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Encounters of binaries – I. Equal energies

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Summary. Gravitational encounters of pairs of binaries have been studied numerically. Various cross-sections have been calculated for qualitative final results of the interaction and for energy transfer between the binding energy and the centre of mass kinetic energy. The distribution of the kinetic energies, resulting from the gravitational collision, were found to be virtually independent of the impact velocity in the case of collision of hard binaries. It was found that one out of five collisions, which are not simple fly-by's, leads to the formation of a stable three-body system.

1 Introduction

In the dynamical theory of dense stellar systems, e.g. the globular clusters, the phenomenon of a core collapse has been studied in recent years (see e.g. Spitzer & Mathieu 1980; Hénon 1972; Heggie 1979). It is not certain whether we see the result of a collapse in real clusters, but it has shown up in Monte Carlo simulations of the globulars. In these calculations the stars are assumed to have smooth orbits except during encounters. However, the encounters considered are usually only two-body encounters. Only Spitzer & Mathieu (1980) tried to take into account encounters between binaries and single stars as well as the binary–binary collisions. However, in the absence of reliable numerical data or theoretical knowledge they could only consider those interactions of pairs of binaries in which one binary is much smaller in size than the other. Therefore they could make use of the knowledge of three-body interactions (see e.g. Saslaw, Valtonen & Aarseth 1974; Heggie 1972, 1975; Valtonen & Heggie 1979; Fullerton & Hills 1982). It has been suggested that the binary formation rate through three-body interactions is negligible in large stellar systems (Spitzer & Hart 1971; Aarseth & Heggie 1976) and therefore binaries can exist only if they were present initially. Whatever the cause for the existence of binaries, we know that binaries exist in our Galaxy, which certainly is a large stellar system. If stellar systems have a large content of binaries, many of the encounters in them will be binary–binary encounters. Consequently the study of interactions between binaries is of utmost importance. The only four-body calculations known to the author are those of Harrington (1974) and Saslaw *et al.* (1974). These authors, however, calculated only a few hundred examples and do not give any cross-sections.

In this paper more detailed information about the results of almost 5000 binary-collision experiments are given.

2 Numerical methods

2.1 INTEGRATION METHODS

The calculations were started using a computer code originally written by S. J. Aarseth and D. C. Heggie. This code uses variable steplength fourth-order polynomial interpolation for the gravitational force (Aarseth 1971) together with a two-body regularization technique (Heggie 1973).

However, it soon became clear that many of the cases were such that a method of this type was not the best possible. This is because after a short period of strong interaction the four-body system develops a hierarchial configuration where, in addition to a close binary, we have two extended, eccentric elliptical orbits. Unlike the problem of three bodies, the four-body system is too complicated to make use of simple Keplerian approximations in such a situation. Because all the 'brute force' methods spend most of the computing time to follow the close binary, the only way out seems to be the use of a variation of constants method. Therefore a completely new code was written. It contains three alternative methods of integration:

(a) the hierarchy method, where one or more of the relative motions in the system are integrated as a perturbed Keplerian orbit, using the variation of parameters method (Herrick 1972, p. 118–120),

(b) direct integration of the Newtonian equations of motion,

(c) the general N-body regularization method of Heggie (1974).

The code attempts to choose the best method for a given configuration. The criteria for choosing the method are as follows:

(a) The system is taken to be hierarchal if the closest pair is elliptic and the ratio of the perturbing acceleration to the two-body acceleration is smaller than a given constant (usually $\sim 3 \times 10^{-4}$). In addition the code checks whether the configuration has two binaries (the other may be hyperbolic) which follow a relative two-body orbit themselves or whether a third star is in a Keplerian orbit relative to the close binary while a fourth star is in motion with respect to the centre of mass of the three other stars. After the treatment of the configuration has been chosen the three orbits are integrated either by brute force or using the variation of parameters of a Keplerian orbit, the selection depending upon the perturbations.

The rather conservative requirement for the smallness of the perturbation and the complication introduced by the choices between brute force and variation of parameters were found necessary because of computational economy. This is due to the fact that the variation of parameters method is so complicated that the steplength must be up to 10 times longer than in other methods before it is advantageous to use it.

(b) Direct integration of the Newtonian equations of motion is employed if the system is not hierarchal and when it does not require regularization (we use the numerical value of the potential energy as the criterion). The advantage of this method is the simplicity of the equations of motion which makes the calculation rapid even if the steps are short.

(c) The global regularization method of Heggie is utilized if the methods described above are not suitable. In addition the code makes use of this method if in one of the other integrators an unacceptable loss of precision occurs.

In each part of the code the configuration is examined after a few integration steps (usually 10) and in branches (a) and (b) the accuracy of the calculation is checked through the conservation of the total energy. If the precision is accepted, the present configuration is saved and computation continues using the method decided by a selection subroutine. If a loss of precision is observed then the calculation returns to the last good configuration and proceeds using the regularization part of the code.

In all the above mentioned parts of the code the actual numerical integration is carried out by means of a Bulirsch–Stoer (1966) method, which is a one-step method and therefore well suited for situations where the integration parameters are frequently changed.

2.2 CLASSIFICATION OF THE FINAL CONFIGURATIONS

The following four types of final motion were assumed possible:

(1) two binaries in a hyperbolic relative orbit,
(2) one escaper and a hierarchal three-body system,
(3) one binary and two escapers,
(4) total disruption into four single stars.

The computational algorithm for discriminating between the various possibilities is as follows:

(a) find the closest pair (say 1 and 2), find the star closer to the first pair (say 3) and, the remaining star is then the star number 4.

(b) Calculate the Keplerian orbital elements of the pairs (1, 2) and (3, 4) and the elements of their relative orbit. If it is found that the pairs have elliptical orbits and the relative orbit between them is hyperbolic, and the distance between the binaries is sufficiently large, then this is classified to be case (1).

(c) Calculate the elements of star 3 with respect to the binary (1, 2) and the elements of the star 4 with respect to the centre of mass of the stars (1, 2, 3). If now (1, 2) has elliptical orbit and the orbit of the star 3 is an ellipse around the binary, and the pericentre distance is sufficiently larger then this is classified to be case (2), provided the orbit of the star 4 is hyperbolic and its distance is sufficiently large. If (1, 2) is a binary, but other orbits are hyperbolic, and the energy of the three-body system formed by the binary and the stars 3 and 4 is positive, then the case is classified to be (3), provided that the separations are large enough to make the asymptotic relation between the total energy of the system and the calculated Keplerian energies valid with a certain tolerance. If all the orbits are hyperbolic, the total energy is positive and the potential energy $\ll$ kinetic energy, then total disruption is deemed to have happened (case 4). If none of the above alternatives is found to be true, then the configuration is classified as unclear (0) and its integration is continued.

Because the treatment of a four-body system by means of three Keplerian orbits is never accurate, the above classification scheme may sometimes give erroneous results. However, it seems that asymptotically the criteria will give correct classes, (i.e. if the systems are integrated over a long enough period of time) the possible exceptions belonging to a set of measure zero. In addition, a misclassification, if it occurs, must be 'nearly correct' in the sense that there are in phase space neighbouring points belonging to the obtained classification. It is easy to understand that in phase space there are points (or hypersurfaces) where very small displacements can alter the qualitative nature of the final state. For example, it is easy to imagine a situation where the body 3 is in a 'branch point' from which it can escape or remain as a companion of either the binary or the escaping body 4.

3 Numerical experiments

3.1 GENERAL

A total of 4800 numerical experiments have been performed. In these experiments all the masses of the stars were equal (= 1), similarly the initial semi-major axes were chosen equal (= 1). The orientations and phases of the colliding binaries were uniformly randomized. For the eccentricities the distribution $\psi(e) = 2e$ was used (see Heggie 1972). The impact parameters (s) of the collision orbits were similarly randomized to given an equal number of impacts on equal areas, i.e. $\psi(s) = 2s/S_{\max}^2$. Here $S_{\max}$ denotes the maximum values used for the impact parameter. The gravitational constant was put equal to one, and therefore the total binding energy of the binaries ($= 2 \times m^2/2a = 1$) is the unit of energy. Computations have been carried out for six values of the impact energy T_∞, which is the kinetic energy of the binaries at infinity (in the centre of mass coordinate system). The values of T_∞ were 0.1, 0.25, 0.5, 0.75, 1.0 and 1.5. Values very close to 0 were not adopted because they would require nearly infinite values for the maximum impact parameter. Also all the cross-sections are very large and sensitive to changes in the neighbourhood of zero. These difficulties can partly be avoided by using instead the combination s^2T_∞. However, this convention was not adopted here.

We shall call the outcome of a four-body experiment either a 'collision' or a 'fly-by'. The latter refers to systems where the two-binary configuration survives, while all other cases are classified as collisions. The fly-by's are mostly weakly interacting systems but include also some strong interactions. In addition, we use the phrase 'strong interaction' for those cases where the two-binary configuration was not produced rapidly (the test for this is explained in the last paragraph of this section).

Two different series of experiments were run. First, 500 experiments for each value of impact energy were computed so that the maximum impact parameter gave a value of 6 length units for the maximum pericentre distance of the relative orbits of the binaries. From these experiments where the outcomes were mostly fly-by's it was possible to deduce the differential cross-sections for the energy transfer between the binding energy of the binaries and the kinetic energy of their centres of mass. Approximate values for the cross-section of the collision phenomenon could also be derived.

Because most of the cases in the first series of experiments were simple fly-by's, another series of calculations was carried out. First an upper bound for the maximum impact parameter at which a collision is possible was estimated (in fact overestimated) from the results of the first series. Using this maximum impact parameter, experiments were carried out until 300 strong interactions were obtained. The immediate fly-by's were rejected, but the total number of experiments was counted, to make possible the evaluation of cross-sections.

In all experiments the integration was started at a distance of 15 units between the binaries, the pericentre time was calculated, and integration was carried out over the time interval of 2.5 times the pericentre time, after which the configuration was examined for the first time to find whether the result was two binaries in a hyperbolic relative orbit (a fly-by) or not. To avoid spending too much computation time in calculating some exceptionally difficult cases, the computations were organized as follows: After the above described starting runs all the configurations were saved and in later runs the yet unclear cases were integrated, one by one, over a given amount of time, after which the present configuration was saved again. The successive runs were stopped when the number of unclear cases was small enough.

3.2 BINARY FLY-BY'S

In this section we study in some detail the first sample of experiments where most of the cases were simple fly-by's.

Table 1 gives the number of cases of various types obtained in this series of experiments. The meaning of the symbols in this stable is as follows: T_∞ = the impact energy; E is total energy; $S_{\max}$ is maximum impact parameter; Type is the type index as explained in Section 2.2; Σ is the total number of experiments.

Table 1. The quantitative distribution of various types of final configurations in the first series of experiments.

T_∞	E	S_{max}	TYPE (0)	(1)	(2)	(3)	(4)	Σ
0.1	-0.9	16.6	7	358	38	97	-	500
0.25	-0.75	11.5	6	387	28	79	-	500
0.50	-0.50	9.2	-	430	5	65	-	500
0.75	-0.25	8.2	2	452	2	44	-	500
1.00	0.00	7.7	1	450	-	49	-	500
1.50	0.50	7.2	-	466	-	31	3	500

Obviously the number of strong interactions in these samples is too small for a detailed study. However, these experiments seem to give material for drawing conclusions about the dynamical behaviour of binaries in distant encounters, and for the comparison of some properties of the fly-by's and collisions. To study to what extent the upper bounds of the impact parameter can be considered sufficiently large, the mean values of the increase of the total binding energy ΔE_B and its maximum and minimum were listed versus the impact area (five zones of equal area). We obtained that in the two last columns the means of ΔE_B were of order $\sim 10^{-3}$, the maxima and minima of order $\sim 10^{-2}$, and in some cases the sign of $\langle \Delta E_B \rangle$ seemed to fluctuate. Therefore we may conclude that the upper bounds of the impact parameter can be considered to be sufficiently large.

The dynamically most interesting quantity is the differential cross-section for the energy transfer between the centres of mass kinetic energy and the binding energy of the binaries. The numerical interpretation of this quantity is a bit difficult for the fly-by cases because, as is clear *a priori,* this cross-section diverges for a small energy transfer. However, a numerical evaluation of the quantity $|\Delta E_B|(d\sigma_1/dE)$ (where $d\sigma_1$ is the above mentioned cross-section) shows immediately that this quantity is regular, and non-zero at small energy transfers, too. Therefore, we can write

$$\frac{d\sigma_1}{d\Delta E_B} = \frac{f(\Delta E_B)}{|\Delta E_B|}, \quad (1)$$

where the function f is bounded and non-zero at $\Delta E_B = 0$. These functions are illustrated in Fig. 1 for the various samples. Here all the two-binary cases are included, even those arising from a strong interaction. This, of course, is in accordance with our definition of a fly-by. Moreover, from the point of view of application to stellar dynamics it is immaterial what are the details of the interaction, only the final results and distributions are of importance. In addition, we give in Table 2 the values of $f(\Delta E_B)$ around zero [$f(0)$], and the

Table 2. Values of $f(\Delta E_B)$ around zero [$f(0)$] and the integrated quantities $\int \Delta E_B d\sigma$. For details see the text.

T_∞	f(0)	$\int \Delta E_B d\sigma_1$	$\int \Delta E_B d\sigma_3$
0.1	49.	+8.8	+74.6
0.25	26.	-0.8	+15.6
0.50	15.	-8.1	+4.5
0.75	16.	-6.3	+6.3
1.0	14.	-5.8	-1.9
1.5	12.	-4.0	-4.3

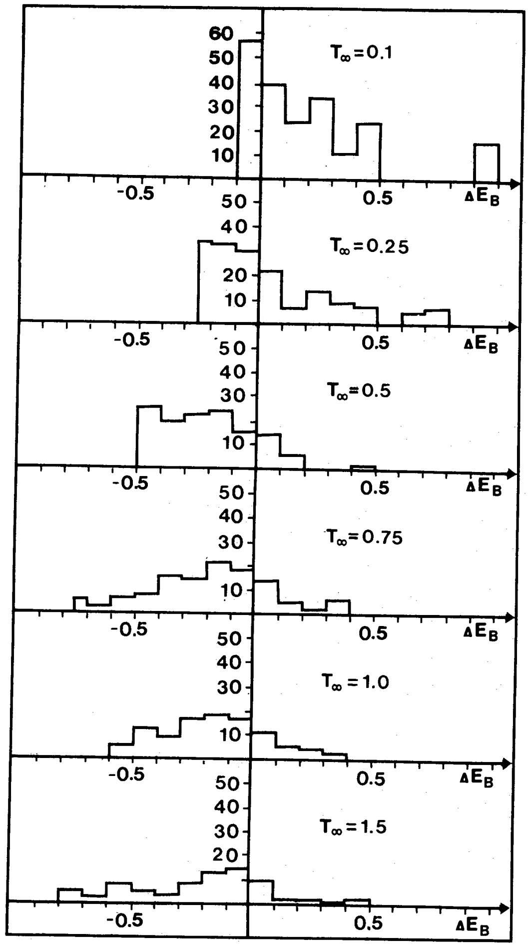

Figure 1. The functions $f(\Delta E_B) = |\Delta E_B| \times d\sigma_1/d\Delta E_B$, where $d\sigma_1$ is the differential cross-section for the increase of the total biding energy. The histograms show the interval means.

quantities $\int \Delta E_B d\sigma_1$ which are proportional to the mean flow of energy from the potential energy of the binaries to the kinetic energy of their centres of mass. For comparison we give the same integral quantities calculated for the cases of two escapers. These quantities are denoted similarly by $\int \Delta E_B d\sigma_3$. If these integrals are adopted as a measure of the energetic importance of the phenomena in question, we see that at low impact energies the collisions are far more important than fly-by's. Thus, for hard binaries the fly-by's are likely to be only of secondary importance in cluster dynamics. However, for the more energetic impacts the two phenomena are rather comparable in importance. It is not completely clear how the numbers behave in the limit $T_\infty \to 0$ because the values $f(0)$ seem to diverge. However, the divergence appears to be slower than T_∞^{-1}, which, as we shall see in the next section, is the divergence rate for the cross-section of a collision.

The qualitative behaviour of both phenomena looks similar in the way that in both cases, the total binding energy seems to increase when the impact velocity becomes small. This result is similar to that obtained by Heggie (1972, 1975) in his study of encounters of binaries and single stars. This is easy to understand, because it is simply a consequence of the conservation of total energy. Namely, if we think of a case of zero impact energy, then any change, whatever its cause, must lead to an increase of binding because the energy conservation makes all other alternatives impossible. The process at hand does not contain singularities, and therefore this behaviour must extend, because of continuity, somewhat to the region of non-zero impact energies, too. The disrupting tendency at large impact energies on the other hand, is understandable in terms of the tendency of energy equipartition between degrees of freedom.

3.3 BINARY COLLISIONS

3.3.1 The cross-section and cut-off of collisions

In this section we study in some detail the samples of 300 strong interactions which were mentioned earlier.

One of the interesting parameters, in addition to the cross-section for the collision, is the maximal value of the impact parameter at which a collision may occur. In Table 3 we see this parameter (S_x), obtained from the actual numerical experiments, together with the product $S_x \sqrt{T_\infty}$. In the same table the values for the cross-section of a collision (σ_c) are given. When evaluating σ_c, all the other cases except those where the outcome was two binaries (which are fly-by's according to our definition) were considered collisions.

Table 3. The cross-section σ_c of a collision, the maximal impact parameter S_x at which a collision occurred and the product $S_x\sqrt{T_\infty}$, for various impact energies T_∞.

T_∞	σ_c	S_x	$S_x\sqrt{T_\infty}$
0.1	251.1	12.75	4.03
0.25	90.3	8.05	4.03
0.50	38.7	6.05	4.28
0.75	23.1	4.48	3.88
1.00	17.5	4.07	4.07
1.50	12.4	3.14	3.85

From the approximate constancy of the numbers in the last column we draw the conclusion that the angular momentum seems to be the important parameter which determines the cut-off of the collision. (It is not simply the pericentre distance because the duration of the encounter is important, too.) Taking into account the fact that the observed values are certainly smaller than the real values of the quantity in question (because of the finite number of experiments) we may write for S_x the simple estimate

$$S_x \simeq 4.3/\sqrt{T_\infty}. \qquad (2)$$

Because the values obtained for S_x are surprisingly small, especially for large values of the impact energy, it may be of some interest to see how the number of collisions behave as a function of the impact distance. Therefore we give in Table 4 the number density of collisions at different impact zones of equal area. This table thus represents the relative collision

Table 4. The number density of the collisions at different impact zones. The total number is normalized to 100 collisions. The maximum impact parameter is calculated from formula (2).

T_∞ / $(S/S_x)^2$	0.–0.1	0.1–0.2	0.2–0.3	0.3–0.4	0.4–0.5	0.5–0.6	0.6–0.7	0.7–0.8	0.8–0.9	0.9–1.0
0.1	21.8	19.9	14.3	14.7	11.3	6.4	6.4	3.0	2.3	0.0
0.25	20.3	24.6	17.0	13.0	8.3	8.7	5.1	2.9	0.7	0.0
0.50	25.4	23.2	14.7	11.4	11.0	8.1	3.7	2.2	0.0	0.4
0.75	28.9	23.7	18.1	7.6	9.2	6.0	3.6	2.4	0.4	0.0
1.0	24.6	25.7	14.3	11.0	11.0	4.4	3.7	2.2	2.9	0.0
1.5	26.9	21.5	17.8	13.5	10.2	4.7	3.3	2.2	0.0	0.0

probability as a function of the impact area. The values used for the maximum impact parameters for these collision runs were in all cases clearly greater than S_x. Thus, because the probability of a collision seems to drop to a very low value at $s \sim S_x$, we conclude that the adopted ranges for the impact parameters were satisfactorily large. However, this does not prove that a collision would be impossible at greater impact distances, but shows that the probability is very small.

The behaviour of the collision probability seems to be similar at all energies. Taking into account the fact that the impact zones in this table are normalized such that the average angular momentum is constant along each column we one again conclude the importance of angular momentum for the collision probability. From formula (2) we can proceed to a conclusion concerning the cross-section (σ_c) of collision. We have the inequality

$$\sigma_c < \pi S_x^2,$$

and therefore we can write

$$\sigma_c = C(T_\infty)/T_\infty, \tag{3}$$

where $C(T_\infty)$ is a bounded function. To get a simple analytical approximation for the cross-section, the function $C(T_\infty)$ was fitted by a simple exponential expression. In fact we have

$$\sigma_c \simeq \frac{17.9}{T_\infty}\,[1 + 0.7\exp(-5.0\,T_\infty)]. \tag{4}$$

This formula fits better than 5 per cent (the statistical uncertainty is of the same order).

3.1.2 Rates of various final configurations resulting from strong interactions

In Table 5 we list the numbers of various classes of final motions resulting from the strong interactions. If these results are applied to cluster dynamics, we may assume that the unclear cases (0) and possibly also the cases of one escaper and a stable three-body system (2) can finally be classified to the class of two escapers (3). Although most of the cases classified to be stable three-body systems really are stable (the separation of inner and outer orbits are large), this is true only in the absence of outer perturbations. In a star cluster the passing stars will modify eccentricities leading finally to disruption and thus to class (3).

3.1.3 The formation of stable three-body systems

It is of interest to note the rather high rate of formation of three-body systems. In this section we now explain in more detail the criteria used to decide when a three-body system is stable.

Table 5. Distribution of the various qualitative classes of final motion in the case of a strong interaction. The symbols are the same as in Table 1.

	TYPE					
T_∞	(0)	(1)	(2)	(3)	(4)	Σ
0.10	14	33	73	180	-	300
0.25	8	22	61	209	-	300
0.50	2	28	35	235	-	300
0.75	2	50	15	233	-	300
1.00	-	28	5	267	-	300
1.50	-	25	2	221	52	300

Table 6. The number of three-body systems of different degrees of stability and the maximum value of the stability index.

T_∞	$N_{2.5}$	$N_{3.5}$	(q_2/a_1) max
0.1	73	57	311.
0.25	61	46	345.
0.50	35	24	312.
0.75	15	8	88.
1.00	5	1	3.9
1.50	2	1	3.6

According to Harrington (1974) the best single index of stability for a three-body system containing one binary is the ratio of the minimum distance q_2 of the outer orbit to the semi-major axis a_1 of the inner orbit, the stability cut-off being somewhere between $q_2/a_1 \sim$ 2.5–3.5, depending upon the orientation of the orbits. Similar results have been obtained by Valtonen (1975). In the computations the value $q_2/a_1 > 2.5$ was chosen as the stability criterion. In Table 6 the number of cases where the index is > 2.5 ($N_{2.5}$) and the number of those cases when it is > 3.5 ($N_{3.5}$) are listed together with the observed maximum value of the ratio q_2/a_1. It is interesting to note that the rate of very stable cases ($N_{3.5}$) drops almost linearly with increasing energy. From the table one can read the rate of formation of stable three-body systems to be $\simeq 1/5$ in collisions of very hard binaries.

3.1.4 Formation of two binaries in strong interactions

It may be of some interest to see what are the properties of the binaries resulting from the strong interactions. Because the number of cases is rather small we cannot present detailed distributions and therefore we tabulate only the mean binding energy $\langle E_B \rangle$ and the number of cases in which exchange of components occurred (N_{ex}). These quantities are given in Table 7. At the smallest impact energy the rate of exchanges is seen to be 50 per cent. Therefore we may conclude that these cases arise from a rather strong interaction. However, it is somewhat surprising that the mean bind energy is still close to unity which is the value of the quantity before the interaction. The decrease of binding with increasing impact velocity and the simultaneous falling off of the rate of exchanges show that at greater energies the cases more and more often are fly-by's where one of the binaries nearly disrupts, whereas the other remains almost the same. Such an occasion can easily happen if the components of one of the binaries are very close to each other at the moment of interaction.

Table 7. The number of two-binary cases (N_B), the mean binding energy $\langle E_B \rangle$ and the number of component exchanges (N_{ex}), observed in the strong interaction process.

T	N_B	$\langle E_B \rangle$	N_{ex}
0.1	33	1.08	16
0.25	22	0.91	8
0.50	28	0.61	10
0.75	50	0.49	9
1.00	28	0.42	6
1.50	25	0.58	2

Because of the low rate of the process (1), these cases are not very important in stellar dynamics. Moreover, in the earlier consideration of the fly-by process all the two-binary cases were included. Therefore the influence of the binary pairs arising from strong interactions is already taken into account in the energy-exchange cross-section presented in Fig. 1.

3.1.5 *The distribution of energies in the case of two escapers*

In case (3) of our classification, the total energy of the system may be divided into four components.

$$E = T_B + T_3 + T_4 - E_B, \tag{5}$$

where T_B is the kinetic energy of the centre of mass of the binary, T_3 and T_4 are the kinetic energies of the escapers (in the centre of mass coordinate system) and E_B is the binding energy of the binary. From the point of view of stellar dynamics the kinetic energies are of prime importance. However, the conservation of total energy together with the value of E_B determines the total kinetic energy, and therefore the distribution of E_B is an interesting

Table 8. The percentage distribution of $x = E_B - E_i$ at various impact energies. The row marked N shows the number of experiments. The last row gives the averages $\langle x \rangle$.

T_∞	0.1	0.25	0.50	0.75	1.00	1.50
Δx/N	277.	288.	300.	277.	316.	252.
0.00-0.10	8.3	9.7	6.7	5.4	1.3	1.6
0.10-0.20	14.8	15.3	15.7	6.5	3.8	3.6
0.20-0.30	17.7	15.3	15.0	10.1	7.0	8.3
0.30-0.40	10.5	12.8	12.0	12.6	8.5	14.3
0.40-0.50	9.7	9.0	7.7	11.9	13.6	15.5
0.50-0.60	6.9	8.0	7.0	9.0	9.8	13.9
0.60-0.70	3.2	4.9	6.3	7.6	10.8	11.5
0.70-0.80	5.4	5.6	5.0	5.1	8.9	5.2
0.80-0.90	3.6	3.5	3.3	5.1	7.3	4.8
0.90-1.00	4.3	1.4	3.7	1.4	4.7	4.4
1.00-1.10	1.8	1.4	1.7	3.6	2.5	3.2
1.10-1.20	1.4	2.4	2.0	2.2	3.5	3.2
1.20-1.30	2.9	0.7	2.0	3.2	2.8	0.8
1.30-1.40	1.4	3.1	0.7	2.9	3.5	0.8
1.40-1.50	1.1	1.0	0.7	1.4	1.6	1.6
1.50-1.60	1.1	1.4	1.7	1.1	1.6	0.4
1.60-1.70	0.4	0.3	1.7	2.5	1.3	2.8
1.70-1.80	1.1	1.4	0.7	1.1	0.3	0.4
1.80-1.90	0.0	0.0	1.3	1.1	1.9	0.0
1.90-2.00	1.1	0.7	0.3	1.4	0.3	0.4
2.00-2.10	0.0	0.7	2.0	1.4	0.0	0.8
2.10-2.20	0.4	0.3	0.0	0.7	0.6	0.8
2.20-2.30	0.0	0.3	0.3	0.7	0.6	0.0
2.30-2.40	1.1	0.0	0.7	0.0	0.3	0.0
2.40-2.50	0.4	0.0	0.0	0.0	0.0	0.0
<x>	0.6168	0.5452	0.6309	0.7491	0.8315	0.6939

quantity. On the other hand, it is convenient to handle the quantity $x = E_B - E_i$, where E_i is the smallest possible value for the binding energy [$E_i = \max(0, -E)$].

For negative total energy our quantity x is thus equal to the total kinetic energy (excluding the relative motion of the binary components). To obtain distributions as reliable as possible the cases of class (3) were collected from both experimental samples. The percentage distributions of x are tabulated in Table 8 where also the averages of x are given. A study of this table shows that the distribution of x seems to approach rapidly to a certain limiting distribution when the impact energy decreases. At the three lowest impact energies we are unable to see, within the statistical fluctuations, any clear differences in the distributions. However, somewhere between the values $T_\infty \simeq 0.5$–0.75 there seems to be a region of transition where the behaviour of the system changes rapidly. From the point of view of stellar dynamics the hard binaries (i.e. collisions with low impact energy) are the most interesting. Therefore we study these low energy collisions in more detail.

To minimize the statistical fluctuations an average distribution of x over the three lowest impact energy samples was computed. This distribution is illustrated in Fig. 2. To get a simple analytical approximation for the distribution, a function of the form $(1 + ax^2)^{-3/4}$ was taken for the probability $P(E_B - E_i > x)$. Here the exponent $-3/4$ was chosen by assuming that the simple phase-space volume estimate for the probability density of the energy of a binary ($\propto E_B^{-2.5}$, see also Heggie 1975) is applicable for large binding energy. Numerical experiments then showed that for the parameter a the value $a \simeq 10$ is optimal. The maximum difference between the sample distribution of $F = (1 + 10x^2)^{-3/4}$ and the corresponding expectation value was ~1.5 times the standard deviation (10 intervals). Our fitting formula may not be perfect but the deviations are not large and the formula has the advantage of simplicity. We see from Table 8 that the mean value of x, as obtained in the numerical calculation, is $\langle x \rangle \simeq 0.6$. However, our fitting formula gives a greater value, $\langle x \rangle \simeq 0.83$. The reason is that the integral for the mean value of x converges slowly and thus the high energy but low probability values contribute much to the analytic mean. Another possible reason is that the asymptotic behaviour of the fitting formula is incorrect.

An important quantity is the mean change of the total binding energy. We obtain a simple expression for it:

$$\langle E_B \rangle = \langle x \rangle - T_\infty. \tag{6}$$

If we keep the distribution function unchanged, then $\langle E_B \rangle$ becomes a linear function of T_∞. Notwithstanding the uncertainty caused by a few energetic cases we conclude that this

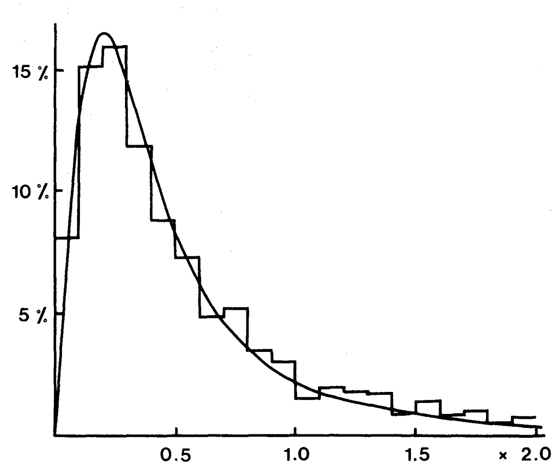

Figure 2. The distribution of the quantity $x = E_B - E_i$. The smooth curve is an analytic fit to a simple probability density function $f(x) = 15x(1 + 10x^2)^{-7/4}$.

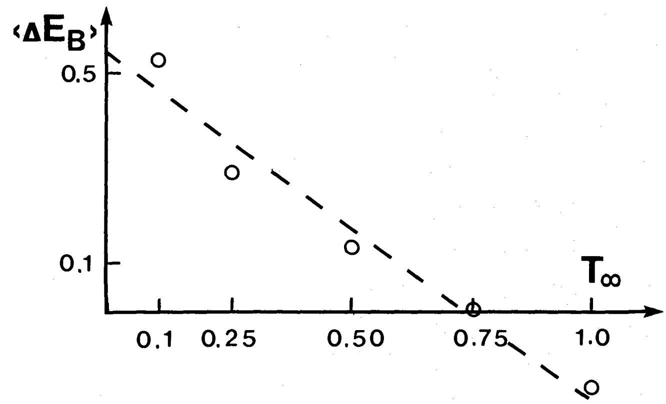

Figure 3. The mean change of total binding energy versus the impact energy T_∞.

may be valid even more generally. It becomes apparent if we look at Fig. 3 where the calculated values of $\langle E_B \rangle$ are plotted. From this figure we see that on the average the collision process becomes disruptive at an impact energy near $T_\infty \simeq 0.7$. This value is comparable with those obtained by Hills (1975) for encounters of binaries and single stars of various mass.

When we consider the application of our statistical results to a cluster simulation, we need in fact the joint distribution of the three kinetic energy quantities appearing on the right-hand side of the equation (5). The two-dimensional distribution of the energies of the escapers (T_3 and T_4) have been calculated for the various impact energies and were found to behave similarly to quantity x studied above; i.e. pronounced differences were not found in the case of the three lowest impact energies. Moreover, these lowest energy samples were joined and the correlations between the distributions of (T_3, T_4) at different values of x investigated. Here again the distributions resembled each other closely and the differences seem irregular. Thus the correlation between x and the pair T_3, T_4 are small, at least too small to be determined with certainty by means of our experimental sample.

What we can do is to assume that, in the first approximation, the sum of the kinetic energies does not correlate with the partition of this sum between the three components T_B, T_3 and T_4. Because the conservation of the linear momentum does not give us an equation, but only a set of inequalities, which can be most easily expressed by the statement that the values of the momenta form a triangle, the distribution function to be obtained is two-dimensional. To avoid inconveniences due to the presence of forbidden regions following from the triangle condition for the momenta, we choose, instead of the energies, certain quantities related directly to the above mentioned momentum triangle. Such quantities, unrestricted by the conservation laws are, e.g. the side representing the momentum of the centre of mass of the binary and the angle (say η) at which the smaller of the two other sides

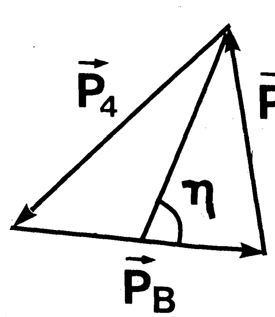

Figure 4. Illustration of the momentum triangle and the choice of the angle η.

is seen from the middle point of the side in question. The momentum triangle is illustrated in Fig. 4. However, it is more convenient to choose the fraction of kinetic energy gained by the binary centre of mass in place of the momentum. Then, if we denote by $\hat{T}_B$, $\hat{T}_3$ and $\hat{T}_4$ the corresponding fractions of $T_B + T_3 + T_4$, the two quantities for which we represent the joint distribution are $\xi = \hat{T}_B$ and

$$\eta = \arccos \{1/2 \, | \hat{T}_4 - \hat{T}_3 | \, [\hat{T}_B(1 - 2\hat{T}_B)]^{-1/2}\}. \tag{7}$$

The correlation of these variables were found to be weak, too. Thus the distribution can be written in the form of a product of the marginal distributions. Because the marginal distributions can be determined fairly accurately we give these in Table 9. Here we see that the distribution of η is almost uniform, except for the absence of very small values. The distribution of ξ is seen to be smooth and almost symmetrical with respect to the middle point of the possible values. We may thus approximate the distributions of η by a constant (where we have left out the underabundance of small values) and the distribution of ξ by a symmetrical second-order polynomial.

We have thus arrive at the result that, in the first approximation, the joint distribution of the variables x, ξ, η can be written in a product form

$$\psi(x, \xi, \eta) \simeq f(x)\, g(\xi)\, h(\eta), \tag{8}$$

where

$$f(x) \simeq 15x(1 + 10x^2)^{-7/4}$$
$$g(\xi) \simeq 0.5 + 18 \times \xi(1 - 2\xi) \tag{9}$$
$$h(\eta) \simeq 2/\pi.$$

Here $g(\xi)$ and $h(\eta)$ are the above mentioned approximations for the functions in Table 9. Even though this is only an approximation, it is simple and therefore useful in a cluster simulation.

4 Transformation of units

To be useful in applications, our results must be scaled to physical units. This can be done by remembering that our unit of distance is the initial semi-major axis of a binary (say a) and the unit of energy is the initial total binding energy of the two binaries (say ΣE_B). For convenience we list here some of the formulae in the 'correct' form:

the differential cross-section for the change of binding energy in a fly-by is

$$d\sigma_1 = a^2 f\left(\frac{T_\infty}{\Sigma E_B}, \frac{\Delta E}{\Sigma E_B}\right) \frac{d\Delta E}{|\Delta E|},$$

where the functions f are those plotted in Fig. 1.

Table 9. The marginal distribution of ξ and η. The distributions are normalized to 100 cases.

$\frac{2}{\pi} \cdot \eta$	.0-.1	.1-.2	.2-.3	.3-.4	.4-.5	.5-.6	.6-.7	.7-.8	.8-.9	.9-1.
distr.	3.6	7.8	9.0	10.4	11.0	13.3	11.8	10.8	10.8	11.4
$2 \cdot \xi$	.0-.1	.1-.2	.2-.3	.3-.4	.4-.5	.5-.6	.6-.7	.7-.8	.8-.9	.9-1.
distr.	5.1	9.7	10.6	12.7	10.8	13.9	13.9	11.0	7.7	4.6

The maximum impact parameter for a collision is given by

$$S_x \simeq 4.3\, a \left(\frac{\Sigma E_B}{T_\infty}\right)^{1/2}.$$

The cross-section for the collision can be written

$$\sigma_c \simeq 17.9 a^2 \left[1 + 0.7 \exp(-5.0\, T_\infty / \Sigma E_B)\right] \frac{\Sigma E_B}{T_\infty}.$$

The probability that the binding energy of the remaining binary, in case of two escapers, is $> |E| + X$ (for $E < 0$) is now given by

$$P(E_B > |E| + X) \simeq \left[1 + 10 \times \left(\frac{X}{\Sigma E_B}\right)^2\right]^{-3/4}.$$

5 Discussion and conclusions

We have found that the encounters of binaries without a strong interaction are less important than the collisions, provided that the binding energy of the colliding binaries exceeds the kinetic energy. Because the low impact energy condition is always satisfied for collisions of hard binaries, and because in dense stellar systems only hard binaries can live long (Heggie 1972), we can in the first approximation neglect the fly-by phenomenon and take into account only the two-escaper case. The fact that the distributions of the kinetic energies resulting from this interaction seem to be virtually independent of the impact velocity (for hard binaries) does simplify the construction of a cluster simulation algorithm.

So far we have studied only energetically similar binaries and equal masses. In the future this work will be extended to different masses and binaries of different size. An important aspect is to check the assumption used by Spitzer & Mathieu (1980) that binaries of very different sizes can be handled by means of the three-body cross-sections. Another important matter will be the study of the possible dynamical role of the stable (or semi-stable) three-body systems resulting from binary collisions. Not much is known about the evolution of such systems in star clusters, but one can speculate that the outer orbit, having a large geometric cross-section for interaction with the surrounding star field may work as an energy pump between the inner very hard binary and the system of field stars. Aarseth (1972, p. 97) has noted the formation of two hierachal three-body systems in his 250-body integrations. In these systems the semi-major axes of the inner orbits were found to be rather stable. Whether this is the general behaviour, remains, however, an open question.

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