Article

Educational and Psychological Measurement
2018, Vol. 78(5) 717–736

DOI: 10.1177/0013164417719947
journals.sagepub.com/home/epm

Empirical Underidentification with the Bifactor Model: A Case Study

Samuel Green[1] and Yanyun Yang[2]

Abstract

Bifactor models are commonly used to assess whether psychological and educational constructs underlie a set of measures. We consider empirical underidentification problems that are encountered when fitting particular types of bifactor models to certain types of data sets. The objective of the article was fourfold: (a) to allow readers to gain a better general understanding of issues surrounding empirical identification, (b) to offer insights into empirical underidentification with bifactor models, (c) to inform methodologists who explore bifactor models about empirical underidentification with these models, and (d) to propose strategies for structural equation model users to deal with underidentification problems that can emerge when applying bifactor models.

Keywords

bifactor model, empirical underidentification, structural equation modeling

In planning studies, researchers should specify the structural equation models (SEMs) of interest and establish mathematically that these models are identified. If a model is identified, all parameters of a model can be estimated uniquely. On the other hand, if the model is underidentified, one or more model parameters cannot be uniquely estimated because there are more unknowns in terms of model parameters than the information provided by the data (e.g., Kenny & Milan, 2012; MacCallum, Wegener, Uchino, & Fabrigar, 1993; Raykov & Marcoulides, 2001; Rindskopf,

[1]Arizona State University, Tempe, AZ, USA
[2]Florida State University, Tallahassee, FL, USA

Corresponding Author:
Samuel Green, Arizona State University, P.O. Box 873701, Tempe, AZ 85287-370, USA.
Email: samgreen@asu.edu

1984). Researchers who fail to consider whether a model is identified in formulating their studies may find out later that they have collected data that do not allow for an empirical investigation of their research hypotheses because the parameters associated with these hypotheses are not uniquely determined. Unfortunately, the derivations required to establish model identification frequently are difficult and tedious, and thus researchers are unlikely to establish identification mathematically. Fortunately, there are heuristic rules to describe certain specifications of models that are identified (e.g., Lee & Hershberger, 1990; MacCallum et al., 1993; Stelzl, 1986). These heuristic rules are applicable for some SEMs, but not others.

Even if a model is identified mathematically, it is still possible that the model is empirically underidentified, that is, not all parameters of a model can be estimated uniquely due to particular characteristics of the data (Kenny & Milan, 2012). For example, one could investigate a correlated two-factor model: X_1 and X_2 are associated with F_1, X_3 and X_4 are associated with F_2, and no cross loadings or covariances among measurement errors are specified. Two simple heuristic rules are applicable if models have no cross loadings or covariances among measurement errors: (a) a model with correlated factors is identified if two or more indicators are associated with each factor and (b) a model with uncorrelated factors is identified if three or more indicators are associated with each factor (Bollen, 1989). For our example, the model is identified mathematically in that it follows the first heuristic identification rule; however, the data indicate that the correlation between the two factors is zero. Therefore, in line with the second heuristic identification rule, empirically we need at least three indicators per factor to have unique parameter estimates. In summary, the data do not allow for unique parameter estimates for the model, and thus, the model is empirically underidentified for these data.

The focus of this article is on empirical underidentification with bifactor models. A bifactor model includes a general factor underlying all indicators and one or more group factors underlying one or more proper subsets of indicators. In addition, the covariances between factors are specified to be zero (Reise, 2012; Rindskopf & Rose, 1988). It is important to understand difficulties that can occur in the application of bifactor models because they are commonly used in the assessment of psychological and educational constructs (Chen, Hayes, Carver, Laurenceau, & Zhang, 2012; Reise, 2012). We consider empirical underidentification problems that are encountered when fitting particular types of bifactor models to certain types of data sets. For simplicity, we restrict our focus to analyses of covariance matrices with bifactor models. Our approach to explore empirical underidentification with the bifactor model can be used to investigate empirical underidentification with other SEMs and with other types of data (e.g., a mean vector as well as a covariance matrix among indicators).

Type of Bifactor Models of Interest

Browne (2001) uses the term *pure-clustered models* to describe models with multiple factors, with each indicator associated with one and only one factor. We coin the term *bifactor pure-clustered models* to designate a model in which all indicators are associated with a general factor, each indicator also is associated with one and only one group factor, and each group factor is associated with a proper subset of indicators. We present an example of a bifactor pure-clustered model in Figure 1. F_G, F_{g_1}, and F_{g_2} are the general factor, the first group factor, and the second group factor, respectively; X_i are the measured variables associated with F_{g_1}, with i varying between 1 and 4; X_j are the measured variables associated with F_{g_2}, with j varying between 5 and 8; E_1 through E_8 are the errors for the eight measures; and the λs are the loadings between the factors and the measures. The factor variances are fixed to 1 to have an identified model. We will consider empirical identification issues for this analysis model, but the conclusions reached with this model generalize to other bifactor pure-clustered models.

We became interested in bifactor pure-clustered models when we encountered estimation problems with generated data for a Monte Carlo study as well as with data collected to address applied psychological questions. The models we explored for these data were similar to the analysis model shown in Figure 1.

When we were empirically investigating bifactor models, the covariance matrix had a pattern similar to the one shown in Table 1: the covariances between indicators associated with the first group factor were equal to each other (denoted as $\sigma_{ii'}$), the covariances between indicators associated with the second group factor were equal to each other (denoted as $\sigma_{jj'}$), and the covariances between the indicators for the first group factor and the indicators for the second group factor were equal to each other (denoted as σ_{ij}). We refer to this type of matrix as a matrix with homogeneous-covariances of measures within group factors and homogeneous-covariances of measures between group factors or, more simply, as a homogeneous-within and homogeneous-between (HWHB) covariance matrix.

The HWHB covariance matrix can be reproduced perfectly based on the factor loadings matrix presented in Table 1. As shown, the factor loadings for the first four indicators on the general factor (λ_{iG}) are the same; the factor loadings for the second four indicators on the general factor (λ_{jG}) are the same; the factor loadings for the first four indicators on first group factor (λ_{ig_1}) are the same; and the factor loadings for the second four indicators on the second group factor (λ_{jg_2}) are the same. All other factor loadings are constrained to zero.

It is important to distinguish between the factor loading matrix for the bifactor analysis model shown in Figure 1 and the one shown in Table 1. We view the model in Figure 1 as the model that would be specified in our analysis (i.e., analysis model) if we postulated a bifactor model with a general factor, a group factor underlying the first four indicators, and a group factor underlying the second four indicators; accordingly, we specify this model to have 16 freely estimated factor loadings. On the other hand, the bifactor factor loading matrix presented in Table 1 reproduces the HWHB covariance matrix, and thus it is one solution that could be obtained in analyzing these data with the model shown in Figure 1. The purpose of this article is to demonstrate that other factor loading matrices can also reproduce the HWHB matrix, that is, the model

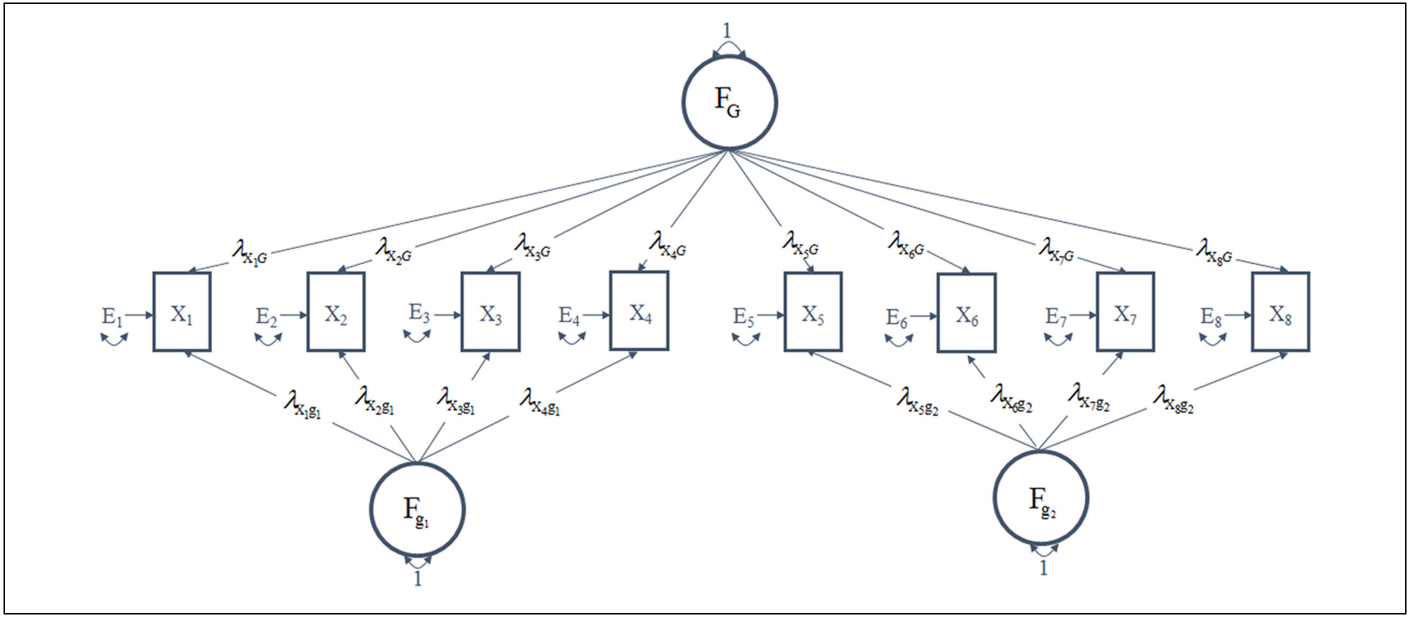

Figure 1. The bifactor pure-clustered model for the presented example.

Table 1. Lower-Left Triangle of the Homogeneous-Within and Homogeneous-Between (HWHB) Covariance Matrix and the Factor Loading Matrix Associated With the HWHB Covariance Matrix.

HWHB covariance matrix (lower left triangle)	Factor loading matrix
$\begin{bmatrix} \sigma^2_{X_1} & & & & & & & \\ \sigma_{ii'} & \sigma^2_{X_2} & & & & & & \\ \sigma_{ii'} & \sigma_{ii'} & \sigma^2_{X_3} & & & & & \\ \sigma_{ii'} & \sigma_{ii'} & \sigma_{ii'} & \sigma^2_{X_4} & & & & \\ \sigma_{ij} & \sigma_{ij} & \sigma_{ij} & \sigma_{ij} & \sigma^2_{X_5} & & & \\ \sigma_{ij} & \sigma_{ij} & \sigma_{ij} & \sigma_{ij} & \sigma_{jj'} & \sigma^2_{X_6} & & \\ \sigma_{ij} & \sigma_{ij} & \sigma_{ij} & \sigma_{ij} & \sigma_{jj'} & \sigma_{jj'} & \sigma^2_{X_7} & \\ \sigma_{ij} & \sigma_{ij} & \sigma_{ij} & \sigma_{ij} & \sigma_{jj'} & \sigma_{jj'} & \sigma_{jj'} & \sigma^2_{X_8} \end{bmatrix}$	$\begin{bmatrix} \lambda_{iG} & \lambda_{ig_1} & - \\ \lambda_{iG} & \lambda_{ig_1} & - \\ \lambda_{iG} & \lambda_{ig_1} & - \\ \lambda_{iG} & \lambda_{ig_1} & - \\ \lambda_{jG} & - & \lambda_{jg_2} \\ \lambda_{jG} & - & \lambda_{jg_2} \\ \lambda_{jG} & - & \lambda_{jg_2} \\ \lambda_{jG} & - & \lambda_{jg_2} \end{bmatrix}$

parameters are not unique when analyzing a HWHB covariance matrix with the bifactor analysis model shown in Figure 1 (i.e., empirical underidentification).

Empirical Underidentification at the Population Level

We initially investigated empirical underidentification at the population level by equating the model-implied covariance matrix for the bifactor model in Figure 1 (as the analysis model) with the HWHB covariance matrix in Table 1. With some algebraic manipulations, we can show that the relationships between the covariances among the measures (i.e., the data) and the model parameters are

$$\begin{aligned} \sigma_{ii'} &= \lambda^2_{iG} + \lambda^2_{ig_1} \\ \sigma_{jj'} &= \lambda^2_{jG} + \lambda^2_{jg_2} \\ \sigma_{ij} &= \lambda_{iG}\lambda_{jG} \end{aligned} \tag{1}$$

Note that we do not show the relationship between the variances for measures and the model parameters because the error variances are adjusted in estimating the model such that these variances plus the model-implied variances of measures have to be equal to the variances of the measures using maximum likelihood methods. Now examining Equation (1), we see that there are three distinct values for the covariances, but four unknown model parameters. Accordingly, parameter estimates based on fitting the bifactor analysis model in Figure 1 to the HWHB covariance matrix are not unique.

Two Nonunique Solutions for Each of the HWHB Covariance Matrices

To further illustrate the underidentification problem at the population level, we present two HWHB covariance matrices and one covariance matrix that is not HWHB in

Table 2. For simplicity, we created covariance matrices with 1s down the diagonal (i.e., correlation matrices). We also present factor loading matrices that could reproduce the covariance matrices, although we know that the factor loading matrices for the two HWHB covariance matrices are not unique. Note that for the factor loading matrices for the two HWHB covariance matrices, we computed the numerical rank based on the singular value decomposition. Both had a column rank of 2, and thus were not of full rank. See the note for Table 2 for more detailed information.

Based on the formulas previously presented, we focused on two of an infinite number of possible nonunique solutions for Covariance Matrices 1 and 2 in Table 2. For the first nonunique solution (A), we made the assumption that all eight loadings on the general factors were equal. For the second nonunique solution (B), we assumed that the loadings for the second group factor were equal to 0. Making these assumptions, we can solve for the remaining model parameters.

Nonunique Solution A:

$$\begin{aligned} &\text{If } \lambda_{i\mathrm{G}} = \lambda_{j\mathrm{G}}, \quad \text{then} \\ &\lambda_{i\mathrm{G}} = \lambda_{j\mathrm{G}} = \sqrt{\sigma_{ij}}, \\ &\lambda_{i\mathrm{g}_1} = \sqrt{\sigma_{ii'} - \sigma_{ij}}, \quad \text{and} \\ &\lambda_{j\mathrm{g}_2} = \sqrt{\sigma_{jj'} - \sigma_{ij}} \end{aligned} \tag{2}$$

Nonunique Solution B:

$$\begin{aligned} &\text{If } \lambda_{j\mathrm{g}_2} = 0, \quad \text{then} \\ &\lambda_{j\mathrm{G}} = \sqrt{\sigma_{jj'}}, \\ &\lambda_{i\mathrm{G}} = \frac{\sigma_{ij}}{\sqrt{\sigma_{jj'}}}, \quad \text{and} \\ &\lambda_{i\mathrm{g}_1} = \sqrt{\sigma_{ii'} - \frac{\sigma_{ij}^2}{\sigma_{jj'}}}. \end{aligned} \tag{3}$$

In that there are two solutions based on different assumptions ($\lambda_{i\mathrm{G}} = \lambda_{j\mathrm{G}}$ vs. $\lambda_{j\mathrm{g}_2} = 0$), parameter estimates are not unique.

For the first example covariance matrix in Table 2, the values of the parameters for the two solutions are as follows:

Nonunique Solution A:

$$\begin{aligned} &\lambda_{i\mathrm{G}} = \lambda_{j\mathrm{G}} = \sqrt{\sigma_{ij}} = \sqrt{.64} = .8 \\ &\lambda_{i\mathrm{g}_1} = \sqrt{\sigma_{ii'} - \sigma_{ij}} = \sqrt{.73 - .64} = .3 \\ &\lambda_{j\mathrm{g}_2} = \sqrt{\sigma_{jj'} - \sigma_{ij}} = \sqrt{.73 - .64} = .3 \end{aligned} \tag{4}$$

Table 2. Example Covariance Matrices and Factor Loading Matrices That Are Consistent With These Covariance Matrices.

Covariance Matrix 1 (HWHB)

$$\begin{bmatrix} 1 & & & & & & & \\ .73 & 1 & & & & & & \\ .73 & .73 & 1 & & & & & \\ .73 & .73 & .73 & 1 & & & & \\ .64 & .64 & .64 & .64 & 1 & & & \\ .64 & .64 & .64 & .64 & .73 & 1 & & \\ .64 & .64 & .64 & .64 & .73 & .73 & 1 & \\ .64 & .64 & .64 & .64 & .73 & .73 & .73 & 1 \end{bmatrix}$$

One set of factor loadings for a bifactor model consistent with Covariance Matrix 1

$$\begin{bmatrix} .8 & .3 & 0 \\ .8 & .3 & 0 \\ .8 & .3 & 0 \\ .8 & .3 & 0 \\ .8 & 0 & .3 \\ .8 & 0 & .3 \\ .8 & 0 & .3 \\ .8 & 0 & .3 \end{bmatrix}$$

Covariance Matrix 2 (HWHB)

$$\begin{bmatrix} 1 & & & & & & & \\ .65 & 1 & & & & & & \\ .65 & .65 & 1 & & & & & \\ .65 & .65 & .65 & 1 & & & & \\ .56 & .56 & .56 & .56 & 1 & & & \\ .56 & .56 & .56 & .56 & .73 & 1 & & \\ .56 & .56 & .56 & .56 & .73 & .73 & 1 & \\ .56 & .56 & .56 & .56 & .73 & .73 & .73 & 1 \end{bmatrix}$$

One set of factor loadings for a bifactor model consistent with Covariance Matrix 2

$$\begin{bmatrix} .7 & .4 & 0 \\ .7 & .4 & 0 \\ .7 & .4 & 0 \\ .7 & .4 & 0 \\ .8 & 0 & .3 \\ .8 & 0 & .3 \\ .8 & 0 & .3 \\ .8 & 0 & .3 \end{bmatrix}$$

Covariance Matrix 3 (not HWHB)

$$\begin{bmatrix} 1 & & & & & & & \\ .72 & 1 & & & & & & \\ .70 & .72 & 1 & & & & & \\ .68 & .72 & .76 & 1 & & & & \\ .25 & .30 & .35 & .40 & 1 & & & \\ .30 & .36 & .42 & .48 & .72 & 1 & & \\ .35 & .42 & .49 & .56 & .70 & .72 & 1 & \\ .40 & .48 & .56 & .64 & .68 & .72 & .76 & 1 \end{bmatrix}$$

Bifactor model consistent with Covariance Matrix 3

$$\begin{bmatrix} .5 & .7 & 0 \\ .6 & .6 & 0 \\ .7 & .5 & 0 \\ .8 & .4 & 0 \\ .5 & 0 & .7 \\ .6 & 0 & .6 \\ .7 & 0 & .5 \\ .8 & 0 & .4 \end{bmatrix}$$

Note. HWHB = homogeneous-within and homogeneous-between. The numerical rank of each of the three factor loading matrices was computed based on the singular value decomposition. The first two factor loading matrices were not of full column rank; both had a column rank of 2. The third factor loading matrix was of full column rank. The specific singular values (to two decimal places) were 2.34, .60, and .00 for the first covariance matrix, 2.24, .72, and .00 for the second covariance matrix, and 2.15, 1.12, and .35 for the third covariance matrix, respectively.

Nonunique Solution B:

$$\begin{aligned} &\lambda_{jg_2} = 0, \\ &\lambda_{jG} = \sqrt{\sigma_{jj'}} = \sqrt{.73} = .8544 \\ &\lambda_{iG} = \frac{\sigma_{ij}}{\sqrt{\sigma_{jj'}}} = \frac{.64}{\sqrt{.73}} = .74906 \\ &\lambda_{ig_1} = \sqrt{\sigma_{ii'} - \frac{\sigma_{ij}^2}{\sigma_{jj'}}} = \sqrt{.73 - \frac{.64^2}{.73}} = .411. \end{aligned} \tag{5}$$

For the second example covariance matrix in Table 2, the values of the parameters for the two solutions are as follows:

Nonunique Solution A:

$$\begin{aligned} \lambda_{iG} &= \lambda_{jG} = \sqrt{\sigma_{ij}} = \sqrt{.56} = .74833 \\ \lambda_{ig_1} &= \sqrt{\sigma_{ii'} - \sigma_{ij}} = \sqrt{.65 - .56} = .3 \\ \lambda_{jg_2} &= \sqrt{\sigma_{jj'} - \sigma_{ij}} = \sqrt{.73 - .56} = .4123 \end{aligned} \tag{6}$$

Nonunique Solution B:

$$\begin{aligned} \lambda_{jg_2} &= 0 \\ \lambda_{jG} &= \sqrt{\sigma_{jj'}} = \sqrt{.73} = .8544 \\ \lambda_{iG} &= \frac{\sigma_{ij}}{\sqrt{\sigma_{jj'}}} = \frac{.56}{\sqrt{.73}} = .6554 \\ \lambda_{ig_1} &= \sqrt{\sigma_{ii'} - \frac{\sigma_{ij}^2}{\sigma_{jj'}}} = \sqrt{.65 - \frac{.56^2}{.73}} = .4695 \end{aligned} \tag{7}$$

For each covariance matrix, the two solutions yield the same model-implied covariance matrix. Other assumptions would yield additional solutions for each covariance matrix when analyzing the data with the analysis model in Figure 1.

Empirically Covariance-Equivalent Models

For each of the two HWHB covariance matrices, the freely estimated loadings between the second group factor and last four indicators are zeros for nonunique Solution B. Because the model yields a fit function value of zero for each of these covariance matrices, we can surmise that a bifactor model with only a single group factor (associated with the first four indictors or, alternatively, with the second four indicators) fits the covariance matrix perfectly too. Provocatively, a correlated two-factor pure-clustered model also evidences perfect fit to the covariance matrix, although it has fewer degrees of freedom (16 vs. 19). These two two-factor models produce equivalent and perfect fit for the two HWHB matrices and, thus, the same model-implied covariance matrices. It is important to note that fitting a bifactor model with a single-group factor or a correlated two-factor pure-clustered model to the third covariance matrix (a non-HWHB covariance) produces a nonzero fit function value.

As described by Hershberger and Marcoulides (2013), two models are covariance-equivalent models if they yield the same model-implied covariance matrix regardless of the data. Alternatively, the term *equivalent models* frequently has additional requirements, including that the two models have the same number of degrees of freedom (Bentler & Satorra, 2010; Hershberger & Marcoulides, 2013). In our examples, the bifactor and correlated factor models produce the same model-implied covariance matrix for the two HWHB covariance matrices, but not for the non-HWHB covariance matrix. Consequently, we refer to these models as empirically covariance-equivalent models with respect to the two HWHB covariance matrices. The bifactor with two-group factors (i.e., model presented in Figure 1) also yields the same implied covariance matrix, but is not a practical alternative in that it is empirically underidentified and, thus, the estimated parameters for any solution is not interpretable.

We wanted to show the generality of these results across different HWHB covariance matrices. To do so, we varied the values of the four types of factor loadings in the factor loading matrix (λ_{iG}, λ_{jG}, λ_{ig_1}, and λ_{jg_2}) from .3 to .8 for each type, excluding combinations that yielded communalities for one or more indicators that were equal to or greater than 1. We then generated HWHB covariance matrices for the 961 valid combinations. For all combinations, the fit function values were zero when fitting a bifactor model with a single group factor or a correlated two-factor pure-clustered model to these HWHB covariance matrices. Thus, it appears that these two models are empirically covariance equivalent for HWHB covariance matrices.

Influence of Start Values on Bifactor Model Results

We were interested in what occurs at the population level when fitting the bifactor model in Figure 1 to the three example covariance matrices using different start values. We first consider fitting the model to the HWHB covariance matrices and then to the non-HWHB covariance matrix.

Results for HWHB Covariance Matrices. Given the model is empirically underidentified for the example HWHB covariance matrices (Covariance Matrices 1 and 2), we expected the model parameter estimates to differ as a function of the start values. We initially conducted these analyses with SAS CALIS and used start values based on the parameter values for nonunique Solutions A and B as well as the default start values provided by CALIS. As reported by the SAS manual (SAS Institute, 2013), CALIS generates start values using a procedure by McDonald and Hartmann (1992) or, if not possible, with ''approximate factor analysis.'' It should be noted that the example covariance matrices are perfectly consistent with the model (i.e., perfect fit), and, given the model is empirically underidentified for these HWHB covariance matrices, the results produced perfect fit for all other possible nonunique solutions. The results for these analyses are in rows labeled first matrix and second matrix of Table 3.

The choice of start values had a strong effect on parameter estimates of our bifactor models. When inputting the values of the parameters for solutions A and B, we obtained parameter estimates for the analysis model that were very close to the inputted start values. The SAS default start values were quite different from the other

Table 3. Results of Analyses for Different Start Values at the Population Level.

	Analyses based on start values from Solution A		Analyses based on start values from Solution B		Analyses based on SAS default start values	
Covariance matrix	Start values[a]	Results[b]	Start values[a]	Results[b]	Start values[a]	Results[b]
First matrix	$\begin{bmatrix} .80 & .30 & 0 \\ .80 & .30 & 0 \\ .80 & .30 & 0 \\ .80 & .30 & 0 \\ .80 & 0 & .30 \\ .80 & 0 & .30 \\ .80 & 0 & .30 \\ .80 & 0 & .30 \end{bmatrix}$	$\begin{bmatrix} .80 & .30 & 0 \\ .80 & .30 & 0 \\ .80 & .30 & 0 \\ .80 & .30 & 0 \\ .80 & 0 & .30 \\ .80 & 0 & .30 \\ .80 & 0 & .30 \\ .80 & 0 & .30 \end{bmatrix}$	$\begin{bmatrix} .75 & .41 & 0 \\ .75 & .41 & 0 \\ .75 & .41 & 0 \\ .75 & .41 & 0 \\ .85 & 0 & .00 \\ .85 & 0 & .00 \\ .85 & 0 & .00 \\ .85 & 0 & .00 \end{bmatrix}$	$\begin{bmatrix} .75 & .41 & 0 \\ .75 & .41 & 0 \\ .75 & .41 & 0 \\ .75 & .41 & 0 \\ .85 & 0 & .00 \\ .85 & 0 & .00 \\ .85 & 0 & .00 \\ .85 & 0 & .00 \end{bmatrix}$	$\begin{bmatrix} .05 & .89 & 0 \\ .05 & .89 & 0 \\ .05 & .89 & 0 \\ .05 & .89 & 0 \\ -.49 & 0 & -.04 \\ -.49 & 0 & -.04 \\ -.49 & 0 & -.04 \\ -.49 & 0 & -.04 \end{bmatrix}$	$\begin{bmatrix} -.75 & .41 & 0 \\ -.75 & .41 & 0 \\ -.75 & .41 & 0 \\ -.75 & .41 & 0 \\ -.85 & 0 & .00 \\ -.85 & 0 & .00 \\ -.85 & 0 & .00 \\ -.85 & 0 & .00 \end{bmatrix}$
Second matrix	$\begin{bmatrix} .75 & .30 & 0 \\ .75 & .30 & 0 \\ .75 & .30 & 0 \\ .75 & .30 & 0 \\ .75 & 0 & .41 \\ .75 & 0 & .41 \\ .75 & 0 & .41 \\ .75 & 0 & .41 \end{bmatrix}$	$\begin{bmatrix} .75 & .30 & 0 \\ .75 & .30 & 0 \\ .75 & .30 & 0 \\ .75 & .30 & 0 \\ .75 & 0 & .41 \\ .75 & 0 & .41 \\ .75 & 0 & .41 \\ .72 & 0 & .41 \end{bmatrix}$	$\begin{bmatrix} .65 & .47 & 0 \\ .65 & .47 & 0 \\ .65 & .47 & 0 \\ .65 & .47 & 0 \\ .85 & 0 & .00 \\ .85 & 0 & .00 \\ .85 & 0 & .00 \\ .85 & 0 & .00 \end{bmatrix}$	$\begin{bmatrix} .66 & .47 & 0 \\ .66 & .47 & 0 \\ .66 & .47 & 0 \\ .66 & .47 & 0 \\ .85 & 0 & .00 \\ .85 & 0 & .00 \\ .85 & 0 & .00 \\ .85 & 0 & .00 \end{bmatrix}$	$\begin{bmatrix} .01 & .86 & 0 \\ .01 & .86 & 0 \\ .01 & .86 & 0 \\ .01 & .86 & 0 \\ .01 & 0 & .89 \\ .01 & 0 & .89 \\ .01 & 0 & .89 \\ .01 & 0 & .89 \end{bmatrix}$	$\begin{bmatrix} .72 & .36 & 0 \\ .72 & .36 & 0 \\ .72 & .36 & 0 \\ .72 & .36 & 0 \\ .78 & 0 & .36 \\ .78 & 0 & .36 \\ .78 & 0 & .36 \\ .78 & 0 & .36 \end{bmatrix}$
Third matrix[c]	$\begin{bmatrix} .65 & .55 & 0 \\ .65 & .55 & 0 \\ .65 & .55 & 0 \\ .65 & .55 & 0 \\ .65 & 0 & .55 \\ .65 & 0 & .55 \\ .65 & 0 & .55 \\ .65 & 0 & .55 \end{bmatrix}$	$\begin{bmatrix} .50 & .70 & 0 \\ .60 & .60 & 0 \\ .70 & .50 & 0 \\ .80 & .40 & 0 \\ .50 & 0 & .70 \\ .60 & 0 & .60 \\ .70 & 0 & .50 \\ .80 & 0 & .40 \end{bmatrix}$	$\begin{bmatrix} .65 & .55 & 0 \\ .65 & .55 & 0 \\ .65 & .55 & 0 \\ .65 & .55 & 0 \\ .65 & 0 & .00 \\ .65 & 0 & .00 \\ .65 & 0 & .00 \\ .65 & 0 & .00 \end{bmatrix}$	$\begin{bmatrix} .50 & .70 & 0 \\ .60 & .60 & 0 \\ .70 & .50 & 0 \\ .80 & .40 & 0 \\ .50 & 0 & .70 \\ .60 & 0 & .60 \\ .70 & 0 & .50 \\ .80 & 0 & .40 \end{bmatrix}$	$\begin{bmatrix} .31 & .84 & 0 \\ .22 & .87 & 0 \\ .11 & .89 & 0 \\ .01 & .90 & 0 \\ -.72 & 0 & .36 \\ -.68 & 0 & .15 \\ -.63 & 0 & -.10 \\ -.56 & 0 & -.20 \end{bmatrix}$	$\begin{bmatrix} -.50 & .70 & 0 \\ -.60 & .60 & 0 \\ -.70 & .50 & 0 \\ -.80 & .40 & 0 \\ -.50 & 0 & .70 \\ -.60 & 0 & .60 \\ -.70 & 0 & .50 \\ -.80 & 0 & .40 \end{bmatrix}$

[a]Start values are given to two decimal places. The exact values of all start values for Covariance Matrices 1 and 2 are provided in Equations (4), (5), (6), and (7). The value of "0" denotes the loading is fixed to 0.
[b]Results are presented to two decimal places. If all loadings on a factor are negatives, they all can be changed to positives. The results for the loadings are essentially the same, except lower values on the former factor are higher values on the latter factor.
[c]Covariance Matrix 3 is not homogeneous-within and homogeneous-between and yields a unique solution. Thus, there are no start values based on Solutions A and B for this matrix. We constructed Solutions A and B start values for this matrix to try to have similar characteristics with the start values based on Solutions A and B for Covariance Matrices 1 and 2.

inputted start values for these two covariance matrices. Nevertheless, for the first covariance matrix, the parameter estimates were essentially the same as those for Solution B. In contrast, for the second covariance matrix, the parameter estimates differed substantially from the others that were obtained with this matrix.

As expected, the model fit perfectly for all results. No error messages occurred for the first covariance matrix with default start values. For all other five analyses, CALIS gave error messages indicating that either the covariance matrix for estimates was not of full rank, or the Moore–Penrose inverse was used in computing the covariance matrix for estimates. The exception was for the Covariance Matrix 1 using the CALIS start values. For this analysis, no error messages were shown. Provocatively the results were essentially comparable with those using Solution B start values. Based on these results, CALIS will indicate that there is a problem with empirically underidentifed models in the population ''most of the time.'' It should be noted that when error messages were given, they indicate a problem, but not that the model is empirically underidentified.

To offer a generalization of our findings, we conducted these nine analyses in M*plus* and EQS. We summarized the similarity and difference in error messages across these three SEM programs in Table A1 in Appendix A as well as presentation of the error messages for the three programs. In summary, for the analyses based on Solutions A and B for the Covariance Matrix 1 and Covariance Matrix 2, the error messages are very similar. Example code is provided in Appendix B for the three programs used to conduct these analyses.

Results for Non-HWHB Covariance Matrices. We also fit the bifactor model in Figure 1 to Covariance Matrix 3, which is not HWHB. These analyses were conducted with CALIS. The parameter estimates are unique for this covariance matrix and consequently there are no nonunique Solutions A and B. Accordingly, we created start values for ''Solutions A and B'' by creating patterns of start values that were similar to these solutions for the other two covariance matrices. Thus, we inputted two sets of start values plus the default start values to assess the impact of start values on estimated factor loadings.

The results for these analyses are in the row labeled third matrix of Table 3. As expected, because the model is identified mathematically and empirically, the parameter estimates of the analysis model were the same regardless of start values. In addition, CALIS produced no error messages.

Empirical Underidentification at the Sample Level

In this section, we investigate at the sample level, empirical underidentification of the bifactor model for the three covariance matrices. We manipulated start values and sample size in this investigation. The three sets of start values were those investigated at the population level: start values based on Solution A, start values based on Solution B, and the CALIS default start values. The sample sizes were 100, 1,000, or

5,000. The number of replications for each of the nine conditions (3 sets of start values $\times$ 3 sample sizes) was 1,000. Sample data were generated using SAS IML based on the model in Figure 1 with the three loading parameters specified in Table 2. The residual variance for an item was fixed at one minus the sum of square of loadings on that item. The mean and variance of factor was fixed at zero and one, respectively. The intercept of each item was zero. All the data were generated to be multivariately normal distributions in the population. Each data set was analyzed with CALIS using the maximum likelihood estimation method. In terms of results, we were interested in the effect of start values and sample size on the percent of replications that converged and percent of replications with converged solutions that produced error messages.

In Table 4, we present the results of convergence rates and error messages rates across the 1,000 replications. Except when the sample size was 100, the convergence rates were close to 100% or 100% for the third covariance matrix (i.e., the matrix that yields a unique solution for the bifactor analysis model at the population level). Even when sample size was small (i.e., 100) for this covariance matrix, the convergence rates were approximately 80% or higher.

In comparison and as expected, the convergence rates were lower for the first two covariance matrices, which yield nonunique solutions for the bifactor analysis model at the population level. The convergence rates for these matrices increased with sample size, but never exceed 77%, even with a sample size of 5,000. The convergence rates were highest for start values based on Solution A (which assumes that all loadings on the general factor are equal) for both covariance matrices. The relative ordering of convergence rates for the other sets of start values differed depending on the covariance matrix that was analyzed. The conclusion appears to be that empirical underidentification at the population level can result in nonconvergence at the sample level, and the choice of start values affects convergence rates.

Overall the percentages of error messages across replications with converged solutions were quite small for all conditions, never exceeding 7.4%. These percentages increased with sample size, which covaried with convergence rates. We interpreted these results to indicate that empirical underidentification at the population level shows through at the sample level with convergence problems and, to the extent that convergence occurs, with error messages.

Conclusions and Implications

Kenny and Milan (2012, p. 145) made the following statement about model identification:

> Identification is perhaps the most difficult concept for SEM researchers to understand. We have seen SEM experts baffled and bewildered by issues of identification. We too have often encountered very difficult SEM problems that ended up being problems of identification. Identification is not just a technical issue that can be left to experts to ponder; if the model is not identified the research is impossible.

Table 4. Effect of Different Start Values and Sample Sizes on Convergence Rates and Percent of Replications With Error Messages for Converged Solutions.

	Covariance Matrix 1		Covariance Matrix 2		Covariance Matrix 3	
Origin of start values for analyses[a]	Convergence rates (%)	Percentage of converged solutions with error messages	Convergence rates (%)	Percentage of converged solutions with error messages	Convergence rates (%)	Percentage of converged solutions with error messages
Sample size of 100						
Solution A	55.7	0.5	58.1	0.2	91.2	0
Solution B	48.2	0	44.4	0	79.9	0
CALIS default	47.8	0.2	48.9	0.2	82.8	0
Sample size of 1,000						
Solution A	68.3	0.7	68.1	0.7	100	0
Solution B	54.2	0.9	48.4	1.2	99.2	0
CALIS Default	51.7	1.5	53.2	0.6	99.8	0
Sample size of 5,000						
Solution A	74.6	2.7	76.8	3.2	100	0
Solution B	56.7	5.6	54.3	3.3	100	0
CALIS default	53.8	7.4	66.0	4.8	100	0

[a]In contrast to Covariance Matrices 1 and 2, Covariance Matrix 3 is not HWHB and yields a unique solution. Thus, there are no start values based on Solutions A and B for this matrix. We constructed Solutions A and B start values for this matrix to try to have similar characteristics with the start values based on Solutions A and B for Covariance Matrices 1 and 2. See Table 3 for start values for these solutions.

Empirical identification is a particular type of model identification, but could be interpreted as even more difficult to understand in that it takes into account the data to be analyzed. An objective of this article was to offer an in-depth analysis of empirical identification/underidentification in the context of a particular example: specification of bifactor pure-clustered models with a HWHB covariance matrix. Through this example, we hope that we have demystified some of the issues surrounding empirical underidentification.

Conclusions for Methodologists

Our research findings are of particular importance to methodologists. In didactic presentations of the bifactor model, we wanted to present simple examples of bifactor models. A natural choice for these examples would be a bifactor pure-clustered model with uniform general factor loadings and uniform factor loadings for each group factor. Although we may present these examples in such a way that empirical underidentification is not an issue, we should consider whether to warn audiences that these models produce data that are empirically underidentified when analyzed with bifactor pure-clustered models. See Reise, Scheines, Widman, and Haviland (2013) for such an example. Also in conducting Monte Carlo studies, methodologists focusing on bifactor models are likely to generate and analyze data using pure-clustered bifactor models that lead to empirical underidentification (as we have done) because of the simplicity of the model structure. Clearly, conditions using these generation and analysis models in combination should be avoided.

Conclusions for Users of SEM

From an applied perspective, we have shown that error messages about the covariance matrix of parameter estimates and lack of model convergence may be diagnostic of empirical underidentification with a bifactor pure-clustered model. But we also have shown that neither of these diagnostics may occur in the presence of empirical underidentification. In addition, we have demonstrated that different start values can lead to different solutions, but not necessarily for any two sets of start values. To be confident that problems in SEM analyses are due to empirical underidentification, we need to conduct a thorough examination of the analysis model in conjunction with the data, as illustrated in this article.

Researchers who apply SEM are not likely to have the time or energy to examine empirical underidentification in the manner that we have. These researchers are more likely to apply heuristic rules, like the ones used to deal with mathematical identification. One applicable heuristic rule that we discussed earlier in the article is that empirical underidentification is more likely to occur for pure-clustered models with two indicators per factor if the correlations between indicators associated with different factors approach zero. A heuristic rule based on this article is that empirical underidentification is more likely to occur for a bifactor pure-clustered model if the covariance matrix to be analyzed approaches a HWHB covariance matrix. Because bifactor models are frequently applied in practice in recent years (Reise, 2012), it is a helpful heuristic rule for researchers to know to explain problems that may occur in conducting their analyses.

Prior to data collection, researchers may design a study to support the importance of a particular bifactor pure-clustered model; however, to avoid empirical underidentification, they have the unenviable job of predicting the type of data they are likely to encounter. If they believe that the data might be a HWHB covariance matrix, then they could take steps to avoid empirical underidentification. One remedy is to include one or more indicators that assess the general factor, but not the group factors. A second remedy is to include indicators that demonstrate different strengths of relationships with the underlying factors. These two remedies may merge into one if the indicators that were chosen to assess only the general factor turn out also to be weakly related to the group factors. Researchers who are unable to anticipate the nature of their covariance matrix prior to data collection may be forced to seek these remedies after finding out that their sample data approximate a HWHB covariance matrix; however, because they had not anticipated this problem when designing their study, they may not have the required indicators.

Our results suggest an alternative approach if one specifies a bifactor pure-clustered model and encounters a HWHB covariance matrix. One view of this outcome is that the bifactor pure-clustered model is too complex for the data, and thus, researchers should seek simpler models to assess. Based on our findings involving empirically equivalent models, researchers should consider two alternative models to investigate: a bifactor model with one fewer group factors and/or a pure-clustered model with correlated factors. These two models should produce approximately the same fit with sample data as the bifactor pure-clustered model, but avoid empirical underidentification. In taking this route, researchers are essentially acknowledging that the design of their study (and specifically the chosen measures) does not permit a test of the hypothesized bifactor pure-clustered model, and thus, they must evaluate less complex models. The choice between simpler models should be dictated by theory and previous research.

Appendix A

Detail Information About Error Messages

In Appendix A, we provide detailed information about the error messages provided by SAS, EQS, and M*plus*. In Table A1, we show when various software programs gave error messages for the different population covariance matrices analyzed.

We next include example error messages provided by the three SEM software programs:

Table A1. Error Messages From SAS Proc CALISb, EQS, and Mplus for the Three Covariance Matrices.

		Analyses based on three sets of start values		
Covariance matrix	Software	Solution A	Solution B	Default start values[a]
Matrix 1	SAS	x	X	No
	EQS	x[b]	X	No
	M*plus*	x	X	X
Matrix 2	SAS	x	X	X
	EQS	x	X	No
	Mplus	x	X	X
Matrix 3	SAS	No	No	No
	EQS	No	X	No
	M*plus*	No	X	No

Note. x denotes that error messages are provided by the software program; "No" indicates that the model converged with no error/warning message provided by the software program.
[a]For analyses based on default starts, we did not expect the three programs to provide identical error messages for Covariance Matrices 1 and 2 because CALIS uses different default start values (as described in the body of the article) from EQS and M*plus* for loadings. The default start value for factor loadings is one in both EQS and M*plus*.
[b]We conducted analyses using both EQS 6.1 and 6.3. The two versions of EQS performed similarly except for the analysis with the superscript *b*. EQS 6.1 provided an error message indicating that the standard errors are not estimable and are fixed as zero. In EQS 6.3, there is no error message, but we noticed that the standard errors for loadings were extremely large and so results were not interpretable.

CALIS.

NOTE: The Moore-Penrose inverse is used in computing the covariance matrix for parameter estimates.
WARNING: Standard errors and t values might not be accurate with the use of the Moore-Penrose inverse.
NOTE: Covariance matrix for the estimates is not full rank.
NOTE: The variance of some parameter estimates is zero or some parameter estimates are linearly related to other parameter estimates as shown in the following equations.

EQS.

PARAMETER CONDITION CODE
V8,F3 LINEARLY DEPENDENT ON OTHER PARAMETERS
MAXIMUM LIKELIHOOD SOLUTION (NORMAL DISTRIBUTION THEORY)
V8,F3 VARIANCE OF PARAMETER ESTIMATE IS SET TO ZERO.

Mplus.

THE STANDARD ERRORS OF THE MODEL PARAMETER ESTIMATES COULD NOT BE COMPUTED. THE MODEL MAY NOT BE IDENTIFIED. CHECK YOUR MODEL.
PROBLEM INVOLVING THE FOLLOWING PARAMETER:
Parameter 10, GROUP2 BY X5

Appendix B

Example Code for Different SEM Software

In Appendix B, we present example code for analyses based on Solution A and Covariance Matrix 1.

CALIS.

```
data Dat(TYPE=COV);
  _type_= 'COV';
INPUT _NAME_ $ x1-x8;

Datalines;
X1   1.00   .73   .73   .73   .64   .64   .64   .64
X2    .73  1.00   .73   .73   .64   .64   .64   .64
X3    .73   .73  1.00   .73   .64   .64   .64   .64
X4    .73   .73   .73  1.00   .64   .64   .64   .64
X5    .64   .64   .64   .64  1.00   .73   .73   .73
X6    .64   .64   .64   .64   .73  1.00   .73   .73
X7    .64   .64   .64   .64   .73   .73  1.00   .73
X8    .64   .64   .64   .64   .73   .73   .73  1.00
;
run;

proc calis cov data=Dat method=max nobs=10000 maxiter=500;
lineqs
x1=a1 (.8) *F1 + b1(.3) *F2 + e1,
x2=a2 (.8) *F1 + b2(.3) *F2 + e2,
x3=a3 (.8) *F1 + b3(.3) *F2 + e3,
x4=a4 (.8) *F1 + b4(.3) *F2 + e4,
x5=a5 (.8) *F1 + c1(.3) *F3 + e5,
x6=a6 (.8) *F1 + c2(.3) *F3 + e6,
x7=a7 (.8) *F1 + c3(.3) *F3 + e7,
x8=a8 (.8) *F1 + c4(.3) *F3 + e8;

std
e1=ve1,e2=ve2,e3=ve3,e4=ve4,e5=ve5,e6=ve6,e7=ve7,e8=ve8,
F1=1,F2=1,F3=1;

cov
F1 F2=0, F1 F3=0, F2 F3=0;
run;
```

EQS.

```
/TITLE
   Matrix 1_Solution A
/SPECIFICATIONS
CASES = 10000; VARIABLES = 8; MATRIX=COVARIANCE;
/LABELS
V1=X1; V2=X2; V3=X3; V4=X4; V5=X5;
V6=X6; V7=X7; V8=X8;
/EQUATIONS
V1=.8*f1 + .3*f2 + e1;
V2=.8*f1 + .3*f2 + e2;
V3=.8*f1 + .3*f2 + e3;
V4=.8*f1 + .3*f2 + e4;
V5=.8*f1 + .3*f3 + e5;
V6=.8*f1 + .3*f3 + e6;
V7=.8*f1 + .3*f3 + e7;
V8=.8*f1 + .3*f3 + e8;

/VARIANCES
f1=1.00; f2=1.00; f3=1.00;
e1=*; e2=*; e3=*; e4=*; e5=*; e6=*; e7=*; e8=*;
/COVARIANCES
f1, f2=0; f1, f3=0; f2, f3=0;
/tech
iter=500;

/MATRIX
1.00
0.73 1.00
0.73 0.73 1.00
0.73 0.73 0.73 1.00
0.64 0.64 0.64 0.64 1.00
0.64 0.64 0.64 0.64 0.73 1.00
0.64 0.64 0.64 0.64 0.73 0.73 1.00
0.64 0.64 0.64 0.64 0.73 0.73 0.73 1.00

/END
```

M*plu*s.

```
Data:
file is matrix1.txt;
```

```
! The file ''matrix1.txt'' contains the lower triangle of the covariance matrix 1
  with 1s in diagonal.
type is covariance;
nobservations=10000;

Variable:
Names are x1-x8;
Usevariables are x1-x8;

Model:
general by x1*.80 x2*.80 x3*.80 x4*.80 x5*.80 x6*.80 x7*.80 x8*.80;
group1 by x1*.30 x2*.30 x3*.30 x4*.30;
group2 by x5*.30 x6*.30 x7*.30 x8*.30;
general@1;group1@1;group2@1;
general with group1@0 group2@0;
group1 with group2@0;
```

Declaration of Conflicting Interests

The author(s) declared no potential conflicts of interest with respect to the research, authorship, and/or publication of this article.

Funding

The author(s) received no financial support for the research, authorship, and/or publication of this article.

References

Bentler, P. M., & Satorra, A. (2010). Testing model nesting and equivalence. *Psychological Methods*, *15*, 111-123.

Bollen, K. A. (1989). *Structural equations with latent variables*. New York, NY: Wiley.

Browne, M. W. (2001). An overview of analytic rotation in exploratory factor analysis. *Multivariate Behavioral Research*, *36*, 111-150.

Chen, F. F., Hayes, A., Carver, C. S., Laurenceau, J. P., & Zhang, Z. (2012). Modeling general and specific variance in multifaceted constructs: A comparison of the bifactor model to other approaches. *Journal of Personality*, *80*, 219-251.

Hershberger, S. L., & Marcoulides, G. A. (2013). The problem of equivalent structural models. In G. R. Hancock & R. O. Mueller (Eds.). *Structural equation modeling: A second course* (2nd ed., pp. 3-40). Greenwich, CT: Information Age.

Kenny, D. A., & Milan, S. (2012). Identification: A non-technical discussion of a technical issue. In R. Hoyle (Ed.), *Handbook of structural equation modeling* (pp. 145-163). New York, NY: Guilford Press.

Lee, S., & Hershberger, S. L. (1990). A simple rule for generating equivalent models in covariance structure modeling. *Multivariate Behavioral Research*, *25*, 313-334.

MacCallum, R. C., Wegener, D. T., Uchino, B. N., & Fabrigar, L. R. (1993). The problem of equivalent models in applications of covariance structure analysis. *Psychological Bulletin*, *114*, 185-199.

McDonald, R. P., & Hartmann, W. M. (1992). A procedure for obtaining initial values of parameters in the RAM model. *Multivariate Behavioral Research*, *27*, 57-76.

Raykov, T., & Marcoulides, G. A. (2001). Can there be infinitely many models equivalent to a given covariance structure model? *Structural Equation Modeling*, *8*, 142-149.

Reise, S. P. (2012). The rediscovery of bifactor measurement models. *Multivariate Behavioral Research*, *47*, 667-696.

Reise, S. P., Scheines, R., Widaman, K. F., & Haviland, M. G. (2013). Multidimensionality and structural coefficient bias in structural equation modeling: A bifactor perspective. *Educational and Psychological Measurement*, *73*, 5-26.

Rindskopf, D. (1984). Structural equation models: Empirical identification, Heywood cases, and related problems. *Sociological Methods & Research*, *13*, 109-119.

Rindskopf, D., & Rose, T. (1988). Some theory and applications of confirmatory second-order factor analysis. *Multivariate Behavioral Research*, *23*, 51-67.

SAS Institute. (2013). *SAS/STAT 13.1 user's guide*. Cary, NC: Author.

Stelzl, I. (1986). Changing a causal hypothesis without changing the fit: Some rules for generating equivalent path models. *Multivariate Behavioral Research*, *21*, 309-331.