EMPIRICAL EVIDENCE OF IDEAL FILTER APPROXIMATION: PERIPHERAL AND SELECTED EU COUNTRIES APPLICATION

Jitka Poměnková, Roman Maršálek[*]

Abstract:

We compare three filters commonly used for business cycle analysis: the Baxter-King, the Christiano-Fitzgerald and the Hamming window filter. Empirical contribution of the paper is numerical evaluation of the approximation of the ideal band-pass filters in the discussion of the filters' theoretical properties (gain and attenuation within the business cycle frequencies, as well as the leakage in the remaining frequencies). We consider the truncation factor for the Baxter-King filter and the sample size for the latter two. We show that the leakage and attenuation of the Christiano-Fitzgerald and the Hamming window filter perform similarly across the range of chosen sample sizes and better than the Baxter-King filter. Moreover, we apply the filters to data of selected EU countries and point out differences in their estimation of growth business cycles. Our findings indicate that Christiano-Fitzgerald filter and the Hamming window both are appropriate for the identification of a business cycle. The Hamming window filter introduces smaller attenuation near the edges but in case of small samples its approximation of ideal filter is very rough.

Keywords: band-pass filters, business cycle, frequency transfer function, gain, attenuation, leakage.
JEL Classification: C15, C02, E32, E37

1. Introduction

Economic literature presents several definitions and methods on how to measure business cycles. Burns and Mitchell (1946) define a classical business cycle concept referring to the cycle in the levels of the output series. Harding and Pagan (2005) develop a business cycle concept distinguishing classical and growth (deviation) cycles. Growth cycles are cycles obtained from an input time series by removing the permanent component (Canova, 1998). The sensitivity of results to the method used for isolating the business cycle from the input data is also discussed (Harding and Pagan, 2005; Canova, 1998). The identification of growth business cycle is in the front of empirical work especially in the context of analysis small samples, such as transition or peripheral economies. The focus is on the process of economic integration, on analysis of business cycle comovement, synchronization of business cycles during the crisis, analysis of international trade linkages and many others.

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Our motivation for evaluating the quality of ideal filter approximation by various band-pass filters derives from the very frequent use of these filters for identifying economic cycles. The aim of this paper follows consequent steps. The first aim/step is to numerically evaluate the quality of the approximation of selected band-pass filters to the ideal filter frequency transfer function. For this we select the Baxter-King (Baxter and King, 1999), the Christiano-Fitzgerald (Christiano and Fitzgerald, 2003) and the Hamming Window filter (Iacobucci and Noulez, 2005). As approved by many empirical studies, these three filters represent common widely use band-pass filters suitable for business cycle analysis. There also exist other methods such as a high-pass Hodrick-Prescott filter, ARMA processes, and first order difference (FOD) or regression functions (Bonenkamp *et al.,* 2001; Poměnková, 2012) for business cycle identification. But those are criticised in the literature as having limited ability to let only the business cycle frequencies pass (Harvey and Jager, 1993; Guy and St-Amant, 2005). Therefore, we do not consider them. The quality of the approximation is assessed by using three metrics: undesired gain (further denoted as gain) in the business cycle region, undesired leakage outside the business cycle region (further denoted as leakage) and attenuation in the business cycle region (further denoted as attenuation).

In the second step we provide an empirical evidence of ideal filter approximation. Here we use application on selected European countries, namely Greece, Ireland, Portugal, Spain, Italy and Austria. The aim is an empirical contribution of obtained knowledge from the step one. Because the transfer function of the Baxter-King filter is influenced by the cut-off parameter *K* while the Christiano-Fitzgerald's and the Hamming Window filter's approximations both depend on the sample size, we also consider the percentage of data loss for several selected sample sizes.

The last third step is focused on the comparison of different filtering techniques appropriate to empirical analysis focussing on time periods affected by global crisis shocks. Therefore, we also conduct a correlation analysis between the business cycles identified for the selected countries and the business cycles identified for Germany. We include Germany as the heart of the euro area and its economically most significant country.

The empirical analysis reveals additional problems such as edge effects of the Hamming window. Note that edge (boundary) effects will be referred to a situation at the end of the sample size where estimated values of filtered time series are biased. Our findings suggest that the Christiano-Fitzgerald and Hamming window filters both are appropriate for identifying business cycles. The Christiano-Fitzgerald filter is suitable even for small sample sizes, while the Baxter-King and the Hamming window filter require a comparably larger sample size. The Hamming window filter also introduces smaller attenuation near the edges but in case of small samples its approximation of ideal filter is very rough.

The paper is organized as follows: in the next section we outline the literature review and consequently methodological background of the three selected filters. The third section contains a description of the chosen data set and evaluation of the filters. In Section 4 we present our results and their practical and theoretical implications. Section 5 is focused on comparison of the results *via* correlation analysis. Section 6 concludes and summarizes the paper.

2. Literature Review

The literature paying attention to the filtering method useful for the business cycle identification is extensive. Generally, we can see two streams which are nicely presented by Canova (1998). He mentioned statistical procedures (polynomial functions of time, first order difference, Beveridge and Nelson's procedure, frequency domain methods, unobserved components model) and economic procedures (a model of common deterministic or stochastic trends, Hodrick-Prescott filter). Even this categorisation application of any method cannot be done without satisfaction of assumption supplemented about economic background.

As we mentioned in the introduction band-pass filters are commonly used for filtering. The suitability of band-pass filters is given by their property of wholly selecting only the data component belonging to a specific frequency band (called pass-band) while eliminating all other components of outside this specific band (Baxter and King 1999). Originally, the use of band-pass filters for business cycle frequency analysis was proposed by Burns and Mitchel (1946). Modern empirical macroeconomics uses a variety of techniques to perform the decomposition of time series into trend components and cyclical components such as deterministic model, stochastic model or filters. Many of those suitable for business cycle analysis are based on the application of a two-sided moving average. A precise (perfect) band-pass filter is an infinite order moving average filter that lets only the components in a given frequency range pass. This is only a theoretical concept and infeasible in practice; such a filter is thus called "an ideal filter". For practical applications it is therefore necessary to use an approximation of this ideal filter. The main problem then becomes constructing the closest possible approximation. As stated by Baxter and King (1999) it is suitable for such an approximation that should let as much as possible of the data of the predefined band of frequency pass, while affecting the other frequencies as little as possible. Then it is the optimal approximation of the ideal filter.

Three band-pass filters commonly used for the business cycle analysis in recent literature are the Baxter-King filter (Baxter and King, 1999), the Christiano-Fitzgerald filter (Christiano and Fitzgerald, 2003) and the Hamming Window filter (Iacoboucci and Noulez, 2005). Guaya and St-Amant (2005) assess the ability of the Baxter-King filter and the Hodrick-Prescott filter to extract the business-cycle component of macroeconomic time series by using two different definitions of the business-cycle component. They show that both filters do relatively well when applied to series that have a peak in their spectra at business-cycle frequencies. But they do poorly with series whose spectra decrease sharply and monotonically at higher frequencies. Therefore, as wrote Harvey and Jager (1993) or Haug and Devald (2004) the Christiano-Fitzgerald filter can be taken as improvement of Baxter-King filter, because it chose the weights of the filter in frequency domain, *i.e.* it uses spectra estimations as weighted function.

Haug and Devald (2012) also use band-pass filters to extract cycles (not even business cycles) in pre-define range (2-8 and 8-40 years) from time series. They use fluctuations for correlation analysis and for assessment of comovement between series. Because the band-pass filters provide possibility to filter predefined frequency range, they distinguish the long-term component and short-term component by specification of filters bands. This approach allows analysis of parts of time series separately. From the group of band-pass filters Haug and Devald (2012) chose the Christiano-Fitzgerald filter. For robustness they

checked their results with the Baxter-King filter. They found that the filtered components were almost identical and that the phase shift (which can occur in the Christiano-Fitzgerald filter case due to its non-symmetry) in the filtered series is likely negligible. According to their findings the Christiano-Fitzgerald filter provides the closest approximation to the ideal filter. Our findings in this article support this statement.

Identification of cyclical fluctuation in the context of convergence analysis is used in Drake and Mills (2011). They focused on examination of properties of GDP in the euro area with the stress to the adoption euro in 1999. Drake and Mills (2011) have particular interest in the time series decomposition into trend and cyclical components using Christiano-Fitzgerald filter, and the Baxter-King filter. They take the Christiano-Fitzgerald filter as superior to the traditional Hodrick-Prescot filter. They support their decision by the fact that the asymmetric version is better at estimating cycle in real time and near at the end of the sample. As they also mentioned, there are two approaches for convergence analysis, comparison of cycles between themselves or comparison of cycles with the benchmark countries such as Germany. We are going to use this idea in different way. On the basis of known empirical results we can evaluate the suitability of selected band-pass filters first according to the level of the measurement for ideal filter approximation and in the context of the results for comovement analysis with Germany.

Croux *et al.* (2001) focused on theory and empirics of comovement of economic variables asking whether it can be explained by large aggregate shocks or if the answer should be found in non-linear propagation mechanism. They propose dynamic correlation and cohesion as the relevant measurement for comovement analysis. Macroeconomic literature often presents standard approach of correlation pre-filtered (high-pass or band-pass filter application) data. Croux *et al.* (2001) discuss the difference between correlation of pre-filtered data and application of dynamic measure. They prefer two-sided filter which eliminates all the inappropriate waves. As inappropriate waves they denote all the waves whose frequency is out of the relevant interval and leaves unchanged the amplitude of the waves within the interval.

From a methodical point of view in the last decade the time domain and the frequency domain (Iacobucci, 2003; Iacobucci and Noullez, 2005) analysis has been extended to an integrated view of the time-frequency domain (Croux *et al.,* 2001; Hallett and Richter, 2007; Rua, 2010; Maršálek *et al.,* 2013). In all these fields the importance of appropriately identifying business cycle phases of fluctuations in economic activity arises. Therefore the ability to precisely identify business cycles can increase the efficiency of economic policy instruments and the assessment of the comovement of economies.

3. Selected Band Pass Filters

We can analyse a time series, y_n, $n = 1, ..., N$ either in the time or the frequency domain. The approach proposed by Baxter and King (1999) or Christiano and Fitzgerald (2003) is to perform the filtering in the time domain, while the requirements are specified in the frequency domain. In the time domain representation of the ideal, though infeasible, two-sided linear filter is given by the infinite moving average producing filtered time series u_n:

$$u_n = \sum_{j=-\infty}^{\infty} b_j y_{n-j}, \tag{1}$$

where y_n is the input time series and b_j are filter weights (Christiano and Fitzgerald, 2003). This linear transform selects only the data components in the specified band of angular frequencies $[\omega_1, \omega_2]$ called pass-band. The components outside of this band are eliminated. The adjective "ideal" corresponds to the requirement of an infinite amount of data. The frequencies specifying the pass-band (the so-called cut-off frequencies) are $\omega_1 = 2\pi / q_1$, $\omega_2 = 2\pi / q_2$, where q_1 and q_2 denote the longest and shortest period of cycles passed through the filter.

In the frequency domain, the ideal band-pass filter is defined by the frequency transfer function $G(\omega)$ equal to 1 for frequencies in the range $[\omega_1, \omega_2]$ and zero for all other frequencies. The power spectrum $S_U, (\omega)$, of the filtered time series can be computed as

$$S_U(\omega) = |G(\omega)|^2 S_y(\omega),\qquad(2)$$

where $S_y, (\omega)$ is the spectrum of the input time series. The filter frequency transfer function can be decomposed as $G(\omega) = |G(\omega)|e^{i\phi(\omega)}$ where the absolute value of $G(\omega)$ denoted as $|G(\omega)|$ is a module characteristic representing how the amplitude of frequency components are altered by the filter, $\phi(\omega)$ is the phase shift caused by the filter and describing how different frequency components are delayed and i is the complex unit. The squared module $|G(\omega)|^2$ thus determines the weights corresponding to the components of the power spectrum $S_y(\omega)$ at the angular frequencies ω. For more details regarding the module and phase characteristics, we refer readers *e.g.* to Pollock (2009). Note that the symmetric filters have linear phase characteristics as a function of frequency. Consequently, a group delay for all frequency components is flat (constant). This is advantageous, as in such a case all components at the filter input, regardless their frequency are delayed by the same amount. Thus the phase distortion is avoided.

The main representatives of filters based on a feasible approximation of the infinite moving average are the Baxter-King, the Christiano-Fitzgerald and the Hamming window filters.

3.1 The Baxter-King (BK) filter

The BK filter is a two-sided linear moving average band-pass filter. In the case of business cycle frequencies its pass-band corresponds to cycle periods between six quarters and eight years. The components outside this range of frequencies are removed. Baxter and King (1999) propose the approximation of an ideal filter by the finite symmetric linear moving average filter of the odd order $M = 2K+1$ such that

$$\hat{u}_n = \sum_{j=-K}^{K} \hat{b}_j y_{n-j}.\qquad(3)$$

The weights of the filter $\hat{b}_j$ are computed in the frequency domain by minimizing the loss function Q (Baxter and King, 1999; Christiano and Fitzgerald, 2003) of the differences between the ideal filter $G(\omega)$ and the feasible filter $H(\omega)$:

$$Q = \frac{1}{2\pi} \int_{-\pi}^{\pi} |G(\omega) - H(\omega)|^2 \, d\omega,\qquad(4)$$

where $H(\omega) = \sum_{j=-K}^{K} b_j e^{-i\omega j}$. According to Kowal (2005), this filter has a number of desirable properties. First, since it is real and symmetric, it does not introduce a phase shift

and leaves the extracted components unaffected except for their amplitudes. Second, being of constant finite length and time-invariant, it is stationary.

3.2 The Christiano-Fitzgerald (CF) filter

Another filter which well approximates the ideal filter is the CF filter (Christiano and Fitzgerald, 2003). A filtered estimate $\hat{u}_n$ of the N-observations long data set y_n can be written as

$$\hat{u}_n = \sum_{j=-f}^{p} \hat{b}_j^{p,f} y_{n-j},$$
(5)

where $f = N - n$ and $p = n - 1$ for $n = 1, ..., N$ and $\hat{b}_j^{p,f}$ are the time-varying filter weights. Similarly to the BK filter, the CF filter weights are designed with the aim of minimizing the mean square error between the output of the ideal filter and its approximation. In the frequency domain this problem can be written in the form

$$Q = \frac{1}{2\pi} \int_{-\pi}^{\pi} \left| G(\omega) - B^{p,f}(\omega) \right|^2 S_y(\omega) d\omega,$$
(6)

where $B^{p,f}(\omega) = \sum_{j=-f}^{p} b_j^{p,f} e^{-i\omega j}$. The filters weights are set with respect to the importance of the spectrum in the given frequency and therefore depend on the property of the analysed time series. If possible, the trend component should be removed from the original time series prior to the CF filter application (Christiano and Fitzgerald, 2003; Haug and Devald, 2004; Iacobucci and Noullez, 2005). We have followed this recommendation in our analysis.

3.3 The Hamming-Window (HW) filter

In the case of finite-length sample sizes it is impossible to design an ideal band-pass filter. An interesting filter that provides very simple (easily applicable) and efficient solution to an ideal filter approximation developed with respect to application on a short time series was proposed by Iacoboucci and Noulez (2005). In contrast to CF filter, the HW filter is symmetric and stationary, in contrast to BK filter its application results in no data loss. As verified further (see Figure 1) its frequency transfer function is much flatter than in the case of CF and BK filters. It consists in smearing the ideal filter response with a selected window and it leads to good attenuation of the spectral power outside the pass-band, allowing almost complete removal of undesired frequency component. The use of a window is advantageous for suppression of a Gibbs phenomenon, Haykin and van Veen (2003), that can result in the appearance of spurious oscillations at the end of time series (edge effects). In contrast to the two above-mentioned (BK, CF) filters, the HW filter is applied in the frequency domain. Thus, although the other two filters can be defined easily in the time domain, the HW filter has to be defined in the frequency domain. In the following we adopted the original HW filter definition provided in Iacoboucci and Noulez (2005). First a time series sy_n is converted to the frequency domain by the calculation of its discrete Fourier transform y_n,

$$Y_k = \sum_{n=0}^{N-1} y_n e^{-i2\pi kn/N}, \qquad k = 0,1...\lfloor N/2 \rfloor.$$
(7)

After that, the Hamming windowed filter is applied according to the formula

$$U_k = (W_k * G_k) Y_k,$$ (8)

where G_k represents the frequency transfer function of the ideal filter, *i.e.* the filter that filters out all the frequency components outside the business cycle band, W_k corresponds to the selected window function, *e.g.* Hamming window (Iacobucci, 2003) and the operator * denotes a linear convolution. Finally, the filtered time series is computed coming back from the frequency domain to the time domain using the Inverse Discrete Fourier Transform, Haykin and van Veen (2003) of U_k

$$\hat{u}_n = \frac{1}{N} \left[U_0 + \sum_{k=0}^{\lfloor N/2 \rfloor} \left(U_k e^{i2\pi kn/N} + U_k^* e^{i2\pi kn/N} \right) \right], \qquad n = 0,1...N-1,$$ (9)

where * in the superscript means complex conjugation.
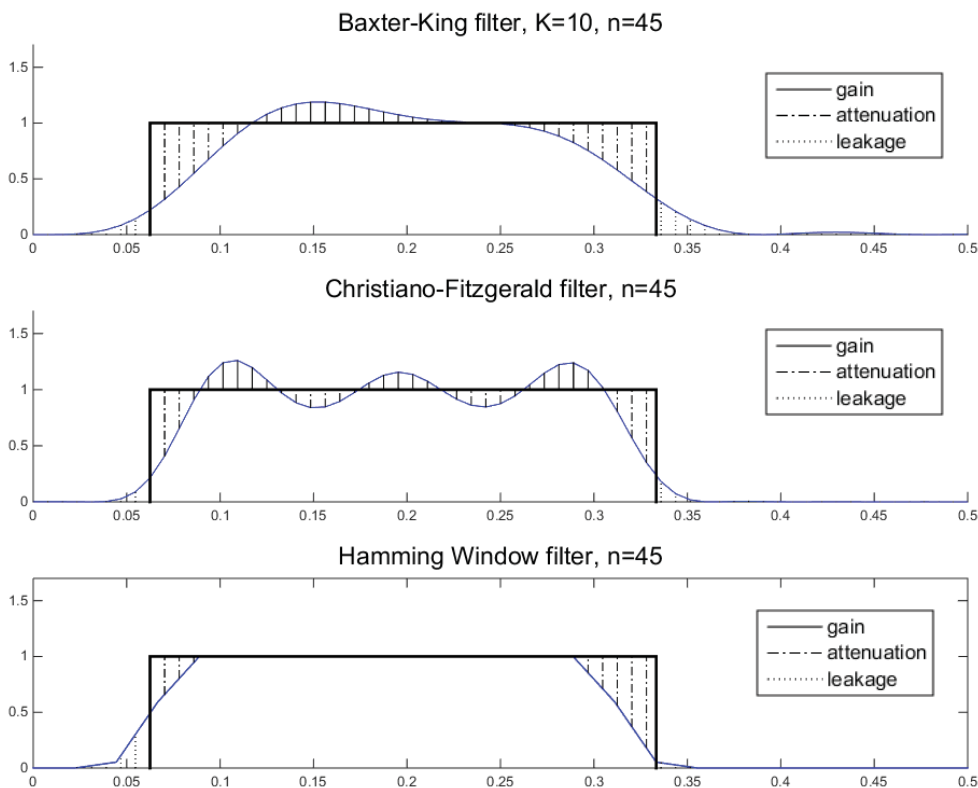
4. Evaluation of the Filters

4.1 Data

The evaluation is done with respect to the influencing factor, which is the truncation factor for the BK filter and the sample size for the CF and HW filter. For the application we use GDP data in quarterly values, denoted in millions of national currency, transformed into chain-linked volumes and based on the reference year 2000 (including 'euro fixed' series for euro area countries). The data are seasonally adjusted and transformed by natural logarithm for business cycle identification. The source of our data is the statistical office of the European Union (Eurostat, 2012). We choose Austria as a benchmark country, representing a stable and developed economy in the EU core, and peripheral euro area countries in available sample sizes. For comparing the approximation results of the filters we also use Germany as another representative of a core EU country. Available sample sizes are: Portugal (N=68) from 1995/Q1–2011/Q4, Ireland (N=59) from 1997/Q1–2011/Q3, Italy (N=84) from 1991/Q1–2011/Q4, Greece (N=45) from 2000/Q1–2011/Q11, Spain (N=68) from 1995/Q1–2011/Q4, Austria (N=96) from 1988/Q1–2011/Q4 and Germany (N=84) from 1991/Q1–2011/Q4. These countries were selected due to topicality of the economic situation during the debt crisis. Another important factor is the different sample size of the selected countries.

4.2 Quality of approximation

As stated in Christiano and Fitzgerald (2003) if the raw data before application of band-pass filters have a non-zero mean or are covariance stationary about a trend, then the trend has to be removed prior to analysis of optimal filter design. This statement is followed by Iacobucci and Noullez (2005) in work focused on a frequency selective filter used for short-length time series. They also recommend as one solution of optimal filter design to subtract deterministic trend before filtering. Therefore, we use for this the high-pass Hodrick-Prescott (HP) filter (Hodrick and Prescott, 1997). The use of the BK filter does not require this step.

The frequency transfer function plots (Figure 1, Figure 3 in Appendix A) are presented in the normalized frequency range (0, 0.5). In the range (0.5, 1) the frequency transfer function of the band-pass filter for business cycle frequencies is zero. Note that the normalized frequency 1 stands for half of the sampling frequency and that there is a straightforward relationship between the normalized frequency and the cyclic component period T: $f_{normalized} = 2/T$. Normalized frequencies of 0.0625 (T_1 = 32 quarters) and 0.33 (T_2 = 6 quarters) correspond to the periods delimitating the business cycle frequency band (Burnsch and Mitchell, 1946).

Figure 1 | Example of the Ideal Filter Approximation Using the Band-Pass Filters, *K*=10, *N*=45 (Solid Line Area: The Gain, Dashed and Dotted Line Area: The Attenuation, Dotted Line Area: The Leakage)

To determine the quality of approximation, we quantify the (undesired) gain of the filter approximation in the business cycle frequency range, the (undesired) attenuation in the business cycle frequency range and the (undesired) leakage of the filter approximation outside the business cycle frequency range. In all cases the metric value is calculated as the area under/above the filter frequency transfer function with respect to the ideal filter. The motivation to select these three metrics is given by the shape of the rectangle of ideal filter. Looking at the Figure 1 we decided to evaluate how big the area of leakage is, because in case of big leakage the filter gives the pass of such cyclical movements close to

the established bands which does not belong to defined range. We also decided to evaluate attenuation and the gain of the cyclical component belonging to the defined range (in our case business cycle range). In case of big gain (attenuation) the filtered time series can excaudate (inhibit) importance of cyclical component. Therefore, in all three cases of approximation of ideal filter rectangle consecutive analysis (comovement analysis *etc.*) using time series filtered by such filter can indicate bias results.

As the transfer function of the CF filter is time variant, these metrics were computed in the time instant $n = N/2$ (the best ideal filter approximation). For the description and illustration of the selected metrics see Figure 1 above Among these three observed quantities, the leakage and the attenuation are more critical for business cycle isolation than the undesired gain in the business cycle frequency range. The gain magnifies the identified components so in some sense it can be beneficial for this application.

4.2.1 Baxter-King filter approximation

The description of the BK filter in Equation 3 shows that both the length of its impulse response and its frequency transfer function as well depend on the truncation factor K only and not on the input sample size. Therefore, the measures of the approximation to the ideal filter using the BK filter will be the same for an arbitrary sample size. The variation of sample size results only in a relative loss of data change. This loss is caused by shortening the analysed time series by K observations from both sides of the data set. Therefore, we define the indicator of relative data loss (RDL) as a ratio between the numbers of reduced data from both sides ($2K$) and the sample size (N):

$$RDL = \tfrac{2K}{N} \cdot 100 . \tag{10}$$

We varied the truncation factor in a range of $K = 5, \ldots, 19$. The results are presented in Table 1.

A detailed graphical representation of the results in Table 1 can be found in Appendix A (Figure 3). As we mentioned above, the frequency transfer function does not depend on the analysed data or on its sample size, but only on the truncation factor K. Thus, the optimum value of the parameter K is the value for which the approximation to the ideal filter results in minimum values of (undesired) gain, attenuation and leakage with respect to minimized data loss. We can state that this condition is fulfilled for the value K=10. For K greater than 10, the value of leakage has a descending tendency, but on the contrary the relative loss of data is ascending. For practical reasons during the application on real data we require a loss of data as small as possible. We can observe that undesired gain is also rising for higher K values and the difference between the leakage of approximation for parameter K=10 (0.0125) and K=11 (0.0107) is not significant. It is thus preferable to accept a marginally higher value of leakage in order to achieve a smaller data loss. Note that this result is in good accordance with the original recommendation for an optimum selection of the K value (Baxter and King, 1999) without considering data loss.

Table 1 | Ideal Filter Approximation of the Baxter-King Filter

Truncation factor K	Gain	Attenuation	Leakage	Relative data loss (%)				
				N=45	N =59	N =68	N =84	N=96
5	0.0260	0.0741	0.0139	22.22%	16.95%	14.71%	11.90%	10.42%
6	0.0402	0.0709	0.0057	26.67%	20.34%	17.65%	14.29%	12.50%
7	0.0407	0.0706	0.0056	31.11%	23.73%	20.59%	16.67%	14.58%
8	0.0382	0.0691	0.0070	35.56%	27.12%	23.53%	19.05%	16.67%
9	0.0273	0.0590	0.0111	40.00%	30.51%	26.47%	21.43%	18.75%
10	0.0119	0.0382	0.0125	44.44%	33.90%	29.41%	23.81%	20.83%
11	0.0106	0.0310	0.0107	48.89%	37.29%	32.35%	26.19%	22.92%
12	0.0122	0.0319	0.0103	53.33%	40.68%	35.29%	28.57%	25.00%
13	0.0127	0.0315	0.0098	57.78%	44.07%	38.24%	30.95%	27.08%
14	0.0175	0.0342	0.0081	62.22%	47.46%	41.18%	33.33%	29.17%
15	0.0177	0.0343	0.0081	66.67%	50.85%	44.12%	35.71%	31.25%
16	0.0145	0.0315	0.0088	71.11%	54.24%	47.06%	38.10%	33.33%
17	0.0146	0.0315	0.0088	75.56%	57.63%	50.00%	40.48%	35.42%
18	0.0126	0.0293	0.0089	80.00%	61.02%	52.94%	42.86%	37.50%
19	0.0093	0.0235	0.0079	84.44%	64.41%	55.88%	45.24%	39.58%

Notes: Own calculations.

4.2.2 Christiano-Fitzgerald and Hamming Window filter approximation

Additionally, we investigate the approximation of the ideal filter using the Christiano-Fitzgerald filter and the Hamming window filter for varying sample size. The results are summarized in Table 2. Focusing on attenuation and leakage, we can see that the BK filter approximation of ideal filter is the worst from all investigated filters. The CF and HW filter's approximation are similar. With growing sample size the value of attenuation decreases. The value of leakage is slightly varying over the sample size, but is still lower for the CF and HW filters than for the BK filter.

Table 2 | Ideal Filter Approximation of all BK, CF and HW Filters

Filter type	Sample size	RDL (%)	Gain	Attenuation	Leakage
BK		K=10; 44.44%	0.0119	0.0382	0.0125
CF	45	0	0.0180	0.0270	0.0051
HW		0	0.0000	0.0228	0.0082
BK		K=10; 33.90%	0.0119	0.0382	0.0125
CF	59	0	0.0198	0.0243	0.0041
HW		0	0.0000	0.0129	0.0067
BK		K=10; 29.41%	0.0119	0.0382	0.0125
CF	68	0	0.0108	0.0156	0.0064
HW		0	0.0000	0.0122	0.0046
BK		K=10; 23.81%	0.0119	0.0382	0.0125
CF	84	0	0.0098	0.0126	0.0036
HW		0	0.0000	0.0115	0.0025
BK		K=10; 20.83%	0.0119	0.0382	0.0125
CF	96	0	0.0173	0.0192	0.0076
HW		0	0.0000	0.0063	0.0067

*Notes*: Own calculations.

5.   Application of the Filters

For evaluating band-pass filter approximation in practical applications, we applied the above-discussed filters on samples of GDP data from the selected countries. For graphical demonstration we use figures in the time domain (Figure 2).

Comparison of the results presented in the Figure 2 reveals several facts of application of selected filters. First, the sample size affects the quality of filtering. Secondly, in some cases some filters can cause edge effects.

A smaller sample size was available for Greece (N=45, Figure 2). At the start of the period (2000–2001) we can see that HW compared to CF filter identify opposite tendencies of growth business cycles. Such behaviour can already been observed on example in Iacobucci and Noullez (2005). The edge effects in boundaries of the sample size are depending on the filter type and amount of data in these regions. In the case of BK filter, the most important effect is the truncation and thus the samples near the time series edges are not available. The output of the CF filter near the edges is attenuated due to the filter nonstationarity (small gain especially at the end of time series), as also stated by Iacobucci and Noullez (2005). The HW filter, as has been also confirmed by our results below, introduces the smallest attenuation (and thus the strongest amplitude fluctuations) near the edges among the selected filters. Nevertheless, in the case of short sample size

the ideal filter characteristic approximation by the HW filter is very rough (in *N/2* points only) with potential residual oscillations due to the Gibbs phenomenon and it is thus not easy to judge which ideal filter approximation provides the best results.

Figure 2 | Growth Business Cycles

Italy

Austria
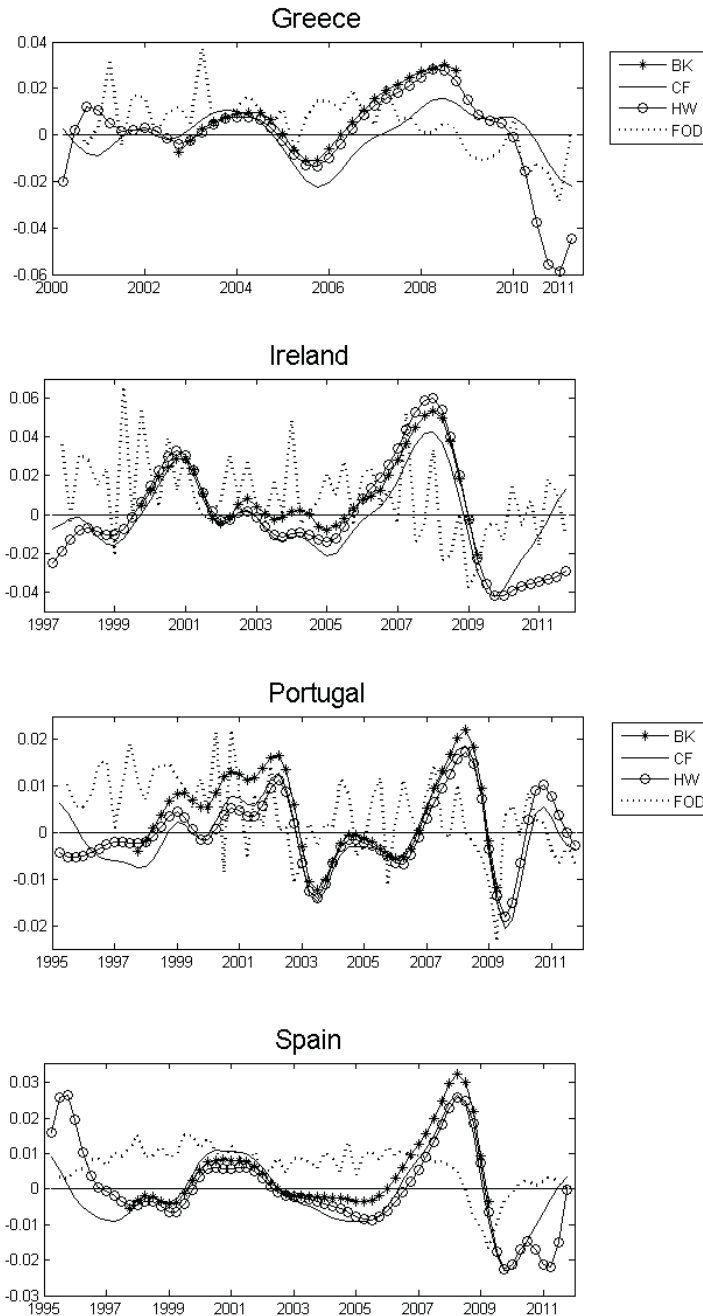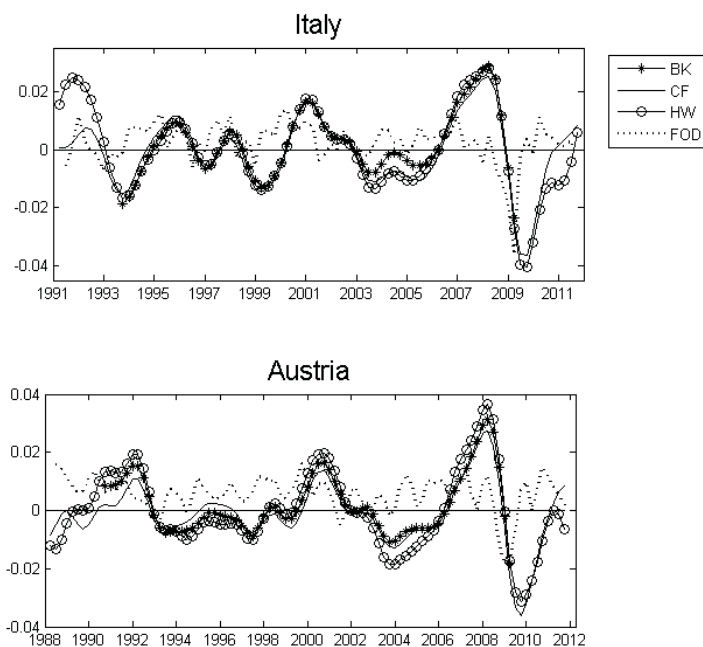
Ireland's sample contains *N*=59 observations. After 2009/Q3 the HW filtered time series indicates a slower expansion phase of the business cycle after a trough than the CF filtered time series (Figure 2). This difference can be caused by the CF filter nonstationary character (higher attenuation near the end). In the middle period from 2003/Q2–2004/Q3 the BK filtered time series shows higher values of the growth business cycle than the HW and the CF filters. We presume this period was one of oscillation in recession with a slow decreasing tendency which the CF and the HW filter were able to capture more accurately. Therefore, we recommend in following analysis such as turning point identification to use advance techniques (Bry-Boschan algorithm or nonparametric estimation of turning points) to avoid misleading results.

In the case of Portugal we see very similar behaviour of all three filters after 2002. Up to this period two situations arise. The first, from 1995–1997, is similar to in the starting period of Greece. The second is that the Baxter-King filtered time series in 1998–2002 has higher estimated values than the other two which can be caused by exaggeration/attenuation of some cycle frequencies in business cycle frequencies as demonstrated in Figure 1. With respect to the worst ideal filter approximation we consider the Baxter-King filter as the worst for application on that sample size. Therefore we recommend using other band-pass filters and also doing additional analyses such as time-frequency representation of a given time series. The sample size of Portugal is the same as that of Spain (Figure 2). For the two countries the resulting differences in the approximation of the three filters are quite close.

At the end of the sample for Spain we can see the "w" situation (HW filtered time series) which according to Romer (2006) occurs after deep recessions, such as after 2008 in the sample. The conclusion for Italy (*N*=84) is similar to the Spain case. Here we can also presume the distorted "w" situation in recession after applying the Hamming window

filter. Let´s note that due its smallest attenuation near the edges, the HW filter is the only one among the investigated filters able to capture these "w" curves. The business cycles identified for Austria, which has the largest sample size (*N*=96) (Figure 2), validate the fact that a larger sample size will usually yield results that are more precise.

To sum up, a disadvantage of the Baxter-King filter is its shortening of the sample size. Hamming window filter provides better approximation of ideal filter in the case of attenuation and as Iacobucci and Noullez (2005) wrote, can be a good choice for empirical study of business cycle, but only in case of large sample size. Extension of applications to the several sample sizes shows some discrepancy of hamming filter the other filters' results as the edge effect at the start of analysed time series, especially for small sample size. In the case of the Christiano-Fitzgerald filter attention should be paid to the end of an analysed time series (as a result of the filter impulse response nonstationarity and lower gain near the end of time series). The tendencies of recession and expansion, which all used filters identify, are similar. From this point of view we can say all filters produce quite close results. We suggest that the Christiano-Fitzgerald filter might be the most appropriate for identifying business cycles even in a short sample size, while the Baxter-King or the Hamming window filter requires a comparably large sample size. The Hamming window filter also introduces smaller attenuation near the edges but in case of small samples its approximation of ideal filter is very rough. A detailed look at each country is possible using additional in-depth analysis. In the case of Greece we can use time-frequency representation to reveal detailed business cycles over time, as introduced by Rua (2010), Halleth and Richter (2007, 2011) and Blumenstein *et al.* (2012). Alternatively, turning points could be identified as described by Harding and Pagan (2002) *via* Markov switching model, Sarlan (2001) using spectrum of turning points chronology and nonparametric kernel estimates as proposed in Poměnková (2010).

6. Comparison of Results via Correlation Analysis

In general, symmetric shocks have asymmetric consequences. Because analysing business cycle synchronization is one of the points most interesting for many economic studies, the comparison of different filtering techniques is appropriate, even if the empirical analysis focuses on periods affected by global crisis shocks. Therefore, we also conduct a correlation analysis between the business cycles identified for the selected countries and the business cycles identified for Germany (similarly to Drake and Mills, 2011 or Fidrmuc and Korhonen, 2006). We include Germany as the heart of the euro area and its economically most significant country. Our conclusions about the band-pass filters approximation are then further examined using the correlation between Germany and the other countries.

In addition to the band-pass filters discussed above (CF, HW and BK), we applied first-order difference (FOD), which are commonly used in the literature. As in Harvey and Jaeger (1993), we select the FOD method as a benchmark for simple de-trending. In particular, the FOD approach does not affect the stochastic properties of time series, while the band-pass filters extract predefined frequency range (business cycle frequencies) and smooth the estimated growth cycle (Baxter and King, 1999; Christiano and Fitzgerald, 2003). Smoother cycles are suitable for further analysis such as turning point identification or analyses of comovement in predefined frequency range. Moreover,

correlation analysis of relatively smooth business cycles results generally in higher degree of synchronization.

Calculations of correlation coefficient were done between Germany and each country with the same filter used for both. The overview of correlation coefficients in Table 3 shows a higher level of correlation for CF and HW filters for FOD. This is probably caused by the smoothing character of the band-pass filters. We can also see that increasing sample size approximates results for CF, HW and BK filters. In the case of Greece, we suppose that HW is not suitable for a small sample size. The resultant cyclical component indicates the significantly highest correlation with Germany. The BK filter produces a cyclical component on the sample size reduced by about 20 observations. Therefore, the correlation inside the observed time is too high and thus demands additional analysis, keeping in mind the reduced sample size. Thus it can be said that the results of our correlation analysis confirm the findings derived from our previous theoretical and empirical analysis.

Table 3 | Correlation between Business Cycles of Germany and Selected Countries

	Greece 2000/Q1-2011/Q1	Ireland 1997/Q1-2011/Q3	Spain 1995/Q1-2011/Q4	Portugal 1995/Q1-2011/Q4	Italy 1991/Q1-2011/Q4	Austria 1991/Q1-2011/Q4
FOD	0.1443	0.3602***	0.4837***	0.3703***	0.6666***	0.5487***
CF	0.1804	0.9230***	0.9082***	0.8115***	0.9533***	0.9180***
HW	0.3750**	0.7655***	0.7329***	0.7075***	0.9202***	0.8918***
	2002/Q3-2008/Q3	1999/Q3-2009/Q1	1997/Q3-2009/Q2	1997/Q3-2009/Q2	1993/Q3-2009/Q2	1993/Q3-2009/Q2
BK, K=10	0.9057***	0.8732***	0.8805***	0.7788***	0.9142***	0.8874***

Source: Own calculations.

Note: Statistically significant at 1 % (***), 5 % (**), 10 % (*).

7.   Conclusion

The motivation of this paper is to compare three filters commonly used for business cycle analysis, namely the Baxter-King, the Christiano-Fitzgerald and the Hamming window filters. The comparison is achieved in two steps. First, we evaluate the quality of the approximation to an ideal frequency transfer function. The considered metrics for the quality of approximation are undesired gain in the business cycle frequency range, leakage outside the business cycle band and attenuation of the business cycle frequencies.

The frequency transfer function for the Baxter-King filter is the same for an arbitrary sample size and depends only on the truncation factor K. Therefore, a disadvantage of the Baxter-King filter is the relative loss of data caused by truncation. This fact is important in the case of small samples of data. We find that the optimum value of the truncation parameter is $K=10$. For this value, the approximation to the ideal filter results in minimum values of (undesired) gain, attenuation and leakage, while still keeping data

loss close to a minimum. This result is in good accordance with the original recommendation for optimum *K* value selection (Baxter and King, 1999). Our theoretical findings show that regarding leakage and attenuation the Christiano-Fitzgerald and the Hamming window filter perform similarly across the range of chosen sample sizes, while yielding better results than the Baxter-King filter.

Secondly, we apply the filters to GDP data of selected EU countries. The empirical analysis reveals additional problems such as edge effects and shape of frequency transfer function of filters. We suggest that the Christiano-Fitzgerald filter might be the most appropriate for the identification of business cycles even for small sample sizes, while the Hamming window filter or Baxter-King especially require a comparably large sample size. Our findings are supported by the results of a correlation analysis between Germany and Ireland, Greece, Spain, Portugal, Italy and Austria.

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Appendix A

Figure 3 | Filter Approximation Using the Baxter-King Filter with Truncation Factor *K* = 5, ..., 19 (Solid Line Area: The Gain, Dashed and Dotted Line Area: The Attenuation, Dotted Line Area: The Leakage), (X-Label: Normalized Frequency, Y-Label: The Frequency Transfer Function)