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END-FAITHFUL SPANNING TREES OF COUNTABLE GRAPHS WITH PRESCRIBED SETS OF RAYS

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Abstract. We prove that a countable connected graph has an end-faithful spanning tree that contains a prescribed set of rays whenever this set is countable, and we show that this solution is, in a certain sense, the best possible. This improves a result of Hahn and Širáň [2, Theorem 1].

1. Introduction

In 1964 Halin [3] introduced the concept of the *end-faithful* subgraph. This is a subgraph H of a graph G such that each end of G contains exactly one end of H as a subset. He proved [3, Satz 3] that any countable connected graph contains an end-faithful spanning tree, and asked if the same holds for any connected graph. This question was answered in the negative in 1991 by Seymour and Thomas [8], and later but independently by Thomassen [10].

In this paper we explore a natural extension of Halin's result by considering the following problem:

Given a countable connected graph G, a set $\mathcal{A}$ of ends of G, and, for each end τ in $\mathcal{A}$, a ray R_τ representing τ, is there an end-faithful spanning tree of G that contains a tail (i.e., a subray) of R_τ for every $\tau \in \mathcal{A}$?

A set $\{R_\tau\colon \tau \in \mathcal{A}\}$, where $R_\tau \in \tau$ for all $\tau \in \mathcal{A}$, will be called a *representing set* of $\mathcal{A}$. Hahn and Širáň [2] already gave a solution to the problem by showing that such a tree exists if every end in $\mathcal{A}$ can separated from the set of all other elements of $\mathcal{A}$ by deleting a finite set of vertices. Note that such a set $\mathcal{A}$ is necessarily countable. We will improve this result by proving the existence of such a tree assuming the countability of $\mathcal{A}$ only. More precisely, we will obtain the following result:

Theorem A. *Let G be a countable connected graph, $\mathcal{A}$ a countable set of ends of G, and Π a representing set of $\mathcal{A}$. Then G has an end-faithful spanning tree which contains a tail of each element of* Π.

In addition we will show that this solution is, in a certain sense, the best possible.

Theorem B. *There exist a countable connected graph G and a set $\mathcal{A}$ of ends of G of cardinality $\aleph_1$ such that, for any representing set Π of $\mathcal{A}$, there is no end-faithful spanning tree of G that contains a tail of each element of* Π.

These two results show in particular that, unless the end set of a graph G is countable, one cannot generally hope to construct inductively an end-faithful tree of G by putting together rays from different ends one by one.

2. Preliminaries

The terminology will be that of [6] and [7]. Moreover, in order to get a more self-contained paper, we will recall the results of [5, 6, 7] that we will need.

2.1. Graphs considered in this paper are undirected and contain neither loops nor multiple edges. For a set A of vertices of a graph G we denote by $G[A]$ the subgraph of G induced by A. If B is any set of vertices and H any graph, we define $G - B := G[V(G) - B]$ and $G - H := G - V(H)$. The *union* of a family $(G_i)_{i \in I}$ of graphs is the graph $\bigcup_{i \in I} G_i$ given by $V\Big(\bigcup_{i \in I} G_i\Big) = \bigcup_{i \in I} V(G_i)$ and $E\Big(\bigcup_{i \in I} G_i\Big) = \bigcup_{i \in I} E(G_i)$. The *intersection* is defined analogously. If $(G_i)_{i \in I}$ is a family of subgraphs of a graph G, the subgraph induced by the union of this family will be denoted by $\bigvee_{i \in I} G_i$. For $x \in V(G)$ the set $V(x; G) := \{y \in V(G) \colon \{x, y\} \in E(G)\}$ is the *neighbourhood* of x in G. If H is a subgraph of G and X a nonempty subgraph of $G - H$, the *boundary of H with X* is the set $\mathcal{B}(H, X) := \{x \in V(H) \colon V(x; G) \cap V(X) \neq \emptyset\}$. The set of components of G is denoted by $\mathcal{C}_G$, and if x is a vertex, then $\mathcal{C}_G(x)$ is the component of G containing x. If H is an induced subgraph of a graph G and N an induced subgraph of a component X of $G - H$, then we set $N + (H) := N \vee G[\mathcal{B}(H, X)]$. A *path* $P = \langle x_0, \ldots, x_n \rangle$ is a graph with $V(P) = \{x_0, \ldots, x_n\}$, $x_i \neq x_j$ if $i \neq j$, and $E(P) = \{\{x_i, x_{i+1}\} \colon 0 \leqslant i < n\}$. A *ray* is a one-way infinite path $R := \langle x_0, x_1, \ldots \rangle$. A subray of a ray R is called a *tail* of R.

2.2. The *ends* of a graph G (a concept introduced by Freudenthal [1] and Hopf [4] to study discrete groups, and independently by Halin [3]) are the classes of the equivalence relation $\backsim_G$ defined on the set of all rays of G by: $R \backsim_G R'$ if and only if there is a ray R'' whose intersections with R and R' are infinite; or equivalently, if and only if $\mathcal{C}_{G-S}(R) = \mathcal{C}_{G-S}(R')$ for any finite $S \subseteq V(G)$ (where $\mathcal{C}_{G-S}(R)$ denotes the component of $G - S$ containing a tail of R). We will denote by $[R]_G$ the class of a ray R of G modulo $\backsim_G$, by $\mathfrak{T}(G)$ the set of all ends of G, and for $\tau \in \mathfrak{T}(G)$ and any finite $S \subseteq V(G)$, by $\mathcal{C}_{G-S}(\tau)$ the component of $G - S$ which contains some ray belonging to τ. Notice that if G is a tree, then two rays of G are equivalent modulo $\backsim_G$ if and only if they have a common tail; hence two disjoint rays of a tree correspond to different ends of the tree.

A subgraph H of G is *end-respecting* (*end-faithful*) if the map $\varepsilon_{HG}\colon \mathfrak{T}(H) \to \mathfrak{T}(G)$ given by $\varepsilon_{HG}([R]_H) = [R]_G$ for every ray R of H is injective (resp. bijective). We denote by $\mathfrak{T}_H(G)$ the image of ε_{HG}, i.e., the set of ends of G having the rays of H as elements. Furthermore, for $\mathcal{A} \subseteq \mathfrak{T}(G)$ we set $\mathcal{A}(H) := \mathcal{A} \cap \mathfrak{T}_H(G)$.

2.3. Throughout this paper we will assume that the end set $\mathfrak{T}(G)$ of a graph G is endowed with the topology, called the *end topology*, for which the closure of a subset $\mathcal{A}$ of $\mathfrak{T}(G)$ is the set

$$\overline{\mathcal{A}} := \{\tau \in \mathfrak{T}(G)\colon \text{ for every finite } S \subseteq V(G) \text{ there is } \tau' \in \mathcal{A} \text{ such that } \mathcal{C}_{G-S}(\tau) = \mathcal{C}_{G-S}(\tau')\},$$

i.e., $\overline{\mathcal{A}}$ is the set of all ends which cannot be separated by a finite $S \subseteq V(G)$ from $\mathcal{A}$.

By [6, Theorem 4.8] the end space $\mathfrak{T}(G)$ of a graph G is scattered (i.e., contains no non-empty subset which is dense in itself) if and only if G has no subdivision of the binary tree as an end-respecting subgraph. Furthermore, by [6, Proposition 4.7], the end space of the binary tree is homeomorphic with the Cantor space 2^ω. Therefore, the cardinality of the end set of a countable graph G is at most $\aleph_0$ or exactly $2^{\aleph_0}$ if $\mathfrak{T}(G)$ is scattered or not, respectively.

2.4. For $\mathcal{A} \subseteq \mathfrak{T}(G)$ we define $m(\mathcal{A}) := \sup\{|\mathcal{R}|\colon \mathcal{R}$ is a set of pairwise disjoint elements of $\bigcup \mathcal{A}\}$. For $\tau \in \mathfrak{T}(G)$ we write $m(\tau)$ for $m(\{\tau\})$, and if H is a subgraph of G, we set $m_H(\tau) := m(\varepsilon_{HG}^{-1}(\tau))$. By the remark in 2.2 about ends of trees, notice that if H is a tree, then H is end-respecting (end-faithful) if and only if $m_H(\tau) \leqslant 1$ (resp. $= 1$) for every end τ of G.

2.5. We will denote by $\mathcal{D}$ (or by $\mathcal{D}_G$ if necessary) the relation between $V(G)$ and $\mathfrak{T}(G)$ defined by $x\mathcal{D}\tau$ if $x \in V(\mathcal{C}_{G-S}(\tau))$ for any finite $S \subseteq V(G-x)$, or equivalently if there exists an infinite set of paths joining x to the vertex set of a ray $R \in \tau$ and having pairwise only x in common. If $x\mathcal{D}\tau$ then we will say that the vertex x *dominates* the end τ, or that τ is *dominated* by x. For $\tau \in \mathfrak{T}(G)$ we will denote by $\mathcal{D}^{-1}(\tau)$ the set of all vertices that dominate τ.

2.6. An infinite subset S of $V(G)$ is *concentrated* in G if there is an end τ such that $S - V(\mathcal{C}_{G-F}(\tau))$ is finite for any finite $F \subseteq V(G)$ (we also say that S is *concentrated in* τ).

For example, the vertex set of any ray of a graph G is concentrated in G. Note that every infinite subset of a concentrated set is also concentrated.

2.7. A set S of vertices of G is *dispersed* if it has no concentrated subset.

2.8. An induced subgraph M of a graph G is called a *multi-ending* of G if it satisfies the following properties:

M1. M is connected.

M2. The boundary of M with every component of $G - M$ is finite.

M3. Any infinite subset of $V(M)$ which is concentrated in G is also concentrated in M.

M4. $\mathcal{D}_M^{-1}(\tau) = \mathcal{D}_G^{-1}(\varepsilon_{MG}(\tau))$ for any end τ of M.

M5. For any family $(R_i)_{i\in I}$ of pairwise disjoint rays of G such that $\{[R_i]_G \colon i \in I\} \subseteq \mathfrak{T}_M(G)$, there is a family $(R_i')_{i\in I}$ of pairwise disjoint rays of M such that $R_i \cap R_i'$ is infinite for every $i \in I$.

By M3, a multi-ending of G is an end-respecting subgraph of G. By M5, $m(\tau) = m(\varepsilon_{MG}(\tau))$ for any end τ of M. A multi-ending which is rayless is called a 0-*ending*. A 0-ending M is then a connected induced subgraph of G whose vertex set is dispersed and whose boundary with any component of $G - M$ is finite. A multi-ending M is an *ending* if $|\mathfrak{T}(M)| = 1$; it is a *discrete* multi-ending if $\mathfrak{T}_M(G)$ is a discrete subspace of $\mathfrak{T}(G)$.

For any subset $\mathcal{A}$ of $\mathfrak{T}(G)$ we denote by $\mathbb{M}(\mathcal{A})$ the set of all multi-endings M of G such that $\mathcal{A} = \mathfrak{T}_M(G)$.

2.9 [7, 6.5.(ii) and 7.9]**.** $\mathbb{M}(\mathcal{A}) \neq \emptyset$ *if and only if* $\mathcal{A}$ *is a closed set.*

In particular, $\mathbb{M}(\{\tau\}) \neq \emptyset$ for every end τ, since the end topology is Hausdorff.

2.10 ([6, 4.15] and [7, 6.11])**.** *Let G be a graph. For any closed discrete subspace Ω of $\mathfrak{T}(G)$ there exists a 0-ending M of G which pairwise separates the elements of Ω, i.e., $\mathcal{C}_{G-S}(\tau) \neq \mathcal{C}_{G-S}(\tau')$ for every pair $\{\tau, \tau'\}$ of distinct elements of Ω.*

2.11. *Let $\tau \in \mathfrak{T}(G)$, $M \in \mathbb{M}(\tau)$ and $R \in \tau$. Then $N := M \cup R$ satisfies* M3.

P r o o f. By M5, since $R \in \tau$, N has exactly one end, $\varepsilon_{NG}^{-1}(\tau)$. Let A be an infinite subset of $V(N)$ which is not concentrated in N. Since N is one-ended and since every infinite subset of $V(R)$ is obviously concentrated, the set $A \cap V(M)$ must be infinite and not concentrated in N, thus not concentrated in M. Therefore, by M3 since M is an ending of G, $A \cap V(M)$, and a fortiori A, is not concentrated in G. $\square$

2.12 [7, 6.10 and 6.15]. *For every induced subgraph H of G satisfying* M3 *there exists a multi-ending M of G which contains H and satisfies $\mathfrak{T}_M(G) = \mathfrak{T}_H(G)$.*

An immediate consequence of this result and the fact that, if some cofinite subset of a set S is concentrated, then S is concentrated as well, is the following.

2.13. *For every multi-ending N of G and every finite $A \subseteq V(G)$ there exists a multi-ending M of G such that $A \cup V(N) \subseteq V(M)$ and $\mathfrak{T}_M(G) = \mathfrak{T}_N(G)$.*

2.14 [7, 6.17]. *Let H be a connected induced subgraph of a graph G whose boundary with any component of $G - H$ is finite. Then any multi-ending of H is a multi-ending of G.*

2.15 [7, 6.19]. *Let M be a multi-ending of a graph G, and X a component of $G - M$. Then any induced subgraph N of X satisfying Axiom* M3 *can be extended to a multi-ending N' of X with the following properties:*
(i) *N' contains a neighbour of each element of $\mathcal{B}(M, X)$;*
(ii) *$\mathfrak{T}_{N'}(G) = \mathfrak{T}_N(G)$;*
(iii) *$N' + (M)$ is a multi-ending of $X + (M)$.*

2.16 [7, 6.18]. *Let N be a multi-ending of G and, for every component X of $G - N$, let N_X be a multi-ending of $X + (N)$ containing $\mathcal{B}(N, X)$. Then $M := N \vee \bigcup_{X \in \mathcal{C}_{G-N}} N_X$ is a multi-ending of G such that $\mathfrak{T}_M(G) = \mathfrak{T}_N(G) \cup \bigcup_{X \in \mathcal{C}_{G-N}} \mathfrak{T}_{N_X}(G)$.*

2.17. An *expansion* of a connected graph G is a sequence $(G_n)_{n \geqslant 0}$ of subgraphs of G satisfying the following conditions. For every $n \geqslant 0$,

E1. $G_n \subseteq G_{n+1}$.

E2. G_n is a multi-ending of G.

E3. G_0 is discrete and, for any component X of $G - G_n$, the subgraph $M := G_{n+1} \cap X$ is a discrete multi-ending of X which contains a neighbour of each element of $\mathcal{B}(G_n, X)$ and with the property that $M + (G_n)$ is a multi-ending of $X + (G_n)$.

E4. $G = \bigcup_{n \geqslant 0} G_n$.

3.1. Hahn and Širáň [2, Theorem 1] proved that, given a countable graph G, if $\mathcal{A}$ is a discrete subspace of $\mathfrak{T}(G)$—which they called a “free set of ends”—and if Π is a representative set of $\mathcal{A}$, then G has an end-faithful spanning tree which contains a tail of each element of Π. Theorem A extends their result to any countable set of ends. To prove it, we will need another lemma.

3.2 [5, 3.2]**.** *Let G be a one-ended connected graph having an end-faithful spanning tree. Then any end-faithful tree of G is included in an end-faithful spanning tree of G.*

Note that this result can also be obtained as a consequence of a result of Širáň [9, Theorem 4].

3.3. Proof of Theorem A. In the following, an end-faithful spanning tree satisfying the condition of the theorem will be called a Π-*end-faithful spanning tree.* Furthermore, for any subgraph X of G that contains a tail of the representing ray $R_\tau \in \Pi$ of each end $\tau \in \mathcal{A}(X)$, we will set

$$\Pi_X := \{R'_\tau \colon \tau \in \mathcal{A}(X)\},$$

where R'_τ is the largest ray contained in $R_\tau \cap X$. Finally, note that, by Halin's theorem [3, Satz 3], since G is countable, any connected subgraph of G has an end-faithful spanning tree.

(a) Let $\mathcal{C}$ be a non-empty closed discrete subspace of $\mathfrak{T}(G)$.

(a.1) We will first show that there exists a multi-ending $M \in \mathbb{M}(\mathcal{C})$ which contains a tail of the representing ray R_τ for each end $\tau \in \mathcal{A} \cap \mathcal{C}$. For each $\tau \in \mathcal{C}$, choose a ray $R_\tau \in \tau$ such that $R_\tau \in \Pi$ if $\tau \in \mathcal{A}$.

By 2.10, there is a 0-ending N (which is empty if $|\mathcal{C}| = 1$) of G which pairwise separates the elements of $\mathcal{C}$. Let Γ be the set of components X of $G - N$ such that $\mathcal{C}(X) \neq \emptyset$. Since N separates the elements of $\mathcal{C}$, $\mathcal{C}(X)$ has a unique element, which will be denoted by τ_X. Since $\mathcal{B}(N, X)$ is finite, X contains a tail of R_{τ_X}. By 2.9, there exists an ending H of X such that $\mathfrak{T}_H(G) = \{\tau_X\}$. By 2.11, $H \vee R_{\tau_X}$ satisfies Axiom M3. Hence, by 2.15, $H \vee R_{\tau_X}$ can be extended to an ending N_X of X which contains a neighbour of each element of $\mathcal{B}(N, X)$, and with the property that $N_X + (N)$ is a multi-ending of $X + (N)$. Then, by 2.16, $M := N \vee \bigcup_{X \in \Gamma} N_X$ is a multi-ending of $N \vee \bigcup_{X\in\Gamma} X$, hence of G by 2.14, such that $\mathfrak{T}_M(G) = \mathfrak{T}_N(G) \cup \bigcup_{X\in\Gamma} \mathfrak{T}_{N_X}(G) = \{\tau_X \colon X \in \Gamma\} = \mathcal{C}$, and which contains a tail of R_τ for each $\tau \in \mathcal{A} \cap \mathcal{C}$. Such a multi-ending will be said to be Π-*compatible.*

(a.2) We now construct a Π_M-end-faithful spanning tree of M. Since N is a 0-ending, it has a rayless spanning tree T_N. Let $X \in \Gamma$. By Halin's result [3, Satz 3] and by 3.2, the ending N_X has an end-faithful spanning tree T_X that contains this tail. Now, denote by e_X an edge joining X with N. Then clearly $T := T_N \vee \bigcup_{X \in \Gamma} T_X \cup \{e_X\}$ is a Π_M-end-faithful spanning tree of M.

(b) We now consider the general case.

(b.1) Let $(\tau_n)_{n \geqslant 0}$ be such that $\mathcal{A} = \{\tau_n \colon n \geqslant 0\}$, and let $(x_n)_{n \geqslant 0}$ be an enumeration of $V(G)$. We will construct an expansion $(G_n)_{n \geqslant 0}$ of G such that G_n is a Π-end-faithful multi-ending with $x_n \in V(G_n)$ and $\tau_n \in \mathfrak{T}_{G_n}(G)$, as follows.

Let $\mathcal{T}_0$ be a closed discrete subspace of $\mathfrak{T}(G)$ that contains τ_0. By (a) and 2.13, there is $G_0 \in \mathbb{M}(\mathcal{T}_0)$ that is Π-compatible and that contains x_0. Suppose that $G_0, \ldots, G_n$ have already been constructed. Let $X \in \mathcal{C}_{G-G_n}$. If $\mathcal{A}(X) = \emptyset$, let $M_X := X$. If $\mathcal{A}(X) \neq \emptyset$, denote by $p(X)$ the least integer p such that $\tau_p \in \mathcal{A}(X)$, and let $\mathcal{T}_X$ be a closed discrete subspace of $\mathfrak{T}(G)$ that contains $\tau_{p(X)}$. Then, by (a.1), there is a Π-compatible multi-ending M_X of X such that $\mathfrak{T}_{M_X}(G) = \mathcal{T}_X$. Moreover, by 2.13 and 2.15, we can choose M_X such that it contains x_{n+1} if $x_{n+1} \in V(X)$, as well as a neighbor of each element of $\mathcal{B}(G_n, X)$, and such that $M_X + (G_n)$ is a multi-ending of $X + (G_n)$. Therefore, by 2.16, $G_{n+1} := G_n \vee \bigcup_{X \in \mathcal{C}_{G-G_n}} M_X$ is a Π-compatible multi-ending of G with $x_{n+1} \in V(G_{n+1})$ and $\tau_{n+1} \in \mathfrak{T}_{G_{n+1}}(G)$.

(b.2) We now construct a Π-end-faithful spanning tree of G. For $n \geqslant 0$, denote by Γ_n the set of components of $G_n - G_{n-1}$ with $G_{-1} := \emptyset$, and let $\Gamma := \bigcup_{n \geqslant 0} \Gamma_n$. By (b.1) $X \in \Gamma_n$ is a multi-ending of $G - G_{n-1}$ which is either discrete and Π-compatible, or such that $\mathcal{A}(X) = \emptyset$. Hence, (a.2) in the first case and [3, Satz 3] in the second imply that X has a Π_X-end-faithful spanning tree T_X. If $X \in \Gamma_n$ for some $n > 0$, denote by e_X an edge of G joining X with $G_{n-1} - \bigcup \{G_i \colon i < n-1 \text{ and } X \notin \mathcal{C}_{G_n - G_i}\}$. Such an edge exists because X contains a neighbour of each element of $\mathcal{B}(G_{n-1}, X)$. Therefore $T := T_{G_0} \vee \bigcup_{X \in \Gamma} T_X \cup \{e_X\}$ is a spanning tree of G which contains a tail of each element of Π.

We have to prove that T is an end-faithful subgraph of G. Let τ be an end of G. If $\tau \in \bigcup_{n \geqslant 0} \mathfrak{T}_{G_n}(G)$, then $\tau \in \mathfrak{T}_X(G)$ for some $X \in \Gamma_n$ and $n \geqslant 0$; thus $m_T(\tau) = 1$. Assume now that $\tau \notin \bigcup_{n \geqslant 0} \mathfrak{T}_{G_n}(G)$, then $\tau \in \overline{\bigcup_{n \geqslant 0} \mathfrak{T}_{G_n}(G)}$ since $G = \bigcup_{n \geqslant 0} G_n$. For all $n \geqslant 0$ there is a unique component Y_n of $G - G_{n-1}$ such that $\tau \in \mathfrak{T}_{Y_n}(G)$. Let $X_n := Y_n \cap G_n$. By the construction of T there is a ray of T originating in G_0 that contains all edges e_{X_n}, $n \geqslant 0$. This ray belongs to the end τ, since the set $\bigcup_{n \geqslant 0} e_{X_n}$ is concentrated in τ by the definition of X_n. Thus $m_T(\tau) \geqslant 1$. Moreover, two rays of T belonging to τ must contain the edges e_{X_n} for all n greater than some integer p.

Hence they have a common tail. This proves that $m_T(\tau) = 1$. Consequently, T is end-faithful, thus it is a Π-end-faithful spanning tree of G. $\square$

3.4. Proof of Theorem B. (a) Let T be the binary tree rooted at a vertex x_0. For every vertex x, denote by T_x the subtree of T induced by the vertices which are greater than or equal to x, with respect to the natural order on $V(T)$, where x_0 is the least element. Furthermore, let $\mathcal{A}_x \subseteq \mathfrak{T}_{T_x}(T)$ be such that $|\mathcal{A}_x| = \aleph_1$. Then the set $\mathcal{A} := \bigcup_{x \in V(T)} \mathcal{A}_x$ of cardinality $\aleph_1$ has the property that $\mathcal{A}(T_x)$ is dense in $\mathfrak{T}_{T_x}(T)$ for every $x \in V(T)$.

Now let $\{R_\tau \colon \tau \in \mathcal{A}\}$ be a representing set of $\mathcal{A}$. Since T is countable and $|\mathcal{A}| = \aleph_1$, there exists a subtree A of T with $A \subseteq \bigcup_{\tau \in \mathcal{A}} R_\tau$ such that $\mathfrak{T}_A(T)$ is uncountable. Thus $|\mathfrak{T}_A(T)| = 2^{\aleph_0}$, i.e., A contains a subdivision of the binary tree (cf. 2.3).

Consider another subset $\mathcal{B}$ of $\mathfrak{T}(T)$ disjoint from $\mathcal{A}$, with $|\mathcal{B}(T_x)| = \aleph_1$ for every $x \in V(T)$. Clearly $\overline{\mathcal{B}(A)} = \mathfrak{T}_A(T)$. Thus, as above, for any representing set $\{R_\tau \colon \tau \in \mathcal{B}\}$ of $\mathcal{B}$ there exists a subtree $B \subseteq \bigcup_{\tau \in \mathcal{B}(A)} R_\tau$ of A which contains a subdivision of the binary tree. Therefore there are $2^{\aleph_0}$ ends of T which have representing rays in each of the subgraphs $\bigcup_{\tau \in \mathcal{A}} R_\tau$ and $\bigcup_{\tau \in \mathcal{B}} R_\tau$.

(b) Now let G be the cartesian product of T with the complete graph K_2. Denote by T_0 and T_1 the two copies of T in G, and let $\mathcal{A}_G$ and $\mathcal{B}_G$ be the sets of ends of G corresponding to the preceding sets $\mathcal{A}$ and $\mathcal{B}$, respectively. Let $\{R_\tau \colon \tau \in \mathcal{A}_G\}$ (resp. $\{R_\tau \colon \tau \in \mathcal{B}_G\}$) be a representing set of $\mathcal{A}_G$ (resp. $\mathcal{B}_G$) with $R_\tau \subseteq T_0$ (resp. $R_\tau \subseteq T_1$) for every $\tau \in \mathcal{A}_G$ (resp. $\tau \in \mathcal{B}_G$). Then, by (a), for any tail R'_τ of R_τ, $\tau \in \mathcal{A}_G \cup \mathcal{B}_G$, there are $2^{\aleph_0}$ ends of G that have representing rays in each of the subgraphs $\bigcup_{\tau \in \mathcal{A}_G} R'_\tau$ and $\bigcup_{\tau \in \mathcal{B}_G} R'_\tau$ of T_0 and T_1, respectively. Consequently, no tree of G that contains a tail of each R_τ, $\tau \in \mathcal{A}_G \cup \mathcal{B}_G$, is end-respecting. $\square$

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