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EMPIRICAL LAPLACE TRANSFORM AND APPROXIMATION OF COMPOUND DISTRIBUTIONS

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SUMMARY

Let $\{X_i\}_1^\infty$ be i.i.d. non-negative r.v.'s with d.f. F and Laplace transform L. Let N be integer valued and independent of $\{X_i\}_1^\infty$. In many applications one knows that for $y \to \infty$ and a function φ

$$P\{\sum_{i=1}^{N} X_i > y\} \sim \varphi(y,\tau,L(\tau),L'(\tau),\ldots)$$

where in turn τ is the solution of an equation

$$\psi(\tau,L(\tau),\ldots) = 0 .$$

On the basis of a sample of size n we derive an estimator τ_n for τ by solving $\psi(\tau_n,L_n(\tau_n),L_n'(\tau_n),\ldots) = 0$ where L_n is the empirical version of L. This estimator is then used to derive the asymptotic behaviour of $\varphi(y,\tau_n,L_n(\tau_n),L_n'(\tau_n),\ldots)$. We include five examples, some of which are taken from insurance mathematics.

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1. GENERAL PROCEDURE

Let X be a non-degenerate random variable with d.f. F on $[0,\infty)$ and with Laplace transform

$$L(s) := E(e^{-sX}) = \int_0^\infty e^{-sx} dF(x)\,. \tag{1}$$

We always assume that L(s) exists in an open neighbourhood of the origin; let $-\sigma$ be the abscissa of convergence of L(s) and put $I = (-\sigma,\infty)$ or $I = [-\sigma,\infty)$ according to the case where L(s) converges at $-\sigma$ or not respectively. The function L(.) is arbitrarily many times differentiable in the interior of I and since $X \geq 0$ it is a decreasing convex function on I. Let int I be the interior of I.

Many applications are concerned with compound distributions for which an approximation is needed. This approximation contains ingredients such as L(s) and derivatives at a point $s = \tau$ where τ is a proper solution of an equation involving in turn L(s) and/or some of its derivatives. Saddle-point type approximations for nonnegative r.v.-s provide a good general example.

In practical situations one often doesn't know the precise form of $L(\cdot)$; one only has a sample version $L_n(\cdot)$ based on a sample of independent observations $X_1,X_2,\ldots,X_n$ of X. Hence

$$L_n(s) = \frac{1}{n}\sum_{j=1}^{n} e^{-sX_j} = \int_0^\infty e^{-sx} dF_n(x) \tag{2}$$

where $F_n(x) = \frac{1}{n}\#\{1 \leq j \leq n : X_j \leq x\}$ is the empirical distribution function of the sample. This *empirical Laplace transform* $L_n(s)$ is a random analytic function for all (complex) values of s. A natural sample-based estimator of the approximation will be obtained by replacing the derivatives of L by those of L_n both in the approximation and in the equation defining τ; the latter will then automatically define an estimator τ_n for τ.

An adaption of the proof of the Proposition in Csörgő (1982) easily yields the following auxiliary result, where $L^{(k)}$ is the k-th derivative of L, $k = 0,1,2,\ldots$.

Proposition 1. If J is any closed interval contained in I, then almost surely as $n \to \infty$ for any bounded $k \in \mathbb{N}$

$$\sup_{s \in J} |L_n^{(k)}(s) - L^{(k)}(s)| \to 0 .$$

This proposition and some elementary analysis imply a crucial consistency result in that $\tau_n \to \tau$ almost surely whenever $\tau \in \text{int } I$.

A more elaborate analysis, based on the defining relation for τ and τ_n, will provide us with a limit in distribution for $\sqrt{n}(\tau_n - \tau)$. The limit in distribution - whenever it exists - will depend on the limits in distribution of some auxiliary r.v.'s that we introduce here for later reference.

For any $s \in \text{int } I$ we define

$$W_{k,n}(s) := \frac{1}{\sqrt{n}} \sum_{j=1}^{n} \{X_j^k e^{-sX_j} - E(X^k e^{-sX})\}. \tag{3}$$

Then it follows from the central limit theorem that

$$W_{k,n}(s) \xrightarrow[n \to \infty]{\mathcal{D}} \mathcal{N}(0, \sigma_k^2(s))$$

where $\xrightarrow{\mathcal{D}}$ denotes convergence in distribution and

$$\begin{aligned} \sigma_k^2(s) &= \text{Var}(X^k e^{-sX}) \\ &= E[X^{2k} e^{-2sX}] - E^2(X^k e^{-sX}) \\ &= (-2)^{-2k} L^{(2k)}(2s) - (L^{(k)}(s))^2 \end{aligned} \tag{4}$$

The latter result only holds if $2s \in \text{int } I$. For later use we quote the next result.

Proposition 2. If $2s \in \text{int } I$ then the r.v.'s defined by (3) satisfy

$$W_{k,n}(s) \xrightarrow[n \to \infty]{\mathcal{D}} \mathcal{N}(0, \sigma_k^2(s))$$

where $\sigma_k^2(s)$ is given by (4).

In many applications we need variables of the type (3) but with s replaced by a random sequence s_n converging almost surely to s. A one term Taylor expansion of the type

$$e^{-s_n X_j} = e^{-sX_j} - (s_n - s)X_j e^{-s_n(j)X_j} \tag{6}$$

introduces random quantities $s_n(j)$ satisfying the inequalities

$$\min(s_n,s) \le s_n(j) \le \max(s_n,s) . \tag{7}$$

We will use the following abbreviations

$$\begin{cases} S_{k,n}(s) = \frac{1}{n} \sum_{j=1}^{n} X_j^k e^{-sX_j} = (-1)^k L_n^{(k)}(s) \\ S_{k,n}(s_n, s) = \frac{1}{n} \sum_{j=1}^{n} X_j^k e^{-s_n(j)X_j} \end{cases} \tag{8}$$

where $s_n(j)$ is determined by (6) and (7).

Combining a strong law of large numbers with proposition 1 we obtain a useful result.

Proposition 3. Let $s \in \text{int } I$; let $s_n \to s$ almost surely. Then for any bounded $k \in \mathbb{N}$ almost surely as $n \to \infty$

(i) $S_{k,n}(s) \to E[X^k e^{-sX}]$

(ii) $S_{k,n}(s_n) \to E[X^k e^{-sX}]$

(iii) $S_{k,n}(s_n,s) \to E[X^k e^{-sX}]$

For easy reference we quote some Slutsky-type results (Serfling (1980, p. 19)), where c is a constant.

<u>Proposition 4.</u> (i) $X_n \xrightarrow{\mathfrak{D}} X,\ Y_n \xrightarrow{\mathfrak{D}} c \Rightarrow X_n + Y_n \xrightarrow{\mathfrak{D}} X + c$

(ii) $X_n \xrightarrow{\mathfrak{D}} X,\ Y_n \xrightarrow{\mathfrak{D}} c \Rightarrow X_n Y_n \xrightarrow{\mathfrak{D}} Xc.$

We now apply the above procedure to a number of specific examples. After shortly introducing the example we then give subsequently

(i) general observations concerning the solution τ

(ii) the asymptotic behaviour of $\sqrt{n}(\tau_n - \tau)$

(iii) the asymptotic behaviour of the estimator for the required approximation.

We always assume $\sigma > 0$, and $2\tau \in \text{int } I$ whenever necessary.

2. COMPOUND PASCAL DISTRIBUTION

Assume $Y = \sum_{i=1}^{N} X_i$ where $P[N = n] = \binom{r+n-1}{n} p^r q^n$ and where N and $\{X_i\}_1^\infty$ are independent; further, $r \in \mathbb{N}$, $0 < p < 1$, $q = 1-p$. Then Y has a *compound Pascal distribution*

$$P\{Y \le y\} = \sum_{n=0}^{\infty} \binom{r+n-1}{n} p^r q^n F^{*n}(y)$$

where F^{*n} is the n-fold convolution of F.

It has been known for a long time that under our assumptions on L(s); i.e. $\sigma > 0$, the tail $P[Y > y]$ has a gamma type approximation. See Beekman (1974, p. 66). In Teugels (1985) and Embrechts e.a. (1985) it is shown that more precisely

$$P[Y > y] \sim \frac{e^{\tau y} p^r y^{r-1}}{\Gamma(r)\,|\tau|\,(L'(\tau))^r}, \quad y \to \infty \tag{9}$$

where $L(\tau) = \frac{1}{q}$. See also Sundt (1982).

(i) First note that as $q < 1$, $\tau < 0$. To have a proper solution within int I we need $L(-\sigma) > \frac{1}{q}$.

(ii) We will estimate τ by solving $L_n(\tau_n) = \frac{1}{q}$. Now $\tau_n \to \tau$ a.s. Furthermore

$$0 = n(L_n(\tau_n) - L(\tau))$$

$$= \sum_{j=1}^{n} (e^{-\tau_n X_j} - E(e^{-\tau X}))$$

and by (6) and (7)

$$= \sum_{j=1}^{n} (e^{-\tau X_j} - E(e^{-\tau X})) - (\tau_n - \tau) \sum_{j=1}^{n} X_j e^{-\tau_n(j) X_j}.$$

Hence

(10) $$\sqrt{n}(\tau_n - \tau) = \frac{V_{0,n}(\tau)}{S_{1,n}(\tau_n, \tau)}.$$

By prop. 2 and 3(iii) $\sqrt{n}(\tau_n - \tau) \xrightarrow{\mathcal{D}} \mathcal{N}(0,\gamma^2)$ where $\gamma^2 = \sigma_0^2(\tau)/E^2[Xe^{-\tau X}]$ if $2\tau \in \text{int } I$. The quantity γ can be consistently estimated by

$$\gamma_n := \frac{1}{-L_n'(\tau_n)} \{L_n(2\,\tau_n) - L_n^2(\tau_n)\}^{1/2}$$

$$= \frac{[S_{0,n}(2\tau_n) - q^{-2}]^{1/2}}{S_{1,n}(\tau_n)}$$

in view of proposition 3; hence almost surely $\gamma_n \to \gamma$.

With Φ denoting the standard normal distribution function, let $z_{\alpha/2}$ be the percentile point for which $\Phi(z_{\alpha/2}) = 1 - \frac{\alpha}{2}$, $0 < \alpha < 1$. Then by the above construction we get an asymptotic confidence interval for τ if $2\tau \in \text{int } I$:

$$\lim_{n \to \infty} P\{\tau_n - z_{\alpha/2} \frac{\gamma_n}{\sqrt{n}} \le \tau \le \tau_n + z_{\alpha/2} \frac{\gamma_n}{\sqrt{n}}\} = 1 - \alpha.$$

(iii) The approximation (9) of $P[Y > y]$ depends upon $C := \dfrac{e^{\tau y}}{|\tau||L'(\tau)|^r}$ which is itself estimated by

$$C_n := \frac{e^{\tau_n y}}{|\tau_n||L_n'(\tau_n)|^r}.$$

From proposition 1 and standard arguments $C_n \to C$ almost surely. We should like to find the limiting distribution of $\sqrt{n}\,(C_n - C)$. Now

$$C_n - C = \frac{e^{\tau y}}{\tau(-L'(\tau))^r} - \frac{e^{\tau_n y}}{\tau_n(-L_n'(\tau_n))^r}$$

or

(11) $$e^{-\tau y}\tau\cdot\tau_n(L'(\tau)L_n'(\tau_n))^r(C_n - C) = \tau_n(-L_n'(\tau_n))^r - \tau(-L'(\tau))^r e^{(\tau_n-\tau)y}.$$

Abbreviate the left hand side by I_n. By a one term Taylor expansion we can write

(12) $$e^{y(\tau_n-\tau)} = 1 + y(\tau_n - \tau)e^{y\theta_n}$$

where $-(\tau_n - \tau)_- \leq \theta_n \leq (\tau_n - \tau)_+$. Hence

(13) $$\begin{aligned} I_n &= \tau_n(-L_n'(\tau_n))^r - \tau(-L'(\tau))^r - \tau y(\tau_n - \tau)(-L'(\tau))^r e^{y\theta_n} \\ &= (\tau_n - \tau)(-L'(\tau))^r + \tau_n\{(-L_n'(\tau_n))^r - (-L'(\tau))^r\} \\ &\quad - \tau y(\tau_n - \tau)(-L'(\tau))^r e^{y\theta_n}. \end{aligned}$$

However

$$(14)\quad (-L_n'(\tau_n))^r - (-L'(\tau))^r = [-L_n'(\tau_n) + L'(\tau)][\sum_{t=0}^{r-1} (-L_n'(\tau_n))^t(-L'(\tau))^{r-1-t}]$$

$$= R_n\{L'(\tau) - L_n'(\tau_n)\}$$

where

$$R_n := \sum_{t=0}^{r-1} (-L'(\tau))^{r-1-t} S_{1,n}^t(\tau_n) \to r(-L'(\tau))^{r-1} =: R$$

almost surely. In turn

$$n\{L'(\tau) - L_n'(\tau_n)\} = \sum_{j=1}^{n} (X_j e^{-\tau_n X_j} - E(Xe^{-\tau X}))$$

$$= \sum_{j=1}^{n} (X_j e^{-\tau X_j} - E(Xe^{-\tau X})) - (\tau_n - \tau) \sum_{j=1}^{n} X_j^2 e^{-\tau_n(j) X_j}$$

by relying on (6). Hence by (3) and (8)

$$(15)\quad \sqrt{n}[L'(\tau) - L_n'(\tau_n)] = W_{1,n}(\tau) - \sqrt{n}(\tau_n - \tau)S_{2,n}(\tau_n,\tau)\,.$$

Combination of (13), (14) and (15) yields

$$\sqrt{n}\, I_n = \sqrt{n}(\tau_n - \tau)\{(1 - \tau y e^{y\theta_n})(-L'(\tau))^r - \tau_n R_n S_{2,n}(\tau_n,\tau)\} + \tau_n R_n W_{1,n}(\tau).$$

We also replace $\sqrt{n}(\tau_n - \tau)$ by its value given by (10). It follows that $\sqrt{n}\, I_n$ is of the form

$$(16)\quad \sqrt{n}\, I_n = U_n W_{0,n}(\tau) + V_n W_{1,n}(\tau)$$

where

$$U_n = \{(1 - \tau y e^{y\theta_n})(-L'(\tau))^r - \tau_n R_n S_{2,n}(\tau_n,\tau)\}/S_{1,n}(\tau_n,\tau)$$

$$V_n = \tau_n R_n .$$

By proposition 3, almost surely

$$U_n \to U := \{(1 - \tau y)(-L'(\tau))^r - \tau R E(X^2 e^{-\tau X})\}/E[Xe^{-\tau X}]$$

and $V_n \to V := \tau R$ where $R = r(-L'(\tau))^{r-1}$. Rewrite (16) in the form

$$\sqrt{n}\, I_n = \frac{1}{\sqrt{n}} \sum_{j=1}^{n} [U(e^{-\tau X_j} - E(e^{-\tau X})) + V(X_j e^{-\tau X_j} - E(Xe^{-\tau X}))]$$

$$+ (U_n - U)W_{0,n}(\tau) + (V_n - V)W_{1,n}(\tau)$$

then proposition 4 can be applied a number of times. We ultimately find

$$\sqrt{n}\, I_n \xrightarrow{\mathfrak{D}} \mathfrak{N}(0,\delta^2)$$

where

$$\delta^2 = \mathrm{Var}[(U + VX)e^{-\tau X}] .$$

Returning to $C_n - C$ we obtain

$$\sqrt{n}(C_n - C) \xrightarrow{\mathfrak{D}} \mathfrak{N}(0,\delta_1^2)$$

where

$$\delta_1^2 = \delta^2/\{e^{-\tau y}\tau^2(L'(\tau))^{2r}\}^2$$

by another application of proposition 4.

3. CHERNOFF BOUNDS

Assume $\sigma > 0$. Let $s < 0$. Then for $y \geq 0$, $L(s) = \int_0^\infty e^{-sx}dF(x) \geq \int_y^\infty e^{-sx}dF(x) \geq e^{-sy}P[X > y]$. Hence we get the *Chernoff bound*

$$P[X > y] \leq \inf_{-\sigma<s<0} e^{sy}L(s) .$$

This means that

$$P[X > y] \leq e^{\tau y}L(\tau) \tag{17}$$

where

$$yL(\tau) + L'(\tau) = 0 . \tag{18}$$

(i) The complete monotonicity of L implies that $\frac{-L'}{L}$ is decreasing for $s > -\sigma$. Hence $-L'(\tau)/L(\tau) > -L'(0) =: \mu$ if $\tau < 0$. Hence we restrict attention in (17) to $y > \mu$.

(ii) We estimate τ by solving $yL_n(\tau_n) + L_n'(\tau_n) = 0$. Now, as before $\tau_n \to \tau$ a.s.. Furthermore

$$\begin{aligned} 0 &= n\{yL_n(\tau_n) + L_n'(\tau_n)\} - n\{yL(\tau) + L'(\tau)\} \\ &= y \sum_{j=1}^{n} (e^{-\tau_n X_j} - E(e^{-\tau X})) - \sum_{j=1}^{n} (X_j e^{-\tau_n X_j} - E(Xe^{-\tau X})) . \end{aligned}$$

By (6) and a little algebra we obtain

$$\sqrt{n}\,(\tau_n - \tau) = \frac{yV_{0,n}(\tau) - W_{1,n}(\tau)}{yS_{1,n}(\tau_n,\tau) - S_{2,n}(\tau_n,\tau)} \to \mathfrak{N}(0,\gamma_1^2) \tag{19}$$

where

$$\gamma_1^2 = \frac{\mathrm{Var}((y - X)e^{-\tau X})}{E^2(X(y - X)e^{-\tau X})}.$$

As before a consistent estimator for γ_1^2 and an asymptotic confidence interval for τ_n can be constructed.

(iii) We estimate the Chernoff bound $C := e^{\tau y}L(\tau)$ by the sample statistic $C_n := e^{\tau_n y}L_n(\tau_n)$. As in (11) using (12)

$$e^{-\tau y}(C_n - C) = e^{y(\tau_n - \tau)}L_n(\tau_n) - L(\tau)$$

$$= \{L_n(\tau_n) - L(\tau)\} + L_n(\tau_n)e^{y\theta_n}y(\tau_n - \tau).$$

Now $\sqrt{n}(L_n(\tau_n) - L(\tau)) = W_{0,n}(\tau) - \sqrt{n}(\tau_n - \tau)S_{1,n}(\tau_n,\tau)$ so that

$$\sqrt{n}\, e^{-\tau y}(C_n - C) = \sqrt{n}(\tau_n - \tau)\{ye^{y\theta_n}L_n(\tau_n) - S_{1,n}(\tau_n,\tau)\} + W_{0,n}(\tau).$$

Introducing (19) into this expression yields after easy algebra

$$\sqrt{n}(C_n - C) = U_n W_{0,n}(\tau) + V_n W_{1,n}(\tau)$$

where

$$U_n = e^{\tau y}\frac{y^2 e^{y\theta_n}L_n(\tau_n) - S_{2,n}(\tau_n,\tau)}{yS_{1,n}(\tau_n,\tau) - S_{2,n}(\tau_n,\tau)} \xrightarrow{\text{a.s.}} e^{\tau y}\frac{y^2 L(\tau) - L''(\tau)}{-yL'(\tau) - L''(\tau)} =: U$$

and

$$V_n = e^{\tau y} \frac{S_{1,n}(\tau_n,\tau) - ye^{y\theta_n}L_n(\tau_n)}{yS_{1,n}(\tau_n,\tau) - S_{2,n}(\tau_n,\tau)} \xrightarrow{\text{a.s.}} e^{\tau y} \frac{-L'(\tau) - yL(\tau)}{-yL'(\tau) - L''(\tau)} = :V .$$

Note however that by (18) $y^2L(\tau) = -yL'(\tau)$ so that $U = e^{\tau y}$ while $V = 0$. Hence

$$\sqrt{n}(C_n - C) \xrightarrow{\mathfrak{D}} \mathfrak{N}(0,e^{2\tau y}\sigma_0^2(\tau)).$$

4. THE CLASSICAL RUIN PROBLEM

Assume that F is the distribution of claim sizes $\{X_i\}_1^\infty$ with $EX = \mu$ in an insurance context where claims arrive according to a Poisson process $\{N(t),t \geq 0\}$ with intensity λ. Starting with initial reserve $x \geq 0$ and with incoming payments in the time interval $[0,t]$ equal to t, the company accumulates the risk reserve

$$Y(t) = x + t - \sum_{i=1}^{N(t)} X_i .$$

The probability of non-ruin with initial reserve x is then

$$W(x) := P\{\inf_{t>0} Y(t) > 0\} .$$

It can be shown (see Takács (1967,p.150) Feller (1971,p.377) or Buhlmann (1970, p. 144)) that, if the expected claims paid per unit time $\rho := \lambda\mu < 1$, then

$$W(x) = \sum_{n=0}^{\infty} (1 - \rho)\rho^n \bar{F}^{*n}(x)$$

where $\bar{F}$ is the equilibrium distribution corresponding to F, i.e.

$$\bar{F}(x) = \frac{1}{\mu} \int_0^x [1 - F(y)]dy .$$

Clearly

$$\bar{L}(s) = \{1 - L(s)\}/\mu s .$$

The famous *Lundberg-Cramér ruin estimate* is given by

$$1 - W(x) \sim \frac{1 - \rho}{\rho|\tau||\bar{L}'(\tau)|} e^{\tau x} \ , \ x \to \infty \tag{20}$$

where $\bar{L}(\tau) = \frac{1}{\rho}$.

(i) Note first that $\tau < 0$ since $\rho < 1$; the existence of τ clearly depends on the condition $\bar{L}(-\sigma) > \frac{1}{\rho}$ or $L(-\sigma) > 1 + \frac{\sigma}{\lambda}$. We rewrite (20) somewhat. Consider $h(s) := sL'(s) - L(s)$; then $h'(s) = sL''(s)$ which is negative for $s < 0$. Hence for $\tau < 0$, $h(\tau) = \tau L'(\tau) - L(\tau) > h(0) = -1$.
Now in (20) $\bar{L}'(\tau) = \{-\tau L'(\tau) - 1 + L(\tau)\}/\mu\tau^2 < 0$ so that (20) is replaced by

$$1 - W(x) \sim \frac{(1 - \rho)\mu}{\rho(-L'(\tau) - \frac{1}{\lambda})} e^{\tau x} \tag{21}$$

where we used $\bar{L}(\tau) = \frac{1}{\rho}$ or $L(\tau) = 1 - \frac{\tau}{\lambda}$.

(ii) We estimate τ by τ_n, solution of $L_n(\tau_n) = 1 - \frac{\tau_n}{\lambda}$ Hence

$$\lambda\{L_n(\tau_n) - L(\tau)\} = (\tau - \tau_n)$$

or with (6) and the usual algebra

$$\sqrt{n}(\tau_n - \tau) = \frac{W_{0,n}(\tau)}{S_{1,n}(\tau_n, \tau) - \frac{1}{\lambda}} \xrightarrow{\mathcal{D}} \mathfrak{N}(0, \zeta^2) \tag{22}$$

where $\zeta^2 = \dfrac{\sigma_0^2(\tau)}{(-L'(\tau) - \frac{1}{\lambda})^2}$.

(iii) To estimate the probability of non-ruin we put $C = e^{\tau x}\{-L'(\tau) - \frac{1}{\lambda}\}^{-1}$ and $C_n = e^{\tau_n x}\{-L_n'(\tau_n) - \frac{1}{\lambda}\}^{-1}$. Then

$$I_n := e^{-\tau x}[L_n'(\tau_n) + \tfrac{1}{\lambda}][L'(\tau) + \tfrac{1}{\lambda}](C_n - C)$$

can be rewritten by (12) in the form

$$I_n = L_n'(\tau_n) - L'(\tau) - (\tau_n - \tau)(L'(\tau) + \tfrac{1}{\lambda})xe^{\theta_n x};$$

with the usual operations and (22) we obtain

$$\sqrt{n}\, I_n = U_n W_{0,n}(\tau) - W_{1,n}(\tau)$$

where

$$U_n = \frac{S_{2,n}(\tau_n,\tau) - (L'(\tau) + \frac{1}{\lambda})xe^{\theta_n x}}{S_{1,n}(\tau_n,\tau) - \frac{1}{\lambda}} \xrightarrow{\text{a.s.}} x - \frac{L''(\tau)}{L'(\tau) + \frac{1}{\lambda}} := U.$$

Hence

$$\sqrt{n}(C_n - C) \xrightarrow{\mathfrak{D}} \mathfrak{N}(0,\zeta_1^2)$$

where

$$\zeta_1^2 = e^{2\tau x}\mathrm{Var}\{(U - X)e^{-\tau X}\}/[L'(\tau) + \tfrac{1}{\lambda}]^4 .$$

5. THE COMPOUND POISSON DISTRIBUTION

Starting with a similar setup as in the previous example one can derive an asymptotic expression for the distribution of the totality of claims incurred up to time t, i.e. $X(t) = \sum_{i=1}^{N(t)} X_i$. Under a number of weak conditions it was shown in Embrechts, Jensen e.a. (1985) that

(23) $$P\{X(t) > y\} \sim \frac{e^{\tau y}[e^{-\lambda t[1 - L(\tau)]} - e^{-\lambda t}]}{|\tau|\sqrt{2\pi\lambda t L''(\tau)}} \ , \ y \to \infty$$

where

(24) $$-\lambda t L'(\tau) = y \ .$$

(i) It is clear that we require $y > \mu\lambda t$ to obtain $\tau < 0$. We also write $z = y/(\lambda t)$ for brevity.

(ii) Again τ_n is estimated by the solution of $L_n'(\tau_n) = -z$. As before

$$\sqrt{n}(\tau_n - \tau) = \frac{V_{1,n}(\tau)}{S_{2,n}(\tau_n,\tau)} \xrightarrow{\mathfrak{D}} \mathfrak{N}(0,\xi^2)$$

where $\xi^2 = \dfrac{\sigma_1^2(\tau)}{(L''(\tau))^2}$.

(iii) Put $C = \dfrac{e^{\tau y}(e^{\lambda t L(\tau)} - 1)}{(-\tau)(L''(\tau))^{\frac{1}{2}}}$ and $C_n = \dfrac{e^{\tau_n y}(e^{\lambda t L_n(\tau_n)} - 1)}{(-\tau_n)(L_n''(\tau_n))^{\frac{1}{2}}}$

and

$$I_n = \tau\tau_n(L''(\tau)L_n''(\tau_n))^{\frac{1}{2}} e^{-\tau y}(C_n - C) \ .$$

We use the identity $\sqrt{a} - \sqrt{b} = (a-b)/(\sqrt{a} + \sqrt{b})$ and a further Taylor expansion

$$e^{\lambda t L_n(\tau_n)} = e^{\lambda t L(\tau)} + \lambda t[L_n(\tau_n) - L(\tau)]e^{\lambda t \Omega_n}$$

where

$$\min(L_n(\tau_n), L(\tau)) \le \Omega_n \le \max(L_n(\tau_n), L(\tau)) .$$

After tedious calculations we arrived at the following expression

$$\sqrt{n}\, I_n = U_n W_{0,n}(\tau) + V_n W_{1,n}(\tau) + W_n W_{2,n}(\tau)$$

where

$$U_n := -\tau\lambda t e^{\lambda t \Omega_n} \sqrt{L''(\tau)} \xrightarrow{\text{a.s.}} -\tau\lambda t e^{\lambda t L(\tau)} \sqrt{L''(\tau)} =: U,$$

$$V_n := \{S_{2,n}(\tau_n,\tau)\}^{-1} \{\sqrt{L''(\tau)}\, [(e^{\lambda t L(\tau)} - 1)(1 - \tau y e^{y\theta_n}) + \tau\lambda t e^{\lambda t \Omega_n} S_{1,n}(\tau_n,\tau)]$$

$$- \frac{-\tau_n(e^{\lambda t L(\tau)} - 1)}{\sqrt{L''_n(\tau_n)} + \sqrt{L''(\tau)}} S_{3,n}(\tau_n,\tau)$$

$$- \frac{\sqrt{L''(\tau)}}{\sqrt{n}} \tau\lambda t y e^{y\theta_n + \lambda t \Omega_n} (W_{0,n}(\tau) - \sqrt{n}(\tau_n - \tau) S_{1,n}(\tau_n,\tau))\}$$

$$\xrightarrow{\text{a.s.}} \frac{1}{L''(\tau)} \{(e^{\lambda t L(\tau)} - 1 + \tau y)\sqrt{L''(\tau)} - \frac{1}{2}(e^{\lambda t L(\tau)} - 1) \frac{\tau L^{(3)}(\tau)}{\sqrt{L''(\tau)}}\} =: V,$$

using (24), and

$$W_n := \frac{\tau_n(e^{\lambda t L(\tau)} - 1)}{\sqrt{L''_n(\tau_n)} + \sqrt{L''(\tau)}} \xrightarrow{\text{a.s.}} \frac{\tau(e^{\lambda t L(\tau)} - 1)}{2\sqrt{L''(\tau)}} =: W .$$

As before we arrive at

$$\sqrt{n}(C_n - C) \xrightarrow{\mathcal{D}} \mathcal{N}(0, \xi_1^2)$$

where

$$\xi_1^2 = \left(\frac{e^{\tau y}}{\tau^2 L''(\tau)}\right)^2 \mathrm{Var}\{(U + VX + WX^2)e^{-\tau X}\} .$$

6. THE COMPOUND PÓLYA DISTRIBUTION

In Embrechts, Jensen e.a. (1985) the *compound Pólya distribution* was approximated as well; here

$$P\{X(t) \le x\} = \sum_{n=0}^{\infty} \binom{k+n-1}{n} \left(\frac{t}{t+k}\right)^n \left(\frac{k}{t+k}\right)^n F^{*n}(x)$$

where $k > 1$.
Under weak conditions it was shown that

$$P[X(t) > y] \sim \frac{k^k e^{-k}}{\Gamma(k)} \frac{e^{\tau y}}{|\tau| y} \left\{\left(\frac{y}{-tL'(\tau)}\right)^k - \left(\frac{k}{t+k}\right)\right)\right)^k\} \quad , \quad y \to \infty \tag{25}$$

where

$$-tL'(\tau) = y\{1 + \tfrac{t}{k}[1 - L(\tau)]\}.$$

(i) Since $y \to \infty$ we can assume that $y > \mu t$ so that the increasing function $1 + \frac{t}{k}[1 - L(s)]$ (with value 1 at $s = 0$) intersects the decreasing function $-\frac{t}{y} L'(s)$ (with value $\frac{\mu t}{y}$ at $s = 0$) at some value $\tau < 0$.
We simplify the notation somewhat by introducing the constants

$$z = \frac{y}{t} \quad , \quad a = \left(\frac{k}{t+k}\right)^k \quad , \quad b = \frac{y}{k}$$

(ii) Again τ_n is defined by $-L_n'(\tau_n) = z + b[1 - L_n(\tau_n)\}$ where $-L'(\tau) = z + b[1 - L(\tau)]\}$ defines τ. Hence elimination of z yields

$$L'(\tau) - L_n'(\tau_n) = b\{L(\tau) - L_n(\tau_n)\}\ .$$

By now the calculations are routine; we obtain

$$\sqrt{n}(\tau_n - \tau) = \frac{W_{1,n}(t) + bW_{0,n}(t)}{S_{2,n}(\tau_n,\tau) + bS_{1,n}(\tau_n,\tau)} \xrightarrow{\mathfrak{D}} \mathfrak{N}(0,K^2)$$

where

$$K^2 = \mathrm{Var}\{(X + b)e^{-\tau X}\}/E^2\{X(X + b)e^{-\tau X}\}.$$

(iii) Put $C = \frac{e^{\tau y}}{|\tau|}\{(\frac{z}{-L'(\tau)})^k - a\}$, $C_n = \frac{e^{\tau_n y}}{|\tau_n|}\{(\frac{z}{-L_n'(\tau_n)})^k - a\}$, and

$$I_n = e^{-\tau y}\tau\tau_n(C_n - C).$$

Again, very much like in the Pascal case, we find

$$\sqrt{n}\, I_n = U_n W_{0,n}(\tau) + V_n W_{1,n}(\tau)$$

where

$$\frac{1}{b} U_n\{S_{2,n}(\tau_n,\tau) + bS_{1,n}(\tau_n,\tau)\} := \left[(\frac{-z}{L'(\tau)})^k - a\right] - y\tau e^{y\theta_n}\left[(\frac{-z}{L'(\tau_n)})^k - a\right]$$

$$+ S_{2,n}(\tau_n,\tau)\frac{\tau(-z)^k}{[L'(\tau)L_n'(\tau_n)]^k} \sum_{m=0}^{k-1} (L_n'(\tau_n))^m (L'(\tau))^{k-1-m}$$

$$\xrightarrow{\text{a.s.}} (1 - y\tau)\left[(\frac{-z}{L'(\tau)})^k - a\right] - \frac{k\tau z^k L''(\tau)}{[-L'(\tau)]^{k+1}}$$

$$=: \frac{1}{b} UE[X(X + b)e^{-\tau X}]$$

and

$$V_n\{S_{2,n}(\tau_n,\tau) + bS_{1,n}(\tau_n,\tau)\} := \left[\left(\frac{-z}{L'(\tau)}\right)^k - a\right] - y\tau e^{y\theta_n}\left[\left(\frac{-z}{L_n'(\tau_n)}\right)^k - a\right]$$

$$-bS_{1,n}(\tau_n,\tau)\frac{\tau(-z)^k}{[L'(\tau)L_n'(\tau_n)]^k}\sum_{m=0}^{k-1}(L_n'(\tau_n))^m(L'(\tau))^{k-1-m}$$

$$\xrightarrow{a.s.} (1 - y\tau)\left[\left(\frac{-z}{L'(\tau)}\right)^k - a\right] + \frac{kb\tau z^k}{[-L'(\tau)]^k} =: VE[X(X+b)e^{-\tau X}].$$

Hence

$$\sqrt{n}(C_n - C) \xrightarrow{\mathfrak{D}} \mathfrak{N}(0,K_1^2)$$

where

$$K_1^2 = e^{2\tau y}\tau^{-4}\mathrm{Var}\{(U + VX)e^{-\tau X}\}.$$

7. REMARKS

As mentioned in the summary, the relationship

$$\text{(25)} \qquad P\{\sum_{i=1}^{N} X_i > y\} \sim \varphi(y,\tau,L(\tau),L'(\tau),\ldots) = :\varphi$$

where $\psi(\tau,L(\tau),\ldots) = 0$, holds for $y \to \infty$. On top of this first approximation we replaced the right hand side of (25) by

$$\varphi_n := \varphi(y,\tau_n,L_n(\tau_n),L_n'(\tau_n),\ldots)$$

where L_n was the empirical version of L and where τ_n was the sample version of the solution of the equation $\psi(\tau_n,L_n(\tau_n),\ldots) = 0$. The reader could wonder how accurate the superimposed approximations will be.

To evaluate the accuracy of (25) in itself is an important but hard problem. In some cases the proof of (25) is easy, like in the Chernoff bounds; however more typically the proof depends on deep theorems from the theory of stochastic processes or on intricate procedures from asymptotic analysis. Hence a second order term or even a series expansion would be desirable. Let us remark that for the compound Pólya and for the compound Poisson case, second and higher order terms are available in Embrechts, Jensen e.a. (1985).

A systematic study of the accuracy of the empirical version φ_n in evaluating φ is an entirely different issue, depending heavily upon simulation studies. As a simple test case we evaluated the Chernoff bound for the exponential distribution with mean 1. Then $L(s) = (1+s)^{-1}$; given $y > 1$, $\tau = y^{-1}-1$ and $C = e^{\tau y} L(\tau) = y\, e^{1-y}$; also $I = (-1,\infty)$. After substantial simulation we could only conclude that the problem is far from trivial. For small y-values the accuracy was always found to be sufficient even though we encountered systematic bias; for large values of y however the accuracy is far less satisfying.

Here are some of our temporary conclusions :

(i) In many examples L(s) and its derivatives have a vertical asymptote at $s = \sigma$; as a result the solution of the implicit equation $\psi(\tau_n, L_n(\tau_n),\ldots) = 0$ for τ_n induces systematic bias. Also the numerical procedure (bisection, Newton-Raphson,...) influences the accuracy but to a lesser extent.

(ii) In all of our examples we made heavy use of proposition 2; a requirement for its applicability was however that $2\tau > \sigma$. In our simulation example, y values greater than 2 caused large fluctuations in the estimation of τ and C. Moreover, the larger y, the closer we get to the singularity of the function L(s).

(iii) Presumably the most important reason for the eventual poor behavior of the sample version τ_n is this : the empirical functions $L_n(s)$ are determined by the positive sample values $X_1, X_2, \ldots, X_n$; in all our formulae however we use the extrapolated values of $L_n(s)$ for $s < 0$; moreover the latter function is entire for every n while L(s) has a singularity at $\sigma < 0$.

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