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EMPIRICAL STIFFNESS MATRIX DERIVATION FOR AN ELASTIC SUBSTRUCTURE

Abdul Hafiez bin Yusoff[1*], Rizal bin Zahari[2] and Faizal bin Mustapha[1]

[1]Aerospace Engineering Department,
Faculty of Engineering,
Universiti Putra Malaysia,
43400 UPM Serdang, Selangor Darul Ehsan. Malaysia

[2]Aerospace Manufacturing Research Centre,
Faculty of Engineering,
Universiti Putra Malaysia,
43400 UPM Serdang, Selangor Darul Ehsan. Malaysia

ABSTRACT

The paper presents the study on the three-dimensional empirical derivations of the static stiffness matrix derivation of a sheet metal substructure based on the basic principles of the finite element method. An experimental procedure and techniques are developed to extract the stiffness coefficients of a 6 by 6 submatrix for a sheet tube made of mild steel ASTM A-500 SHS. The submatrix obtained from this experiment is then included in the finite element software NASTRAN as a new element. Comparison of results between the experimental and the finite element analysis is carried out via a test case to validate the method employed. Excellent agreement with the experimental results has been observed which confirms the accuracy of the approached employed.

Keywords: *Stiffness Matrix, Finite Element Analysis, Direct Stiffness Method, GENEL*

1.0 INTRODUCTION

The experimental stiffness matrix extraction for a complex structural component is sometimes desired in order to obtain accurate finite element model. Previously, the stiffness matrix was generated through computer program and numerical computation, none of work related to experimental stiffness matrix found in literature review. Hoa and Sankar [1] suggested a computer program to generate stiffness and mass matrices automatically in finite element analysis. Palaninathan and Chandrasekharan [2] introduced a program subroutine, NEWCBM for the stiffness matrix formulation of curved beams that has been written in FORTRAN which can be added to the element library of general purpose computer programs like SAP-IV and its improved versions. Chen et al. [3] derived the dual boundary integral formulation to determine the stiffness and flexibility matrices for rods and beams by using the direct and indirect methods through the concept of boundary element method. Flexibility matrix is the inverse of stiffness matrix; Felipa and Park [4] introduced free-free flexibility matrices as duals of free-free stiffness matrices. One of the user defined element existed in NASTRAN is general element (GENEL), the GENEL entry is used to define general elements whose property are defined in terms of deflection influence coefficients or stiffness matrices which can be connected between any number of grid points. This paper concentrates on GENEL to represent a structure by means of experimentally measured stiffness coefficients of a sheet metal joint.

Corresponding author : abdhafiez@gmail.com

Although the stiffness matrix of most structures can be readily obtained via numerical or computational methods, there are cases where due the complexity of the material properties or geometry of structure, it is difficult and time consuming to model using numerical techniques accurately. One of the ways to overcome this problem is obtaining the structures' stiffness experimentally. The objective of this study is to develop an experimental technique to extract three-dimensional stiffness coefficients of a structure.

In structural analysis, the direct stiffness method is recognized as a powerful method for computer programming. The direct stiffness method could be easily formulated into computer programming and become the dominating approach in finite element analysis (FEA) because they are a powerful tool.

1.1 Direct Stiffness Method

When computers came into use for structural analysis, it was soon recognized that the displacement method or direct stiffness method could be easily formulated for computer programming, and it has become the dominating approach in finite element methods [5]. The basic approach undertaken in carrying out the analysis for the modeling and analysis of a sheet metal joint is direct stiffness method. Load displacement relationship for a general individual structure member which behave in elastic manner such as shown in Figure 1 given by

$$P = K\delta \tag{1}$$

Where P is the loads in global coordinate, K is the element stiffness in global coordinate, and δ is the displacement in global coordinate. Equation (1) can be represented in matrix form as in Equation (2).

$$\begin{bmatrix} P_{x1} \\ P_{y1} \\ P_{z1} \\ M_{x1} \\ M_{y1} \\ M_{z1} \\ P_{x2} \\ P_{y2} \\ P_{z2} \\ M_{x2} \\ M_{y2} \\ M_{z2} \end{bmatrix} = \begin{bmatrix} k_{11} & k_{12} & k_{13} & k_{14} & k_{15} & k_{16} & k_{17} & k_{18} & k_{19} & k_{110} & k_{111} & k_{112} \\ k_{21} & k_{22} & k_{23} & k_{24} & k_{25} & k_{26} & k_{27} & k_{28} & k_{29} & k_{210} & k_{211} & k_{212} \\ k_{31} & k_{32} & k_{33} & k_{34} & k_{35} & k_{36} & k_{37} & k_{38} & k_{39} & k_{310} & k_{311} & k_{312} \\ k_{41} & k_{42} & k_{43} & k_{44} & k_{45} & k_{46} & k_{47} & k_{48} & k_{49} & k_{410} & k_{411} & k_{412} \\ k_{51} & k_{52} & k_{53} & k_{54} & k_{55} & k_{56} & k_{57} & k_{58} & k_{59} & k_{510} & k_{511} & k_{512} \\ k_{61} & k_{62} & k_{63} & k_{64} & k_{65} & k_{66} & k_{67} & k_{68} & k_{69} & k_{610} & k_{611} & k_{612} \\ k_{71} & k_{72} & k_{73} & k_{74} & k_{75} & k_{76} & k_{77} & k_{78} & k_{79} & k_{710} & k_{711} & k_{712} \\ k_{81} & k_{82} & k_{83} & k_{84} & k_{85} & k_{86} & k_{87} & k_{88} & k_{89} & k_{810} & k_{811} & k_{812} \\ k_{91} & k_{92} & k_{93} & k_{94} & k_{95} & k_{96} & k_{97} & k_{98} & k_{99} & k_{910} & k_{911} & k_{912} \\ k_{101} & k_{102} & k_{103} & k_{104} & k_{105} & k_{106} & k_{107} & k_{108} & k_{109} & k_{1010} & k_{1011} & k_{1012} \\ k_{111} & k_{112} & k_{113} & k_{114} & k_{115} & k_{116} & k_{117} & k_{118} & k_{119} & k_{1110} & k_{1111} & k_{1112} \\ k_{121} & k_{122} & k_{123} & k_{124} & k_{125} & k_{126} & k_{127} & k_{128} & k_{129} & k_{1210} & k_{1211} & k_{1212} \end{bmatrix} \begin{bmatrix} \delta_{x1} \\ \delta_{y1} \\ \delta_{z1} \\ \theta_{x1} \\ \theta_{y1} \\ \theta_{z1} \\ \delta_{x2} \\ \delta_{y2} \\ \delta_{z2} \\ \theta_{x2} \\ \theta_{y2} \\ \theta_{z2} \end{bmatrix} \tag{2}$$

Equation (1) and (2) valid for static and linear analysis and the stiffness matrix must symmetrically according to Maxwell's Reciprocal Theorem. Since three-dimensional substructure is considered, a complete set of 12 by 12 matrix written in global coordinate. $\boldsymbol{K_{11}}$, $\boldsymbol{K_{12}}$, $\boldsymbol{K_{21}}$, $\boldsymbol{K_{22}}$ are submatrices with six degrees of freedom but only submatrix $\boldsymbol{K_{22}}$ is needed for GENEL in Bulk Data Section.

$$K_{22}=\begin{bmatrix} \frac{P_{x2}}{\delta_{x2}} & \frac{P_{y2}}{\delta_{x2}} & \frac{P_{z2}}{\delta_{x2}} & \frac{M_{x2}}{\delta_{x2}} & \frac{M_{y2}}{\delta_{x2}} & \frac{M_{z2}}{\delta_{x2}} \\ \frac{P_{x2}}{\delta_{y2}} & \frac{P_{y2}}{\delta_{y2}} & \frac{P_{z2}}{\delta_{y2}} & \frac{M_{x2}}{\delta_{y2}} & \frac{M_{y2}}{\delta_{y2}} & \frac{M_{z2}}{\delta_{y2}} \\ \frac{P_{x2}}{\delta_{z2}} & \frac{P_{y2}}{\delta_{z2}} & \frac{P_{z2}}{\delta_{z2}} & \frac{M_{x2}}{\delta_{z2}} & \frac{M_{y2}}{\delta_{z2}} & \frac{M_{z2}}{\delta_{z2}} \\ \frac{P_{x2}}{\theta_{x2}} & \frac{P_{y2}}{\theta_{x2}} & \frac{P_{z2}}{\theta_{x2}} & \frac{M_{x2}}{\theta_{x2}} & \frac{M_{y2}}{\theta_{x2}} & \frac{M_{z2}}{\theta_{x2}} \\ \frac{P_{x2}}{\theta_{y2}} & \frac{P_{y2}}{\theta_{y2}} & \frac{P_{z2}}{\theta_{y2}} & \frac{M_{x2}}{\theta_{y2}} & \frac{M_{y2}}{\theta_{y2}} & \frac{M_{z2}}{\theta_{y2}} \\ \frac{P_{x2}}{\theta_{z2}} & \frac{P_{y2}}{\theta_{z2}} & \frac{P_{z2}}{\theta_{z2}} & \frac{M_{x2}}{\theta_{z2}} & \frac{M_{y2}}{\theta_{z2}} & \frac{M_{z2}}{\theta_{z2}} \end{bmatrix} \quad (3)$$

Livesely [5] has developed equilibrium matrix to determine the sub-matrices $[K_{11}]$, $[K_{12}]$, and $[K_{21}]$; the force P_1 and P_2 that connecting point 1 and point 2 respectively can be written in the matrix form as $P_1 + H_{12}P_2 = 0$, or alternatively as $H_{21}P_1 + P_2 = 0$, where $H_{21} = H_{12}^{-1}$. H could be determine by establishing the equilibrium equations for the substructure by considering the equilibrium of forces and moments about the axes of point 1. Then, all the equilibrium equations of sheet metal joint concerned could be written in matrix form as

$$\begin{bmatrix} P_{x1} \\ P_{y1} \\ P_{z1} \\ M_{x1} \\ M_{y1} \\ M_{z1} \end{bmatrix} + \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & -L_z & 0 & 1 & 0 & 0 \\ L_z & 0 & -L_x & 0 & 1 & 0 \\ 0 & L_x & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} P_{x2} \\ P_{y2} \\ P_{z2} \\ M_{x2} \\ M_{y2} \\ M_{z2} \end{bmatrix} = 0 \quad (4)$$

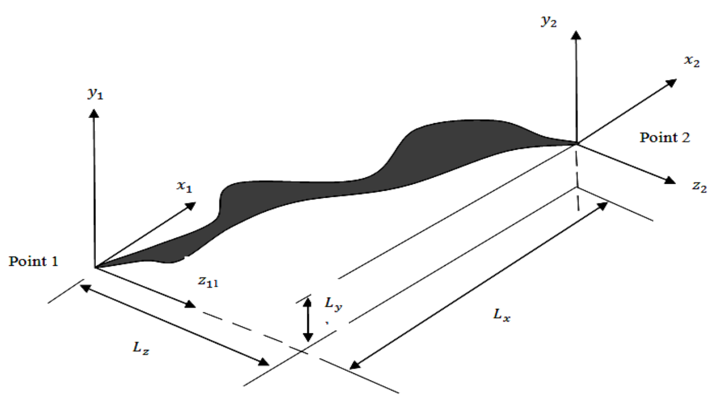

Figure 1: General member in space

For the sheet metal joint used, the equilibrium matrix $\boldsymbol{H}$ is determined by establishing equilibrium equation and the remaining submatrices $[K_{11}]$, $[K_{12}]$ and $[K_{21}]$ can be obtained by the following equation:

$$[K_{11}] = [H][K_{22}][H]^T \tag{5}$$

$$[K_{12}] = -[H][K_{22}] \tag{6}$$

$$[K_{21}] = -[K_{22}][H]^T \tag{7}$$

1.2 General Element

A NASTRAN program statement to generate the finite element geometry model by using stiffness matrix has been identified, format of the program can be found in MD Nastran R3 Quick Reference Guide. General element (GENEL) would contain entries that specify model geometry, element connectivity, and material properties which belong to Pre-processing Phase in NASTRAN-PATRAN workflow. The method used for finite element analysis in this study needs to write input data in NASTRAN Bulk Data Section based on experimentally measured data.

GENEL is an element not in the same sense as the CBAR or CQUAD4 element because there are no properties explicitly defined. The GENEL is used to describe a substructure that has an arbitrary number of connection grid points or scalar points.

2.0 EXPERIMENTAL SET UP

In the analysis, the length of the sheet metal joint is assumed to be 100mm. The joint's cross section is shaped like square hollow with 19mm wide and 1mm thickness. The geometrical model of the sheet metal joint was developed using SolidWorks as exhibited in Figure 2 and 3.

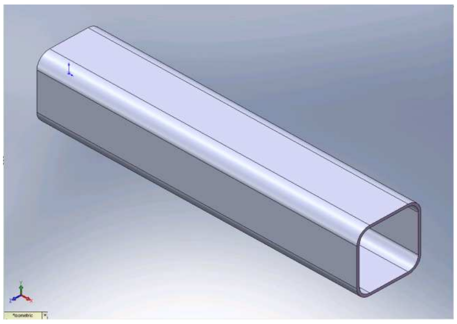

Figure 2: Three-dimensional sheet metal joint

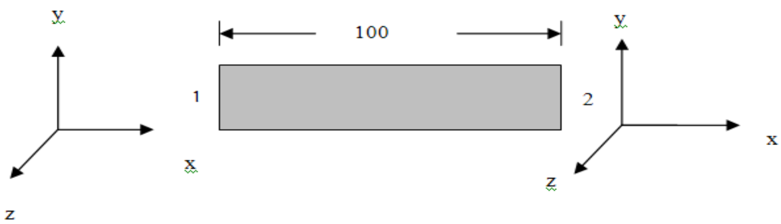

Figure 3: Sheet metal joint axes

Stiffness coefficients were extracted experimentally in static and linear region based on the direct stiffness method and included in NASTRAN as GENEL. During experiment, the substructure was fully constrained at one end while at the other end all degrees of freedom are constrained except in the direction where the load is applied. Forces and moments correspond to the free degree of freedom are then applied in turn to obtain the respective reaction forces and moments at the clamped end (constrained point 1). Schematics drawing of the principle of extracting stiffness coefficients based on direct stiffness method depicted in the following Figure 4.

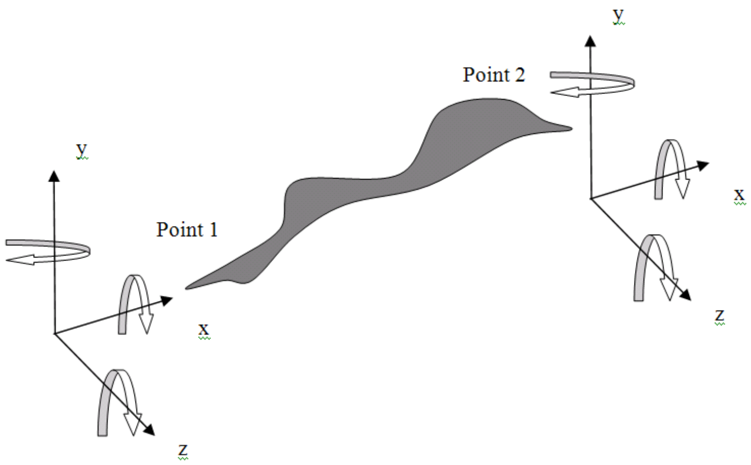

Figure 4: The degree of freedom of an element

In order to apply the principles of extracting the stiffness matrix experimentally, several test fixtures are designed to be mounted to the available testing machines. Figures 5 to 8 depict the attachment of the substructure with relevant jigs and test rig concerned for experimental purposes. Functions of each of the test rig components are tabulated in Table 1 and Table 2.

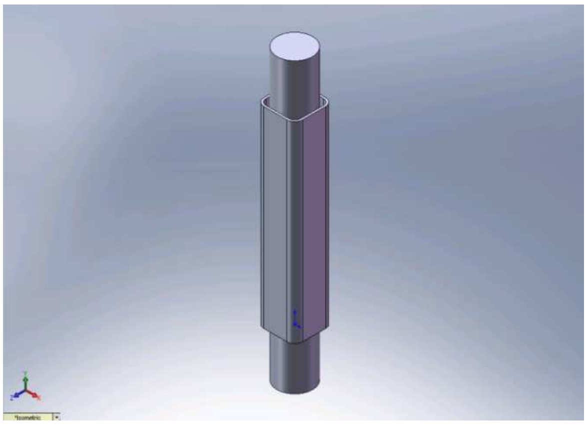

Figure 5: View of specimen with jig for x-axis applied force

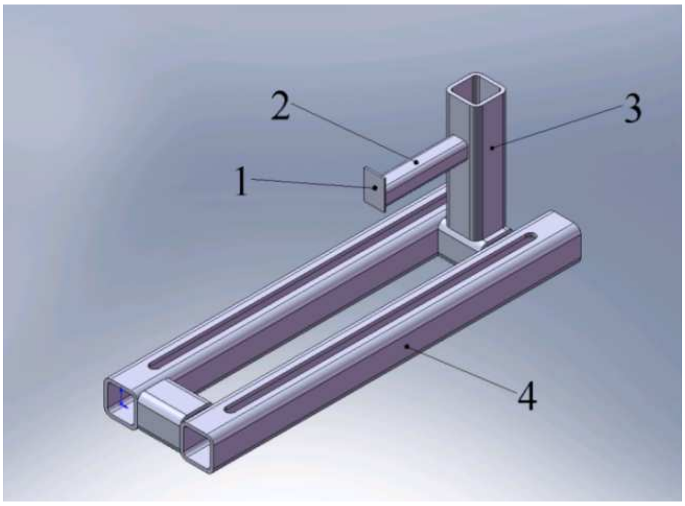

Figure 6: View of test rig for y axis applied force

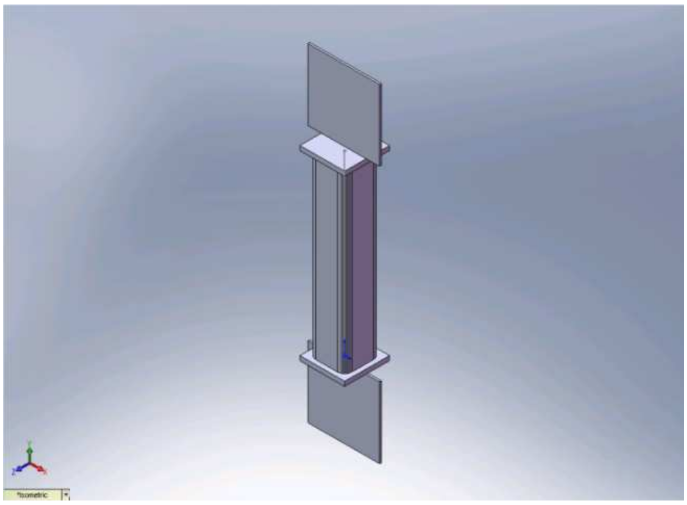

Figure 7: View of specimen with jig for x axis moment

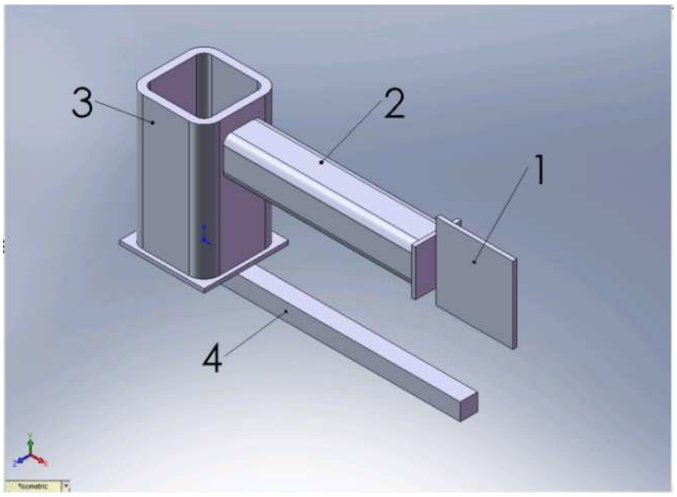

Figure 8: View of test rig for y-axis and z-axis bending moment

Table 1 Description of translational test rig component

Part number	Component	Description
1	Translational Support Plate	It functions to allow only force at y or z direction applied and prevents the remaining five degree of freedom motion.
2	Specimen	Sheet metal substructure used in this study to extract stiffness coefficients.
3	Holder	Holds the specimen, and fixed the end of specimen through welding.
4	Base	Act as support ground so that all remain rigid on test area.

Table 2 Description of rotational test rig component

Part number	Component	Description
1	Rotational Support Plate	It functions to allow only moment at y axis applied and prevent the remaining five degree of freedom motion.
2	Specimen	Sheet metal joint used in this study to extract stiffness coefficient.
3	Holder	Holds the specimen, and fixed the end of specimen through welding.
4	Base	Act as support ground so that all remain rigid on test area.

The experimentally measured data were collected using Universal Testing Machine Instron 3382 and Instron 8874. Translational displacement measured data were obtained by Instron 3382 and rotational displacement measured data by Instron 8874.

3.0 RESULTS AND DISCUSSION

From experimental data, force and moment reading of each axis was selected for stiffness coefficients development and tabulated in stiffness matrix, K_{22} as shown by Equation (8). The stiffness matrix is diagonal matrix and symmetrical according to Maxwell's Reciprocal Theorem.

$$K_{22} = \begin{bmatrix} 3.200E4 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 \\ 0.000 & 1.400E3 & 0.000 & 0.000 & 0.000 & 0.000 \\ 0.000 & 0.000 & 1.400E3 & 0.000 & 0.000 & 0.000 \\ 0.000 & 0.000 & 0.000 & 1.000E6 & 0.000 & 0.000 \\ 0.000 & 0.000 & 0.000 & 0.000 & 5.600E6 & 0.000 \\ 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 5.600E6 \end{bmatrix} \quad (8)$$

3.1 Validation of results

The stiffness coefficients of the stiffness matrix of the sheet metal joint were extracted by the universal testing machine experimentally. The stiffness coefficients are needed as input for GENEL to seek the behaviour of the specimen under various loadings. Six analyses regarding forces and moments had been done to validate GENEL translations and rotations experimentally and were compared against simulation. The experimental method to determine the translational and rotational displacement is similar to the stiffness coefficients extracting method. Figures 9 to 11 provide the comparison load-deflection curves of between experiment and simulation (GENEL).

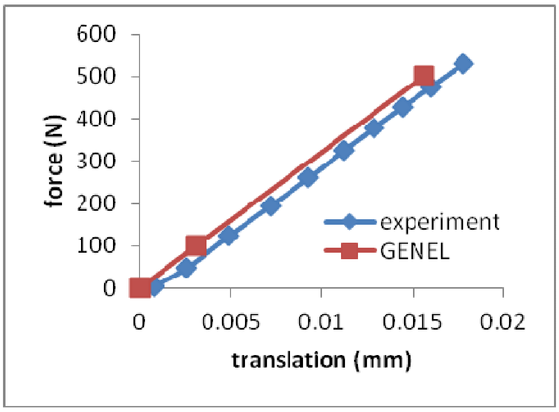

Figure 9: Translation in x direction

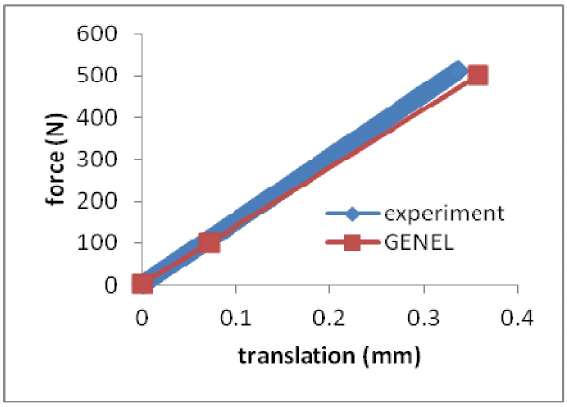

Figure 10: Translation in y direction

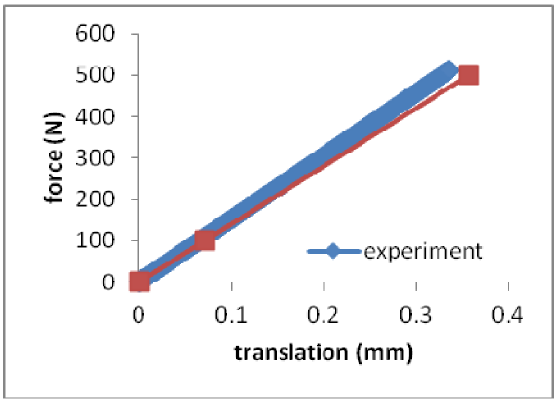

Figure 11: Translation in z direction

From the load-deflection curves, the maximum translational axis percentage error for x direction is found to be 4.17% and 8.4% for both y and z direction. Linear behavior for both the experimental and simulation had been observed although there is a slight deviation between the two curves for y and z directions. These slight discrepancies in the results may be attributed to the heterogeneous behavior of the specimen and also due to small error in the readings taken from the test equipments.

Figures 12 to 14 illustrate the bending moments against rotational displacement behavior between GENEL and experimental procedure.

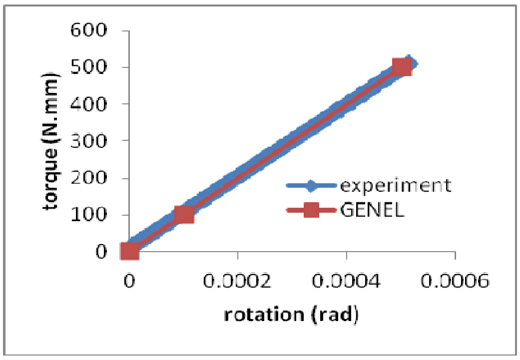

Figure 12: Rotation in x direction

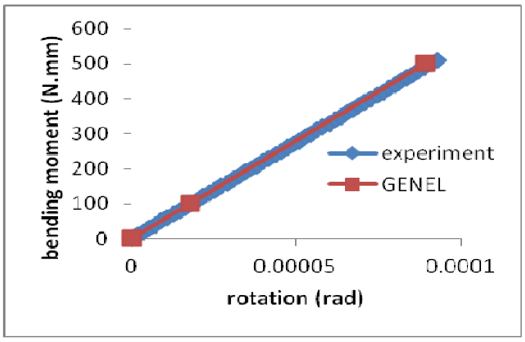

Figure 13: Rotation in y direction

Torque or bending moment vs. rotational displacement curves had been plotted for both the simulation and experimental works. Excellent agreement between the GENEL and experimental results are observed for all three rotational degrees of freedom.

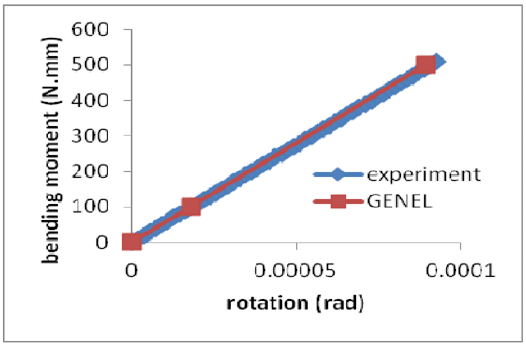

Figure 14: Rotation in z direction

3.2 Case Study: Stadium roof

A stadium roof design with sheet metal joint was presented in the following figures as a case study. Comparison of design between structure with and without GENEL had been made. In the later case, a beam element has been used to represent the substructure.

The design of stadium roof with 14 CBEAM elements and one GENEL is presented by the following figure. A 500N force was applied at one tip and the base of stadium roof was constrained in six degree freedom (DOF).

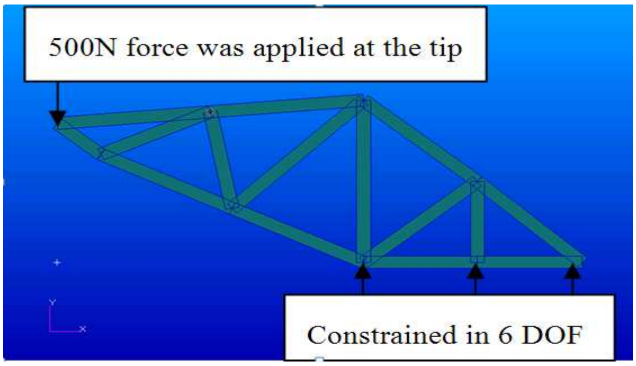

Figure 9: Stadium roof design modeling geometry

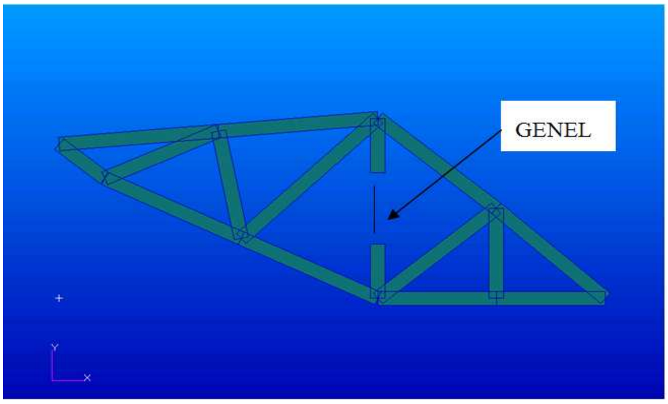

Figure 10: Stadium roof with GENEL design modeling geometry

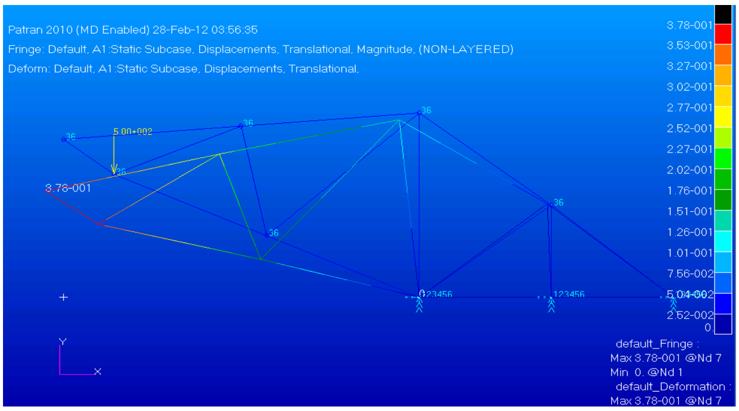

Figure 11: Stadium roof displacement magnitude (deflection)

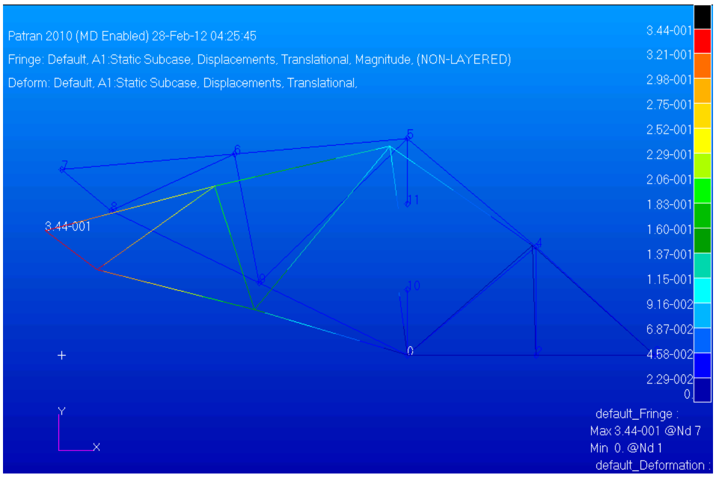

Figure 12: Stadium roof displacement magnitude (deflection)

Table 3: Stadium roof displacement in y-axis direction

Stadium roof design	Displacement
Without GENEL	0.378mm
With GENEL	0.344mm

From Table 3 it can be seen that the design of stadium roof without GENEL is 9.0% less stiff than design with GENEL. This proofs that the flexibility of the sheet metal joint plays an important role in producing reliable results.

4.0 CONCLUSION

A systematic approach of stiffness matrix extraction of a sheet metal substructure has been studied and developed. The methodology of extraction stiffness coefficients experimentally has also been developed on the basis of the direct stiffness method. A validation of results for simple test cases between the new element via the finite element method and the results obtained experimentally confirms the accuracy of the developed method. From the case studies, structures without GENEL are stiffer than structures with GENEL because the stiffness matrix extracted represents real material behavior and cause variation between simulation and experimental results. In addition, the experimental method developed from this project may serve as the basis for further development in the test rig for extracting a more geometrically complex substructure.

5.0 ACKNOWLEDGEMENT

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