JOURNAL OF RESEARCH of the National Bureau of Standards
Volume 82, No. 3, November 1977

Enclaveless Sets and MK-Systems*

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(August 10, 1977)

A hypergraph $H = (X, \mathscr{E})$ is called a Menger System if the maximum cardinality of a family of pairwise disjoint edges ($\nu_1(H)$) is equal to the minimum cardinality of a subset of vertices which meets every edge ($\tau_0(H)$). A set $S \subseteq X$ is defined to be enclaveless if each vertex in S is adjacent to at least one vertex in $X - S$. A parameter π_0 related to the formation of maximal enclaveless sets is defined, and it is shown that if H has no singleton edges then $\nu_1(H) \leq \pi_0(H)$. MK-Systems are defined to be those hypergraphs H without singleton edges for which $\nu_1(H) = \pi_0(H)$; simple graphs which are Menger Systems are shown also to be MK-Systems.

Key words: Dominating set; enclaveless set; graph; hypergraph; König System; KM-System; Menger System; MK-System.

1. Introduction

Prior to his presentation of the following problems in the Graph Theory Unsolved Problem Session of the National Meeting of the American Mathematical Society in January, 1975, Alan J. Goldman explained to me that his motivation was to show that questions of a social-policy nature can, when given a mathematical interpretation, lead to technically interesting problems and, perhaps, some worthwhile insights into the motivating questions. He considered a society (be it a country, a club, a school, or a corporation) in which there are two distinct groups of people, say black and white or mathematician and nonmathematician. It is assumed that each individual, while being uncomfortable if he is not in contact with at least one member of his own group, should for reasons of "integration" be in contact with at least one member of the other group.

For example, consider a corporation in which the technical staff has r mathematicians, s nonmathematicians and $r + s$ individual offices in which to locate them. Construct a graph G on $r + s$ vertices corresponding to the $r + s$ offices in which two vertices are adjacent if the relative positions of the associated offices are such that the occupants are likely to encounter each other. One would like to be able to partition $V(G)$ into sets R and S with $|R| = r$ and $|S| = s$ such that each $v \in V(G)$ is adjacent to some $v_R \in R$ and to some $v_S \in S$, where v is not v_R or v_S since we will not allow loops in G.

More generally, in the terminology of Berge [1],[1] if $H = (X, \mathscr{E})$ is a hypergraph one might wish to partition the vertex set X into n sets $S_1, S_2, \ldots, S_n$. If $v \in X$, let $\Gamma(v)$ denote the set of neighbors of v, that is, $\Gamma(v) = \{w \in X: w \neq v, \{w, v\} \subseteq E_i$ for some $E_i \in \mathscr{E}\}$. If $v \in S \subseteq X$ call v an *isolate* of S if $\Gamma(v) \cap S = \phi$, and call v an *enclave* of S if $\Gamma(v) \cap (X\text{-}S) = \phi$.

The questions proposed by Goldman were the following. If G is a graph and $V(G)$ is partitioned into sets S_1 and S_2, then let i_1 and i_2 (respectively, e_1 and e_2) denote the number of isolates (respectively, enclaves) of S_1 and S_2, respectively. Find an algorithm to partition $V(G)$ into sets S_1 and S_2 with $|S_1| = r$ and $|S_2| = s$ (where $|V(G)| = r + s$) so that

(1) $i_1 + i_2$ is minimized, or

(2) $e_1 + e_2$ is minimized, or

(3) $\alpha(i_1 + i_2) + \beta(e_1 + e_2)$ is minimized where α and β are the positive "penalty values" associated with each isolate and enclave, respectively.

The question concerning isolates led to the work of Maurer [7], "Vertex Colorings without Isolates," and indirectly to [8] and [9]. In [7] the following conjecture is made.

AMS Subject Classification: 05C99.

* An invited paper. This work was supported by the U.S. Energy Research and Development Administration (ERDA) under Contract No. AT(29-1)-789.

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[1] Figures in brackets indicate the literature references at the end of this paper.

CONJECTURE (Maurer): *If* G *is an* n*-connected graph with* $|V(G)| = r = r_1 + r_2 + \ldots + r_n (r_i \geq 1)$ *then it is possible to partition* V(G) *into* n *sets* $S_1, \ldots S_n$ *so that for all* i *one has* $|S_i| = r_i$ *and the induced subgraph* $\langle S_i \rangle$ *is connected.*

Thus, for $r_i \geq 2$, S_i will have no isolates. A stronger version of this conjecture, in which for any designated n vertices $v_1, v_2, \ldots, v_n$ it is also required that $v_i \in S_i$, was made independently by András Frank [2]. This stronger version (presumably not derived from a social question) has been proven by Lovasz [6] and Győry [4].

To my knowledge, however, no work pertaining to the second or third of Goldman's questions has been submitted for publication. Call vertex set $S \subseteq X$ *enclaveless* in H if there are no enclaves of S. (The complement of S in X may, or may not, contain enclaves.) This paper will be concerned with two different parameters arising from the construction of enclaveless sets. Following some comments about Menger and König systems in the next section, section 3 deals with the obvious relation of enclaveless sets to dominating sets and introduces the concept of an MK-System. Section 4 comprises a brief consideration of the "dual properties" and Dual MK-Systems which will be called KM-Systems.

2. Menger and König Systems

Suppose $H = (X, \mathscr{E})$ is a hypergraph with $X = \{x_1, x_2, \ldots x_n\}$ and $\mathscr{E} = \{E_1, E_2, \ldots E_m\}$. (By the definition of "hypergraph," each $E_i \neq \phi$, and each x_j is contained in some E_i.) Let $\nu_1(H)$ denote the maximum number of pairwise disjoint members of $\mathscr{E}$; let $\tau_0(H)$ denote the minimum number of elements of X in a transversal of $\mathscr{E}$, that is, the minimum size of a subset T of X such that $T \cap E_i \neq \phi$ for $1 \leq i \leq m$; let $\beta_0(H)$ denote the maximum size of an $\mathscr{E}$-independent set, that is, the maximum size of a subset I of X such that $|I \cap E_i| \leq 1$ for $1 \leq i \leq m$; and let $\alpha_1(H)$ denote the minimum size of an $\mathscr{E}$-cover of X, that is, the minimum number of members in a subcollection $\mathscr{G}$ of $\mathscr{E}$ such that $\cup_{E_i \in \mathscr{G}} E_i = X$. Clearly $\nu_1(H) \leq \tau_0(H)$ and $\beta_0(H) \leq \alpha_1(H)$, and H is called a *Menger System* if $\nu_1(H) = \tau_0(H)$ and is called a *König System* if $\beta_0(H) = \alpha_1(H)$. An excellent survey paper on Menger and König systems is that of Woodall [11].

The incidence matrix $M_H = [a_{ij}]$ of H is an m by n matrix with $a_{ij} = 1$ if $x_j \in E_i$ and $a_{ij} = 0$ if $x_j \notin E_i$. Thus the columns of M_H correspond to the vertices in X, and the rows of M_H correspond to the edges in $\mathscr{E}$. The hypergraph H^* for which $M_H^* = M_H^T$, the transpose of M_H, is called the *dual hypergraph of* H. Thus $H^{**} = H$.

Note that the maximum number of rows of M_H for which no column contains a 1 in two or more of these rows is the maximum number of columns of M_{H^*} for which no row contains a 1 in two or more of these columns. Thus $\nu_1(H) = \beta_0(H^*)$, so that ν_1 and β_0 are "dual parameters." Similarly τ_0 and α_1 are dual parameters. Thus H is a Menger System if and only if H^* is a König System.

In [11] there are several examples of Menger Systems, three of which will be presented here. Two of these provide the motivation for the terminology.

1. If G is a bipartite graph with vertex set V and edge set E, let H be the hypergraph with $X = V$ and with $\mathscr{E}$ comprised of the edges in E considered as pairs of vertices. Then H is a Menger System. That this H^* is a König System is König's Theorem [5].
2. If $G = (V, E)$ is a bipartite graph, let H be the hypergraph with $X = E$ and with $\mathscr{E}$ being the family of vertex coboundaries in G. Then H is a Menger System.
3. If $G = (V, E)$ is a graph containing vertices u and v, let H be the hypergraph with $X = V - \{u, v\}$ and with $\mathscr{E}$ being the family of vertex sets of the open u-v paths. That this H is a Menger System is Menger's Theorem [10].

THEOREM 1: *(Gallai [3]) For any simple graph* G *without isolated vertices let* $H = (X, \mathscr{E})$ *with* $X = V(G)$ *and* $\mathscr{E} = \{\{u, v\}: (u, v)$ *is an edge of* $G\}$. *Then* $\tau_0(H) + \beta_0(H) = \alpha_1(H) + \nu_1(H) = |X|$.

For contrast with the later Theorem 2, observe that Theorem 1 is not true for all hypergraphs. This can be seen by letting $H_p = (X_p, \mathscr{E}_p)$ with $X_p = \{1, 2, \ldots, p\}$ and $\mathscr{E}_p = \{X_p\}$. (That is, $|X_p| = p$ and $|\mathscr{E}_p| = 1$.) Consider $p \geq 3$. To show that the first equation also need not hold for hypergraphs, let $H = (X, \mathscr{E})$ where $X = \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$; and $\mathscr{E} = \{\{1, 2\}, \{1, 3\}, \{2, 3\}, \{4, 5\}, \{4, 6\}, \{5, 6\}, \{7, 8\}, \{7, 9\}, \{8, 9\}, \{1, 4, 7\}, \{2, 5, 8\}, \{3, 6, 9\}\}$. Then $\tau_0(H) + \beta_0(H) = 6 + 3 \neq 3 + 4 = \alpha_1(H) + \nu_1(H)$.

3. Enclaveless Sets and MK-Systems

If $H = (X, \mathscr{E})$ is a hypergraph with $S \subseteq X$, then S will be said to be *maximally enclaveless* if S is enclaveless and for each $v \in X - S$ one has that $S \cup \{v\}$ is not enclaveless. Let $\mu_0(H)$ denote the size of the largest (maximally) enclaveless set of H, and let $\pi_0(H)$ denote the size of the smallest maximally enclaveless set of H. Also, $S \subseteq X$ is called a *dominating set* if for each $v \in X - S$ one has $\Gamma(v) \cap S \neq \phi$, and S will be said to be an *irredundant dominating set* if it is a dominating set and for each $v \in S$ one has that $S - v$ is not dominating. Now $\gamma_0(H)$ denotes the size of the smallest (irredundant) dominating set, and $\delta_0(H)$ will denote the size of the largest irredundant dominating set.

THEOREM 2: *If* $H = (X, \mathscr{E})$ *is any hypergraph, then* $\mu_0(H) + \gamma_0(H) = \pi_0(H) + \delta_0(H) = | X |$.

PROOF: If F is any family of subsets of X, let

$$M(X, F) = \max \{| S |: S \in F\},$$

and

$$m(X, F) = \min \{| S |: S \in F\}.$$

Families F_1 and F_2 of subsets of X will be called *complement-related* if $S \in F_1$ if and only if $X - S \in F_2$. Suppose F_1 and F_2 are complement-related. Since the complement of any set in F_1 is in F_2, $m(X, F_2) \leq |X| - M(X, F_1)$; since the complement of any set in F_2 is in F_1, $M(X, F_1) \geq |X| - m(X, F_2)$. Thus $M(X, F_1) + m(X, F_2) = |X|$. Since the families of enclaveless sets and dominating sets in X are complement-related, $\mu_0(H) + \gamma_0(H) = |X|$. (The same argument establishes Theorem 1's relation $\tau_0(H) + \beta_0(H) = |X|$ for simple graphs.)

Let F^+ denote the family of those members of F which are set-theoretically maximal with respect to membership, and F^- those which are minimal. It is easily seen that if F_1 and F_2 are complement-related, then so are F_1^+ and F_2^-. Hence $m(X, F_1^+) + M(X, F_2^-) = |X|$. Now if F_1 is the family of enclaveless sets in H and F_2 is the family of dominating sets in H, then $\pi_0(H) = m(X, F_1^+)$ and $\delta_0(H) = M(X, F_2^-)$. Thus $\pi_0(H) + \delta_0(H) = |X|$.

THEOREM 3: *If* G *is a simple graph, then* $\pi_0(G) \leq \tau_0(G)$.

PROOF: Let $S \subseteq V(G)$ be a transversal (a vertex cover of $E(G)$) of size $\tau_0(G)$. Observe that S cannot contain any isolated vertices. (That is, S contains no vertices of degree zero. There may indeed exist an isolate of S.) Assuming that vertex v is an enclave of S, then any edge incident with v has both endpoints in S, but this would imply that $S - v$ is a smaller transversal than S is. To show that S is a maximally enclaveless set, let $v \in V(G) - S$. Since every edge incident with v must have its other endpoint in S, v would be an enclave in $S \cup \{v\}$. Thus $\pi_0(G) \leq |S| = \tau_0(G)$.

The example in the last paragraph of this section shows that Theorem 3 does not extend to general hypergraphs.

A *singleton edge* of hypergraph H is an edge of H that contains exacty one vertex.

THEOREM 4: *If* H *is a hypergraph without singleton edges, then* $\nu_1(H) \leq \pi_0(H)$.

PROOF: Suppose $F_1, F_2, \ldots, F_k$ are pairwise disjoint edges of H with $k = \nu_1(H)$. Note that all $|F_i| \geq 2$. Let S be a maximally enclaveless set with $|S| = \pi_0(H)$. We wish to prove that $k \leq |S|$.

Number the F_i's so that $F_i \subseteq S$ if and only if $i \leq j(1)$, $|F_i \cap S| \geq 2$ if and only if $i \leq j(2)$, and $F_i \cap S \neq \phi$ if and only if $i \leq j(3)$. Thus $0 \leq j(1) \leq j(2) \leq j(3) \leq k$. Let $F = \cup_{i=1}^{k} F_i$, and set $d = |S \cap (X - F)|$. Thus

$$d = |S| - \sum_{i=1}^{j(1)} |F_i| - \sum_{i=j(1)+1}^{j(2)} |F_i \cap S| - (j(3) - j(2)).$$

Because S is maximally enclaveless, for each vertex $v \in \cup_{j(3)+1}^{k} F_i$ there must be an enclave v' in $S \cup \{v\}$. We will show (a) that $v' \in (S \cap (X - F)) \cup \left(\cup_{i=1}^{j(1)} F_i\right)$ and (b) that the vertices v' are distinct. This implies that

$$d + \sum_{i=1}^{j(1)} |F_i| = \left|(S \cap (X - F)) \cup \left(\cup_{i=1}^{j(1)} F_i\right)\right| \geq \left|\cup_{i=j(3)+1}^{k} F_i\right|.$$

The assumption $|S| < k$, however, would yield

$$\begin{aligned}\left|\bigcup_{i=j(3)+1}^{k} F_i\right| &\geq 2(k - j(3)) \\ &> 2|S| - 2j(3) \\ &= 2\left(d + \sum_{i=1}^{j(2)} |F_i \cap S| + j(3) - j(2)\right) - 2j(3) \\ &\geq 2\left(d + \sum_{i=1}^{j(1)} |F_i| + 2(j(2) - j(1)) - j(2)\right) \\ &\geq 2d + \sum_{i=1}^{j(1)} |F_i| + 2j(1) + 4(j(2) - j(1)) - 2j(2) \\ &= 2d + \sum_{i=1}^{j(1)} |F_i| + 2(j(2) - j(1)) \\ &\geq d + \sum_{i=1}^{j(1)} |F_i|.\end{aligned}$$

Thus the assumption leads to a contradiction, implying the desired result $k \leq |S|$.

To prove that $v' \in (S \cap (X\text{-}F)) \cup (\bigcup_{i=1}^{j(1)} F_i)$, first suppose $v' \in X - F$. Then $v' \neq v$; since $v' \in S \cup \{v\}$, it follows that $v' \in S$ and thus that $v' \in S \cap (X-F)$. Next suppose $v' \in F$, i.e. $v' \in F_h$ for some h with $1 \leq h \leq k$. Since v' is an enclave of $S \cup \{v\}$, it follows that $F_h \subseteq S \cup \{v\}$. Because $|F_h| \geq 2$, this implies that $F_h \cap S \neq \phi$, and thus that $h \leq j(3)$. Since the F_i's are pairwise disjoint and $v \in \bigcup_{j(3)+1}^{k} F_i$, it follows that $v \notin F_h$, and so $F_h \subseteq S$, implying $h \leq j(1)$. Thus $v' \in \bigcup_{i=1}^{j(1)} F_i$.

To prove that the vertices v' are distinct, it suffices to show that $\Gamma(v') \cap \left(\bigcup_{i=j(3)+1}^{k} F_i\right) = \{v\}$. For this purpose, note first that since v' is an enclave of $S \cup \{v\}$,

$$\begin{aligned}\Gamma(v') \cap \left(\bigcup_{i=j(3)+1}^{k} F_i\right) &\subseteq (S \cup \{v\}) \cap \left(\bigcup_{i=j(3)+1}^{k} F_i\right) \\ &= \{v\} \cap \left(\bigcup_{i=j(3)+1}^{k} F_i\right) = \{v\}.\end{aligned}$$

The conclusion of the last paragraph implies $v' \neq v$; thus $v' \in S$, and since $\Gamma(v') \subseteq S \cup \{v\}$ and S is enclaveless, it follows that $v \in \Gamma(v')$, completing the proof.

Call hypergraph H an *MK-System* if H has no singleton edges and $\nu_1(H) = \pi_0(H)$. For graph G_1 in figure 1, $\{\{1, 2\}, \{3, 4\}, \{5, 6\}, \{7, 8\}, \{9, 10\}, \{11, 12\}\}$ is a family of pairwise disjoint edges and $\{1, 4, 5, 8, 9, 12\}$ is a maximal enclaveless set. By Theorem 4 one has $\nu_1(G_1) = \pi_0(G_1) = 6$. Since $\tau_0(G_1) = 7$, G_1 is an MK-System but not a Menger System. Theorems 3 and 4 combine to produce the next corollary.

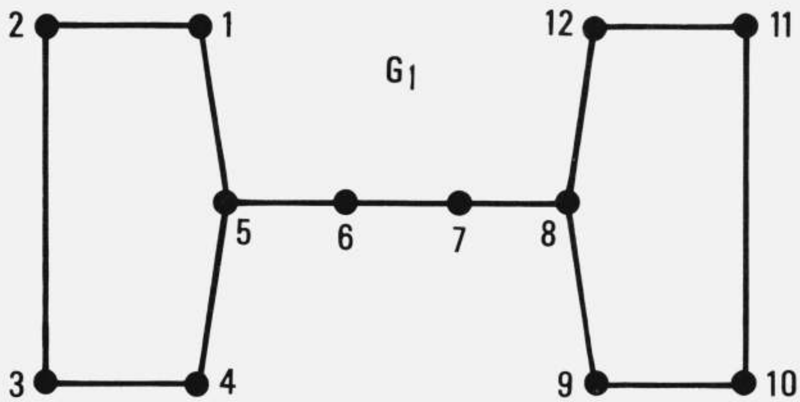

FIGURE 1. *A simple graph with $\nu_1 = \pi_0 = 6$ and $\tau_0 = 7$.*

COROLLARY. *Any simple graph which is a Menger System is also an MK-System.*

Considering example 3 of section 2 as applied to graph G_2 of figure 2, one obtains hypergraph $H = (X, \mathscr{E})$ where $X = \{1, 2, 3, 4\}$ and $\mathscr{E} = \{\{1, 3\}, \{1, 4\}, \{2, 3\}, \{2, 4\}, \{1, 3, 4\}, \{2, 4, 3\}, \{1, 2, 4\}, \{2, 1, 3\}, \{1, 2, 4, 3\}\}$. Since $\nu_1(H) = \tau_0(H) = 2$ and $\pi_0(H) = 3$, this is an example of a Menger System which is not an MK-System. Indeed, most examples of Menger Systems described in [11] (such as examples 2 and 3 of section 2 in this paper) are, in general, not MK-System.

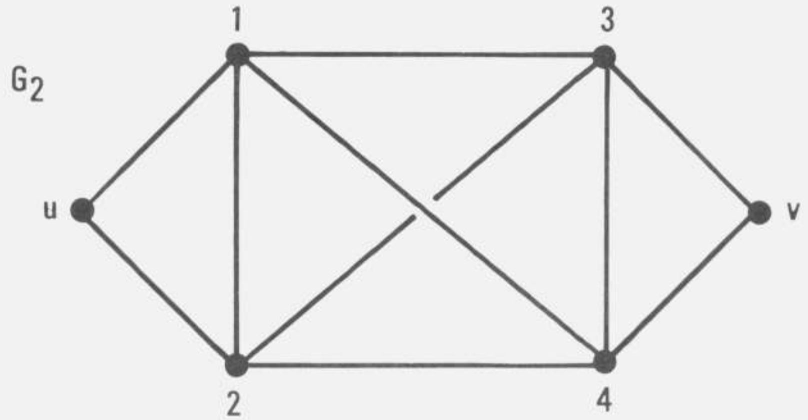

FIGURE 2. *A graph whose open* u-v *paths create a Menger System but not an MK-System.*

4. Dual Concepts

As noted, $\nu_1(H) = \beta_0(H^*)$ and $\tau_0(H) = \alpha_1(H^*)$, giving us two sets of dual parameters. One can also describe the duals of the parameters defined in section 3. Let $\alpha_{11}(H)$ denote the minimum number of edges in a set $\mathfrak{F} \subseteq \mathscr{E}$ such that (*): if $E_1 \in \mathscr{E} - \mathfrak{F}$ then there is an edge $F_1 \in \mathfrak{F}$ for which $E_1 \cap F_1 \neq \phi$. Let $\beta_{11}(H)$ denote the maximum number of edges in a set $\mathfrak{F} \subseteq \mathscr{E}$ such that $\mathfrak{F}$ has property (*) and for each $F \in \mathfrak{F}$ property (*) does not hold for $\mathfrak{F} - F$. Then $\gamma_0(H) = \alpha_{11}(H^*)$ and $\delta_0(H) = \beta_{11}(H^*)$.

If v is a vertex in hypergraph H, call v a *singleton vertex* if it is contained in only one edge. If $F \in \mathfrak{F} \subseteq \mathscr{E}$, call F an *enclosure* of $\mathfrak{F}$ if $E \in \mathscr{E} - \mathfrak{F}$ implies that $E \cap F = \phi$, and $\mathfrak{F}$ will be called *enclosureless* if it contains no enclosures. Also, $\mathfrak{F}$ is *maximally enclosureless* if it is enclosureless and for each $F \in \mathscr{E} - \mathfrak{F}$ one has that $\mathfrak{F} \cup \{F\}$ is not enclosureless. Let $R_1(H)$ denote the size of the largest (maximally) enclosureless set of H, and let $\rho_1(H)$ denote the size of the smallest maximally enclosureless set of H. Then $\mu_0(H) = R_1(H^*)$ and $\pi_0(H) = \rho_1(H^*)$.

The theorems which follow are simply the duals of Theorems 2 and 4. First note that graph G_3 of figure 3 demonstrates that Theorem 3 does not dualize since the dual hypergraph of a simple graph is not necessarily a simple graph. One has $\alpha_1(G_3) = 4$ and $\rho_1(G_3) = 6$.

THEOREM 5: *If* $\mathrm{H} = (\mathrm{X}, \mathscr{E})$ *is any hypergraph, then* $\mathrm{R}_1(\mathrm{H}) + \alpha_{11}(\mathrm{H}) = \beta_{11}(\mathrm{H}) + \rho_1(\mathrm{H}) = |\mathscr{E}|$.

THEOREM 6: *If* H *is a hypergraph without singleton vertices, then* $\beta_0(\mathrm{H}) \leq \rho_1(\mathrm{H})$.

In view of the previous terminology, it is natural to call hypergraph H a *Dual MK System* or a *KM-System* if H has no singleton vertices and $\beta_0(H) = \rho_1(H)$.

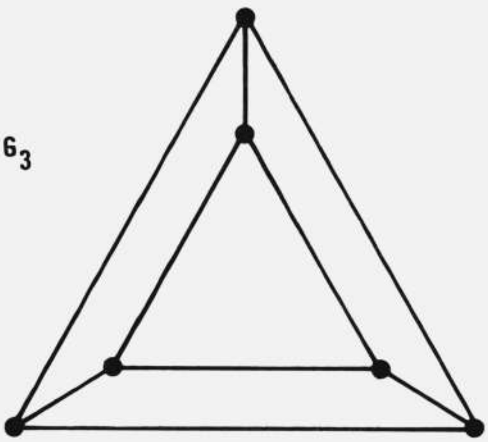

FIGURE 3. *A simple graph with $\rho_1 > \alpha_1$.*

From August, 1974, to August, 1975, I worked as an NRC/NBC Postdoctoral Research Associate in the Applied Mathematics Division at the National Bureau of Standards. While expressing appreciation to all my associates there, special thanks (for different reasons) are due to Lambert Joel and Alan Goldman.

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