End spaces and spanning trees

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We determine when the topological spaces $|G|$ naturally associated with a graph G and its ends are metrizable or compact.

In the most natural topology, $|G|$ is metrizable if and only if G has a normal spanning tree. We give two proofs, one of them based on Stone's theorem that metric spaces are paracompact.

We show that $|G|$ is compact in the most natural topology if and only if no finite vertex separator of G leaves infinitely many components. When G is countable and connected, this is equivalent to the existence of a locally finite spanning tree. The proof uses ultrafilters and a lemma relating ends to directions.

1. Introduction

Our aim in this note is to point out a surprising connection between two seemingly unrelated sets of questions about infinite graphs, and in doing so give answers to both.

The first of these concerns the end space of a graph, more precisely, the topological space $|G|$ associated with a graph and its ends. When G is locally finite, this is the well-known Freudenthal compactification of G [11]. For arbitrary G, little is known in general; indeed, there is more than one natural way to topologize $|G|$. For each of these, we characterize by a natural structural condition the graphs G for which $|G|$ is metrizable or compact.

The other set of questions concerns normal spanning trees of graphs. These are the infinite analogues of depth-first search trees: a spanning tree is *normal* if all the edges of the graph run along its branches, never across. (More formally: the endvertices of any edge of G must be comparable in the associated tree-order.) Normal spanning trees are an important tool for infinite graphs (see e.g. [4, 6, 7, 10]), but they do not always exist. Characterizations of the graphs that do admit normal spanning trees have been given by Jung [13] and in [9]. In this paper we give a third, in terms of the end space $|G|$.

If G has a normal spanning tree then $|G|$, suitably topologized, ought to be metrizable: one of the main features of normal spanning trees is that their end space (ends only) is canonically homeomorphic to that of their host graph, and the end space of a tree is clearly metric. The converse, however, comes as a surprise: one can use Jung's characterization to show that every connected graph G with a metrizable space $|G|$ has a normal spanning tree, which makes its metric visible in a very tangible structural way.

Another question that is interesting from the spanning tree point of view is: When does a graph G have a locally finite spanning tree, perhaps even a normal one? If it does, its end space $|G|$ is easily seen to be compact in another natural (only slightly different) topology. We shall prove that, conversely, every countable graph with a compact end space has a locally finite spanning tree, even a normal one. Thus, in particular, if a graph has any locally finite spanning tree at all it also has a normal one. For uncountable G, we prove that $|G|$ with this topology is compact if and only if no finite vertex separator splits G into infinitely many components.

Finally, a word about the (perhaps at first unsettling) prospect that we are going to consider not one but three possible topologies for $|G|$. This is by necessity, not design, and the three topologies are very similar. They all induce the same topology on the subspace of ends (only), and if G is locally finite they all agree with its Freudenthal compactification (see [8]). However, when G is not locally finite, then each of these topologies provides the 'right' setting for an important problem and thus has its place: the first has just the right fineness for metrizability and yields the surprising characterization mentioned above; the second is coarse enough to make the compactness problem interesting (but its coarseness unnecessarily precludes metrizability in many cases); the third has been used in most existing applications of the end space but is unnecessarily fine in our context, precluding both metrizability and compactness as soon as any vertex has infinite degree. We shall concentrate on the interesting cases, but for completeness mention the solutions for the others in passing.

2. Tools and terminology

Basic terminology can be found in [2]. A 1-way infinite path is called a *ray*, a 2-way infinite path is a *double ray*, and the subrays of a ray or double ray are its *tails*. Two rays in a graph $G = (V, E)$ are *equivalent* if no finite set of vertices separates them; the corresponding equivalence classes of rays are the *ends* of G. The set of these ends is denoted by $\Omega = \Omega(G)$. A vertex v *dominates* and end ω if G contains infinitely many paths, disjoint except in v, between v and some (equivalently: every) ray in ω. If $S \subseteq V$ is finite and $\omega \in \Omega$, there is a unique component of $G - S$ that contains a tail of every ray in ω; we denote this component by $C(S, \omega)$, and the set of all ends ω' whose rays have a tail in $C(S, \omega)$ by $\Omega(S, \omega)$.

We now define three topologies on G together with its ends, to be called TOP, MTOP and VTOP in order of decreasing fineness. We first define TOP, which is the topology used in [5, 6, 7].

We begin by viewing G itself (without ends) as a 1-complex. In this topology, every edge is homeomorphic to the real unit interval $[0, 1]$, and for a vertex v the unions of half-open partial edges $[v, z)$, one for every edge e at v with z an inner point of e, form a neighbourhood basis of open sets for v. Note that, in this topology, a vertex of infinite degree has no countable neighbourhood basis: if $O_1, O_2, \ldots$ are open neighbourhoods of v and $e_1, e_2, \ldots$ are its incident edges, choose $z \in \mathring{e}_n$ inside O_n to obtain a union of half-open partial edges as above that contains none of the O_n. To extend this topology to Ω, we add as basic open sets all sets of the form

$$\widehat{C}_*(S,\omega) := C(S,\omega) \cup \Omega(S,\omega) \cup \mathring{E}_*(S,\omega)\,,$$

where $\omega \in \Omega$ and S is a finite subset of V, and $\mathring{E}_*(S,\omega)$ is any union of half-open partial edges $(z, v]$, one from every S–$C(S,\omega)$ edge uv with $v \in C(S,\omega)$.

To define MTop, we again begin with G as the point set of a 1-complex, but we take a coarser topology. As before, every edge is a copy of the real interval $[0,1]$, and we give it the corresponding metric and topology. For a vertex v, however, we take as a neighbourhood basis only the open stars of radius $1/n$ around v, where distances are measured individually in the relevant edges. To extend this topology to Ω, we add as open sets all sets of the form

$$\widehat{C}_\epsilon(S,\omega) := C(S,\omega) \cup \Omega(S,\omega) \cup \mathring{E}_\epsilon(S,\omega)\,,$$

where $\omega \in \Omega$ and S is a finite subset of V, and $\mathring{E}_\epsilon(S,\omega)$ is the set of all inner points of S–$C(S,\omega)$ edges at distance less than $\epsilon = 1/n$ from their endpoint in $C(S,\omega)$. Our metrizability theorem will have its natural setting in MTop.

Finally, we define VTop just as MTop, except that as a neighbourhood basis of open sets for an end ω we only take the sets

$$\widehat{C}(S,\omega) := C(S,\omega) \cup \Omega(S,\omega) \cup \mathring{E}(S,\omega)\,,$$

where $\mathring{E}(S,\omega)$ is the set of all inner points of S–$C(S,\omega)$ edges. Note that while Top and MTop are clearly Hausdorff, VTop is Hausdorff only if no end is dominated: for a vertex and an end it dominates there are no disjoint open neighbourhoods in VTop. (This is deliberate: in some contexts it would be artificial to distinguish a vertex from an end from which it cannot be finitely separated, just as two ends are the same if no finite separator can distinguish them.) Our compactness theorem will have its natural setting in VTop.

In all three cases, we write $|G|$ for the topological space on the point set $V \cup \Omega \cup \bigcup E$ thus defined. Note that every ray converges in $|G|$ to the end of which it is an element.

A *direction* on G is a function f that assigns to every finite $S \subseteq V$ one of the components of $G - S$ so that $f(S) \supseteq f(S')$ whenever $S \subseteq S'$. For every end ω, the map $S \mapsto C(S,\omega)$ is easily seen to be a direction. In fact, one can show that every direction is defined by an end in this way:

Lemma 2.1. [8] *For every direction f on a graph G there is an end ω such that $f(S) = C(S,\omega)$ for every finite $S \subseteq V(G)$.*

Let $T \subseteq G$ be a spanning tree of G, with a root r. The *tree-order* on V associated with T and r is defined by setting $u \leqslant v$ if u lies on the unique path rTv from r to v in T. Given $k \in \mathbb{N}$, the kth *level* of T is the set of vertices at distance k from r in T. The *down-closure* of a vertex v is the set $\lceil v \rceil := \{ u \mid u \leqslant v \}$; its *up-closure* is the set $\lfloor v \rfloor := \{ w \mid v \leqslant w \}$. The down-closure of v is always a finite chain, the vertex set of the path rTv. A *normal ray* in T is one that starts at the root. If $R \subseteq T$ is a normal ray and v is any vertex, then the *R-height* of v is the unique maximal vertex in $V(R) \cap \lceil v \rceil$. The R-height of a set of vertices is *unbounded* if R lies in the union of their down-closures.

T is a *normal spanning tree* in G if the endvertices of every edge of G are comparable in this tree-order. The following basic properties of normal spanning trees are easily proved:

Lemma 2.2. [2, 10] *Let T be a normal spanning tree of G.*

(i) *Any two vertices $x, y \in V$ are separated in G by the set $\lceil x \rceil \cap \lceil y \rceil$.*

(ii) *If $S \subseteq V(T)$ is down-closed, then the components of $G - S$ are spanned by the sets $\lfloor x \rfloor$ with x minimal in $T - S$.*

(iii) *Every end of G contains exactly one normal ray of T.*

(iv) *A vertex dominates an end ω in G if and only if its neighbours in G have unbounded $R(\omega)$-height, where $R(\omega)$ is the unique normal ray in ω.*

By a theorem of Halin [12], connected graphs not containing a subdivision of an infinite complete graph have normal spanning trees. In particular:

Lemma 2.3. *If G is connected and none of its ends is dominated, then G has a normal spanning tree.*

Jung [13] has characterized the graphs admitting a normal spanning tree by a condition which is particularly simple to express in our topology. Note that, in each of our three topologies, a set $U \subseteq V$ is closed in $|G|$ if and only if every ray can be separated from U by a finite set of vertices.

Theorem 2.4. (Jung [13])
A connected graph has a normal spanning tree if and only if its vertex set is a countable union of closed sets.

By Lemma 2.2, we may take as the closed sets required for the forward implication of Theorem 2.4 the levels of the normal spanning tree T assumed to exist. Indeed, if U is a level of T (other than the root) and Q is a ray in G, with end ω say, let $R := R(\omega)$, let u be the unique vertex of R in U, and let v be the predecessor of u on R. Then $\lfloor u \rfloor$ spans the component $C(\lceil v \rceil, \omega)$ of $G - \lceil v \rceil$ (Lemma 2.2 (ii)), and the finite set $\lceil v \rceil \cup V(Q - \lfloor u \rfloor)$ separates Q from U in G. See [9] for more on normal spanning trees, including a forbidden-minor characterization of the graphs admitting one.

A *filter* on a set A is a non-empty set $\mathcal{F}$ of subsets of A such that $\emptyset \notin \mathcal{F}$, any superset of an element of $\mathcal{F}$ is in $\mathcal{F}$, and $\mathcal{F}$ is closed under finite intersection.

By Zorn's lemma, every filter on A extends to a maximal filter on A, which is called an *ultrafilter*. If $\mathcal{U}$ is an ultrafilter on A and $A' \subseteq A$, then exactly one of A' and $A \setminus A'$ is an element of $\mathcal{U}$. Hence if we partition A into finitely many sets, then exactly one of these lies in $\mathcal{U}$.

In one of our two proofs of Theorem 3.1 (i), we shall use the well-known metrizability theorem of Bing and Nagata-Smirnow. (As the other proof shows, the theorem is not needed; but its interplay with Jung's theorem throws an interesting light on the metric of $|G|$.) We shall use the metrization theorem in the direction that follows from the (non-trivial) theorem of Stone [15] that metric spaces are paracompact. Call a set $\mathcal{A}$ of subsets of a topological space X *locally finite* if every point in X has a neighbourhood that meets only finitely many sets in $\mathcal{A}$.

Lemma 2.5. *The topology of any metric space has a basis that is a countable union of locally finite sets (of open sets).*

3. Metrizability

Pointing out the surprisingly intimate connection between the metrizability of $|G|$ and normal spanning trees, as expressed in Theorem 3.1 (i) below, is the main purpose of this paper. The statement can be read both as a structure theorem for graphs with a metric end space—where the aim would be to make this metric visible by some structure to be found in the graph—and as a topological characterization of the graphs admitting a normal spanning tree.

Theorem 3.1. *Let G be a connected graph.*

(i) *In* MTop, *$|G|$ is metrizable if and only if G has a normal spanning tree.*

(ii) *In* VTop, *$|G|$ is metrizable if and only if none of its ends is dominated.*

(iii) *In* Top, *$|G|$ is metrizable if and only if G is locally finite.*

Proof. (i) Let T be a normal spanning tree of G; we define a metric on $|G|$ explicitly and show that it induces the topology of $|G|$. We begin with T itself. T comes with a metric in which every edge has length 1, being a copy of $[0, 1]$. We scale this metric linearly in every edge so that edges between levels $n-1$ and n get length $1/2^n$. Now every finite or infinite path in T has a finite length, the sum of the lengths of its edges. (Normal rays, for example, have length 1.) We may thus extend our metric to $T \cup \Omega(G)$ by defining the distance between an end ω and a vertex v as the length of the unique ray of ω in T that starts at v (cf. Lemma 2.2 (iii)), and the distance between two ends as the length of the unique double ray in T that has tails in both these ends. Finally, we extend our metric to $|G|$ by scaling the remaining edges of G linearly to the length of the unique path in T between their endvertices.

It is easy to check that this is indeed a metric; what remains to be shown is that the topology it induces on $|G|$ is MTop. This, too, is straightforward (and we omit the details) from the following facts, which follow from the definition of a normal spanning tree and Lemma 2.2. First, the lengths of edges incident with any given vertex, at level n say, are bounded below by $1/2^{n+1}$ and above by 1; this implies that vertices have equivalent neighbourhood bases in our metric and in MTop. Similarly, if $\widehat{C}_\epsilon(S,\omega)$ is a basic open neighbourhood of an end ω in MTop and n is the highest level of a vertex in S, then $\widehat{C}_\epsilon(S,\omega)$ contains the open $(\epsilon/2^{n+1})$-ball around ω, because every S–$C(S,\omega)$ edge has length at least $1/2^{n+1}$. Conversely, the open ϵ-ball around ω contains $\widehat{C}_{\epsilon/2}(\lceil v \rceil,\omega)$ for any vertex v on $R(\omega)$ at distance less than $\epsilon/2$ from ω (Lemma 2.2 (ii)).

For the converse implication we offer two proofs. In both of these, we assume that $|G|$ is metrizable and find closed subsets $V_1, V_2, \dots$ of $V(G)$ whose union is $V(G)$; by Jung's theorem, this will imply that G has a normal spanning tree. Since $|G|$ is metrizable, so is its subspace $X := V(G) \cup \Omega(G)$. Note that X is closed in $|G|$, and that singleton vertex sets $\{v\}$ are open (as well as closed) in the subspace topology which MTop induces on X.

For the first proof, we apply Lemma 2.5 to the metric space X. The lemma implies that X has a basis of the form $\mathcal{O}_1 \cup \mathcal{O}_2 \cup \dots$ such that, for each n, every $\omega \in \Omega(G)$ has a basic open neighbourhood $\widehat{C}_\epsilon(S,\omega)$ in $|G|$ that contains only finitely many vertices v with $\{v\} \in \mathcal{O}_n$. Adding these vertices to S, we may assume even that $\widehat{C}_\epsilon(S,\omega)$ contains no such vertex v. This shows that

$$V_n := \{\, v \mid \{v\} \in \mathcal{O}_n \,\}$$

is closed in X, and hence in $|G|$. Moreover, since singleton vertex sets $\{v\}$ are open they must lie in our basis, so $V_1 \cup V_2 \cup \dots = V(G)$.

For the second proof, we take as V_n the set of vertices that have distance at least $1/n$ from every end. These sets are closed, because they are intersections of complements of open balls around ends:

$$V_n = \bigcap_{\omega \in \Omega(G)} \left(X \setminus B(\omega, 1/n) \right).$$

To show that $V_1 \cup V_2 \cup \dots = V(G)$, consider a vertex v. As the set $\{v\}$ is open in X it contains an open ball in X around v, in the metric that we assume defines our given topology on X. If this ball has radius $1/n$, say, then $v \in V_n$.

(ii) If $|G|$ is metrizable it must be Hausdorff, which under VTop implies that no end can be dominated. If no end of G is dominated, then by Lemma 2.3 G has a normal spanning tree T. Using T, we define a metric on $|G|$ as in (i). Since VTop is coarser than MTop, its open sets are open also in MTop, and hence by (i) also in the metric. Conversely, we have to show that a given open ball $B_\epsilon(\omega)$ around an end ω in the metric contains a VTop-neighbourhood of ω. Put $R := R(\omega)$, choose a vertex v on R inside $B_\epsilon(\omega)$, and let u be its upper neighbour on R. Then $\lfloor u \rfloor \subseteq B_\epsilon(\omega)$; recall that the diameter of $\lfloor u \rfloor$ is no greater than the length of the ray vR, since half of this length accounts for the edge vu. Since ω is not dominated, Lemma 2.2 (iv) implies that there is a strict upper bound $x \in R$ on the R-heights of the neighbours of the vertices in $\lceil v \rceil$. Then $\lfloor x \rfloor \subseteq \lfloor u \rfloor$, and there is no edge between $\lfloor x \rfloor$ and $\lceil v \rceil$. Hence all neighbours of $\lfloor x \rfloor$ lie on the path $uRx \subseteq \lfloor u \rfloor \subseteq B_\epsilon(\omega)$. Thus for $S := \lceil x \rceil \setminus \{ x \}$ we have $V(C(S,\omega)) = \lfloor x \rfloor$ by Lemma 2.2 (ii) and $\widehat{C}(S,\omega) \subseteq B_\epsilon(\omega)$, as desired.

(iii) Since vertices of infinite degree do not have a countable neighbourhood basis in TOP, G must be locally finite if $|G|$ is metrizable. Conversely, if G is locally finite it has a normal spanning tree by Lemma 2.3. As TOP = MTOP for locally finite graphs, the corresponding metric as defined in the proof of (i) induces TOP. □

4. Compactness

If G is locally finite, then $|G|$ is well known to be compact (see e.g. [2] for a proof), and we recall that our three topologies for $|G|$ coincide in this case. If G has a vertex v of infinite degree, it cannot be compact in either TOP or MTOP. (Indeed, take an open cover that consists of an open subset of every edge at v and one large open set covering the rest but missing an inner point of every edge at v; this clearly has no finite subcover.) We therefore consider only VTOP in this section.

Given a finite set $S \subseteq V(G)$ and a component C of $G - S$, let us write $\widehat{C}$ for the union of C with the set (possibly empty) of ends that have a ray in C and the set of all inner points of C–S edges. This is always an open set: if C contains a ray, then $\widehat{C}$ is a basic open neighbourhood of the end of that ray, while if C contains no ray then $\widehat{C}$ is simply a union of its (open) edges and open stars around its vertices.

Theorem 4.1. *The following statements are equivalent in* VTOP *for any graph $G = (V, E)$.*

(i) *$|G|$ is compact.*

(ii) *For any finite $S \subseteq V$ the graph $G - S$ has only finitely many components.*

(iii) *Every closed set of vertices is finite.*

Proof. (i)→(iii): If $|G|$ is compact then so are its closed subsets. As every set of vertices is discrete, a compact set of vertices must be finite.

(iii)→(ii): If $G - S$ has infinitely many components, we can pick a vertex from each to obtain an infinite closed set of vertices.

(ii)→(i): Let an open cover of $|G|$ be given. Given a finite set $S \subseteq V$, call a component C of $G - S$ *bad* if $\widehat{C}$ lies in none of the cover sets. If, for some S, no component of $G - S$ is bad, we can take a finite subcover for $G[S]$ (which is compact because S is finite) and one cover set for each of the finitely many $\widehat{C}$ with C a component of $G - S$ to obtain a finite subcover for $|G|$. We may thus assume that $G - S$ always has a bad component, for every finite $S \subseteq V$.

Our aim now is to define a direction on G that points to just one bad component of $G - S$ for every S. Then the cover set for the end that defines this direction will contain some of those bad components, a contradiction.

As a first step, we assign to every finite $S \subseteq V$ the union D_S of all the bad components of $G - S$. These D_S already have the defining property of directions: if $S \subseteq S'$ then $D_S \supseteq D_{S'}$, because every bad component of $G - S'$ lies inside one of $G - S$. Hence, the intersection of any two of these D_S contains a third: $D_{S \cup S'} \subseteq D_S \cap D_{S'}$.

The supersets of the D_S (that is, the subsets of $|G|$ containing some D_S) thus form a filter; let $\mathcal{U}$ be an ultrafilter containing it. Note that $\mathcal{U}$ contains no (point set of a) finite subgraph $G[S]$, because the complement of $G[S]$ contains the filter set D_S and hence lies in $\mathcal{U}$. Moreover, since $G - S$ has only finitely many components C, one of the corresponding sets $\widehat{C}$ (and clearly only one) must lie in $\mathcal{U}$: otherwise $G[S]$ and these sets would form a finite partition of $|G|$ into sets not in $\mathcal{U}$, which is impossible. Define $f(S)$ as the unique component C of $G - S$ such that $\widehat{C} \in \mathcal{U}$. Note that $f(S)$ is bad: since $D_S \in \mathcal{U}$ we must have $f(S) \cap D_S \neq \emptyset$, and hence $f(S) \subseteq D_S$. Since every two ultrafilter sets meet, f is a direction.

By Lemma 2.1, there is an end ω such that $f(S) = C(S, \omega)$ for every S. Let O be a set in our open cover that contains ω. Since O is open, it contains a basic open neighbourhood $\widehat{C}(S, \omega)$ of ω. Then $f(S) = C(S, \omega) \subseteq O$, so $f(S)$ is not bad, a contradiction. □

For countable connected graphs,* Theorem 4.1 can be rephrased in terms of spanning trees. The equivalence of (ii), (iii) and (iv) below is a windfall that appears to have gone unnoticed before; note that its one-line proof does not rely on Theorem 4.1.

Corallary 4.2. *Let G be any graph that has a normal spanning tree. Then the following assertions are equivalent:*

(i) *$|G|$ is compact in* VTOP.

(ii) *G has a locally finite spanning tree.*

(iii) *G has a locally finite normal spanning tree.*

(iv) *Every normal spanning tree of G is locally finite.*

(v) *For no finite $S \subseteq V(G)$ does $G - S$ have infinitely many components.*

Proof. The implications (ii)→(v)→(iv)→(iii)→(ii) are trivial or follow from Lemma 2.2 (i). The equivalence between (i) and (v) is Theorem 4.1. □

* These have normal spanning trees by Jung's theorem; a direct proof is given in [2].

In some contexts, such as the duality of infinite planar graphs [1], the natural space in which to embed G is not $|G|$ but a quotient space of $|G|$, defined as follows. Assume that no two vertices of G can be linked by infinitely many independent paths. (In particular, no end is dominated by more than one vertex.) Let $\tilde{G}$ be the quotient space obtained from $|G|$ by identifying every vertex with all the ends it dominates.

Thus in $\tilde{G}$, the undominated ends (which are precisely its topological ends in Freudenthal's sense, see [8]) appear as extra points added to G as before, while every dominated ray converges to the unique vertex that dominates it.

The space $\tilde{G}$ was introduced in [7] on the basis of TOP, which unfortunately does not in general make it compact. When based on VTOP, however, $\tilde{G}$ is compact. Moreover, just as under TOP it is Hausdorff, even if $|G|$ is not:

Corollary 4.3. *Let G be 2-connected, and such that no two vertices are linked by infinitely many independent paths. If $|G|$ is endowed with* VTOP, *$\tilde{G}$ is a compact Hausdorff space.*

Proof. Since G is 2-connected, no finite set S of vertices can leave infinitely many components in $G - S$: otherwise, infinitely many of these would send edges to the same two vertices $u, v \in S$, yielding infinitely many u–v paths in G, contrary to our assumption. Hence $|G|$ is compact by Theorem 4.1, and $\tilde{G}$ is compact as a continuous image of $|G|$.

The disjoint open neighbourhoods of two given points in $\tilde{G}$ needed to show that $\tilde{G}$ is Hausdorff are the same as those constructed in [7, Theorem 4.7], where $\tilde{G}$ was shown to be Hausdorff under TOP. □

Finally, we remark that our proof of Theorem 4.1 re-establishes the following corollary of Polat's theorem that the topology which TOP induces on Ω is induced by a complete uniform space (see [14]). As our three topologies agree on Ω, we need not specify which of them we use:

Corollary 4.4. *The subspace $\Omega(G)$ of $|G|$ is compact if and only if for every finite $S \subseteq V(G)$ only finitely many components of $G - S$ contain a ray.* □

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