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Empirical Likelihood Inference for the Mean Past Lifetime Function

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EMPIRICAL LIKELIHOOD INFERENCE FOR THE MEAN PAST LIFETIME FUNCTION

by

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Under the Direction of Yichuan Zhao, PhD

ABSTRACT

In several fields, such as survival analysis, reliability theory, and forensic science, the mean past lifetime (MPL), also known as the expected inactivity time function, plays a vital role. For inference on the MPL function, some procedures have been proposed in the literature, based on a Central Limit Theorem result for the MPL function's estimator. In this thesis, an empirical likelihood (EL) inference procedure of the MPL function is proposed. In addition to that, we obtain the adjusted EL and mean EL confidence interval for the MPL function. The proposed confidence intervals are compared through simulation studies in terms of coverage probability and the average length of the confidence interval. The simulation studies showed that the proposed EL methods have better coverage probability and shorter average lengths than the normal approximation result. Finally, the proposed methods are illustrated by two real data analyses.

INDEX WORDS: Confidence interval, Estimating equation, Empirical likelihood, Adjusted empirical likelihood, Mean empirical likelihood, Wilks' theorem.

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DEDICATION

This research work is dedicated to my family, friends, course mates, and the memory of my loving grandfather.

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- MPL - Mean past lifetime
- EL - Empirical likelihood
- MEL - Mean empirical likelihood
- AEL - Adjusted empirical likelihood
- NA - Normal approximation
- LLR - Log likelihood ratio

1 INTRODUCTION

The mean past lifetime (MPL), also known as the expected inactivity time, remains significant in survival analysis and reliability theory. The inactivity time of a component, corresponds to the time elapsed from the failure of the component, given that its lifetime T, is less than or equal to t, denoted by $k_t = (t - T \mid T \leq t)$, (see Ruiz and Navarro (1996)). The reversed mean past life (or mean residual life) of a system is well known to be defined as the expected value of the remaining lifetime of a system, provided that it has survived up to time t, that is, $m(t) = E(T - t \mid T > t)$, at time $t \geq 0$. Conversely the MPL, following from the inactivity time and mean residual life, is defined as the expected value of the conditional random variable $k(t) = E(t - T \mid T \leq t)$ which is the mean time elapsed since the failure time of T, given that $T \leq t$, at time $t \geq 0$.

Over the years, the mean past lifetime has been an area of increasing interest among researchers. Navarro et al. (1997) first studied the stochastic order of the reverse mean residual life order (RMRL). In parallel systems, the stochastic comparisons of residual lifetime and inactivity time were discussed by Li and Lu (2003). For aging properties of the residual lifetime and past lifetime, Gupta et al. (2012) examined the conditions sufficient for series and parallel systems. To establish various properties of the stochastic comparisons in both reliability theory and survival analysis, the MPL function was also studied by Kayid and Ahmad (2004). The relationship between the reversed hazard rate (the ratio of the density to the distribution function) with the RMRL was also studied by Li and Lu (2003). Nanda and Kundu (2008) showed that the one partial moment and second-order moment of inactivity time determines the distribution uniquely. Kundu et al. (2010) considered $m(t)$ for characterizations of quite a few distributions, which can be established to the $k(t)$ proposed by Chandra and Roy (2001).

The MPL function is crucial for characterizing aging and reliability properties, and it has been extensively studied in the literature by Jayasinghe and Zeephongsekul (2013).

Regarding statistical estimation, Asadi and Berred (2012) investigated the properties of the MPL along with other reliability measures, taking into account the empirical estimator and proved that it is a consistent estimator. The importance of the hazard rate, mean residual life, and the reversed hazard rate in studying the properties of the MPL function was also established. Ortega (2009) explained the relationship between the MPL order and the mean residual life order. Furthermore, Nanda et al. (2003) obtained some results on the order of the MPL. Finkelstein (2002) has shown that the reversed hazard rate ordering implies the MPL ordering. Jayasinghe and Zeephongsekul (2013) used a local polynomial fitting technique to obtain several nonparametric estimators for the MPL function. It was also shown by Jayasinghe and Zeephongsekul (2013) that the proposed estimators are asymptotically unbiased, consistent and also when they are standardized, have an asymptotic normal distribution.

The MPL can also be used to predict the time of occurrence of an event such as time of death, particularly in forensic science and insurance (Gupta and Nanda, 2001). Another significant usage of the MPL function is to investigate the aging properties of a system. Goliforushani and Asadi (2008) discussed the use of the MPL on the reliability of a system. The MPL function is also useful in biomedicine to study the incubation times of diseases and maintenance policies in reliability. For example, Eeckhoudt and Gollier (1995) studied this reliability measure and applied it to risk theory and econometrics. The MPL has also proven useful in determining maintenance policies in the reliability of a system (Finkelstein, 2002). More so, in a medical test to test for the novel coronavirus, one may mainly be interested in knowing how much time has elapsed since being infected by the virus.

For additional applications of the MPL, we refer the reader to Li and Zuo (2004), Kayid and Ahmad (2004), Tavangar and Asadi (2010), Jayasinghe and Zeephongsekul (2013), etc.

1.1 The review of the empirical likelihood

There are a variety of ways to compute the MPL function in parametric and nonparametric settings. The drawbacks associated with the parametric approach are enormous. To the best of our knowledge and based on the literature by Jayasinghe and Zeephongsekul (2012), the empirical likelihood, which is a nonparametric method, outperforms the parametric approach. The empirical likelihood in survival analysis was first introduced by Thomas and Grunkemeier (1975). Owen (1998, 1990) introduced the empirical likelihood confidence regions. The empirical likelihood method is a robust nonparametric method that requires no parametric assumptions. Inference based on the EL method does not require estimation of variance and has unique properties of transformation-preserving, Bartlett correctability, and demonstrates better coverage probability for small sample sizes, see (Zhao and Qin, 2006). A review of inference based on the EL can be found in Owen (2001).

An extensive application of the EL approach exists. For instance, Yang and Zhao (2012) proposed EL methods for the semiparametric linear transformation model. Chen et al. (2009) proposed an EL confidence interval for copulas. In regression analysis, where observations can be missing not at random, the EL approach has also been considered based on the rank-based gradient function (Bindele and Zhao, 2018). Further studies have also been done using the jackknife empirical likelihood (JEL), introduced by Jing et al. (2009). JEL methods are known to provide superior coverage probabilities, even for the skewness and kurtosis coefficients (Zhao et al., 2018). Zhao et al. (2015) extended the JEL to construct confidence intervals for the mean absolute deviation. Furthermore, Cheng and Zhao (2018) introduced the Bayesian jackknife empirical likelihood to replace the likelihood component with the JEL. Also, Wang and Zhao (2016) derived the JEL for the difference of two Gini indices.

Zardasht (2017) proposed an EL methodology for the mean past lifetime function based on random censorship. For the MPL function, the variance estimate is too conservative and leads to heavy overcoverage in the normal approximation based confidence intervals from Zardasht (2017). The procedure in this thesis primarily follows from Zhao and Qin (2006), where the EL method was applied to the mean residual life function.

1.2 The review of the adjusted empirical likelihood

Chen et al. (2008) introduced the adjusted empirical likelihood to solve the problem posed by the non-existence of solutions while computing the profile empirical likelihood. Most importantly, the adjusted empirical likelihood upholds the optimal asymptotic properties of the empirical likelihood, and its associated confidence regions have demonstrated to be closer in terms of coverage probabilities of confidence intervals. Furthermore, for small sample sizes, the EL method is affected by the low precision of the chi-square approximation. Also, Liu and Chen (2010) showed that if the dimension of the accompanying estimating function is high, the EL method tends to be affected. However, the adjusted empirical likelihood can address this problem successfully. This thesis proposes the AEL method for the mean past lifetime function.

1.3 The review of the mean empirical likelihood

To further overcome the poor accuracy, particularly for small sample sizes and multi-dimensional situations of the EL method, Liang et al. (2019) proposed a novel approach known as the mean empirical likelihood (MEL) method. Currently, the MEL approach has proven to yield much more accurate confidence regions and coverage probabilities. This method achieves its merits by using pairwise-mean data. It should be noted that the Hodges–Lehmann sign-based estimator, by Hodges and Lehmann (1963), uses a similar mean-pair data concept, which gives much more reliable nonparametric estimators compared to the standard median estimator. In this thesis, the MEL method for the mean past lifetime function is also proposed.

1.4 Purpose of study

The main aim of this thesis is to estimate the mean past lifetime function and make inference via constructing confidence intervals, using both normal approximation and empirical likelihood methods. The empirical likelihood method is then extended to the adjusted empirical likelihood and the mean empirical likelihood. This thesis is organized as follows. In Chapter 2, we review the estimation of the mean past lifetime function in detail. In Chapter 3, confidence bands of the mean past lifetime for the empirical likelihood, adjusted empirical likelihood, and mean empirical likelihood methods are proposed. We conduct extensive simulation studies to evaluate the proposed methods in Chapter 4. In Chapter 5, an application to two real datasets is provided. Some concluding remarks are given in Chapter 6.

2 ESTIMATION OF THE MEAN PAST LIFETIME

In this chapter, we review the normal approximation method for the mean past lifetime for completeness. We adopt similar notations as in Asadi and Berred (2012). Let T be a lifetime random variable with distribution function F. The survival function is $\bar{F} = 1 - F$, density function is $f(t)$, MPL is $k(t)$ and such that $E(T) < \infty$. Denote $\tau_0 = \inf\{t : F(t) > 0\}$ and $M_F = \sup\{t : F(t) < 1\}$.

The mean past lifetime (MPL) function $k(t)$ of T at time $t \geq \tau_0$ is

$$\begin{aligned} k(t) = E(t - T | T \leq t) &= \frac{\int_0^t F(x)\, dx}{F(t)} \\ &= t - \frac{\int_0^t x f(x) dx}{F(t)}, \end{aligned}$$

for t, such that $F(t) > 0$.

Let $T_1, \ldots, T_n$ be a random sample drawn from a random variable T having a continuous distribution function F. We consider the empirical MPL $k_n(t)$ as

$$\begin{aligned} k_n(t) &= \frac{\int_0^t F_n(x)\, dx}{F_n(t)} I_{[t \geq T_{1:n}]}, \\ &= \sum_{k=1}^{n} \left(t - \frac{1}{k} \sum_{j=1}^{k} T_{1:j} \right) I_{[T_{1:k} \leq t < T_{k+1:n}]}, \end{aligned}$$

where $T_{n+1:n} = M_F = \sup\{t : F(t) < 1\}$ by the convention, and $T_{1:n} < \ldots < T_{n:n}$ are the ordered statistics.

In some situations, the observations may contain ties. Hence, the empirical estimator needs to be adjusted by taking into account the number of tied observations. See the discussions in Jayasinghe and Zeephongsekul (2013) about ties.

The mean of the empirical estimator $k_n(t)$ is

$$E(k_n(t)) = k(t)(1 - \bar{F}^n(t)).$$

The variance of the empirical estimator proposed by Asadi and Berred (2012), is

$$V(k_n(t)) = k^2(t)(1-\bar{F}^n(t))F^n(t) + \upsilon_n(t)\left(2\frac{\rho_1(t)}{F(t)} - (k(t))^2\right),$$

where the survival function $\bar{F}(t) = 1 - F(t)$, $\upsilon_n(t) = \sum_{i=1}^{n} \frac{\bar{F}^{n-i}(t)(1-\bar{F}^i(t))}{i}$ and $\rho_1(t) = \int_a^t (t-z)\, F(z)\, dz$. Since $0 < \bar{F}^n(t) < 1$, it follows that $k_n(t)$ is asymptotically unbiased for $k(t)$ as $n \to \infty$. For $\tau_0 < s \leq t < M_F$, Asadi and Berred (2012) also showed the covariance as

$$\operatorname{Cov}[k_n(s), k_n(t)] = k(s)\,k(t)\left(1-\bar{F}^n(t)\right)F^n(s) + \upsilon_n(s,t)\left(2\frac{\rho_1(t)}{F(t)} - k^2(t)\right) +$$

$$\frac{\bar{F}^n(s) - \bar{F}^n(t)}{\bar{F}(s) - \bar{F}(t)}\rho_0(s)\left(t - s - \frac{\rho_0(t) - \rho_0(s)}{F(s)}\right),$$

where $\upsilon_n(s,t) = \sum_{i=1}^{n} \frac{\bar{F}^{n-i}(s)(1-\bar{F}^i(t))}{i}$.

Thus, an asymptotic $100(1-\alpha)\%$ normal approximation (NA) confidence interval for $k(t)$ at the fixed time $t > \tau_0$ is given by (see Asadi and Berred (2012))

$$I_1(t) = \left\{k(t) : n\left(\hat{k}(t) - k(t)\right)^2 \leq \widehat{V}(k_n(t))\chi^2_{1,\alpha}\right\},$$

where $\chi^2_{1,\alpha}$ is the upper α-quantile of the chi-square distribution with one degree of freedom and $\widehat{V}(k_n(t))$ is a consistent estimator of $V(k_n(t))$.

3 CONFIDENCE BANDS USING EMPIRICAL LIKELIHOOD METHODS

In this chapter, we propose empirical likelihood, mean empirical likelihood, and adjusted empirical likelihood methods for the mean past lifetime function. The procedure for the empirical likelihood in this chapter uses the approach by Zhao and Qin (2006).

3.1 The empirical likelihood method for $k(t)$

Let $T_1, \ldots, T_n$ be independent and identically distributed (i.i.d.) samples of T with distribution function $F(t)$. Under the EL approach, the estimation equation is defined at the fixed time $t > \tau_0$ as follows:

$$U(k(t)) = \int_0^t \frac{1}{n} \sum_{i=1}^n I\left(T_i \leq u\right) du - k\left(t\right) \frac{1}{n} \sum_{i=1}^n I\left(T_i \leq t\right). \tag{3.1}$$

Also, it can be easily inferred that the equation $U(k(t)) = 0$ has the solution $\hat{k}(t)$. Let the true value of $k(t)$ at time t be $k_0(t)$. By the estimation from Eq. (3.1) it can be further proven that $E[U(k_0(t))] = 0$. For $1 \leq i \leq n$, we can then define for each fixed time $t > \tau_0$:

$$\begin{aligned} W_i(k(t)) &= \int_0^t I(T_i \leq u) du - k(t) I(T_i \leq t) \\ &= I(T_i \leq t)(t - T_i - k(t)). \end{aligned} \tag{3.2}$$

Then, the EL at $k(t)$ is given by

$$L\left(k(t)\right) = \sup \left\{ \prod_{i=1}^n p_i : \sum_{i=1}^n p_i W_i(k(t)) = 0,\ p_i \geq 0,\ \sum_{i=1}^n p_i = 1,\ i = 1, \ldots, n \right\},$$

where $\mathbf{p} = (p_1, \ldots, p_n)$ is a probability vector, i.e., $\sum_{i=1}^n p_i = 1$ and $p_i \geq 0$ for $1 \leq i \leq n$.

The EL ratio at $k(t)$ is defined by

$$R\left(k(t)\right) = \sup\left\{\prod_{i=1}^{n} np_i : \sum_{i=1}^{n} p_i W_i(k(t)) = 0,\ p_i \geq 0,\ \sum_{i=1}^{n} p_i = 1,\ i = 1, \ldots, n\right\}.$$

We know by using Lagrange multipliers that $R(k(t))$ is maximized. Hence, we have

$$-2\log R\left(k\left(t\right)\right) = 2\sum_{i=1}^{n} \log\left\{1 + \lambda(t) W_i(k(t))\right\},$$

where $\lambda(t)$ satisfies the following nonlinear equation

$$\frac{1}{n}\sum_{i=1}^{n} \frac{W_i(k(t))}{1 + \lambda(t) W_i(k(t))} = 0.$$

We may find the value of $\lambda(t)$ by numerical search. (See Hall and La Scala (1990) and Owen (2001)). Let $k_0(t)$ be the true value of $k(t)$. We establish the main result for the EL ratio statistic as follows:

Theorem 3.1. Suppose $E(T^2) < \infty$ and $\Gamma(t) = \mathrm{Var}(W_1(k_0(t))) = \mathrm{E}[(t - T - k_0(t))^2 I(T \leq t)] > 0$ for each fixed time $t > \tau_0$. $-2\log R(k_0(t))$ converges in distribution to χ_1^2, that is,

$$-2\log R(k_0(t)) \xrightarrow{\mathcal{D}} \chi_1^2\ ,$$

where χ_1^2 is a chi-square distribution with one degree of freedom.

Thus, an asymptotic $100(1-\alpha)\%$ pointwise confidence band for $k(t)$ in $t \in [a, b]$, $a > \tau_0$ can be written from Theorem 3.1 as

$$I_2\left(\alpha\right) = \left\{k\left(t\right) : -2\log R\left(k\left(t\right)\right) \leq \chi_1^2\left(\alpha\right)\right\}\ ,$$

where $\chi_{1,\alpha}^2$ is the upper α-quantile of the distribution of χ_1^2.

In addition, we can establish simultaneous confidence bands as follows:

Theorem 3.2. Suppose $E(T^2) < \infty$ and $\Gamma(t) > 0$ for $t \in [a,b]$, $\tau_0 < a < b < M_F$. $-2\log R(k_0(t))$ weakly converges to $U^2(t)/\Gamma(t)$ in $D[a,b]$, where $U(t)$ is a Gaussian process with zero mean and covariance function

$$\mathrm{Cov}(U(s),U(t)) = E[(s-T-k_0(s))(t-T-k_0(t))I(T\le s)]$$

where $a \le s \le t \le b$.

Thus, an asymptotic $100(1-\alpha)\%$ simultaneous confidence band for $k(t)$ in $t \in [a,b], a > \tau_0$ is

$$I_3(\alpha) = \{k(t) : -2\log R(k(t)) \le c(\alpha), t \in [a,b]\},$$

where $c(\alpha)$ is the upper α-quantile of the distribution of $\sup\limits_{a\le t\le b}\{U^2(t)/\Gamma(t)\}$.

3.2 The adjusted empirical likelihood method for $k(t)$

The empirical likelihood poses a zero convex hull problem. To deal with this problem, Chen et al. (2008) proposed the adjusted empirical likelihood (AEL). The proposed method attempts to solve the problem by adding one more data point to $W_i(k(t))$, and this is applied to the empirical likelihood.

From the generated $W_i(k(t))$'s in Eq. (3.2), the extra data point is obtained by using the proposed convention;

$$W_{n+1}(k(t)) = -a_n\frac{\sum_{i=1}^n W_i(k(t))}{n}\ .$$

In the above equation, a_n is a positive constant. By following the recommendation by Chen et al. (2008), to choose the value of a_n, we thereby use

$$a_n = \max(1, \log(n)/2).$$

With the $(n+1)$ data points, the empirical likelihood ratio at $k(t)$ is

$$R^{adj}(k(t)) = \sup\left\{\prod_{i=1}^{n+1}(n+1)\,p_i : \sum_{i=1}^{n+1} p_i W_i(k(t)) = 0, p_i \geq 0, \sum_{i=1}^{n+1} p_i = 1,\ i = 1, 2, \ldots, n+1\right\} .$$

The adjusted empirical likelihood ratio at $k(t)$ is given by

$$l^{adj}(k(t)) = -2\log R^{adj}\,(k(t)) = 2\sum_{i=1}^{n+1}\log\left\{1 + \lambda^a\,(t)\,W_i(k(t))\right\}\ ,$$

where λ^a is the solution to the following equation

$$\sum_{i=1}^{n+1}\frac{W_i(k(t))}{1+\lambda^a\,(t)\,W_i(k(t))} = 0\ . \tag{3.3}$$

We can establish the following Wilks' theorem for the adjusted empirical likelihood (AEL) at $k_0(t)$.

Theorem 3.3. Under regularity conditions as in Theorem 3.1, the $l^{adj}(k_0(t))$ converges in distribution to a chi-square distribution with one degree of freedom given as χ_1^2 as $n \to \infty$.

From Theorem 3.3, we can then construct the asymptotic $100\,(1-\alpha)\,\%$ adjusted empirical likelihood confidence interval for $k_0(t)$ as

$$I_4\,(\alpha) = \{k\,(t) : -2\log R^{adj}\,(k\,(t)) \leq \chi_1^2\,(\alpha)\}.$$

3.3 The mean empirical likelihood method for $k(t)$

The accuracy of the empirical likelihood confidence interval for small sample sizes can be further improved. To construct more accurate confidence intervals, the mean empirical likelihood (MEL) proposed by Liang et al. (2019), is further used to assess the performance since their proposed MEL method provides higher accuracy than that of other empirical likelihood methods. The main idea of the MEL method is to generate a pseudo data set using the means of observation values and apply the empirical likelihood to the new data set. We denote the pairwise-mean data set as follows,

$$V(k(t)) = \left\{ \frac{W_i(k(t)) + W_j(k(t))}{2} : 1 \le i \le j \le n \right\},$$

which can be written as

$$V(k(t)) = \{V_1(k(t)), V_2(k(t)), \dots, V_N(k(t))\} \ ,$$
$$N = n(n+1)/2 \ .$$

With the pseudo data set, the empirical likelihood ratio at $k(t)$ is

$$R^m(k(t)) = \sup \left\{ \prod_{i=1}^{N} N p_i : \sum_{i=1}^{N} p_i V_i(k(t)) = 0, p_i \ge 0, \sum_{i=1}^{N} p_i = 1, \ i = 1, 2, \dots, N \right\}.$$

The mean empirical likelihood ratio at $k(t)$ is given as

$$\begin{aligned} l^m(k(t)) &= \frac{-2 \log R^m(k(t))}{n+1} \\ &= \frac{2}{n+1} \sum_{i=1}^{N} \log \{1 + \lambda^m(t) V_i(k(t))\} \ , \end{aligned} \tag{3.4}$$

where λ^m is the solution to the following equation

$$\sum_{i=1}^{N} \frac{V_i(k(t))}{1 + \lambda^m(t) V_i(k(t))} = 0 \ .$$

We establish Wilks' theorem for the mean empirical likelihood as follows.

Theorem 3.4. Under regularity conditions as in Theorem 3.1, the $l^m(k_0(t))$ converges in distribution to a χ_1^2 as $n \to \infty$.

From Theorem 3.4, we can then construct the asymptotic $100\,(1-\alpha)\,\%$ mean empirical likelihood (MEL) confidence interval for $k_0(t)$ as

$$I_5\,(\alpha) = \{k\,(t) : -2\log R^m\,(k\,(t)) \leq \chi_1^2\,(\alpha)\}.$$

4 SIMULATION STUDY

Simulation studies were carried out to assess the performance of the normal approximation (NA), empirical likelihood (EL), adjusted empirical likelihood (AEL), and mean empirical likelihood (MEL) methods. Three distributions of T were used to conduct the simulations. Each simulation was repeated 2000 times. Under each distribution, the MPL function, 95% confidence interval, coverage probabilities, and average lengths were computed and summarized. The 95% confidence intervals were used to obtain the coverage probabilities, that is, the proportion of the confidence intervals, which contain the true value $k_0(t)$.

Given that T follows a uniform distribution on the interval $(0,1)$, then at time t, its mean past lifetime function $k_0(t)$ is $t/2$. Table 4.1, gives the coverage probabilities and average lengths at time $(0.2, 0.4, 0.6, 0.8)$ on 30, 50 and 100 sample sizes (n).

Table 4.1: Empirical coverage probabilities and (average lengths) of 95% confidence intervals for the NA, EL, AEL, and MEL methods for the uniform distribution.

n	Method	$\boldsymbol{t = 0.2}$		$\boldsymbol{t = 0.4}$		$\boldsymbol{t = 0.6}$		$\boldsymbol{t = 0.8}$	
30	NA	0.997	(0.315)	0.999	(0.409)	1.000	(0.489)	1.000	(0.583)
	EL	0.923	(0.057)	0.932	(0.123)	0.938	(0.154)	0.945	(0.179)
	AEL	0.956	(0.070)	0.959	(0.151)	0.963	(0.172)	0.961	(0.194)
	MEL	0.954	(0.066)	0.956	(0.137)	0.964	(0.167)	0.960	(0.190)
50	NA	0.999	(0.234)	1.000	(0.313)	1.000	(0.376)	1.000	(0.433)
	EL	0.914	(0.059)	0.934	(0.095)	0.944	(0.121)	0.944	(0.141)
	AEL	0.964	(0.091)	0.955	(0.107)	0.960	(0.130)	0.954	(0.148)
	MEL	0.945	(0.069)	0.957	(0.104)	0.960	(0.127)	0.953	(0.146)
100	NA	0.999	(0.157)	1.000	(0.218)	1.000	(0.264)	1.000	(0.304)
	EL	0.946	(0.046)	0.949	(0.066)	0.944	(0.082)	0.958	(0.099)
	AEL	0.963	(0.052)	0.958	(0.071)	0.954	(0.086)	0.962	(0.103)
	MEL	0.964	(0.051)	0.956	(0.069)	0.951	(0.085)	0.961	(0.102)

Given that T follows an exponential distribution, with mean 1. At time t, it can be readily shown that its corresponding mean past lifetime function is given by,

$$k_0(t) = \frac{t + \exp(-t) - 1}{1 - \exp(-t)} \ \ I(t > 0) \ .$$

The time points taken into account to conduct the simulation are $(0.5, 1, 2, 4)$, and the results are summarized in Table 4.2.

Table 4.2: Empirical coverage probabilities and (average lengths) of 95% confidence intervals for the NA, EL, AEL, and MEL methods for the exponential distribution.

n	Method	$\boldsymbol{t = 0.5}$		$\boldsymbol{t = 1}$		$\boldsymbol{t = 2}$		$\boldsymbol{t = 3}$	
30	NA	0.980	(0.207)	0.983	(0.315)	0.992	(0.662)	1.000	(2.866)
	EL	0.928	(0.151)	0.947	(0.245)	0.943	(0.394)	0.938	(0.513)
	AEL	0.957	(0.181)	0.963	(0.272)	0.962	(0.426)	0.953	(0.551)
	MEL	0.954	(0.169)	0.962	(0.266)	0.961	(0.421)	0.955	(0.551)
50	NA	0.998	(0.229)	0.998	(0.348)	1.000	(0.607)	1.000	(3.747)
	EL	0.941	(0.123)	0.943	(0.192)	0.939	(0.308)	0.947	(0.400)
	AEL	0.965	(0.137)	0.956	(0.205)	0.953	(0.324)	0.958	(0.419)
	MEL	0.963	(0.133)	0.955	(0.202)	0.953	(0.321)	0.955	(0.417)
100	NA	0.996	(0.148)	0.999	(0.226)	0.999	(0.356)	1.000	(2.692)
	EL	0.952	(0.088)	0.949	(0.138)	0.950	(0.220)	0.949	(0.284)
	AEL	0.962	(0.093)	0.959	(0.143)	0.958	(0.226)	0.954	(0.291)
	MEL	0.960	(0.092)	0.956	(0.141)	0.957	(0.224)	0.953	(0.289)

If T follows a a Pareto distribution, with scale parameter 2, and shape parameter 1, then at time t its mean past lifetime function can be defined as,

$$k_0(t) = \frac{t(t-1)}{t+1} \ \ I(t > 1) \ .$$

Table 4.3: Empirical coverage probabilities and (average lengths) of 95% confidence intervals for the NA, EL, AEL, and MEL methods for the Pareto distribution.

n	Method	$\boldsymbol{t = 1.5}$		$\boldsymbol{t = 2}$		$\boldsymbol{t = 2.5}$		$\boldsymbol{t = 3}$	
30	NA	0.999	(0.260)	0.999	(0.288)	0.980	(0.348)	0.976	(0.462)
	EL	0.939	(0.129)	0.934	(0.213)	0.941	(0.285)	0.931	(0.350)
	AEL	0.962	(0.145)	0.953	(0.233)	0.952	(0.308)	0.947	(0.377)
	MEL	0.960	(0.141)	0.956	(0.230)	0.953	(0.307)	0.950	(0.378)
50	NA	0.993	(0.227)	0.987	(0.253)	0.983	(0.329)	0.979	(0.438)
	EL	0.948	(0.102)	0.948	(0.168)	0.944	(0.224)	0.953	(0.274)
	AEL	0.961	(0.110)	0.958	(0.177)	0.957	(0.236)	0.962	(0.287)
	MEL	0.961	(0.108)	0.961	(0.176)	0.959	(0.235)	0.963	(0.287)
100	NA	0.962	(0.202)	0.973	(0.239)	0.980	(0.311)	0.986	(0.452)
	EL	0.953	(0.073)	0.956	(0.120)	0.953	(0.159)	0.949	(0.195)
	AEL	0.962	(0.076)	0.962	(0.123)	0.958	(0.164)	0.952	(0.200)
	MEL	0.961	(0.075)	0.960	(0.122)	0.958	(0.163)	0.951	(0.199)

It can be observed from the simulation studies in Tables 4.1, 4.2, and 4.3 that the coverage probabilities based on the NA method far exceed the nominal level of 0.95. This indicates that the NA based method leads to high overcoverage. The proposed EL methods appear to be consistent with the reviewed literature as it produces better coverage than the normal approximation method. For the sample size of 30, the EL method falls below the nominal level of 0.95 but still better in comparison to the coverage probabilities based on the NA method.

At the base time points, for all distributions, the AEL and MEL results are almost close to the nominal level and even get better as the time increases. The average lengths of the confidence intervals for the EL, AEL, and MEL methods appear to decrease as the sample size increases. However, the average lengths for the AEL and MEL under all distributions, are larger than the EL method. Consequently, the average length for the MEL method under all distributions is shorter than or equal to the AEL method.

5 REAL DATA ANALYSIS

In this chapter, the performance of the proposed EL methods is further assessed by considering real data in survival analysis and reliability theory. These applications are demonstrated by obtaining the confidence intervals and average lengths for the MPL function by using the NA, EL, AEL, and MEL methods. These applications are critical since the simulation study has proven the EL-based methods outperform the NA-based methods. Hence, an application to real data would be considered sufficient to show how the performance of the proposed methods. Therefore, an application to a burn dataset is considered in the first example. In the second example, an application in reliability theory using an insulating fluid data is illustrated.

An application using burn data:

A dataset from the study (Ichida et al., 1993) on burn infection of 154 patient records and charts is reviewed in the first example. A burn wound infection is a common complication resulting in extended hospital stays and the death of severely burned patients. Table 5.1 summarizes the estimated MPL function of the infection times with its corresponding 95% confidence intervals based on the NA, EL, AEL, and MEL methods at time points $(15, 25, 35, 50)$. Furthermore, the confidence intervals and length of the intervals are illustrated in Figures 5.1 and 5.2, respectively.

Table 5.1: Estimated MPLs, 95% confidence intervals and length of the intervals of the normal approximation (NA), empirical likelihood (EL), adjusted empirical likelihood (AEL), and mean empirical likelihood (MEL) methods using the burn dataset.

t	MPL	NA		EL		AEL		MEL	
15	6.52	(3.660, 9.380)	(5.720)	(5.694, 7.552)	(1.858)	(5.662, 7.585)	(1.923)	(5.675, 7.574)	(1.899)
25	12.43	(7.569, 17.291)	(9.722)	(11.477, 13.858)	(2.381)	(11.448, 13.887)	(2.439)	(11.460, 13.875)	(2.415)
35	19.44	(10.474, 28.407)	(17.933)	(18.206, 21.242)	(3.036)	(18.174, 21.272)	(3.098)	(18.186, 21.258)	(3.072)
50	30.96	(14.640, 47.280)	(32.640)	(28.887, 32.888)	(4.001)	(28.849, 32.923)	(4.074)	(28.862, 32.905)	(4.043)

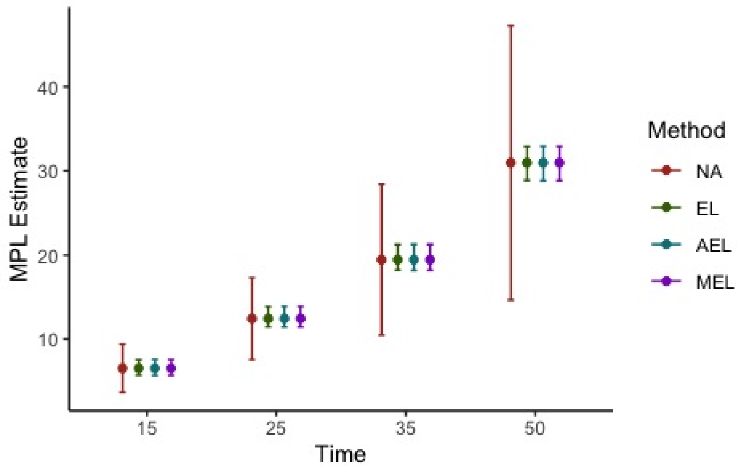

Figure 5.1: Confidence intervals for the estimated MPLs for the burn data.

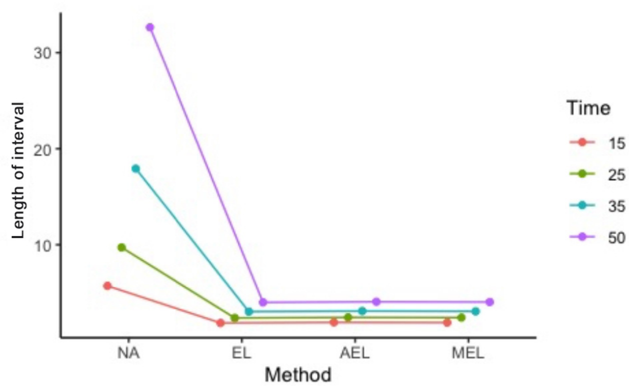

Figure 5.2: Length of intervals for the time (in minutes) for the burn data.

An application using insulating fluid data:

In the second example, an application of the MPL function in reliability theory is considered. We refer to Hettmansperger and McKean (2011) for a study on the breakdown time of an electrical insulating fluid on different levels of voltage stress. The purpose of an electrical insulating fluid is to prevent electric discharges. Hence, they are primarily used in high voltage machines to give electrical insulation. Its breakdown can have a tremendous effect on thermal stability and cost. The MPL function is estimated at different time points t and is summarized in Table 5.2, along with the 95% confidence intervals and average lengths for the NA, EL, AEL, and MEL based methods. Figure 5.3 shows the confidence intervals, and Figure 5.4 shows the length of the intervals.

Table 5.2: Estimated MPLs, 95% confidence intervals and length of the intervals of the normal approximation (NA), empirical likelihood (EL), adjusted empirical likelihood (AEL), and mean empirical likelihood (MEL) methods using the insulating fluid dataset.

t	MPL	NA		EL		AEL		MEL	
7.5	5.009	(3.213, 6.805)	(3.592)	(3.934, 5.865)	(1.931)	(3.828, 5.955)	(2.127)	(3.814, 5.903)	(2.089)
21.5	16.151	(8.994, 23.308)	(14.314)	(13.514, 18.005)	(4.491)	(13.318, 18.152)	(4.834)	(13.221, 18.062)	(4.841)
35.5	26.942	(13.619, 40.265)	(26.646)	(22.842, 29.949)	(7.107)	(22.570, 30.159)	(7.589)	(22.491, 30.028)	(7.537)
55.5	43.951	(20.736, 67.166)	(46.430)	(38.737, 47.912)	(9.175)	(38.418, 48.164)	(9.746)	(38.314, 48.009)	(9.695)
100.5	87.198	(37.465, 136.931)	(99.466)	(80.373, 91.882)	(11.509)	(79.969, 92.174)	(12.205)	(79.310, 91.989)	(12.679)

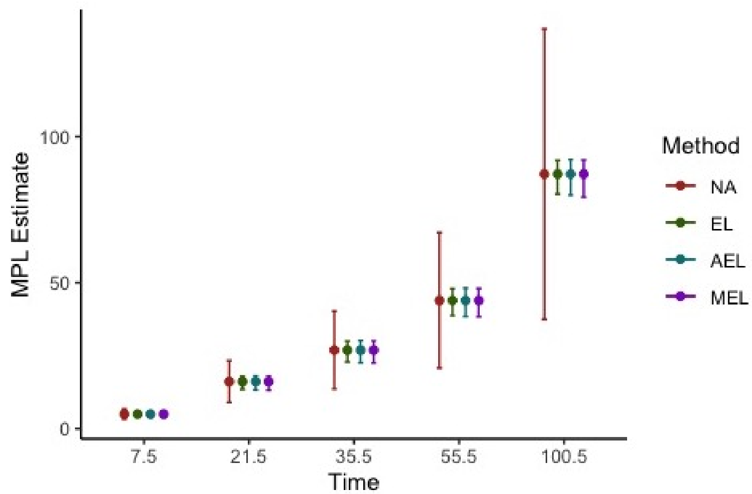

Figure 5.3: Confidence intervals for the estimated MPLs for the insulating fluid data.

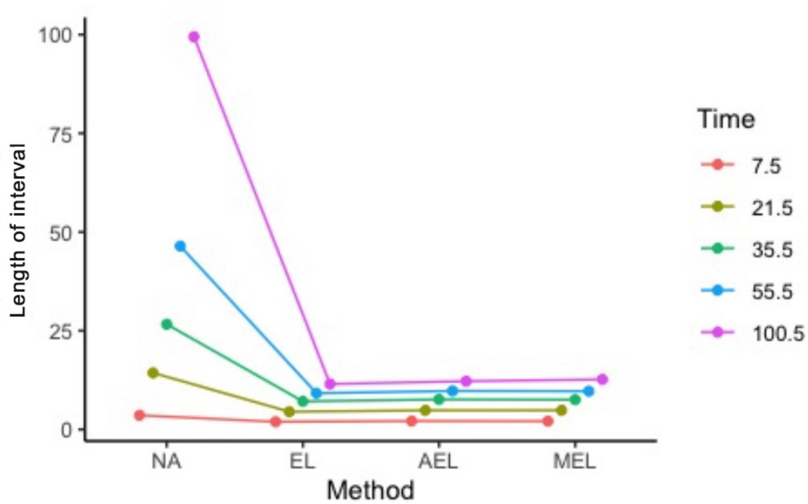

Figure 5.4: Length of intervals for the time (in minutes) for the insulating fluid data.

Summary of the real data analysis:

Employing an application of the proposed EL, AEL, and MEL methods to real data, it is evident that the methods perform better than the NA method. In the case of the burn data, the MPL estimator was computed by using the MPL function for tied observations since there were tied observations present in the data. However, the MPL estimator for the insulating fluid data was computed using the MPL function with no tied observations as the data had no ties. As the time t increases, the MPL estimator correspondingly increases. For the NA-based confidence intervals in Table 4.3, the length of the intervals tends to become two times more than the previous time point. Thus, the length of the intervals increases with time t. This may be attributed to the conservative behavior of the variance of the MPL function. This behavior may also be seen in Table 5.1 for the insulating fluid data.

The length of the intervals for the EL, AEL, and MEL appear to be consistent from Figures 5.2 and 5.4. Their trend lines appear smooth, although the confidence intervals for the AEL and MEL methods are longer than the EL method in Figures 5.1 and 5.3. Overall, the NA-based confidence intervals and lengths have high overcoverage compared to the EL, AEL, and MEL methods, which produce more accurate confidence intervals with shorter lengths.

6 CONCLUSIONS

In this thesis, we proposed empirical likelihood methods to construct confidence intervals for the mean past lifetime (MPL) function. The confidence interval for the MPL function was first considered using the normal approximation (NA) method followed by the empirical likelihood (EL), adjusted empirical likelihood (AEL), and mean empirical likelihood (MEL) approach. Extensive simulation experiments were performed to assess the performance in terms of coverage probability and average lengths of the NA, EL, AEL, and MEL confidence intervals.

The simulation studies show that overall, the EL interval estimates have more accurate coverage probability and shorter average lengths than intervals based on normal approximation. Although the proposed AEL and MEL methods produced longer average lengths than the EL method, they outperformed the NA method in terms of coverage probability and shorter average length. The efficiency of the EL, AEL, and MEL methods was illustrated using a real dataset to construct confidence intervals and was compared with the NA-based method in terms of confidence intervals and average lengths.

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APPENDICES

Appendix A: Proofs of theorems

The proof follows from Zhao and Qin (2006). Although the proofs of Theorem 3.1. is straightforward, it is helpful to understand the proof of Theorem 3.2 in space $D[a,b]$. The following lemma is helpful for the proof of the theorems.

Lemma 1. *Let $\tau_0 < a < b < M_F$. Assume $E(T^2) < \infty$. We have*

(i) $n^{-1/2}\sum_{i=1}^{n} W_i(k_0(t)) \xrightarrow{D} U(t)$ *in* $D[a,b]$,

(ii) $n^{-1}\sum_{i=1}^{n} W_i^2(k_0(t)) \xrightarrow{\mathcal{P}} \Gamma(t)$ *uniformly over* $t \in D[a,b]$.

We state a proposition in order to prove Lemma 1 (cf. Theorem 3 of van der Vaart and Wellner, 2000).

Proposition 1 Suppose that $F_1,\ F_2, \ldots,\ F_k$ are P-Glivenko-Cantelli classes of functions and that $\phi : R^k$ to R is a continuous function. Then the class of functions $H = \phi(F_1,\ F_2, \ldots,\ F_k)$ is P-Glivenko-Cantelli, provided it has an integrable envelope function. The class H is the collection of all functions $h(u)$ which are of the form $h\left(u\right) = \phi(f_1\left(u\right),\ f_2\left(u\right), \ldots,\ f_k(u))$, where f_i is in F_i.

Proof of Lemma 1. We note that $n^{-1/2}\sum_{i=1}^{n} W_i(k_0(t)) = \hat{F}\left(t\right) n^{1/2}(k_n(t) - k_0(t))$. By Theorem 2.1 of Zardasht (2017), Lemma 1 (i) follows. For the proof of Lemma 1 (ii), we follow the similar argument of Zhao and Qin (2006). Take F_1 to be the class $I\left(T \le t\right)$ for $\tau_0 < a \le t \le b < M_F$, and F_i to be the class $t - T - k_0\left(t\right),\ i = 2, 3$. Then, F_1 is P-Glivenko-Cantelli by the usual Glivenko-Cantelli theorem, and F_i is P-Glivenko-Cantelli by the strong law of large numbers for $i = 2, 3$. Take the function $\phi\left(b, c, d\right) = bcd$, which is continuous. Recall that $\tau_0 < a < b < M_F < \infty$ and $[a, b]$ is a compact interval. We have $0 \le t + k_0\left(t\right) \le M$ for $t \in [a, b]$, where M is a constant. Finally, the class of functions $H : \{I(T \leq t)(t - T - k_0(t))^2 : t \in [a,b]\}$ certainly has an integrable envelope function $(|T| + M)^2$. Because $I(T \leq t)(t - T - k_0(t))^2 \leq (|T| + M)^2$ for $t \in [a,b]$ and $E(|T| + M)^2 < \infty$ by the assumption $E\left(T^2\right) < \infty$. Thus, Lemma 1 (ii) follows from Proposition 1. $\square$

Lemma 2. *Recall that* $Var(W_i(k_0(t))) = \Gamma(t)$*. We have*

(i) $\max\limits_{1 \leq i \leq N} |V(k_0(t))| = o_p(n^{1/2})$,

(ii) $\dfrac{1}{N} \sum\limits_{i=1}^{N} (V_i(k_0(t)))^2 = \dfrac{\Gamma(t)}{2} + o_p(1)$.

Proof of Lemma 2. We give the proof of Lemma 2, following the argument from Liang et al. (2019).

(i) Since $\text{Var}(W_i(k_0(t))) = \Gamma(t)$ exists, we have $\max_i |W_i(k_0(t))| = o_p(n^{1/2})$ and

$$
\begin{aligned}
\max_{1 \leq i \leq N} |V_i(k_0(t))| &= \max_{i \leq j} \left| \frac{W_i(k_0(t)) + W_j(k_0(t))}{2} \right| \\
&\leq \frac{1}{2} \left(\max_{1 \leq i \leq n} |W_i(k_0(t))| + \max_{1 \leq j \leq n} |W_j(k_0(t))| \right) \\
&= o_p(n^{1/2}).
\end{aligned}
$$

(ii) We notice that

$$
\begin{aligned}
\frac{1}{N} \sum_{i=1}^{N} (V_i(k_0(t)))^2 &= \frac{1}{2N} \left[\sum_{i=1}^{n} \sum_{j=1}^{n} \left(\frac{W_i(k_0(t)) + W_j(k_0(t))}{2} \right)^2 + \sum_{i=1}^{n} (W_i(k_0(t))^2 \right] \\
&= \frac{1}{2(n+1)} \left(\frac{1}{\sqrt{n}} \sum_{i=1}^{n} (W_i(k_0(t)))^2 \right) + \frac{n+2}{2(n+1)} \left(\frac{1}{n} \sum_{i=1}^{n} (W_i(k_0(t)))^2 \right) \\
&= \frac{\Gamma(t)}{2} + o_p(1). \qquad \square
\end{aligned}
$$

Proof of Theorem 3.1. Throughout the proof, we fix $t \in (\tau_0, \infty)$. Recall that $E\left(W_i^2(k_0(t))\right) <$

∞ for fixed $t \in [\tau_0, \infty)$. Following the proof of Lemma 3 of Owen (1990) we have

$$\max_{1 \le i \le n} |W_i(k_0(t))| = o_p\left(n^{1/2}\right).$$

Let

$$\hat{\Gamma}(t) = \frac{1}{n} \sum_{i=1}^{n} W_i^2(k_0(t)).$$

Note that for the fixed t, $W_1(k_0(t)), \ldots, W_n(k_0(t))$ are i.i.d random variables and

$$\operatorname{Var}(W_i(k_0(t))) = E\left[(t_i - T_i - k_0(t))^2 I(T_i \le t)\right] = \Gamma(t).$$

Note that $E\left[W_i^2(k_0(t))\right] = \Gamma(t)$ and $n^{-1}\sum_{i=1}^{n} W_i^2(k_0(t)) \xrightarrow{\mathcal{P}} \Gamma(t)$ follows from the Law of Large Numbers. Thus, we have

$$\hat{\Gamma}(t) = \Gamma(t) + o_p(1).$$

By the Central Limit Theorem, $n^{-1/2}\sum_{i=1}^{n} W_i(k_0(t)) \xrightarrow{D} N(0, \Gamma(t))$ follows. Similar to the proof of Zhao and Qin (2006), we have

$$\begin{aligned}
-2\log R(k_0(t)) &= \sum_{i=1}^{n} \lambda(t) W_i(k_0(t)) + o_p(1) \\
&= \left(n^{-1/2}\sum_{i=1}^{n} W_i(k_0(t))\right)^2 \left(n^{-1}\sum_{i=1}^{n} W_i(k_0(t)) W_i(k_0(t))\right)^{-1} + o_p(1) \\
&\xrightarrow{D} \chi_1^2. \qquad \square
\end{aligned}$$

Proof of Theorem 3.2. Recall that $E\left(T^2\right) < \infty$. We have

$$\max_{1 \le i \le n} |T_i| = o_p\left(n^{1/2}\right).$$

Therefore, we have

$$\max_{1\le i\le n}|W_i(k_0(t))| \le \max_{1\le i\le n}|T_i| + (t + k_0\,(t))$$
$$\le \max_{1\le i\le n}|T_i| + M,$$

for $t \in [a,b]$, where M is a constant which satisfies that $t + k_0\,(t) \le\ M$ for $t \in [a,b]$.

Hence,

$$\max_{1\le i\le n}|W_i(k_0(t))| = o_p\left(n^{1/2}\right),$$

uniformly over $t \in [a,b]$. Then by Lemma 1 and the proof of Theorem 3.1, we have

$$\begin{aligned}
-2\log R\,(k_0\,(t)) &= \sum_{i=1}^{n} \lambda\,(t)\,W_i(k_0(t)) + o_p(1) \\
&= \left(n^{-1/2}\sum_{i=1}^{n} W_i(k_0(t))\right)^2 \left(n^{-1}\sum_{i=1}^{n} W_i(k_0(t))W_i(k_0(t))\right)^{-1} + o_p(1) \\
&\xrightarrow{D} \frac{U^2(t)}{\Gamma(t)}
\end{aligned}$$

in $D[a,b]$, where $o_p(1)$ is uniformly over $t \in [a,b]$. □

Proof of Theorem 3.3. We first prove that $|\lambda^a| = O_p(n^{-1/2})$. From Eq. (3.3), we can write

$$\begin{aligned}
0 &= \frac{1}{n}\left|\sum_{i=1}^{n+1} W_i(k_0(t)) - \lambda^a \sum_{i=1}^{n+1} \frac{W_i^2(k_0(t))}{1+\lambda^a\,(t)\,W_i(k_0(t))}\right| \\
&\ge \frac{|\lambda^a|}{n}\sum_{i=1}^{n+1} \frac{W_i^2(k_0(t))}{1+\lambda^a\,(t)\,W_i(k_0(t))} - \frac{1}{n}\left|\sum_{i=1}^{n+1} W_i(k_0(t))\right| \\
&\ge \frac{|\lambda^a| s_n^2(t)}{1+|\lambda^a(t)|W_i^*(k_0(t))} - \left|\frac{1}{n}\sum_{i=1}^{n} W_i(k_0(t))\right|\left(1-\frac{a_n}{n}\right),
\end{aligned}$$

where $s_n^2(t) = n^{-1}\sum_{i=1}^{n} W_i^2(k_0(t))$ and $W_n^*(k_0(t)) = \max_{1\le i\le n}|W_i(k_0(t))|$. By Lemma 1 (i), $\overline{W}_n(k_0(t)) = n^{-1}\sum_{i=1}^{n} W_i(k_0(t)) = O_p(n^{-1/2})$. By Lemma 1 (ii), $s_n^2(t) = \Gamma(t) + o_p(1)$ and by the result (i) in the proof of Theorem 3.1, we have $W_n^*(k_0(t)) = o_p(n^{1/2})$. These results coupled with $a_n = o_p(n)$, give $|\lambda^a| = O_p(n^{-1/2})$. Hence, $\lambda^a = \overline{W}_n(k_0(t))/s_n^2(t) + o_p(n^{-1/2})$. Finally, by replacing λ^a in the Taylor expansion of $l^{adj}(k_0(t))$, we have,

$$
\begin{aligned}
l^{adj}(k_0(t)) &= 2\sum_{i=1}^{n+1}\left(\lambda^a W_i(k_0(t)) - \frac{1}{2}(\lambda^a)^2 W_i(k_0(t))^2\right) + o_p(1) \\
&= \frac{n\overline{W}^2(k_0(t))}{s_n^2(t)} + o_p(1).
\end{aligned}
$$

By Lemma 1, $l^{adj}(k_0(t))$ converges to χ_1^2. □

Proof of Theorem 3.4. The proof of Theorem 3.4 follows the proof of Theorem 3.3. From the arguments used in Owen (1990) and Liang et al. (2019), we have $|\lambda^m(t)| = O_p(n^{-1/2})$. Now, with the following equation

$$
\begin{aligned}
0 &= \frac{1}{N}\sum_{i=1}^{N}\frac{V_i\left(k_0(t)\right)}{1+\lambda^m(t)V_i\left(k_0(t)\right)} \\
&= \frac{1}{N}\sum_{i=1}^{N} V_i(k_0(t)) - \frac{\lambda^m}{N}\sum_{i=1}^{N} V_i(k_0(t))^2 + \frac{1}{N}\sum_{i=1}^{N}\frac{(\lambda^m)^2 V_i(k_0(t))^3}{1+\lambda^m(t)V_i(k_0(t))} \\
&= \frac{1}{N}\sum_{i=1}^{N} V_i(k_0(t)) - \frac{\lambda^m}{N}\sum_{i=1}^{N} V_i(k_0(t))^2 + z_n \ ,
\end{aligned}
$$

where

$$
\begin{aligned}
|z_n| &\leq (\lambda^m)^2 \max_{i\leq i\leq n}|V_i(k_0(t))|\frac{1}{N}\sum_{i=1}^{N}(V_i(k_0(t))^2\frac{1}{1+\lambda^m V_i(k_0(t))} \\
&= O_p(n^{-1})o_p(n^{1/2})O_p(1)O_p(1) \\
&= o_p(n^{-1/2}).
\end{aligned}
$$

Then, we obtain

$$
\lambda^m = \left(\sum_{i=1}^{N}(V_i(k_0(t)))^2\right)^{-1}\left(\sum_{i=1}^{N} V_i(k_0(t))\right) + o_p(n^{-1/2}).
$$

Applying Taylor's expansion to Eq. (3.4), we have

$$l^m(k_0(t) = \frac{2}{n+1}\sum_{i=1}^{N}\left(\lambda^m V_i(k_0(t)) - \frac{1}{2}(\lambda^m(V_i(k_0(t))^2\right) + \frac{r_N}{n+1}, \tag{6.1}$$

where

$$\begin{aligned} |r_N| \leq\ & C\,|\lambda^m\,(t)|^3 \max_{1\leq i\leq N}|V(k_0(t))| \sum_{i=1}^{n}|V_i(k_0(t))\,|^2 \\ &= O_p(n^{-3/2})o_p(n^{1/2})O_p(n^2) \\ &= o_p(n). \end{aligned}$$

Substituting λ^m into Eq. (6.1), and noticing that

$$\begin{aligned} \frac{1}{N}\sum_{k=1}^{N}V_k(k_0(t)) &= \frac{1}{2N}\left(\sum_{i=1}^{n}\sum_{j=1}^{n}\frac{W_i(k_0(t)) + W_j(k_0(t))}{2} + \sum_{i=1}^{n}W_i(k_0(t))\right) \\ &= \frac{n+1}{2N}\sum_{i=1}^{n}W_i(k_0(t)) = \overline{W}_n(k_0(t))\ , \end{aligned}$$

Hence, we have that

$$\begin{aligned} l^m(k_0(t)) &= \frac{1}{n+1}\left(\sum_{i=1}^{N}V_i(k_0(t))\right)^2\left(\sum_{i=1}^{N}V_i^2(k_0(t))\right)^{-1} + o_p(1) \\ &= \frac{1}{n+1}\left(\sum_{i=1}^{N}V_i(k_0(t))\right)^2\left(\frac{N\Gamma(t)}{2}\right)^{-1} + o_p(1) \\ &= \frac{2}{N(n+1)(\Gamma(t))}\left(\sum_{i=1}^{N}V_i(k_0(t))\right)^2 + o_p(1) \\ &= \frac{2N^2}{N(n+1)(\Gamma(t))}\left(\frac{1}{N}\sum_{i=1}^{N}V_i(k_0(t))\right)^2 + o_p(1) \\ &= \frac{2N}{(n+1)(\Gamma(t))}\overline{W}_n(k_0(t))^2 + o_p(1) \\ &= \frac{n\overline{W}_n(k_0(t))^2}{\Gamma(t)} + o_p(1) \\ &\xrightarrow{D} \chi_1^2. \end{aligned}$$

$\square$