way, using Eq. (24) to perturb Eq. (11) and replacing the above relation, we acquire

$$
\delta\ddot{a} - \left(\frac{(1 + 3w)k\rho_{ES} + 4\alpha k A_2 \rho_{ES}^2}{a_{ES}^2(\rho_{ES} + 2\alpha A_1)}\right)\delta a = 0 .
\tag{27}
$$

Now Eq. (27) allows us to consider the stability of the CPs extracted previously, via an alternative way i.e. a small perturbation around CPs. As a cross-check by turning off the coupling parameter α in Eq. (27), we recover what is expected in the context of standard GR [27]. In Figs. 4 and 5, we represent the evolution of the scale factor derived from Eq. (27) corresponding to CPs $C_\pm$ and $D_\pm$ with some optional parameters consistence with Figs. 1 and 2. We observe the universe displays small oscillations around the relevant ESUs.
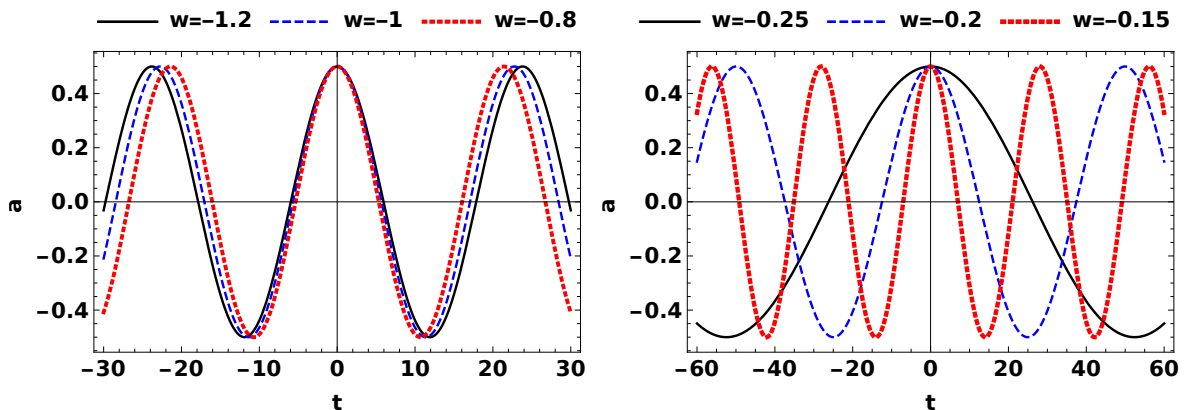

FIG. 4: The evolutionary curves of scale factor corresponding to CPs C_- (left panel) and C_+ (right panel) in terms of time with different values of w and numerical values $\Lambda = 0.5$, $\alpha = -1$ and $\Lambda = 0$, $\alpha = -1$, respectively.
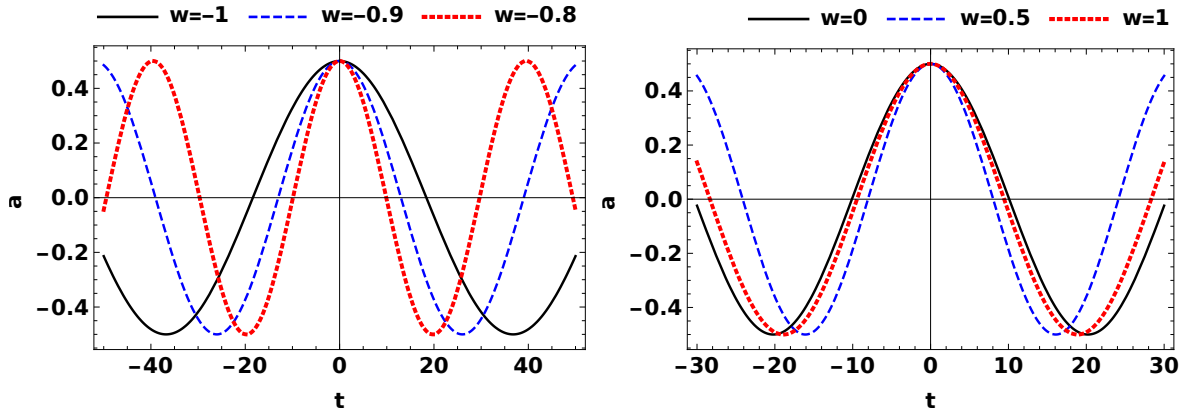
FIG. 5: The evolutionary curves of scale factor corresponding to CPs D_- (left panel) and D_+ (right panel) in terms of time with different values of w and numerical values $\Lambda = -1$, $\alpha = 2$ and $\Lambda = -0.5$, $\alpha = 5$, respectively.

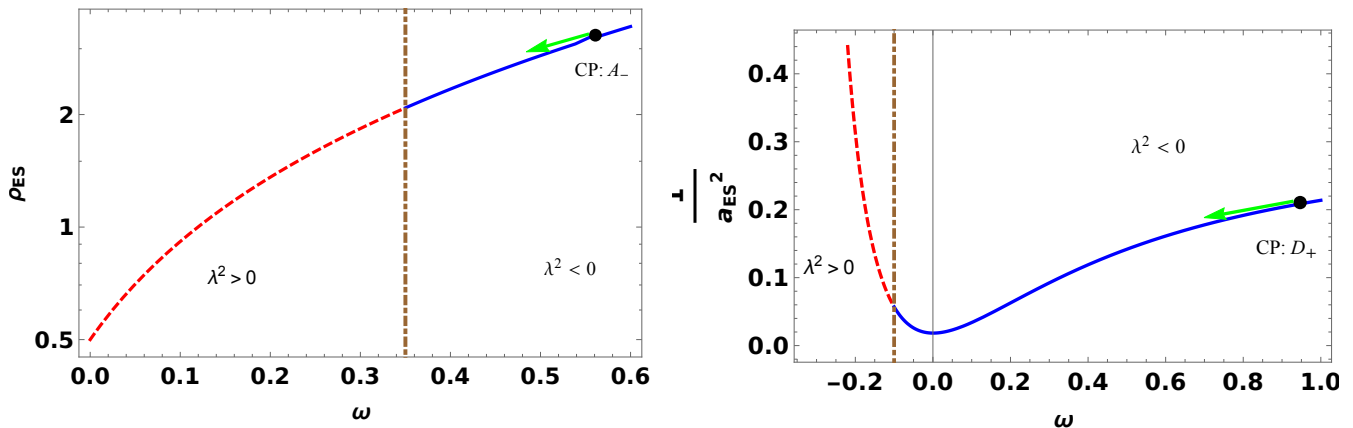FIG. 6: The general trend of transition of CPs A_- corresponding to $\Lambda = 0$, $\alpha < 0$ (left panel) and D_+ corresponding to $\Lambda < 0$, $\alpha > 0$ (right panel) from stable region with $\lambda^2 < 0$ towards the unstable region with $\lambda^2 > 0$, as ω decreases. The vertical brown dashed-dotted line represents the critical value of ω where this transition occurs.

IV. THE REALIZATION OF EMERGENT COSMOLOGY SCENARIO

In this section, we intend to find a graceful exit mechanism for the stable CPs in order to enter the standard thermal history of the universe by joining the CPs to the inflationary period. More precisely, by slowly reducing the universe equation of state parameter w, we look for a phase transition from a stable state to an unstable one for the aforementioned ESU solutions (CPs). In this way, we deal with a non-singular early universe in a stable state which finally will emerge in a standard expanding universe.

We start with CPs derived for the spatially flat case i.e. $A_\pm$. To extract a stable to unstable phase transition we need to know under what conditions the underlying CPs are both physical and unstable. Hence, for CPs $A_\pm$, we demand the conditions $\lambda^2 > 0$ as well as $\rho_{ES} \geq 0$. This will yield the following

constraints on the set parameters (w, α, Λ)

$$
\begin{aligned}
&\Lambda < 0, \quad \alpha < \tfrac{3-\sqrt{13}}{4\Lambda} \,, \quad \left(-\sqrt{\tfrac{26\alpha\Lambda+3}{18\alpha\Lambda}} - \tfrac{4}{3} \le w < -\tfrac{4+\sqrt{13}}{3} \text{ or } -\tfrac{4+\sqrt{13}}{3} < w < -1 \text{ or} \right. \\
&\sqrt{\tfrac{4}{54\alpha\Lambda+9}} - \tfrac{1}{3} < w < \tfrac{\sqrt{13}-4}{3} \text{ or } \tfrac{\sqrt{13}-4}{3} < w \le \sqrt{\tfrac{26\alpha\Lambda+3}{18\alpha\Lambda}} - \tfrac{4}{3} \Big) \\
&\Lambda < 0, \quad \alpha = \tfrac{3}{4\Lambda} - \tfrac{\sqrt{13}}{4|\Lambda|}, \quad \left(-\sqrt{\tfrac{26\alpha\Lambda+3}{18\alpha\Lambda}} - \tfrac{4}{3} \le w < -\tfrac{4+\sqrt{13}}{3} \text{ or } -\tfrac{4+\sqrt{13}}{3} < w < -1 \text{ or} \right. \\
&\tfrac{\sqrt{13}-4}{3} < w \le \sqrt{\tfrac{26\alpha\Lambda+3}{18\alpha\Lambda}} - \tfrac{4}{3} \Big) \\
&\Lambda < 0 \,, \quad \tfrac{1}{2\Lambda} < \alpha < 0, \quad \left(-\sqrt{\tfrac{26\alpha\Lambda+3}{18\alpha\Lambda}} - \tfrac{4}{3} \le w < \tfrac{1}{3}, \text{ or } -\tfrac{\sqrt{13}+4}{3} < w < -1 \right) \\
&\Lambda < 0, \quad 0 < \alpha \le -\tfrac{3}{26\Lambda}, \quad \left(w < -\tfrac{\sqrt{13}+4}{3} \text{ or } -\tfrac{\sqrt{13}+4}{3} < w < -1 \right) \\
&\Lambda < 0, \quad -\tfrac{3}{26\Lambda} < \alpha < -\tfrac{1}{8\Lambda} \,, \quad \left(w < -\tfrac{4+\sqrt{13}}{3} \text{ or } -\tfrac{4+\sqrt{13}}{3} < w \le -\sqrt{\tfrac{26\alpha\Lambda+3}{18\alpha\Lambda}} - \tfrac{4}{3} \text{ or } \sqrt{\tfrac{26\alpha\Lambda+3}{18\alpha\Lambda}} - \tfrac{4}{3} \le w < -1 \right) \\
&\Lambda < 0, \quad \alpha = -\tfrac{1}{8\Lambda}, \quad \left(w < -\tfrac{4+\sqrt{13}}{3} \text{ or } -\tfrac{4+\sqrt{13}}{3} < w < -\sqrt{\tfrac{26\alpha\Lambda+3}{18\alpha\Lambda}} - \tfrac{4}{3} \right) \\
&\Lambda < 0, \quad -\tfrac{1}{8\Lambda} < \alpha < \tfrac{3}{4\Lambda} + \tfrac{\sqrt{13}}{4|\Lambda|}, \quad \left(w < -\tfrac{3}{4\Lambda} + \tfrac{\sqrt{13}}{4|\Lambda|} \text{ or } -\tfrac{4+\sqrt{13}}{3} < w < -\sqrt{\tfrac{4}{54\alpha\Lambda+9}} - \tfrac{1}{3} \right) \\
&\Lambda < 0, \quad \alpha = \tfrac{3}{4\Lambda} + \tfrac{\sqrt{13}}{4|\Lambda|}, \quad w < -\tfrac{4+\sqrt{13}}{3} \\
&\Lambda < 0, \quad \tfrac{3}{4\Lambda} + \tfrac{\sqrt{13}}{4|\Lambda|} < \alpha < -\tfrac{1}{6\Lambda}, \quad w < -\sqrt{\tfrac{4}{54\alpha\Lambda+9}} - \tfrac{1}{3}
\end{aligned}
\tag{28}
$$

and

$$
\begin{aligned}
&\Lambda = 0, \quad \alpha < 0, \quad \left(w < -\tfrac{4+\sqrt{13}}{3} \text{ or } \tfrac{\sqrt{13}-4}{3} < w < \tfrac{1}{3} \right) \\
&\Lambda > 0, \quad \alpha \le -\tfrac{1}{6\Lambda}, \quad \left(w < -\tfrac{4+\sqrt{13}}{3} \text{ or } w > \tfrac{\sqrt{13}-4}{3} \right) \\
&\Lambda > 0, \quad -\tfrac{1}{6\Lambda} < \alpha < 0, \quad \left(w < -\tfrac{4+\sqrt{13}}{3} \text{ or } \tfrac{\sqrt{13}-4}{3} < w < \sqrt{\tfrac{4}{54\alpha\Lambda+9}} - \tfrac{1}{3} \right) \\
&\Lambda > 0, \quad \alpha > 0, \quad -1 < w < -\sqrt{\tfrac{4}{54\alpha\Lambda+9}} - \tfrac{1}{3} \\
&\Lambda < 0, \quad \alpha < \tfrac{1}{2\Lambda}, \quad \left(-\sqrt{\tfrac{26\alpha\Lambda+3}{18\alpha\Lambda}} - \tfrac{4}{3} \le w < -\tfrac{4+\sqrt{13}}{3} \text{ or } \tfrac{\sqrt{13}-4}{3} < w \le \sqrt{\tfrac{26\alpha\Lambda+3}{18\alpha\Lambda}} - \tfrac{4}{3} \right) \\
&\Lambda < 0, \quad \alpha = \tfrac{1}{2\Lambda}, \quad \left(-\sqrt{\tfrac{26\alpha\Lambda+3}{18\alpha\Lambda}} - \tfrac{4}{3} \le w < -\tfrac{4+\sqrt{13}}{3} \text{ or } \tfrac{\sqrt{13}-4}{3} < w < \sqrt{\tfrac{26\alpha\Lambda+3}{18\alpha\Lambda}} - \tfrac{4}{3} \right) \\
&\Lambda < 0, \quad \tfrac{1}{2\Lambda} < \alpha < 0, \quad \left(-\sqrt{\tfrac{26\alpha\Lambda+3}{18\alpha\Lambda}} - \tfrac{4}{3} \le w < -\tfrac{4+\sqrt{13}}{3} \text{ or } \tfrac{\sqrt{13}-4}{3} < w < \sqrt{\tfrac{4}{54\alpha\Lambda+9}} - \tfrac{1}{3} \right) \\
&\Lambda < 0, \quad 0 < \alpha \le -\tfrac{3}{26\Lambda}, \quad -\sqrt{\tfrac{4}{54\alpha\Lambda+9}} - \tfrac{1}{3} < w < -1 \\
&\Lambda < 0, \quad -\tfrac{3}{26\Lambda} < \alpha < -\tfrac{1}{8\Lambda}, \quad \left(-\sqrt{\tfrac{4}{54\alpha\Lambda+9}} - \tfrac{1}{3} < w \le -\sqrt{\tfrac{26+3}{18\alpha\Lambda}} - \tfrac{4}{3} \text{ or } \sqrt{\tfrac{26\alpha\Lambda+3}{18\alpha\Lambda}} - \tfrac{4}{3} \le w < -1 \right)
\end{aligned}
\tag{29}
$$

respectively. Concerning the realization of the emergent universe scenario via the exit from the ESU, the CPs $A_\pm$ show some promising behavior.

In general, we expect this transition to be achieved in two ways, as the universe equation of state parameter slowly drops from a constant value in past $(t \to -\infty)$. Either via changing the stability status of each of CPs separately from stable to unstable or exchanging a stable CP with its unstable counterpart so that finally we will deal with an unstable CP [35, 39, 43, 45, 46]. Note that in both ways, it is expected that the phase transition occurs for a critical value of ω.

Concerning the former, one can find from terms in Eqs. (22) and (28) that for the CP A_+ with negative cosmological constant and negative values for the modified parameter α, there is the possibility of the transition discussed above. By a closer inspection after comparing the first cases in Eqs. (22) and (28), we note that by setting values $\Lambda = -1$, $\alpha = -1$, for spatially flat geometry, we can achieve a stable ESU filled by a matter with the equation of state parameter w around -1. However, the CP A_+ is no longer stable after gradually dropping w below -1, meaning that it undergoes a stable-unstable phase
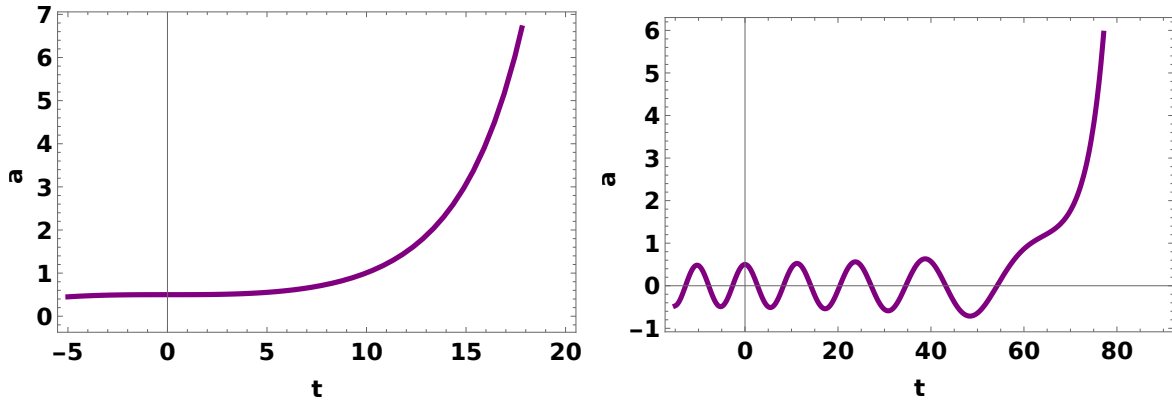
FIG. 7: The graceful exit of ESU from a stable state to an inflationary epoch by supposing a gradually decreasing equation of state parameters $w(t) = 0.33 - 0.001t$ and $w = -0.3 - 0.01t$ for the EMSG model with flat and spatially open geometries from left to right panels, respectively. We set numerical values ($\alpha = -1$, $\Lambda \geq 0$) and ($\alpha = 0.5$, $\Lambda = -1$) with common initial conditions $a(0) = 0.5$ and $\dot{a}(0) = 0$ for left and right panels, respectively.
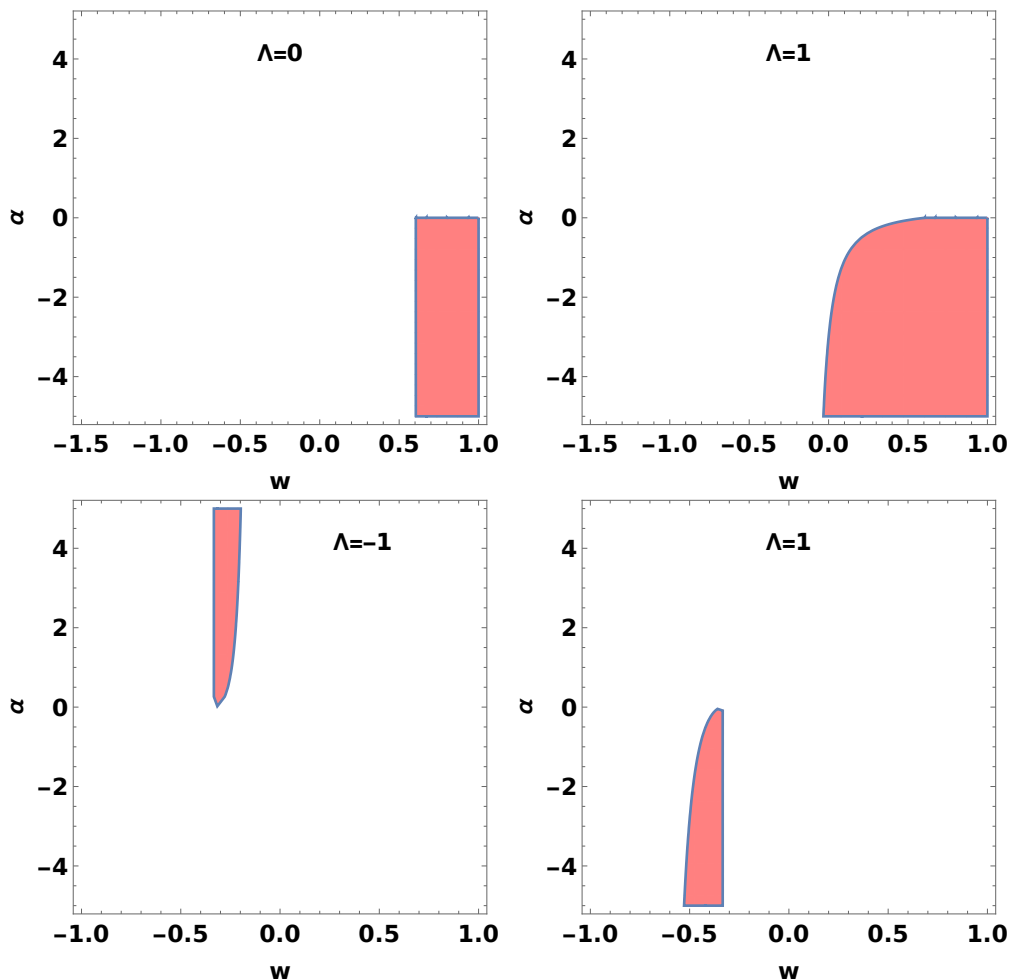

FIG. 8: Regions of existence in the (w, α) parameter space for the unstable CPs C_- (up row) and D_+ (bottom row).

transition. Comparing the third cases in Eqs. (22) and (28), we can also have a stable ESU, but this time filled by a matter with the equation of state parameter w around 0. After gradually decreasing w below
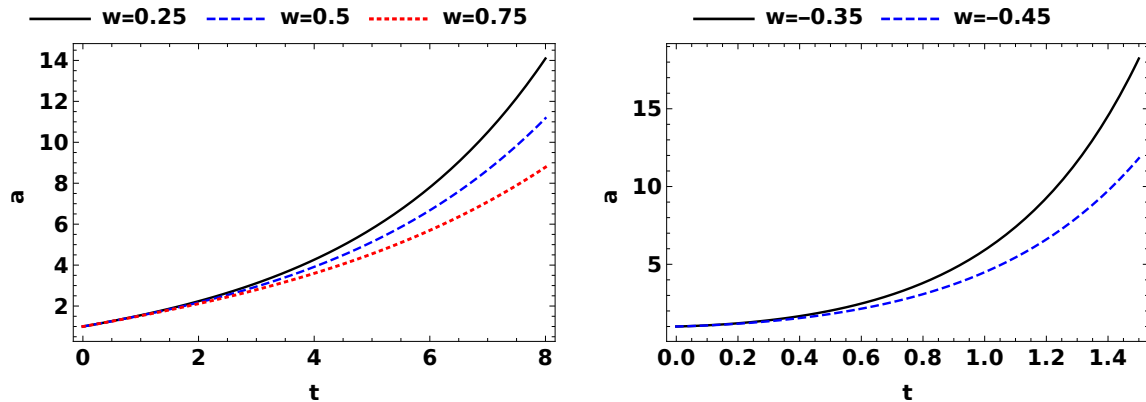
FIG. 9: The increasing exponential behavior of the scale factor corresponding to the unstable CPs C_- (left panel) and D_+ (right panel) with different values of ω and other numerical values related to Fig. 8.

0, the CP A_+ exits from its stable status (for example, by setting values $\Lambda = -1$, $\alpha = -1/4$). It turns out that this does not happen for positive values of α.

Concerning CP A_-, we see from comparison of the first cases in Eqs. (23) and (29) that if $\Lambda = 0$, $\alpha < 0$ then the phase transition does occur for a ESU filled by normal matter with the equation of state parameter w around $1/3$. The third cases in Eqs. (23) and (29) along with fifth item in Eq. (23) and the seventh in Eq. 29, signal the relevant phase transition for $\Lambda > 0$ and $\Lambda < 0$ but with $\alpha < 0$, respectively. Given the admitted non-negative values for the cosmological constant by observation, the CPs relevant to $\Lambda \geq 0$ with $\alpha < 0$ seem to be favored. Besides, the constraint obtained from the neutron stars on the coupling parameter α implies that the EMSG model with $\alpha < 0$, also has the potential to adapt with observation [57]. Also, newly in the light of observational data analysis of cosmic chronometers and Supernovae Type Ia data, shown that the flat EMSG model with $\alpha < 0$ result in a favor and consistency cosmology in the absence of cosmological constant [62]. In [55], also by setting the opposite sign convention to action (1) i.e. $R - \alpha T_{\mu\nu} T^{\mu\nu}$, it is clearly discussed that in de Sitter universe with a flat geometry to have a well-behaved cosmology in late-time which is stable, it is required that $\alpha < 0$. As a result of this discussion, the cases $\Lambda \geq 0$, $\alpha < 0$ are favored. In the left panel of Fig. 6, we schematically show the general trend of the transition of CP A_-, corresponding to $\Lambda = 0$, $\alpha < 0$, from a stable region to an unstable one, as ω decreases and reaches a critical value.

Now we focus on the latter possibility for realization of phase transition. Comparing the third item in Eq. (22) with the seventh in Eq. (29), one finds that by setting values $\Lambda < 0$, $\frac{1}{2\Lambda} < \alpha < 0$ for spatially flat geometry we can have a stable ESU addressed by CP A_+ filled by a matter with the equation of state parameter w around -1. However, by gradually decreasing w from -1 then the stable CP A_+ is exchanged with the unstable CP A_-. This describes a stable-unstable phase transition. The sixth and fifth (also sixth) items in Eqs. (23) and (28) respectively show that there is also the possibility of transition from stable CP A_- to unstable CP A_+ for positive values of α but with $\Lambda < 0$. If the an anti-de Sitter universe finds any support by evidence, this case with $\alpha > 0$ can be favored. Left panel in Fig. 7 qualitatively displays the graceful exit from a stable state to inflation for the most popular possible case between mentioned above cases (i.e $\Lambda \geq 0$ and $\alpha < 0$).

Considering the non-flat spatial geometries, let us start with closed universe ($k = +1$). By demanding instability conditions $\lambda^2 > 0$, $\rho_{ES} \geq 0$ and $\frac{1}{a_{ES}^2} > 0$ at the same time for the CPs $C_\pm$, we deal with allowed regions in (w, α) parameter space (Fig. 8). First of all, it is required that the scale factor no longer oscillates but grows exponentially to reach the inflationary phase. This can be done by setting the numerical values related to the allowed regions of Fig. 8 into the perturbed Eq. (27), as displayed in Fig. 9.

Interestingly, we find that the CP C_+ with non of the values of a cosmological constant admits these conditions. However, this is not the case for the CP C_- with $\Lambda \geq 0$, as one can see in Fig. 8 (up

row). By comparing these plots with its stable counterpart in Fig. 2 (up row), one recognizes that the stable-unstable phase transition is not possible as a result of the gradual decline of the equation of state parameter. However, by going to the spatially open geometry case, we realize that just for the CP D_+ the underlying phase transition will happen. More explicitly, the bottom row plots in Figs. 1 and 8, address a stable ESU full of normal matter filled with $w \geq 0$ for case of $\Lambda = -1$ and $\alpha > 0$ which with the passage of time and decline of w, it is finally converted to an unstable state describing a phase transition. The right panel of Fig. 6, schematically shows the general trend of a transition from a stable region to an unstable one, as w decreases. The right panel of Fig. 7 also qualitatively shows the graceful exit from a stable state to inflation for the CP D_+, as the time passes.

V. DISCUSSION AND CONCLUSIONS

Aiming to circumvent the initial singularity issue in the FLRW cosmology, we investigate one of the scenarios under the spotlight in recent years named the Emergent Universe Scenario. Based on this scenario, our universe does not stem from an origin as a big bang singularity, but it comes from an Einstein static state in an infinite past and finally joins the inflationary period. The original framework for implementing this scenario was GR. Because GR does not admit a stable static universe, this led to the failure of this scenario in the first steps. Subsequent studies have shown that taking into account some mechanisms, such as effects of modified gravity, quantum gravity and extra dimensions, could improve original results in favor of the realization of Emergent Universe Scenario.

In this paper, we have implemented the Emergent Universe Scenario considering a modification of Einstein gravity known as Energy-Momentum Squared Gravity (EMSG) which is distinguished from its standard counterpart by the correcting term $T_{\mu\nu}T^{\mu\nu}$ in the action, that is considering a self-interaction of stress-energy tensor. By applying this theory into cosmology, some modified terms, addressed by the coupling parameter α, appear in the Friedmann equations describing the dynamics of universe. These new terms can affect the original Emergent Universe Scenario.

As a first step, we started our analysis by extracting the static critical points known as Einstein Static Universe as a central concept in the study of the Emergent Universe Scenario. By employing a first order dynamical analysis within the phase space parameterized by α, we found some stable static critical points for any three possible spatial geometries ($k = 0, \pm 1$) in the absence and presence of cosmological constant $\Lambda > 0 \ (< 0)$.

In the next step, by finding a phase transition from a stable to an unstable state for these extracted static critical points, we investigated the graceful exist of the Einstein Static Universe superseded by an inflationary epoch. More precisely, the main idea is that for $t \to -\infty$, the value of ω is constant and the universe, in essence, has been eternally stuck in the stable Einstein static state until it finally comes out naturally from this state and evolves into an inflationary period, if ω drops slowly as time goes forward. It is worth noticing that when we say the Einstein Static Universe is stable or unstable, the statement is true only within a specified range of w. Besides, in the downward trend of w, at times when the cosmic w approaches to critical values, some stable static solutions find the chance to exit from the allowed range of w and, in this way, they enter into an unstable range of w, meaning that the standard universe evolution takes place. This can be seen schematically in Fig. 6 for the static solutions A_- and D_+. As a result, this transition from stable to unstable Einstein Static Universe can be imagined natural in the sense that it occurs in two separated ranges of w, as the parameter decreases according to the time evolution.

Interestingly, we found that, in the context of EMSG, unlike the original idea in the standard general relativity, the realization of an emergent universe scenario does not impose a positive spatial curvature ($k = +1$). In the other words, our analysis have revealed that here the possibility of having an emergent universe for a closed universe, is ruled out. However, for both spatially flat $k = 0$ and open $k = -1$ cases, under some constraints on the parameters (Λ, ω, α), there is the possibility of a perfect realization for the Emergent Universe Scenario. From three aspects, the most impressive output is related to the spatially flat case with $\Lambda = 0$ and $\alpha < 0$.

First, the spatially flat universe seems to be favored by cosmological data, also if the new Planck

measurements could question this statement according to the discussion in [22, 23]. We leave its confirmation/rejection to the next data in the future.

Second, the Emergent Universe Scenario works in the absence of a cosmological constant. This fact could be relevant in view of solving the cosmological constant problem [96].

Third, it seems that the negative values of free model parameter α is supported by some observational constraints derived in [57, 62].

Our analysis, meanwhile, have shown that, in the spatially flat case with $\alpha > 0$, there exists the possibility of a successful realization for the emergent universe if $\Lambda < 0$, which despite its major role in AdS/CFT correspondence and a fundamental theory such as string, however, it is not supported by the cosmological observations. According to our analysis, by the Emergent Universe Scenario in the context of EMSG, it is possible to bypass the initial singularity considering some favorable conditions as well as in absence of quantum corrections. Results of our analysis become remarkable considering those reported in [69] where it is shown that the initial bounce, expected from EMSG [55], is not viable as a regular bounce. This means that finding a bounce solution is not a sufficient condition to remove the singularity. On the other hand, we have shown that it is possible via a full realization of the Emergent Universe Scenario.

A further comment is worth at this point. The analysis of Emergent Universe Scenario can become richer, if one takes into account the presence of scalar fields in the EMSG Lagrangian density. This is due to the fact that EMSG not only changes the gravitational sector also modifies dynamics of all involved matter fields. So, by considering scalar fields in the Lagrangian density, due to the higher derivative terms $T_{\mu\nu}T^{\mu\nu}$ in (1), non-trivial effects come out in the Emergent Universe Scenario.

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