Encoded recoupling and decoupling: An alternative to quantum error-correcting codes applied to trapped-ion quantum computation

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A recently developed theory for eliminating decoherence and design constraints in quantum computers, ‘‘encoded recoupling and decoupling,’’ is shown to be fully compatible with a promising proposal for an architecture enabling scalable ion-trap quantum computation [D. Kielpinski *et al.*, Nature (London) **417**, 709 (2002)]. Logical qubits are encoded into pairs of ions. Logic gates are implemented using the Sørensen-Mølmer (SM) scheme applied to pairs of ions at a time. The encoding offers continuous protection against collective dephasing. Decoupling pulses, that are also implemented using the SM scheme directly to the encoded qubits, are capable of further reducing various other sources of qubit decoherence, such as due to differential dephasing and due to decohered vibrational modes. The feasibility of using the relatively slow SM pulses in a decoupling scheme quenching the latter source of decoherence follows from the observed $1/f$ spectrum of the vibrational bath.

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I. INTRODUCTION

In the quest to construct a scalable quantum computer, a recent proposal [1] advocating an array-based approach to quantum computing (QC) with trapped ions appears particularly promising. Ions are stored for later processing in a memory region, then transported to an interaction region, where pairs are coupled in order to enact quantum logic gates. This proposal overcomes some of the design constraints that plagued the original Cirac-Zoller (CZ) ion-trap QC proposal [2], which prevented the latter from becoming fully scalable. In the new proposal an encoding of a single logical qubit into the states $\{|0_L\rangle=|\downarrow\uparrow\rangle,|1_L\rangle=|\uparrow\downarrow\rangle\}$ of two trapped-ion (physical) qubits is used. Quantum logic gates are implemented using the Sørensen-Mølmer (SM) scheme [3,4] (see also related schemes by Milburn *et al.* in Refs. [5,6]), which has the advantage of reduced sensitivity to motional state heating compared to the CZ proposal. The encoding into $\{|\downarrow\uparrow\rangle,|\uparrow\downarrow\rangle\}$ is useful because these states form a decoherence-free subspace (DFS) [7–10] with respect to collective dephasing, a process whereby the environment introduces identical random phase modulations on groups of physical qubits [11]. In the context of the ‘‘quantum charge coupled device’’ (QCCD) proposed in Ref. [1], such a process is one of two dominant sources of decoherence. The DFS encoding reduces the collective dephasing problem by several orders of magnitude [12]. A method to perform universal QC using the SM scheme on these DFS qubits was proposed in Ref. [1], and independently in Ref. [13].

The DFS encoding $\{|\downarrow\uparrow\rangle,|\uparrow\downarrow\rangle\}$ is well known [9,14–16], and its utility against collective dephasing was demonstrated experimentally using photons [17] and trapped ions [18]. The notion of universal QC using a DFS has been explored, in general, in Refs. [16,19–21] for Hamiltonians that always preserve the DFS; in [22–24] using a combination of DFS and active quantum error correction methods [25–31]; and in Ref. [32], using an approach wherein transitions out of the code space are suppressed by continuous observation. Still more generally, the notion of *universal QC, while overcoming decoherence as well as design constraints* has been explored by us and coworkers at a theoretical level in a series of recent papers [33–41]. This theory uses a combination of qubit encoding into a DFS with selective recoupling [34] and dynamical decoupling [42–50] operations, so we refer to it as ‘‘encoded recoupling and decoupling’’ (ERD). The utility of ERD as a general method for quantum simulation, universal QC, and decoherence suppression has also been stressed and explored by Viola [51].

The main focus of ERD so far was on solid state [33–41] and NMR [51,52] QC proposals. Here, we present a *unified treatment* of the ERD ideas, and show that they apply also in an atomic physics setting, namely, the QCCD ion-trap proposal [1]. Specifically, we show here how to perform universal QC on DFS qubits comprised of pairs of trapped ions, by using the SM scheme for quantum logic, in a manner that involves manipulating only pairs of ions at any given time, while always perfectly preserving the DFS encoding (see Brown *et al*. [53] for an interesting alternative set of ion-pairs-only logic gates, which, however, does not preserve the DFS at all times). By applying strong and fast dynamical decoupling (‘‘bang-bang’’ [42,43]) SM pulses we show how to further drastically reduce sources of decoherence beyond collective dephasing. While a qubit is being *stored* an example of such a source of decoherence is deviations from the collective dephasing approximation. While a qubit is being *manipulated* for the purposes of QC, coupling to vibrational modes is necessary [2–6], and decoherence of these vibrational modes due to patch-potential noise is the second dominant source of qubit decoherence [18,54–56]. A method to suppress vibrational mode decoherence (as well as heating, which is not an as serious problem when the SM scheme is used), employing a version of the dynamical decoupling method known as ‘‘parity kicks,’’ was proposed and discussed in detail by Vitali and Tombesi (VT) [46,47]. This method uses a fast and strong modulation of the trapping potential. We present here an alternative decoupling method for suppressing decoherence of ion-trap qubits due to their coupling to decohered vibrational modes, that operates directly on the qubit (spin) states. The feasibility of this scheme, in spite of the relative slowness of the SM pulses, follows from the observed $1/f$ spectrum of the vibrational bath [56,57]. The concentration of most of the bath spectral density in the vicinity of the low, rather than the high-frequency cutoff, implies much relaxed constraints on the decoupling pulses compared to those usually assumed [42,43,46,47]. A full analysis of this result is presented in Ref. [58].

More generally, we show here how *all* sources of decoherence beyond collective dephasing can, in principle, be suppressed using sufficiently strong and fast SM pulses. This includes bath-induced "leakage errors," wherein the system-bath coupling induces transitions into or out of the qubit subspace [37,39,45]. We provide feasibility estimates for the decoupling pulses and find that they are within current experimental reach. The overall picture emerging from this work is that ERD provides a means for a robust, decoherence-resistant implementation of universal QC with trapped ions. Experimental implementation of the ERD method should be possible with current ion-trap technology and we suggest a few experiments.

In this work, we assume that our pulse sequences can be implemented perfectly. This clearly is an assumption that must be relaxed in a more realistic treatment. In a future publication, we will address the problem of imperfect pulses in terms of the theory of composite pulses [59,60]. This theory has been developed in NMR, where experiments involving thousands of (imperfect) pulses are common, and it provides a robust method for dealing with such imperfections (both off-resonant effects and pulse-length errors) in a systematic manner. Another possibility is to consider concatenation with quantum error correcting codes, as done in Refs. [15,22–24] for DFSs, so as to benefit from the fault tolerant implementation of such codes [26,30,31].

The structure of the paper is as follows. In Sec. II, we review the DFS encoding into two spins and the associated logic gates. We show how our previous formulation thereof can be reinterpreted in the context of acting on pairs of trapped ions within the SM scheme. We also present a method for coupling pairs of encoded qubits using pulses that involve controlling only pairs of ions at a time, while always preserving the DFS encoding. In the subsequent sections, we discuss how to reduce decoherence. In Sec. III, we review the decoupling method, emphasizing its application to trapped ion arrays. We then proceed to apply the ERD method: in Sec. IV, we show how to eliminate the residual differential dephasing contribution to decoherence using SM pulses; and, in Sec. V, we discuss how to reduce all further sources of decoherence, including the component that arises due to coupling to decohered motional states. Then, in Sec. VI, we show how to fully implement ERD, i.e., we show how to combine universal QC via recoupling over DFS-encoded qubits with decoherence suppression via encoded decoupling. To make ERD fully effective for trapped ions we suggest to combine it with the VT potential-modulation method. Concluding remarks are presented in Sec. VII.

II. ENCODED UNIVERSAL LOGIC GATES IN ION TRAPS

To fix terminology we first connect the methods developed in Refs. [33–35] to the gates proposed for trapped ions in Ref. [1]. Let X_i, Y_i, Z_i denote the standard Pauli matrices $\sigma_i^x, \sigma_i^y, \sigma_i^z$, acting on the ith physical qubit (we will use both notations interchangeably). In Ref. [33], it was shown that for the code $\{|0_L\rangle = |\downarrow\uparrow\rangle, |1_L\rangle = |\uparrow\downarrow\rangle\}$ the encoded logical operations (involving the first two physical qubits) are

$$\bar{X}_{12} = \frac{1}{2}(X_1X_2 + Y_1Y_2),$$

$$\bar{Y}_{12} = \frac{1}{2}(Y_1X_2 - X_1Y_2),$$

$$\bar{Z}_{12} = \frac{1}{2}(Z_1 - Z_2). \tag{1}$$

These operations form an su(2) algebra (i.e., we think of them as Hamiltonians rather than unitary operators). We use a bar to denote logical operations on the encoded qubits. In Refs. [33–35] these logical operations were denoted by T_{12}^{α}, $\alpha \in \{x, y, z\}$, and a detailed analysis was given on how to use typical solid-state Hamiltonians (Heisenberg, XXZ, and XY models) to implement quantum logic operations using this DFS encoding. E.g., the term $X_1X_2 + Y_1Y_2$ is the spin-spin interaction in the XY model, and $Z_1 - Z_2$ represents a Zeeman splitting. A static Zeeman splitting and a controllable XY interaction can be used to generate a universal set of logic gates, a result that has very recently been applied in the context of spin-based QC using semiconductor quantum dots and cavity quantum electrodynamics [61]. Similar conclusions hold when the XY interaction is replaced by a Heisenberg [34,62,63] or XXZ interaction [35], or even by a Heisenberg interaction that includes an anistropic spin-orbit term [38]. We remark that, as first shown in Refs. [16,19], the various types of exchange interactions can be made universal also without any single-qubit terms (such as a Zeeman splitting), by encoding into three or more qubits [16,64–67], a result that has been termed "encoded universality" [68].

A. Logic gates on two ions encoding a single logical qubit

Sørensen and Mølmer proposed a quantum logic gate that couples two ions via a two photon process that virtually populates the excited motional states of the ions [4]. The SM scheme works well even for ions in thermal motion, while the CZ scheme requires cooling the ions to their motional ground state. The SM scheme involves applying two lasers of opposite detuning δ to the two ions. Ideally, the Lamb-Dicke limit should be satisfied

$$(n+1)\eta^2 \ll 1, \tag{2}$$

where η is the Lamb-Dicke parameter and n is the mean vibrational number. Deviations from the Lamb-Dicke limit lead to fidelity reduction that is proportional to η^4 [4]. The time required to prepare a maximally entangled state using the SM scheme is

$$\tau_{\mathrm{SM}}=\frac{\pi}{\eta\Omega}\sqrt{K}, \tag{3}$$

where Ω is the Rabi frequency and K is an integer [4]. For realistic parameters, in the strong-field limit ($K=1$ in Eq. (12) of Ref. [4]), τ_{SM} can be made as short as 1 μsec.

In Ref. [1] it was shown that the SM two-ion gate can be expressed as follows. The unitary gate $U_2(\theta,\phi_1,\phi_2)$ was introduced, which we here rename $U_{ij}(\theta,\phi_i,\phi_j)$,

$$\begin{aligned}U_{ij}(\theta,\phi_i,\phi_j)&\equiv\exp(i\theta X_{\phi_i}X_{\phi_j})\\&=\cos(\theta)I_iI_j+i\sin(\theta)X_{\phi_i}X_{\phi_j},\end{aligned} \tag{4}$$

where I is the identity operator and

$$X_\phi\equiv X\cos\phi+Y\sin\phi.$$

The phase ϕ_i is the phase of the driving laser at the ith ion, while $\theta\propto\Omega$ and can be set over a wide range of values [4,69]. Introducing the operators

$$\tilde{X}_{ij}\equiv\frac{1}{2}(X_iX_j-Y_iY_j),\quad \tilde{Y}_{ij}\equiv\frac{1}{2}(Y_iX_j+X_iY_j) \tag{5}$$

(denoted R_{ij}^x, R_{ij}^y, respectively in Refs. [33–35]), we can express

$$\begin{aligned}U_{ij}(\theta,\phi_i,\phi_j)=\cos(\theta)\bar{I}+i\sin(\theta)[&\cos(\Delta\phi_{ij})\bar{X}_{ij}\\&+\sin(\Delta\phi_{ij})\bar{Y}_{ij}+\cos(\Phi_{ij})\tilde{X}_{ij}\\&+\sin(\Phi_{ij})\tilde{Y}_{ij}],\end{aligned} \tag{6}$$

where $\Phi_{ij}=\phi_i+\phi_j$. It is simple to check that $\tilde{X}_{ij}$ and $\tilde{Y}_{ij}$ annihilate the code subspace $\{|0_L\rangle=|\downarrow\uparrow\rangle,|1_L\rangle=|\uparrow\downarrow\rangle\}$ and have nontrivial action (as encoded X and Y) on the orthogonal subspace $\{|\downarrow\downarrow\rangle,|\uparrow\uparrow\rangle\}$. Therefore, as also observed in Refs. [1,13], upon restriction to the DFS we can write

$$\begin{aligned}U_{ij}(\theta,\phi_i,\phi_j)&\stackrel{\mathrm{DFS}}{\mapsto}\bar{U}_{ij}(\theta,\Delta\phi_{ij})\\&=\exp(i\theta\bar{X}_{\Delta\phi_{ij}})=\cos(\theta)\bar{I}+i\sin(\theta)\bar{X}_{\Delta\phi_{ij}}.\end{aligned} \tag{7}$$

The fact that $\bar{U}_{ij}$ depends only on the relative phase $\Delta\phi_{ij}$ is crucial: this quantity can be controlled by adjusting the angle between the driving laser and the interatomic axis, as well as by small adjustments of the trap voltages (which, in turn, control the trap oscillation frequency, and hence the ion spacing), whereas it is much harder to control the absolute phase ϕ_i [1,69,70], and hence also Φ_{ij}. This is why the code subspace $\{|\downarrow\uparrow\rangle,|\uparrow\downarrow\rangle\}$ enjoys a preferred status over the subspace $\{|\downarrow\downarrow\rangle,|\uparrow\uparrow\rangle\}$. A thorough theoretical analysis of the approximations leading to the gate $U_{ij}(\theta,\phi_i,\phi_j)$ is given in Ref. [4] (see also Ref. [13] for an abbreviated exposition that emphasizes the connection to computation in a DFS).

Let us establish the connection between the seemingly distinct sets of logic operations in Eqs. (1) and (4). To do so, we only need to use Eqs. (6) and (7), while ignoring the component that annihilates the DFS $(\tilde{X},\tilde{Y})$. Then

$$\begin{aligned}\exp(i\theta\bar{X}_{12})&=U_{12}(\theta,\phi,\phi)=\bar{U}_{12}(\theta,0),\\\exp(i\theta\bar{Y}_{12})&=U_{12}\left(\theta,\phi,\phi+\frac{\pi}{2}\right)=\bar{U}_{12}\left(\theta,\frac{\pi}{2}\right),\\\exp(i\theta\bar{Z}_{12})&=\exp\left(i\frac{\pi}{4}\bar{Y}_{12}\right)\exp(i\theta\bar{X}_{12})\exp\left(-i\frac{\pi}{4}\bar{Y}_{12}\right)\\&=\bar{U}_{12}\left(\frac{\pi}{4},\pi/2\right)\bar{U}_{12}(\theta,0)\bar{U}_{12}\left(-\frac{\pi}{4},\pi/2\right).\end{aligned} \tag{8}$$

The third line follows from the elementary operator identity

$$X_\phi=X\cos\phi+Y\sin\phi=e^{-i\phi Z/2}Xe^{i\phi Z/2}, \tag{9}$$

which holds for any su(2) angular momentum set $\{X,Y,Z\}$, i.e., operators that satisfy the commutation relation $[X,Y]=2iZ$ (and cyclic permutations thereof), in particular, also the encoded operators $\{\bar{X},\bar{Y},\bar{Z}\}$.

This proves the equivalence of the two sets of operators. Using these results and Eq. (4), a more direct connection can be written in terms of the Hamiltonians:

$$\bar{X}_{12}\Leftrightarrow X_\phi X_\phi \tag{10}$$

$$\bar{Y}_{12}\Leftrightarrow X_\phi X_{\phi+\pi/2}, \tag{11}$$

where the equivalence is meant in terms of a projection of the right-hand-side Hamiltonians to the DFS. In the context of ion-trap QC the logic gate $\bar{U}(\theta,\Delta\phi)$ can be performed directly as an SM gate, so it may be more intuitively useful than the $\{\bar{X},\bar{Y},\bar{Z}\}$ set. Equations (8) show that by properly choosing θ and $\Delta\phi_{ij}$ all single DFS-encoded qubit gates can be performed.

B. Entangling gate between pairs of encoded qubits involving four ions

In Ref. [1] the following unitary gate was introduced:

$$\begin{aligned}U_4&=\exp\left(-i\frac{\pi}{4}X_{\phi_1}X_{\phi_2}X_{\phi_3}X_{\phi_4}\right)\\&=\frac{1}{\sqrt{2}}(I_1I_2I_3I_4-iX_{\phi_1}X_{\phi_2}X_{\phi_3}X_{\phi_4})\\&\stackrel{\mathrm{DFS}}{\mapsto}\frac{1}{\sqrt{2}}(\bar{I}_{12}\bar{I}_{34}-i\bar{X}_{\Delta\phi_{12}}\bar{X}_{\Delta\phi_{34}})\\&=\exp\left(-i\frac{\pi}{4}\bar{X}_{\Delta\phi_{12}}\bar{X}_{\Delta\phi_{34}}\right),\end{aligned} \tag{12}$$

where the last equality holds in the subspace spanned by $\{|01\rangle,|10\rangle_{12}\}\times\{|01\rangle,|10\rangle_{34}\}$. This gate, also considered in slightly less general form in Ref. [13], can be used to entangle two DFS qubits using the SM scheme. It involves simultaneous control over two phase differences $\Delta\phi_{12},\Delta\phi_{34}$, and thus control over the motion of two pairs of ions. The case $\Delta\phi_{12}=\Delta\phi_{34}=0$ was used in [70] to demonstrate entanglement of four trapped-ion qubits, but this choice is not unique.

We now come to an important point that was not addressed in Ref. [1]: in order for the DFS encoding to offer protection against collective dephasing during the exection of the entangling gate, *collective dephasing conditions must prevail over all four ions*. To see this, note that a differential dephasing term such as $(Z_1-Z_3)\otimes B$ (where B is a bath operator) does not commute with U_4, so that if such a term exists during gate execution then the DFS will not be preserved, according to a theorem in Ref. [19]. On the other hand, collective dephasing over all four ions, expressed by a system-bath coupling of the form $(\Sigma_{i=1}^4 Z_i)\otimes B$, does commute with U_4, so that in this case the DFS is preserved [19]. While deviations from collective dephasing over pairs of ions have been shown experimentally to be small [18], this may not be the case over the length scales involving four ions [12]. We discuss in Sec. IV, how to create such extended collective dephasing conditions.

Taken together, the results in this section show how universal QC can be implemented using trapped ions by applying the SM scheme to pairs of ions at a time, each encoding a DFS qubit. The DFS encoding takes care of protecting the encoded information against collective dephasing. We now move on to a discussion of how to reduce additional source of decoherence.

III. DYNAMICAL DECOUPLING PULSES AND THEIR APPLICATION TO TRAPPED IONS

Let us briefly review the decoupling technique, as it pertains to our problem (for an overview see, e.g., Ref. [51]). Decoupling, as proposed by Viola and Lloyd [42,43], relies on the ability to apply *strong and fast* pulses, in a manner which effectively averages the system-bath interaction Hamiltonian H_{SB} to zero. A quantitative analysis was first performed in Refs. [42,43] for pure dephasing in the linear spin-boson model (which corresponds to the ohmic case of the Caldeira-Leggett model [71]): $H_{SB}=\gamma\sigma^z\otimes B$, where B is a Hermitian boson operator. The analysis was recently extended to the nonlinear spin-boson model, with similar conclusions about performance [49]. Imperfections in the pulses were considered in [48], and it was shown that an optimal value for the pulse period can be found. Since the decoupling pulses are *strong* one ignores the evolution under H_{SB} while the pulses are on, and since the pulses are *fast* one ignores the evolution of the bath under its free Hamiltonian H_B during the pulse cycle. The simplest example of eliminating an undesired unitary evolution $U=\exp[-it(H_{SB}+H_B)]$, is the "*parity-kick sequence*" [42,43,46]. Suppose, we have at our disposal a fully controllable interaction generating a gate R such that "*R conjugates U*": $R^\dagger UR=U^\dagger$. Then, the sequence $UR^\dagger UR=I$ serves to eliminate U. A simple example of a parity kick sequence is the following. Assume we can turn on the single-qubit Hamiltonian ΩX_j for a time $\pi/2\Omega$. This generates the single-qubit gate $X_j=i\exp[-i(\pi/2)X_j]$. Suppose that $H_{SB}=\Sigma_{i=1}^N\Sigma_{\alpha\in\{x,y,z\}}\gamma_i^\alpha\sigma_i^\alpha\otimes B_i^\alpha$. Each term in H_{SB} either commutes or anticommutes with X_j. If a term A in H_{SB} anticommutes with X_j then the evolution under it will be conjugated by the gate X_j: $X_j\exp(-iA\Delta t)X_j=\exp(-iX_jAX_j\Delta t)=\exp(iA\Delta t)$. This allows for selectively removing this term using the parity-kick cycle, which we write as $[\Delta t,X_j,\Delta t,X_j]$. Reading from right to left, this notation means: apply X_j pulse, free evolution for time Δt, repeat. Suppose that we can also apply the single-qubit gate Y_j. Then, since every system factor in the above H_{SB} contains a single-qubit operator, it follows that we can selectively keep or remove each term in H_{SB} by using the parity-kick cycle. Note, however, that without additional symmetry assumptions, this procedure, if used to eliminate *all errors*, requires a number of pulses that is exponential in the number of qubits N [44,48]. The reason is that without symmetry assumptions we will need at least two noncommuting single-qubit operators per qubit (e.g., X, Y), and we will need to concatenate decoupling pulse sequences. Below we show how to dynamically generate such symmetries, in a way that avoids this exponential scaling (for a discussion of this point see the Conclusions section). Note that in the analysis of the parity-kick cycle we ignored H_{SB} and H_B, while R was operating; this is justified by the *strength* assumption. The bath Hamiltonian H_B commutes with the applied pulses, but its effect is very important since $[H_B,H_{SB}]\neq 0$, in general. Therefore, if the bath has spectral components at frequencies higher than the inverse of the interval between decoupling pulses, then the bath density matrix will be modulated by phases that are essentially random, and this effect will show up as decoherence (for a quantitative analysis see Refs. [42,43,47–49]. Hence, it is commonly assumed that the pulse interval Δt should be small compared to the inverse of the high-frequency cutoff ω_c of the bath spectral density $I(\omega)$ [42,43], which also sets the scale of the bath-induced noise correlation time t_c (the *speed* assumption). It can be shown that the overall system-bath coupling strength γ_{SB} is then renormalized by a factor $\Delta t\omega_c$ after a cycle of decoupling pulses [44], or that the bath-induced error rate is reduced by a factor proportional to $(\Delta t/t_c)^2$ [48]. Using a Magnus expansion [60], it can be shown that there is a hierarchy of decoupling schemes, whereby γ_{SB} is renormalized by a factor $(\Delta t\omega_c)^k$, where $k\geqslant 1$ is the order of the decoupling scheme [44]. The implication for single-qubit dephasing, $H_{SB}=\frac{1}{2}\gamma_{SB}Z\otimes B$ (B is a dimensionless bath operator), is that the dephasing time T_2 is increased by a factor $1/(\Delta t\omega_c)^{2k}$ [72]. Thus, it seems crucial to be able to apply pulses at intervals $\Delta t\ll 1/\omega_c$. However, as shown first by Viola and Lloyd [42], and then by VT in their quantitative analysis of a vibrational mode linearly coupled to a boson bath, a finite-temperature bath sets another, thermal time scale that must be beat in order for the decoupling method to work [47]. Let the system-bath coupling be

$$H_{SB}^{\mathrm{vib}}=\gamma\sum_k (ab_k^\dagger+a^\dagger b_k), \tag{13}$$

where a (b_k) is an annihilation operator for the system (kth bath) vibrational mode, and γ is the (for simplicity uniform) energy damping rate. In the context of trapped ions the bath is provided by fluctuating patch-potentials (due, e.g., to randomly oriented domains at the surface of the electrodes or adsorbed materials on the electrodes) [56]. Then VT showed that the decoupling pulse interval (in fact, the entire cycle time) must be shorter also than the thermal decoherence time

$$t_{\mathrm{dec}}(T)=\{\gamma[1+2n(T)]\}^{-1},$$

where $n(T)=[e^{\hbar\omega_0/k_BT}-1]^{-1}$ is the mean vibrational number of the system oscillator at thermal equilibrium with temperature T, and ω_0 is the frequency of the oscillator, i.e., the system is described by the harmonic-oscillator Hamiltonian $H_S=\hbar\omega_0 a^\dagger a$. Thus, the time scale condition for successful decoupling is

$$\Delta t \ll \min\{1/\omega_c, t_{\mathrm{dec}}(T)\}.$$

As shown in the VT analysis, the time scale $t_{\mathrm{dec}}(T)$ is especially relevant for vibrational mode decoherence in ion traps, which as already mentioned above, is responsible for qubit decoherence during quantum logic-gate operations.

However, for trapped ions experimental evidence so far points to a $1/f^\alpha$ spectrum for the vibrational bath over a range $1-100$ MHz ([56] p. 5), implying that there is no clear high-frequency cutoff ω_c. Encouragingly, in a recent experiment involving a charge qubit in a small superconducting electrode (Cooper-pair box), a version of parity-kick decoupling was successfully used to suppress low-frequency energy-level fluctuations (causing dephasing) due to $1/f$ charge noise [73]. This suggests that decoupling can help even in the absence of a clear cutoff frequency. Recent theoretical results strongly support this observation [58]: Since for $1/f$ noise most of the bath spectral density $I(\omega)$ is concentrated in the low, rather than the high end of the frequency range, it turns out that dynamical decoupling depends more sensitively on the lower than on the upper cutoff. In particular, it is shown in Ref. [58] that the suppression of dephasing is more effective when the noise originates in a bath with $1/f$ spectrum than in the Ohmic case, owing to the abundance of infrared modes in a bath with $1/f$ spectrum.

In spite of the apparent $1/f^\alpha$ spectrum in trapped ions, VT used a cutoff estimate of $\omega_c \leqslant 100$ MHz [47], and showed that suppression of vibrational decoherence can be accomplished by *pulsing the oscillation frequency* ω_0 *of the ion chain* (i.e., by pulsing the trapping potential), provided $\Delta t < 1/\omega_c \sim 10$ ns, *and* $T \leqslant 10$ mK. It should be emphasized that there is currently no relevant experimental data to support the 10 ns figure. We use it in our discussion below as an example, rather than an estimate.

Given the estimate in Sec. II A of $\tau_{\mathrm{SM}} \gtrsim 1$ μs for the SM gate, it is clear that we cannot hope to satisfy the strict $\Delta t < 10$ ns time scale requirement which would be needed in order to use decoupling directly on the qubit, rather than the vibrational modes, assuming the VT value of ω_c. However, the theoretical analysis [58] and the success of parity-kick decoupling in the presence of $1/f$ noise in the charge qubit case [73] suggests that it may well be worthwhile to apply decoupling pulses on the qubit *in addition* to, or perhaps instead of, the VT trapping-potential-modulation scheme.

Now let us comment on the strength assumption. Here, we must make sure that the amplitude of the decoupling pulses, Ω, can be made much stronger than the system-bath interaction γ in Eq. (13). The heating rate $t_{\mathrm{dec}}(T)$ from the vibrational ground state of the ion chain is experimentally measurable, and $\gamma=[(1+2n(T))t_{\mathrm{dec}}(T)]^{-1}$. Typical values of $n(T)$ range from 10^2 to 10^4 as T ranges from 10 mK to 1 K [47]. Experimental measurements of $1/t_{\mathrm{dec}}(T)$ are very sensitive to trap geometry, secular frequency, and size [56], and range from a few Hz to a few tens of KHz [56,74]. On the other hand, one can have Ω as high as 1 MHz [75]. Thus, the strength assumption can be comfortably satisfied. This does come at a price, however, since in the strong-field limit the SM gate is perturbed by a term that yields direct, off-resonant coupling of the qubit $|\uparrow\rangle$ and $|\downarrow\rangle$ states without changes in the vibrational motion [4]. This is a *unitary* gate error that decreases the gate fidelity by $(N/2)(\Omega/\delta)^2$, where N is the number of ions participating in the gate ([4] Table II). This forces us to be in a parameter regime, where $\Omega \ll \delta$. In principle, it is possible to exactly cancel this effect if the duration of the laser pulses is chosen so that both Eq. (3) and the condition $\tau_{\mathrm{SM}}=K'\pi/\delta$ are satisfied, where K' is an integer and δ is the detuning. However, in the context of decoupling we will also need to satisfy conditions such as $\Omega\tau_{\mathrm{SM}}=\pi/m$, where m is an integer. Putting these conditions together yields

$$\Omega\frac{K'\pi}{\delta}=\pi/m \Rightarrow \delta=mK'\Omega,$$

$$\Omega\frac{\pi}{\eta\Omega}\sqrt{K}=\pi/m \Rightarrow \eta=m\sqrt{K}.$$

While there is no problem with the first of these, the second condition implies that we cannot be in the Lamb-Dicke limit, Eq. (2). Therefore exact cancellation is not possible in our case, and we must resort to $\Omega \ll \delta$ in order to keep the fidelity reduction to a minimum. On the other hand, the kind of unitary error that is caused by off-resonant coupling can be corrected by optimal control pulse-shaping methods [76], resonant cancellation [77], and by a ‘‘dressed qubit’’ method [78].

Finally, we note that fluctuations of various experimental parameters, such as intensity and phase fluctuations of the exciting lasers, can cause pure dephasing of the vibrational modes, in addition to the dissipative coupling described above [79]. Clearly, the success of decoupling strategies hinges on strong suppression of such fluctuations, as in the threshold theorem of fault tolerant quantum error correction [25,26,30,31].

To conclude, the discussion in this section indicates that the experimental viability of decoupling schemes in ion traps is rather promising, although it is hard to quantitatively estimate their success at this point. In the following sections, the analysis will be carried out at a more abstract level, emphasizing the algebraic conditions for a successful implementation of ERD. In the end, it will be up to an experiment to decide the usefulness of the proposed schemes.

IV. CREATING COLLECTIVE DEPHASING CONDITIONS USING DECOUPLING PULSES: REDUCING DECOHERENCE DURING STORAGE

One of the important advantages of the DFS encoding $\{|\downarrow\uparrow\rangle,|\uparrow\downarrow\rangle\}$ is that it is immune to collective dephasing. However, other sources of decoherence inevitably remain. In this and the following section, we algebraically classify all additional decoherence effects and show how they can be eliminated. It should be noted that the methods we propose in this section require that ions are kept cold and are accessible to lasers also during the storage period, which entails a modification of the QCCD design of Ref. [1].

A. Creating collective dephasing on a pair of ions

First, let us analyze the effect of breaking the collective dephasing symmetry, by considering a system-bath interaction of the form

$$H_{SB}^{\mathrm{deph}(2)}=Z_1\otimes B_1^z+Z_2\otimes B_2^z,$$

where B_1^z,B_2^z are arbitrary bath operators. This describes a general dephasing interaction on two qubits, and we can expect this to be the case during *storage* of trapped-ion qubits in the QCCD proposal. The source of such dephasing during storage is long wavelength, randomly fluctuating ambient magnetic fields [18], that randomly shift the relative phase between the qubit $|\uparrow\rangle$ and $|\downarrow\rangle$ states through the Zeeman effect. The interaction can be rewritten as a sum over a collective dephasing term Z_1+Z_2 and another, differential dephasing term Z_1-Z_2, that is responsible for errors on the DFS:

$$H_{SB}^{\mathrm{deph}(2)}=(Z_1+Z_2)\otimes B_{\mathrm{col}}^z+(Z_1-Z_2)\otimes B_{\mathrm{dif}}^z.$$

Here, $B_{\mathrm{col}}^z=(B_1^z+B_2^z)/2$ and $B_{\mathrm{dif}}^z=(B_1^z-B_2^z)/2$. If B_{dif}^z were zero then there would only be collective dephasing and the DFS encoding would offer perfect protection [80]. However, in general, $B_{\mathrm{dif}}^z\neq 0$, and the DFS encoding will not suffice to offer complete protection.

The crucial observation is that, since $Z_1-Z_2\propto\overline{Z}_{12}$ [recall Eq. (1)], the offending term causes *logical* errors on the DFS [37]. As shown in Refs. [36,51], then the problem of $B_{\mathrm{dif}}^z\neq 0$ can be solved using a series of pulses that symmetrize $H_{SB}^{\mathrm{deph}(2)}$ such that only the collective term remains [81]. To do so note that since the offending term $\propto\overline{Z}_{12}$, it anticommutes with $\overline{X}_{12}=\frac{1}{2}(X_1X_2+Y_1Y_2)$. At the same time $\overline{X}_{12}$ commutes with Z_1+Z_2. This allows us to flip the sign of the offending term by using a pair of $\pm\pi/2$ pulses in $\overline{X}_{12}$, while leaving only the collective term. Evolution with the flipped sign followed by unaltered evolution leads to cancellation of the offending term. Specifically [36]

$$e^{-iH_{SB}\tau}e^{-i(\pi/2)\overline{X}_{12}}e^{-iH_{SB}\tau}e^{i(\pi/2)\overline{X}_{12}}=e^{-i(Z_1+Z_2)\otimes B_{\mathrm{col}}^z 2\tau},$$

or, in ion-trap terms

$$e^{-iH_{SB}\tau}\overline{U}_{12}\left(-\frac{\pi}{2},0\right)e^{-iH_{SB}\tau}\overline{U}_{12}\left(\frac{\pi}{2},0\right)=e^{-i(Z_1+Z_2)\otimes B_{\mathrm{col}}^z 2\tau}, \tag{14}$$

where $\overline{U}_{ij}(\theta,\Delta\phi_{ij})$ was defined in Eq. (7), and we used the identification found in Eq. (8). This equation means that the system-bath coupling effectively looks like collective dephasing at the end of the pulse sequence. Thus, the system is periodically (every 2τ) projected into the DFS.

In order for the procedure described in Eq. (14) to work, the SM gate $\overline{U}_{12}(\pm\pi/2,0)$ must be executed at a time scale faster than the cutoff frequency associated with the fluctuating magnetic fields causing the differential dephasing term in $H_{SB}^{\mathrm{deph}(2)}$. This cutoff has not yet been characterized experimentally, but the decay rate of the DFS-encoded state of two ions has been measured to be 2.2 KHz [18]. Using this as a rough estimate for the cutoff frequency, we see that the procedure of Eq. (14) is likely to be attainable with fast ($\tau_{\mathrm{SM}}\approx 1\ \mu$s) SM pulses.

B. Creating collective dephasing on a block of four ions

So far we have discussed creation of collective dephasing conditions on a single DFS qubit. However, as mentioned in Sec. II B, it is essential for the reliable execution of an entangling logic gate to have collective dephasing over all four ions participating in the gate, even if only two are coupled at a time. A procedure for creating collective *decoherence* conditions over blocks of 3, 4, 6, and 8 qubits was given in Ref. [36]. Here, we show how to do the same for a block of 4 qubits with collective dephasing.

Let us start with a general dephasing Hamiltonian on N ions, and rewrite it in terms of nearest-neighbor sums and differences,

$$\begin{aligned}H_{SB}^{\mathrm{deph}}&=\sum_{i=1}^{N} Z_i\otimes B_i\\&=\sum_{j=1}^{N/2}(Z_{2j}+Z_{2j-1})\otimes B_{2j}^{+}+(Z_{2j}-Z_{2j-1})\otimes B_{2j}^{-},\end{aligned}$$

where $B_{2j}^{\pm}\equiv(B_{2j}\pm B_{2j-1})/2$. As noted above, $Z_{2j}-Z_{2j-1}\propto\overline{Z}_{2j-1,2j}$, so that to eliminate all nearest-neighbor differences of the form $(Z_{2j}-Z_{2j-1})$ we can use the collective decoupling pulse $X_{nn}=\otimes_{j=1}^{N/2}e^{i(\pi/2)\overline{X}_{2j-1,2j}}$,

$$\begin{aligned}&e^{-iH_{SB}\tau}X_{nn}e^{-iH_{SB}\tau}X_{nn}^{\dagger}\\&\quad=\exp\left[-i2\tau\sum_{j=1}^{N/2}(Z_{2j}+Z_{2j-1})\otimes B_{2j}^{+}\right],\end{aligned}$$

or, in ion-trap terms

$$e^{-iH_{SB}\tau}\left[\bigotimes_{j=1}^{N/2}\overline{U}_{2j-1,2j}\left(-\frac{\pi}{2},0\right)\right]e^{-iH_{SB}\tau}$$
$$\times\left[\bigotimes_{j=1}^{N/2}\overline{U}_{2j-1,2j}\left(\frac{\pi}{2},0\right)\right]$$
$$=\exp\left[-i2\tau\sum_{j=1}^{N/2}(Z_{2j}+Z_{2j-1})\otimes B_{2j}^{+}\right].$$

The next step is to eliminate next-nearest-neighbor differential terms. To this end, let us rewrite the outcome of the X_{nn} pulse in terms of sums and differences over blocks of four ions

$$\sum_{j=1}^{N/2}(Z_{2j}+Z_{2j-1})\otimes B_{2j}^{+}$$
$$=\sum_{j=1}^{N/2}[Z_{2j+2}+Z_{2j+1}+Z_{2j}+Z_{2j-1}]\otimes B_{2j}^{+,+}$$
$$+\sum_{j=1}^{N/2}[(Z_{2j+2}-Z_{2j})+(Z_{2j+1}-Z_{2j-1})]\otimes B_{2j}^{+,-},$$

where $B_{2j}^{+,\pm}\equiv(B_{2j+2}^{+}\pm B_{2j}^{+})/2$. The first term on the right hand side contains only the desired block-collective dephasing over four ions. The last term contains undesired differential dephasing terms that we wish to eliminate. But these terms once again have the appearance of encoded Z operators, between next-nearest-neighbor ion pairs. Therefore, we need to apply a second collective pulse $X_{nnn}=\otimes_{j=1}^{N/2}e^{i(\pi/2)\overline{X}_{2j-1,2j+1}}e^{i(\pi/2)\overline{X}_{2j,2j+2}}$, that applies encoded X operators on these ion pairs. At this point, we are left just with collective dephasing terms on blocks of four ions, as required

$$\exp\left[-i2\tau\sum_{j=1}^{N/2}(Z_{2j}+Z_{2j-1})\otimes B_{2j}^{+}\right]$$
$$\times\left[\bigotimes_{j=1}^{N/2}\overline{U}_{2j-1,2j+1}\left(-\frac{\pi}{2},0\right)\overline{U}_{2j,2j+2}\left(-\frac{\pi}{2},0\right)\right]$$
$$\times\exp\left[-i2\tau\sum_{j=1}^{N/2}(Z_{2j}+Z_{2j-1})\otimes B_{2j}^{+}\right]$$
$$\times\left[\bigotimes_{j=1}^{N/2}\overline{U}_{2j-1,2j+1}\left(\frac{\pi}{2},0\right)\overline{U}_{2j,2j+2}\left(\frac{\pi}{2},0\right)\right]$$
$$=\exp\left[-i4\tau\sum_{j=1}^{N/2}(Z_{2j+2}+Z_{2j+1}+Z_{2j}+Z_{2j-1})\otimes B_{2j}^{+,+}\right].\qquad(15)$$

This pulse sequence is important to ensure that collective dephasing conditions will prevail during the execution of logic gates between DFS qubits, as emphasized in Sec. II B.

To conclude, the procedures discussed in this section provide a means for *engineering collective dephasing conditions in an ion-trap experiment*. We propose here to implement these symmetrization schemes experimentally. Moreover, we propose to combine them with the logic gates described in Sec. II. How to do this efficiently is discussed in Sec. VI below.

V. REDUCTION OF ALL REMAINING DECOHERENCE ON A SINGLE DFS QUBIT DURING LOGIC-GATE EXECUTION

The reduction of differential dephasing errors, as in the preceding section, is particularly relevant for storage errors. However, this is only the first step. Additional sources of decoherence may take place during storage, and, in particular, during the execution of logic gates, the most dominant of which is qubit decoherence due to coupling to decohered vibrational modes, as discussed above. It is useful to provide a complete algebraic classification of the possible decoherence processes. This will allow us to see what can be done using SM-decoupling pulses. To this end, let us now write the system-bath Hamiltonian on two physical qubits in the general form

$$H_{SB}=H_{\mathrm{Leak}}+H_{\mathrm{Logi}}+H_{\mathrm{DFS}},$$

where

$$H_{\mathrm{DFS}}=\mathrm{Span}\left\{\frac{ZI+IZ}{2},\frac{XY+YX}{2},\frac{XX-YY}{2},ZZ,II\right\},$$
$$H_{\mathrm{Leak}}=\mathrm{Span}\{XI,IX,YI,IY,XZ,ZX,YZ,ZY\},$$
$$H_{\mathrm{Logi}}=\mathrm{Span}\left\{\overline{X}=\frac{XX+YY}{2},\overline{Y}=\frac{YX-XY}{2},\overline{Z}=\frac{ZI-IZ}{2}\right\},\qquad(16)$$

where I is the identity operator, $XZ\equiv X_1Z_2$ (etc.), and where Span means a linear combination of those operators tensored with bath operators. The 16 operators in Eq. (16) form a complete basis for all two-qubit operators. This classification, first introduced in Ref. [37], has the following significance. The operators in H_{DFS} either vanish on the DFS, or are proportional to identity on it. In either case their effect is to generate an overall phase on the DFS, so they can be safely ignored from now on. The operators in H_{Leak} are the *leakage errors*: terms that cause transitions between states inside and outside of the DFS. A universal and efficient decoupling method for eliminating such errors, for arbitrary numbers of (encoded) qubits was given in Ref. [39]. Finally, the operators in H_{Logi} have the form of logic gates on the DFS. However, these are undesired logic operations, since they are coupled to the bath, and thus cause decoherence.

It is worthwhile to already emphasize that the operator $YI+IY\in H_{\mathrm{Leak}}$ is of particular importance: As shown in Ref.

[4], Eq. (43), this is the operator that describes qubit decoherence due to motional decoherence during application of the SM gate.

In the preceding section, we showed how to eliminate the logical error $\bar{Z}$, but we see now that this was only one error in a much larger set. To deal with the additional errors it is useful at this point to introduce a more compact notation for the pulse sequences. We denote by $[\tau]$ a period of evolution under the free Hamiltonian, i.e., $U(\tau)\equiv \exp(-iH_{SB}\tau)\equiv[\tau]$, and further denote

$$P\equiv \bar{U}_{12}\left(-\frac{\pi}{2},0\right)=\exp\left(-i\frac{\pi}{2}\bar{X}_{12}\right).$$

Thus, Eq. (14) can be written as

$$\exp[-i(B_1^z+B_2^z)(Z_1+Z_2)\tau]=[\tau,P,\tau,P^\dagger].$$

Notice that this is an example of a parity-kick pulse sequence.

As a first step in dealing with the additional errors, note that the symmetrization procedure $[\tau,P,\tau,P^\dagger]$ can in fact achieve more than just the elimination of the differential dephasing Z_1-Z_2 term. Since $\bar{X}_{12}$ also anticommutes with $\bar{Y}_{12}=\frac{1}{2}(Y_1X_2-X_1Y_2)\in H_{\mathrm{Logi}}$, if such a term appears in the system-bath interaction it too will be eliminated using the same procedure.

So far we have used a $(\pi/2)\bar{X}_{12}$ pulse. Interestingly, the Hamiltonian $\bar{X}_{12}$ can also be used to eliminate all leakage errors [37]. To see this, note that $\bar{U}_{12}(\pm\pi,0)=\exp(\pm i\pi\bar{X}_{12})=Z_1Z_2$. This operator anticommutes with *all* terms in H_{Leak}. Hence, it too can be used in a parity-kick pulse sequence, that will eliminate all the leakage errors. In particular, this pulse sequence will eliminate qubit decoherence due to motional decoherence, i.e., the error $YI+IY\in H_{\mathrm{Leak}}$.

At this point we are left with just a single error: $\bar{X}_{12}\otimes B$ itself, in H_{Logi}. Clearly, we cannot use a pulse generated by $\bar{X}_{12}$ to eliminate this error. Instead, to deal with this error we need to introduce one more pulse pair that anticommutes with $\bar{X}_{12}$, e.g., $\exp[\pm i(\pi/2)\bar{Y}_{12}]=\bar{U}_{12}(\pm\pi/2,\pi/2)$.

Let us now see how to combine all the decoherence elimination pulses into one efficient sequence. First, we introduce the abbreviations

$$\Pi\equiv\bar{U}_{12}(\pm\pi,0)=\exp(\pm i\pi\bar{X}_{12})=\Pi^\dagger=PP,$$

$$Q\equiv\bar{U}_{12}\left(-\frac{\pi}{2},\frac{\pi}{2}\right)=\exp\left(-i\frac{\pi}{2}\bar{Y}_{12}\right),$$

$$\Lambda\equiv\bar{U}_{12}\left(\pm\pi,\frac{\pi}{2}\right)=\exp(\pm i\pi\bar{Y}_{12})=\Lambda^\dagger=QQ. \quad (17)$$

As argued above, the π pulse Π eliminates H_{Leak}:

$$\exp[-i(H_{\mathrm{Logi}}+H_{\mathrm{DFS}})2\tau]=[\tau,\Pi,\tau,\Pi].$$

This may be sufficient in practice, since as argued above this pulse sequence eliminates the $YI+IY$ term, and the DFS encoding takes care of collective dephasing. Thus, we expect that using cycles of two pulses we can almost entirely eliminate the two most important sources of decoherence. This expectation, of course, depends on the time scale requirement for decoupling being satisfied, as discussed in detail in Sec. III above. In practice, it may well be advantageous to combine the DFS encoding and $[\tau,\Pi,\tau,\Pi]$ pulse sequence with the VT method of pulsing the trapping potential [46,47].

Now let us discuss adding the extra pulses needed to achieve full decoherence elimination. The $\pi/2$ pulse P eliminates $\bar{Y}$ and $\bar{Z}$ in H_{Logi}. Combining this with the sequence for leakage elimination we have the sequence of four pulses

$$e^{-i(H_{\mathrm{DFS}}+\bar{X}\otimes B_{\bar{X}})4\tau}=[U(\tau)\Pi U(\tau)\Pi]P^\dagger[U(\tau)\Pi U(\tau)\Pi]P$$
$$=[\tau,\Pi,\tau,P,\tau,\mathbf{\Pi},\tau,P^\dagger] \quad (18)$$

(where we have used $\Pi P^\dagger=P$, $\Pi P=P^\dagger$).

If we wish to entirely eliminate decoherence then we are left just with getting rid of the logical error due to $\bar{X}$. To eliminate it we now combine with the $\bar{Y}$ direction, $\pi/2$ pulse, Q,

$$e^{-iH_{\mathrm{DFS}}8\tau}$$
$$=[U(\tau)\Pi U(\tau)PU(\tau)\Pi U(\tau)P^\dagger]$$
$$\times Q^\dagger[U(\tau)\Pi U(\tau)PU(\tau)\Pi U(\tau)P^\dagger]Q$$
$$=[\tau,\Pi,\tau,P,\tau,\Pi,\tau,P^\dagger,Q^\dagger,\tau,\Pi,\tau,P,\tau,\Pi,\tau,P^\dagger,Q], \quad (19)$$

which takes ten pulses. Unfortunately, it is not possible to compress this further, since $P^\dagger Q=(i\bar{X})(-i\bar{Y})=i\bar{Z}$ and $P^\dagger Q^\dagger=-i\bar{Z}$, neither of which cannot be generated directly (in one step) from the available gate $\bar{U}_{ij}(\theta,\Delta\phi_{ij})=\cos\theta\bar{I}+i\sin\theta\bar{X}_{\Delta\phi_{ij}}$. Finally, note that, in principle, the last pulse sequence is applicable also to other QC proposals, such as NMR and quantum dots.

One important caveat (mentioned in Sec. III above) is that, because we need very strong and fast pulses, our gate operation may become imperfect. Specifically, off-resonant coupling and deviations from the Lamb-Dicke approximation may become important. The former introduces a term $XI+IX$ into the Hamiltonian generating the $U_{ij}(\theta,\phi_i,\phi_j)$ gate ([4] Sec. III A), which can cause *unitary* leakage errors from the DFS. These can in turn be reduced using the methods in Refs. [76–78]. Whether the decoupling method we have proposed offers an improvement will have to be put to an experimental test.

VI. COMBINING LOGIC GATES WITH DECOUPLING PULSES

So far we have discussed computation using the encoded recoupling method (Sec. II), and encoded decoupling (Secs IV and V). We now put the two together in order to obtain the full ERD picture. At least two methods are available for combining quantum computing operations with the sequences of decoupling pulses we have presented above. For a general analysis of this issue see Ref. [21].

A. Method of fast and strong gates

The decoupling pulse sequences given in Sec. V "stroboscopically" create collective dephasing conditions at the conclusion of each cycle. As noted above, this is equivalent to a periodic projection into the DFS. This property allows for "stroboscopic" quantum computation at the corresponding projection times [21]. Here, the computation pulses need to be synchronized with the decoupling pulses, and inserted at the end of each cycle. The amount of time available for implementation of a logic gate is no more than the bath correlation time $\tau_c = 2\pi/\omega_c$. Assuming the dominant decoherence contributions not accounted for by the DFS encoding to come from differential dephasing (setting the τ_c time scale) and $1/f$ noise, and that we already assumed that we can use pulses with interval $\Delta t \ll \tau_c$, it is consistent to assume that we can then also perform logic gates on the same time scale.

B. Method of fast and weak gates

There may be an advantage to using fast but weak pulses for the logic gates, while preserving the fast and strong property of the decoupling pulses. To see how to combine logic gates with decoupling in this case, let us denote by $H_S = X_{\phi_i} X_{\phi_j}$ the controllable system Hamiltonian that generates the entangling gate $U_{ij}(\theta,\phi_i,\phi_j)$ [recall Eq. (4)]. Suppose first that we turn on this logic-gate generating Hamiltonian in a manner that is neither very strong nor very fast, so that the system-bath interaction is not negligible, while H_S is on (this obviously puts less severe demands on experimental implementation). Then, the corresponding unitary operator describing the dynamics of system plus bath is

$$\tilde{U}(t) = \exp[-it(H_S + H_{SB} + H_B)].$$

Now, *if we choose H_S so that it commutes with the decoupling pulses*, then we can show that after decoupling

$$\tilde{U}(t) \mapsto \exp[-i2t(H_S + H_B)], \tag{20}$$

provided t is sufficiently small. Tracing out the bath then leaves a purely unitary, decoherence-free evolution on the system. To prove this, assume we have chosen t' and the decoupling Hamiltonian H'_S so that (i) $\exp(-it'H'_S) H_{SB} \exp(it'H'_S) = -H_{SB}$ (the parity-kick transformation) and (ii) $[H'_S, H_S] = 0$. Then

$$\begin{aligned}
&\tilde{U}(t) e^{-it'H'_S} \tilde{U}(t) e^{-it'H'_S} \\
&= \tilde{U}(t) e^{-it[H_S + e^{-it'H'_S} H_{SB} e^{it'H'_S} + H_B]} \\
&= e^{-it(H_S + H_{SB} + H_B)} e^{-it(H_S - H_{SB} + H_B)} \\
&= e^{-\{2it(H_S + H_B) + t^2([H_{SB}, H_S] + [H_{SB}, H_B]) + O(t^3)\}},
\end{aligned}$$

where we have used the Baker-Campbell-Hausdorff formula, $\exp(\alpha A)\exp(\alpha B) = \exp\{\alpha(A+B) + (\alpha^2/2)[A,B] + O(\alpha^3)\}$.

Setting $H_S = \Omega S$ and $H_{SB} = \gamma_{SB} S' \otimes B$, we have the condition $t \ll 1/\sqrt{\Omega \gamma_{SB}}$, in order to be able to neglect the $O(t^2)$ term $[H_{SB}, H_S]$. Using $\Omega \sim 1$ MHz, $\gamma_{SB} \sim 10$ KHz as in Sec. III, we find $t \ll 10\ \mu$s. However, the more stringent constraint comes from the $[H_{SB}, H_B]$ term, since H_B is not bounded for a harmonic oscillator. A more careful analysis then shows the familiar conclusion, that the bath should not be allowed to evolve for longer than its correlation time [42,43,47]. Hence, the actual requirement may still the far more stringent condition $t \ll 1/\omega_c \leq 10$ ns for the decoupling pulse interval; see Sec. III. This cannot be satisfied with SM pulses, but in this case we can resort to the VT potential modulation method. When we do this in conjunction with SM decoupling pulses we can be sure that Eq. (20) is an excellent approximation. On the other hand, the requirements for a $1/f$ bath spectral density are far less stringent and may be satisfied even with SM pulses alone [58]. Furthermore, for the rotation angle $\theta = \Omega t$ describing the computation we have $\theta \ll \sqrt{\Omega/\gamma_{SB}} \leq 10$, which means that there is no restriction on applying large rotations.

Let us now show how to efficiently combine logic operations and decoupling pulses. For simplicity consider only the case where we can neglect the $\bar{X}$ error, i.e., our decoupling sequence is the four pulse one given in Eq. (18). Suppose we wish to implement a logical X operation, i.e., $\exp(-i\theta\bar{X}_{12})$. Recall [Eq. (10)] that this involves turning on the Hamiltonian $H_S^X = \Omega_X X_\phi X_\phi \overset{\mathrm{DFS}}{\mapsto} \Omega_X \bar{X}_{12}$ between two physical qubits. Because the decoupling pulses $P = \exp[-i(\pi/2)\bar{X}_{12}]$ and $\Pi = \exp(\pm i\pi\bar{X}_{12})$ are generated in terms of the same Hamiltonian, they commute with H_S^X, while eliminating H_{SB} (except for the terms in H_{SB} that have trivial action on the DFS). Thus, the conditions under which Eq. (20) holds are satisfied. This allows us to insert the logic gates into the four free evolution periods involved in the pulse sequence of Eq. (18). Thus, the full pulse sequence that combines creation of collective dephasing conditions with execution of the logic gate is

$$e^{-it(\Omega_X \bar{X}_{12} + H_{\mathrm{DFS}})} = \tilde{U}(t/4)\Pi\tilde{U}(t/4)P\tilde{U}(t/4)\Pi\tilde{U}(t/4)P^\dagger, \tag{21}$$

with $\tilde{U}(t) = \exp[-it(H_S^X + H_{SB} + H_B)]$, and which, using the DFS encoding, is equivalent to the desired $\exp(-i\theta\bar{X}_{12})$. This involves eight control pulses, four of which are of the fast- and strong type (those involving P and Π), and four of which must be fast, but need not be so strong that we can neglect H_{SB}.

If we wish to implement logical Y operation, i.e., $\exp(-i\theta\bar{Y}_{12})$, then we cannot now use P and Π, since they anticommute with $\bar{Y}_{12}$ and will eliminate it. Instead we should use decoupling pulses generated in terms of $\bar{Y}_{12}$, which will also have the desired effect of eliminating H_{Leak}, as well as $\bar{X}$ and $\bar{Z}$ logical errors, while commuting with the $\bar{Y}$ logic operations (and for this reason, of course, cannot eliminate $\overline{Y}$ errors). These are just the Q and Λ pulses defined in Eq. (17). In ion-trap terms this implies [recall Eq. (11)] turning on the Hamiltonian $H_S^Y = \Omega_Y X_\phi X_{\phi+\pi/2} \overset{\mathrm{DFS}}{\mapsto} \Omega_Y \overline{Y}_{12}$ between two physical qubits. Thus

$$e^{-it(\Omega_Y \overline{Y}_{12} + H_{\mathrm{DFS}})} = \widetilde{U}(t/4)\Lambda \widetilde{U}(t/4) Q \widetilde{U}(t/4)\Lambda \widetilde{U}(t/4) Q^\dagger, \tag{22}$$

with $\widetilde{U}(t) = \exp[-it(H_S^Y + H_{SB} + H_B)]$, and which, using the DFS encoding, is equivalent to the desired $\exp(-i\theta\overline{Y}_{12})$.

Finally, to generate single DFS-qubit rotations about an arbitrary axis we can combine Eqs. (21) and (22) according to the Euler angles construction. Given that Eqs. (21) and (22) each take eight pulses, the Euler angle method will generate an arbitrary DFS-qubit rotation in at most 24 pulses.

Concerning gates that entangle two DFS qubits, the situation is more involved, since now the next-nearest-neighbor pulses in Eq. (15), that create the collective dephasing conditions on four ions, do not all commute with the U_4 gate of Eq. (12). Therefore, here we must resort to the strong and fast method of the previous section, i.e., we need to synchronize the U_4 pulses with the end of the decoupling pulse sequence.

Taken together, the methods described in this section provide an explicit way to implement universal QC using trapped ions in a manner that offers protection against all sources of qubit decoherence, using a fast and strong (or fast and weak) version of the SM scheme, possibly in combination with the VT potential modulation method.

VII. DISCUSSION AND CONCLUSIONS

We have proposed a method of encoded recoupling and decoupling for performing decoherence-protected quantum computation in ion traps. Our method combines the Sørensen-Mølmer scheme for quantum logic gates with an encoding into ion-pair decoherence-free subspaces (each pair yielding one encoded qubit), and sequences of recoupling and decoupling pulses. The qubit encoding protects against collective dephasing processes, while the decoupling pulses symmetrize all other sources of decoherence into a collective dephasing interaction. The recoupling pulses are used to implement encoded quantum logic gates, either during or in between the decoupling pulses. All pulses are generated directly using the SM scheme. We have provided numerical estimates of the feasibility of our scheme, which seem quite favorable. In order to achieve full protection against all decoherence it may be necessary to supplement ERD with the potential modulation method due to Vitali and Tombesi, in order to reduce vibrational mode decoherence. However, it may be worthwhile to test ERD without potential modulation first, as a significant reduction in decoherence can already be expected according to the results presented here. This is so because the vibrational bath has been found experimentally to have a $1/f^\alpha$ spectral density [57], and theory predicts that in such a case decoupling may be possible under moderate timing constraints [58].

As mentioned in Sec. III, the dynamical decoupling method requires an exponential number of pulses if the most general form of decoherence is to be suppressed, that can couple arbitrary numbers of qubits to the environment (total decoherence [10]). This exponential scaling is avoided here by focusing on decoherence elimination inside blocks of *finite* size (e.g., at most four ions), where arbitrary decoherence is allowed. However, we have implicitly assumed that there are no decoherence processes coupling different blocks. This is a reasonable assumption for trapped ions, where the different blocks can be kept sufficiently far apart until they need to be brought together in order to execute interblock logic gates. When this happens, ERD can still be efficiently applied on the temporarily larger block.

It may be questioned whether there is any advantage in using ERD compared to methods of active quantum error correcting codes (QECC). Both ERD and QECC are capable of dealing with arbitrary decoherence processes, and are fully compatible with universal quantum computation. There are two main advantages to ERD: First, we need only two ions per qubit, compared to a redundancy of five ions per qubit to handle all single-qubit errors in QECC [27]. So far experiments involving trapped ions have used up to four ions [70], so that this encoding economy is a distinct advantage for near-term experiments. Second, our method is directly compatible with the SM scheme for logic gates in ion traps. On the other hand, it is not clear how to directly use SM gates for QECC. These are general features of ERD: economy of encoding redundancy and use of only the most easily controllable interactions. The disadvantage of ERD compared to QECC is that there does not exist, at this point, a result analogous to the threshold theorem of fault tolerant quantum error correction. This means that we cannot yet guarantee full scalability of ERD as a stand-alone method, because we do not yet know how to compensate for imperfect pulses. However, in principle it is always possible to concatenate ERD with QECC, as done, e.g., for DFS with QECC in Refs. [15,22–24], and then the standard fault tolerance results apply. As mentioned in the Introduction, we expect that the theory of composite pulses [59,60] will also play a key role in this further development of ERD.

Finally, we note that ERD is a general method, that is not limited to trapped ions. We hope that the methods proposed here will inspire experimentalists to implement encoded recoupling and decoupling in the lab, thus demonstrating the possibility of fully decoherence-protected quantum computation, in particular, using trapped ions.

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