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Endogenous Debt Maturity: Liquidity Risk vs. Default Risk

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Abstract

We study the endogenous determination of corporate debt maturity in a setting with default risk. We assume that firms must access the bond market and they issue debt with a flexible structure (coupon, face value, and maturity). Initially, the firm is in a low growth/illiquid state that requires debt refinancing if it matures. Since lenders do not refinance projects with positive but small net present value, firms may be forced to default in the first phase. We call this *liquidity risk*. The technology is such that earnings can switch to a higher (but riskier) level. In this second phase firms have access to the equity market but they may default if this is the best option. We call this *strategic default risk*. In the model optimal maturity balances these two risks. We show that firms with poor prospects and firms in more unstable industries will **choose** shorter maturities even if it is feasible to issue longer debt. The model also offers predictions on how asset maturity, asset salability, and leverage influence maturity. Even though our model is extremely stylized we find that the predictions are roughly consistent with the evidence. Moreover, it offers some insights into the factors that determine the structure of the debt.

JEL: G23, G32, G33,
Keywords: Bonds, Debt, Maturity, Default, Bankruptcy, Leverage, Risk, Liquidity.

1 Introduction

The endogenous determination of the maturity structure of debt is a topic that interests both financial and macro economists. In the case of business firms—the case mostly analyzed by financial economists—early work by Diamond (1991) and by Leland and Toft (1996) has been followed by substantial work on the optimal structure of debt. In the macro literature, Lucas and Stokey (1983) highlighted the role that debt maturity has in supporting an optimal policy. Angeletos (2009) and Buera and Nicolini (2004) study the case of non-contingent debt while Shin (2007) allows for a maturity structure that changes depending on the state of the economy. In all cases, there is no default in equilibrium. In the case of sovereign debt, Arellano and Ramanarayanan (2012) develop a theory of debt maturity based on the relative costs of different types of debt which is consistent with evidence that in low-output periods countries have shorter-term debt. Aguiar et al. (2016) use the framework of Arellano and Ramanarayanan (2012) to argue that when a country must reduce debt under risk of default, it is optimal to remain passive in the long-term bond market (let those bonds mature) and use the short-term bond market to roll over part of such debt. Hatchondo et al. (2016) quantify the impact of reducing the ability of a country to dilute existing debt on average maturity and find it substantial. Fernandez and Martin (2015) highlight the trade-off between illiquidity and maturity when the country chooses between two types of debt and there is an incentive for debt dilution. Similarly, Sánchez et al. (2018) present a model of endogenous determination of sovereign debt maturity, but focus on the term structure of interest rate spreads and the impact of alternative debt rescheduling arrangements.

In this paper, we study a model of a risk-neutral agent—which we interpret as a firm—that chooses the type of debt that it will issue to finance an initial investment. We assume that there are two phases. In the first phase, the earnings-generating process (i.e. the technology) is in a "low growth" regime with steady but low earnings. During this phase, the firm has access only to the bond market, which is used to finance the initial investment and rollover debt. Because of the limited access to external financing, we also refer to this phase as "illiquid regime." The regime can switch stochastically to a "high growth" regime with a more uncertain —but potentially much higher— earnings process. In this phase the firm has access to perfect capital markets and, therefore, we also refer to this phase as the "liquid regime."

On the financing side, we allow for a general menu of noncontingent securities. We assume that the firm can choose the structure of its debt from a continuum of securities. These securities—that we view as bonds—are completely summarized by three parameters:

the coupon rate, b, that must be paid in every period, the face value (or the value that is due at maturity), K, and a Poisson parameter, η, that determines the stochastic maturity of the bond. The expected maturity of the bond is then given by $1/\eta$.[1] Upon maturity, the debt can be refinanced.[2] This flexible specification of the structure of debt has two major advantages over the constant maturity approach pioneered by Leland and Toft (1996) and then adopted by He and Xiong (2012) and Diamond and He (2014). First, it allows us to study debt that can be either front or backloaded, and it is a useful setting to capture the tradeoff between expected maturity and face value of a bond. Second, at refinancing time, the firm is completely free to change the structure of the debt. This is essential to understanding how changes in the prospects of the firm get reflected in changes in maturity.[3]

We assume that both the firm and credit market are risk neutral and have the same discount rate. We purposely abstract away from the strategic aspects associated with issuing short and long-term debt simultaneously since there is no risk of debt dilution. Moreover, we also ignore how the liquidity of the secondary market influences the choice of maturity which has been the focus of the recent literature. Our emphasis is on the non-strategic determinants of debt maturity and the role played by technology that we take as exogenously varying on the structure of debt. We view this as an interesting exercise since our "project/technology" could be, in some applications, viewed as the outcome of a market equilibrium. In this case exogeneity of some features (e.g. variability of the growth rate of earnings) is a reasonable assumption.

To highlight some of the economic forces that underlie the choice of maturity we first discuss a simple version of the model that rules out refinancing.[4] In this setting, we show that the market value of the debt relative to its safe value depends on two risk prices (or discounts). One of them captures the risk that the debt matures during the illiquid regime, while the other measures the cost of strategic default. We show how the state of the firm and the economic environment affect these two prices of risk, and we find that disentangling these effects provides intuition about the optimal maturity choice.

We tackle the analysis of the general model in the context of a quantitative exercise. We show that the structure of the debt depends both on the level of potential output and on features of the project/technology. Firms whose potential output (that is, the output

[1] We assume that maturity is stochastic to keep the model stationary.

[2] In our simple setting without an exogenous reason for issuing debt, the firm will never choose to refinance in the high growth/liquid regime. In this regime, the optimal policy once the debt matures is to either pay it off or default.

[3] In a recent paper, Chen et al. (2012) use a shortcut to allow for a change in maturity when the aggregate state changes within the constant maturity model.

[4] The limit to refinancing is an extreme form of rollover risk: the probability of rolling over the debt is effectively zero.

that would be obtained if the project moved from the low growth/illiquid regime to the high growth/liquid regime) is low—and hence more likely to default for strategic/solvency reasons—choose *shorter* bonds. As potential output increases, the risk of strategic default decreases and the optimal bond is chosen so as to decrease the risk of default associated with the bond maturing in the low growth regime. This is accomplished by increasing expected maturity of the bond at the cost of a higher face value and, hence, of a higher premium paid for strategic default.

Consider what happens when a firm (or a country) has to refinance its debt. If at the time of refinancing the firm's prospects are poor the market charges a high price for strategic risk while the level of earnings has no impact on the price of illiquidity risk in our setting. A firm that wants to decrease the implicit price it pays for strategic risk —at the cost of increasing the price of illiquidity risk— can decrease the expected maturity of the debt. Our model implies that the lower the (optimal) value of the firm the lower its optimal choice of maturity. An outside observer without a full understanding of the problem solved by the firm might be tempted to reverse the direction of causality and conclude that the firm made a "mistake" issuing short debt. This, in turn, would exacerbate the chances of default.

The (fixed) properties of the technology have an impact on the choice of maturity: firms that operate in more unstable environments choose shorter maturities and we find that the expected maturity-output relationship is flatter the higher the uncertainty about growth rates. Thus, the model not only implies that firms in high uncertainty environments borrow using more short-term debt than similar firms in a low uncertainty environment, but also that the cross-sectional dispersion of expected maturities is smaller in the high uncertainty case.

Our results show that the degree of salability of the firm's assets (or, alternatively the cost of fire sales) also influences the choice of the optimal financing structure. We find that higher post-default value of the assets results in longer expected maturities. The result is intuitive: A higher recovery rate—conditional on default—reduces the cost of strategic default and hence its shadow price. The firm then chooses to reduce the higher price risk—the risk of illiquidity driven default—by choosing a bond with longer duration.

For the examples that we study, we find that higher levered firms (i.e. firms with higher financing needs) choose shorter maturities while, at the same time, selling bonds with similar face values. Thus, unlike other features of the environment, changes in leverage have a large impact on expected maturity and a small impact on the face value of the optimal bond.

Not surprisingly, we find that the optimal choice of debt structure cannot be summarized as the minimization of some simple measure of cost like the excess yield. More specifically, we show that the optimal choice often does not imply that the excess yield is minimized.

1.1 Review of Empirical Findings

Although our theory presents a potentially important trade-off determining debt maturity, we do not expect that it will account for all the variation in debt maturity in the data. Reviewing empirical studies, however, it is useful to illustrate how our simple theory can account for some key findings. Since the literature is very broad, we provide here a short description of the studies with determinants that are captured by our theory.[5] In Appendix A, we reproduce some of those findings using non-financial and non-regulated firms in Standard & Poor's Compustat.[6]

Barclay and Smith (1995) uses Compustat data to study debt maturity. They restrict their analysis to firms only industrial corporate sector. Debt maturity is measured as the share of the firm's total debt with a maturity longer than 3 years, which is a very common choice given the information provided by Compustat. We focus our attention here on their finding of a negative relationship between R&D investment / firm value and debt maturity. This result seems to contradict our theory, which predicts that firms with higher potential growth (as measured by the expected growth rate of earnings) would prefer to issue longer term debt. Guedes and Opler (1996) reproduce some of the findings in Barclay and Smith (1995) using an alternative dataset with information about debt issuances. However, they find a positive relationship between R&D investment-to-sales ratio and debt maturity. Our findings in Appendix A, using a broader sample of Compustat firms, are in line with Guedes and Opler (1996).

Stohs and Mauer (1996) also use a broad sample of Compustat firms to analyze the determinants of debt maturity. They find that less risky firms use longer-term debt. A similar finding is reported by Okzan (2002) for UK firms. In Appendix A, we also find that firms in more volatile environments tend to issue shorter-term debt. They argue that this may be because riskier firms need to re-balance their capital structure to moderate expected bankruptcy costs. We will provide an alternative explanation for this finding.

Guedes and Opler (1996), Stohs and Mauer (1996) and Okzan (2002) also show that firms with longer-term asset maturities use longer-term debt. This relationship is expected from the idea of "matching principle," where firms would like to match the maturity of their assets to the maturity of their debt in order to avoid defaulting when the debt matures earlier. In Appendix A, we also find that firms with a higher proportion of assets maturing at a long-term period also issue debts with longer-maturity, though this coefficient is not

[5] Some articles that we do not include here empirically analyze debt maturity in the context of the capital structure of the firm. See, for instance, Titman and Wessels (1988).

[6] Most of the empirical literature on determinants of debt maturity uses data on publicly traded companies. Ideally, one would like to analyze data on all firms but available balance sheet data on non-traded firms is very limited.

statistically significant.

Demirgüç-Kunt and Maksimovic (1999) find that firms with higher asset salability, thus higher post-default value of assets, are associated with longer maturity. This result is also consistent with the findings of Benmelech (2009), who reports that American railroads which used rolling stock that could be more easily redeployed (higher resale value) tended to issue longer term bonds. Our results in Appendix A are also in line with this finding.

Barclay et al. (2003) find that leverage is negatively associated with debt maturity. In contrast, Johnson (2003) find a positive relationship between maturity and leverage, and argues that firms with high leverage choose longer term debt to avoid liquidation. This effect is probably more important for firms with a very high risk of default. In fact, our results in Appendix A using Compustat firms suggest that firm with higher leverage (actually, with more reliance on external financing) prefer shorter term debt. In our theory, firms with higher leverage would choose shorter debt maturity.

2 Model

In this section, we describe the simplest version of the model. We study the case of a risk-neutral firm that has to incur an irreversible cost to implement an investment project. We assume that the returns of the project are completely described by a few parameters that determine the associated stochastic processes for output (or earnings). We assume that the project is initially in the **low growth/illiquid regime** and has constant earnings given by z. The amount of time spent in this regime is random and distributed exponentially with parameter υ. Denote by T_υ the time spent in the low growth regime. At time T_υ the technology switches from the **low growth/illiquid regime** to the **high growth/liquid regime** and net earnings/output are given by a stochastic process $x_t + \theta z$, where $\theta \in (0,1)$ and x_t satisfies

$$dx_t = \mu x_t dt + \sigma x_t dW_t,$$

where W_t is a Wiener process or Brownian motion, and μ ("the percentage drift") and σ ("the percentage volatility") are constants. This formulation captures the idea that some managerial resources must be allocated to the x_t-dimension and, hence, the net output from the low growth activity decreases (as captured by $\theta < 1$). In what follows we view the technology as a triplet (υ, μ, σ).

We assume that x_t is known at all times. That is, the firm—and the market—know its potential output if the technology switched to the high growth regime. We restrict the feasible technological choices to lie in a set indexed by the current potential productivity, that is, $(\upsilon, \mu, \sigma) \in \Lambda(x)$. We assume an extreme form of moral hazard: In the illiquid regime no outsider can operate the project and, hence the firm cannot issue equity. In the high growth regime, the firm can issue equity and its market price would be the expected present value of earnings.

2.1 Debt Financing

In this section, we assume that the firm has to finance either partially or totally the cost of the project by issuing non-contingent debt. The firm has access to a risk-neutral credit market that will price any debt issued by the firm using the same discount rate, r. Thus, none of our results are driven by either difference in discount factor or difference in the curvature of direct payoffs.

We study a market wherein the set of feasible debt instruments is completely described by three parameters (b, K, η), where b is the coupon rate, K is the payment due at maturity (which does not necessarily coincide with the initial value of the debt) and η is the parameter of a Poisson process that determines the maturity of the bond. Thus, maturity is stochastic and the maturity time is denoted T_η. The expected maturity of a bond is $1/\eta$.[7] Given that the length of the illiquid phase is uncertain, even if maturity was not stochastic by choosing higher maturity firms would be choosing a lower probability that the debt matures in the illiquid phase.

We assume that if the firm defaults on its debt in the low growth/illiquid regime, bondholders receive D_B.[8] If the firm defaults in the high growth regime bondholders receive D_G.[9]

Our simple version of the technology puts some restrictions on the type of bonds that the firm can issue. Since we assumed that the firm cannot issue equity and that earnings are z in the low growth regime, it follows that it cannot commit to paying a coupon rate that exceeds this value. Thus, the only feasible choice is $b \leq z$.

If the debt matures in the illiquid regime it must be refinanced. If the firm cannot issue debt that raises enough revenue to pay off the original debtholders, the firm will default. This is the *illiquidity risk* associated with the bond. In case of default, the payoff to bondholders is denoted D_B and shareholders receive zero. The cost of refinancing (a fixed cost) is denoted C_F. Given that in the event that the debt has to be refinanced there is no other debt outstanding, the firm chooses a new debt structure, (b', K', η') subject only to the restriction

[7] We only consider bonds that make payments that are not contingent on the state of the firm. In particular, we rule out callable bonds.

[8] For example, this includes the case in which debtholders receive a certain fraction of the value of the certain stream z, denoted by $(1 - \delta_L)z/r$. We could interpret δ_L as a measure of the value of the firm in secondary markets including the changes in values associated with the probability of fire sales.

[9] As in the case of the residual value of the debt in the low growth regime D_G captures a variety of factors including the probability of fire sales, the strength of debtors rights and costs of bakruptcy. In our quantitative exercise we let this value depend on the value of the assets held by the firm.

that the market value of the debt must be at least as high as the amount to be refinanced. Note that the firm can hedge the risk of not being able to refinance when it first issues debt by choosing the original (b, K, η) bond such that its face value, K, is small.

In the high growth/liquid regime, depending on the values of b and θz the firm has incentives to strategically default either before the debt matures (when x is low) or at maturity. Formally, this corresponds to the case in which in the high growth regime the firm can issue equity to pay its creditors. We refer to this case as strategic default risk, or just *default risk*.

As it is standard in these models the existence of debt financing could potentially lead to the liquidation of the firm and this is inefficient given our assumptions about the stochastic process for earnings. Thus, debt financing can potentially lead to inefficient liquidation, as in as in Diamond (1991) and Leland (1994).

2.1.1 The Value of the Firm in the Liquid Regime

The market value of a productive firm that has issued a (b, η, K) bond, denoted by $T(x; z, b, \eta, K)$, is

$$T(x; z, b, \eta, K) = \Big\{ E \int_0^{T_s \hat{} T_\eta} e^{-rs} \left(x_{t+s} + \theta z - b\right) ds$$
$$+ e^{-rT_\eta} \max\left(\frac{x_{T_\eta}}{r-\mu} + \frac{\theta z}{r} - K, 0\right) \aleph_{[T_\eta < T_s]} \mid x_t = x \Big\}.$$

Here $T_s = \inf\{t : x_t \leq x^*\}$ is an endogenously chosen stopping time that determines default before the debt matures, and $\aleph_{[T_\eta < T_s]}$ is the indicator function of the event $[T_\eta < T_s]$. The first term is the value of net output until default or maturity, while the second term is the residual value of the firm. The optimal default rule at maturity is very simple: Default if and only

$$\frac{x_{T_\eta}}{r-\mu} + \frac{\theta z}{r} < K,$$

that is, if the value of the assets falls short of the face value of the outstanding debt.

Standard arguments show that $T(x)$ satisfies the following Hamilton–Jacobi–Bellman (HJB) equation (where the arguments after the semicolon are suppressed to keep the notation simple):

$$rT(x) = x + \theta z - b + \frac{\partial T}{\partial x}(x)\mu x + \frac{\partial^2 T}{\partial x^2}(x)\frac{\sigma^2}{2}x^2 + \eta[\max(\frac{x}{r-\mu} + \frac{\theta z}{r} - K, 0) - T(x)], \tag{1}$$

and the boundary conditions are (if $b = 0$)

$$\lim_{x\to 0} T(x) = 0, \text{ and } \lim_{x\to\infty} \frac{T(x)}{x} = 0,$$

while if $b > 0$, the optimal default before the debt matures is the first time that the process $\{x_t\}$ hits x^*, where x^* satisfies the standard value matching and super-contact conditions given by,

$$T(x^*) = 0, \text{ and } T'(x^*) = 0.$$

Let's denote the lowest value of x that triggers bankruptcy **at maturity** by $\bar{x}$. Thus,

$$\bar{x}(K) = (r - \mu)\left(K - \frac{\theta z}{r}\right).$$

Of course, if the face value of the debt is sufficiently small, the above formula implies that $\bar{x}(K) < 0$ and there is no default at maturity. Appendix B discusses the details of the solution to equation (1) but, in general, it is possible —depending on the choice of debt structure— to observe the following cases:

- The debt is safe. This implies no default at any time.
- The debt is safe until it matures. This means that the firm will choose to honor the coupon but potentially default at maturity.
- The debt is not safe in any state. In this case—depending on x—the firm will choose to default before the debt matures—standard strategic default—or at maturity (a type of solvency default).

2.1.2 The Value of the Debt in the Liquid Regime

The value of a (b, η, K) bond (again suppressing the constants from the notation) satisfies

$$\begin{aligned} rB(x) = b + \frac{\partial B}{\partial x}(x)\mu x + \frac{\partial^2 B}{\partial x^2}(x)\frac{\sigma^2}{2}x^2 \\ + \eta[\Pi(x, K) - B(x)], \end{aligned} \tag{2}$$

where

$$\Pi(x, K) = \begin{cases} K & if\ x \geq \bar{x}(K) \\ D_G & if\ x < \bar{x}(K) \end{cases}.$$

Moreover, if $b > 0$, then the choice of bankruptcy by the firm implies that

$$B(x^*) = D_G.$$

If $b = 0$, the value of the debt must satisfy $\lim_{x\to 0} B(x) = 0$. Since the value of risk free debt is given by

$$B^* = \frac{b + \eta K}{r + \eta},$$

then the no-bubble condition is simply

$$\lim_{x\to\infty} [B(x) - B^*] \leq 0.$$

2.1.3 Valuations in the Illiquid Regime

In order to describe the value of the firm and the debt in the illiquid regime it is necessary to take a stand on the possibility that the firm has to refinance any debt that matures in the illiquid state. Let $M(x;K)$ be the value of a firm that has just refinanced a maturing bond with face value K.

Consider now the value of a firm that has issued a (b, η, K) bond and has the function $M(x;K)$ —which must be determined endogenously— as its continuation value after refinancing. We take that $M(x;K) = 0$ corresponds to default.

Let $V(x; b, \eta, K; M(x;K))$ be the value of the firm in the illiquid regime when (potential) earnings are given by x and the firm has debt characterized by the triplet (b, η, K). Note that potential earnings also depend on z, but since z is a constant we suppress it to simplify the notation. Then,

$$\begin{aligned} V(x; b, \eta, K; M(x;K)) = E\left\{\int_0^{T_\upsilon \wedge T_\eta} e^{-rt}(z-b)dt + e^{-rT_\eta} M(x_{T_\eta}; K)\aleph_{[T_\eta < T_\upsilon]} + \right. \\ \left. e^{-rT_\upsilon} T(x_{T_\upsilon})\left(1 - \aleph_{[T_\eta < T_\upsilon]}\right)] \mid x_0 = x, \right\}, \end{aligned} \quad (3)$$

where $\aleph_{[T_\eta < T_\upsilon]}$ is the indicator function of the event $\{T_\eta < T_\upsilon\}$. The first term corresponds to the expected present value of net earnings, $z - b$, until the time when either the debt matures, $T_\eta < T_\upsilon$, and the firm's continuation value is $M(x_{T_\eta}; K)$, or the project switches to the liquid regime $T_\upsilon < T_\eta$) and the value of the firm is given by its valuation in this regime, $T(x_{T_\upsilon})$.

The valuation of the firm, V, satisfies:

$$\begin{aligned} rV(x) = z - b + \frac{\partial V}{\partial x}(x)\mu x + \frac{\partial^2 V}{\partial x^2}(x)\frac{\sigma^2}{2}x^2 \\ +\eta\left[\max\left(M(x,K) - C_F, 0\right) - V(x)\right] + \upsilon\left[T(x) - V(x)\right]. \end{aligned} \quad (4)$$

In addition, the solution must satisfy the appropriate boundary conditions to rule out bubble solutions. An important property of the solution to this valuation problem is that the function $V(x;b,\eta,K;M(x;K))$ is increasing in $M(x;K)$ for any given debt structure.

We now describe the market value of the debt, $L(x;b,\eta,K;M(x,K))$. It is given by

$$L(x;b,\eta,K;M(x,K)) = E\left\{\int_0^{T_\upsilon \wedge T_\eta} e^{-rt}bdt + e^{-rT_\eta}[\Lambda(x_{T_\eta},K)\aleph_{[T_\eta<T_\upsilon]} + e^{-rT_\upsilon}B(x_{T_\upsilon})\left(1-\aleph_{[T_\eta<T_\upsilon]}\right)] \mid x_0 = x\right\}. \tag{5}$$

The value of the bond in the event that it matures in the illiquid regime depends on the value to the firms of refinancing. Formally,

$$\Lambda(x,K) = \begin{cases} K & \text{if } M(x,K) - C_F \geq 0 \\ D_B & \text{otherwise} \end{cases},$$

that is, the bond gets repaid if the value to the firm of refinancing, net of refinancing costs C_F, is positive. If that is not the case, the firm defaults and the post-default value of the debt is denoted D_B

Standard arguments (if the function L is C^2) show that the solution to equation (5) also satisfies the following HJB equation[10]:

$$\begin{aligned} rL(x) &= b + \frac{\partial L}{\partial x}(x)\mu x + \frac{\partial^2 L}{\partial x^2}(x)\frac{\sigma^2}{2}x^2 \\ &\quad + \eta\left[\Lambda(x,K) - L(x)\right] \\ &\quad + \upsilon\left[B(x) - L(x)\right]. \end{aligned} \tag{6}$$

Since increases in the refinancing value of the firm, $M(x;K)$, increase the chance that a bond maturing in the illiquid regime will be repaid, it follows that the market value of the bond, $L(x;b,\eta,K;M(x,K))$, is increasing in $M(x,K)$.

A firm that needs to refinance a bond with face value K faces the following constraints. It can issue a new debt, (b',η',K') that raises K. If the market is unwilling to lend the necessary amount the firm must default. Thus, a first constraint is that the market value of the new debt (b',η',K') must be greater than or equal to the amount to be refinanced plus the refinancing cost. This is simply

$$L(x;b',\eta',K';M(x,K')) \geq K + C_F. \tag{7}$$

[10]The Feynman-Kac theorem can be used to show that, for fixed stopping times, L is C^2 even though $\Lambda(x,K)$ is an indicator function.

Since in this sequential setting the possibility of repeated refinancing can give rise to a bubble-like solution, we also impose that a debt that has no chance of maturing during the illiquid regime and that it gets repaid with probability one, cannot exceed the value of the project if it was debt free. This vaue is denoted as $\hat{L}(b', \eta', K')$. Formally, we impose that

$$\frac{x}{r-\mu} + \frac{r+\upsilon\theta}{r+\upsilon}\frac{z}{r} \geq \hat{L}(b', \eta', K') = \frac{b'}{r+\upsilon} + \frac{\upsilon}{r+\upsilon}\frac{b'+\eta K'}{r+\eta}. \tag{8}$$

The left side of equation (8) is the value of a firm that has zero debt, while the right side is the value of a bond with no default risk but no option to be refinanced.

We can now describe the problem faced by a firm that has to choose a new debt structure to refinance maturing debt with face value K. The firm solves the following problem,

$$\sup_{(b',\eta',K')\in\Sigma(x,K,M)} V(x; b', \eta', K'; M(x; K')) \tag{9}$$

subject to

$$\Sigma(x, K, M) \equiv \{(b', \eta', K') : L(x; b', \eta', K') \geq K + C_F \tag{10}$$
$$\text{and } \frac{x}{r-\mu} + \frac{r+\upsilon\theta}{r+\upsilon}\frac{z}{r} \geq \frac{b'}{r+\upsilon} + \frac{\upsilon}{r+\upsilon}\frac{b'+\eta K'}{r+\eta}\}.$$

We assume that if $\Sigma(x, K, M)$ is empty then the value of the firm is zero.

We find it useful to view the maximized value in equation (9) as defining an operator H. Thus,

$$H(M)(x, K) = \sup_{(b',\eta',K')\in\Sigma(x,K,M)} V(x; b', \eta', K'; M(x; K')). \tag{11}$$

The the equilibrium refinancing function —which is also the value of the firm— is a fixed point of the operator H.

Proposition 1 *There exists at least one fixed point of H.*

Proof. See Appendix G ■

The reason we cannot replace the sup operator with a max operator is that the function $M(x; K')$ is not necessarily continuous.[11]

[11] We can show that H maps lower semi-continuous functions into themselves with a discontinuity when there is default. In the continuous region the standard argument shows that the max is well defined.

2.2 Choosing the Debt Structure

The problem faced by a firm that has to finance its investment issuing debt is similar to the problem faced by the firm who refinances existing debt. If the cost of the project is C and the firm has resources given by $\overline{O}$ then the problem of the firm is equivalent to the defined by equations (9) and (10) with $K = C - \overline{O}$.

3 The One-Bond Example: Illiquidity Risk vs. Strategic Risk

In order to highlight the interplay between technological parameters and the choice of financing, we describe some results in a special case of the general model which provide insights into the factors that influence the choice of maturity. We study two deviations relative to the general model: no refinancing of the initial debt and no safe income ($\theta = 0$) in the liquid regime. We relax these assumptions in Section 5, where we present the numerical results.

Finally, to simplify the algebra, we also assume that, upon default, bondholders get zero.[12] The relevant events in this case are:

Illiquid Regime

- Firm has enough (safe) income to pay the coupon b. [Formally, $z \geq b$.]
- If the debt matures in this regime it will be defaulted. This is the *illiquidity risk*.

Liquid Regime

- Depending on the structure of the bond, that is, the values of (b, η, K), the firm will choose to default strategically:
  - Potentially before the debt matures (low K). [This corresponds to $x^* > \bar{x}(K)$.]
  - Either before or at the time that the debt matures (high K). [This corresponds to $x^* < \bar{x}(K)$.]

The previous two scenarios correspond to what we label as *strategic risk*.

[12] Assuming specific rules for the default payoff of the bonds simply adds a number of constants and makes the algebra messier.

It is possible to model the market value of the debt in the illiquid regime as being determined by a discount over the value of a similar (b, η, K) debt with zero risk, that is, debt with no liquidity or strategic risk.

Formally, let the market value of the debt be denoted $L(x; b, \eta, K)$. Without loss of generality, we can define two "prices" or "discount factors" which we view as capturing the way the market prices the illiquidity risk and the strategic risk. We denote by Q our measure of the *price of illiquidity risk*, and we label S, our measure of the *price of strategic risk*.

The market price of a bond (b, η, K) in the absence of illiquidity and strategic risk, that is, if $Q = S = 0$, is

$$B^* = \frac{b + \eta K}{r + \eta}.$$

Given this definition we can view the market value of the debt as a discount relative to the riskless version. Thus, our risk prices Q and S are such that

$$L(x; b, \eta, K) = B^* \left(1 - Q(b, \eta, K) - S(x; b, \eta, K)\right),$$

where, anticipating the results that follow, we specify that the price of illiquidity risk —the risk that the bond matures in the illiquid regime— is independent of potential earnings x.

There is a tight connection between these risk prices and the excess yield of the debt issued by the firm. To see this, let us denote the yield of debt with market value $L(x; b, \eta, K)$ as the value of the discount rate, $\tilde{r}$, that would make the value of a riskless version of the debt using that discount rate equal its market value. Thus,

$$\frac{b + \eta K}{\tilde{r} + \eta} = L(x; b, \eta, K) = B^* \left(1 - Q(b, \eta, K) - S(x; b, \eta, K)\right).$$

It follows that —omitting the arguments in the functions—

$$\tilde{r} - r = \frac{(r + \eta)\left(Q + S(x)\right)}{\left(1 - \left(Q + S(x)\right)\right)}.$$

The excess yield of the debt issued by the firm depends on the sum of the two risk prices. Any factors that increase this sum will result in higher excess yield.

3.1 Measuring Risk Prices

We define the value of a (b, η, K) debt that has **no illiquidity risk** as the value of the same (b, η, K), denoted $L_{NI}(x; b, \eta, K)$, except that, if it matures in the illiquid regime, debtholders receive K. This is unlike the payoff of the debt issued by the firm that in that event (maturity in the illiquid regime) pays zero to debtholders. Simple calculations show that

$$L_{NI}(x; b, \eta, K) = L(x; b, \eta, K) + \frac{\eta K}{r + \eta + \upsilon},$$

and we view **the market value of the illiquidity risk** as the difference between the value of the debt with no illiquidity risk and the debt that has illiquidity risk, that is

$$L_{NI}(x; b, \eta, K) - L(x; b, \eta, K) = \frac{\eta K}{r + \eta + \upsilon}.$$

It follows that the price of illiquidity, Q, is

$$Q(b, \eta, K) = \frac{\eta K}{b + \eta K} \frac{r + \eta}{r + \eta + \upsilon}. \tag{12}$$

Using the definition of the risk prices, the price of strategic risk is

$$S(x; b, \eta, K) = \frac{b}{b + \eta K} + \frac{\eta K}{b + \eta K} \frac{\upsilon}{r + \eta + \upsilon} - \frac{r + \eta}{b + \eta K} L(x; b, \eta, K). \tag{13}$$

3.2 Properties of Risk Prices

Price of Illiquidity Risk There are two elements that influence the price of illiquidity risk in equation (12). First, the degree of backloading of debt payments —as measured by the ratio $\eta K/(b + \eta K)$— has a positive impact on the market price of risk. Second, the maturity of the debt —as measured by η— and the maturity structure of the underlying assets —as measured by υ— also influence the market price of illiquidity risk. It is straightforward to see that:

- **Expected Maturity of the Debt.** Increases in the expected maturity of the debt (decreases in η) decrease the price of illiquidity risk. In particular, $\lim_{\eta \to \infty} Q = 1$ and $\lim_{\eta \to 0} Q = 0$. This signifies that short term debt bears a high implicit price of illiquidity risk while consols are priced as if there is no illiquidity risk.

- **Asset Maturity**. Increases in the expected duration of the illiquid regime ($1/\upsilon$) have similar effects: when the illiquid regime is arbitrarily short ($\upsilon \to \infty$), then $Q = 0$. At the other end when the illiquid regime lasts forever ($\upsilon = 0$) then the price of illiquidity risk is proportional to the degree of backloading.

Before discussing the price of strategic risk it is useful to be explicit about the valuation of the debt. As in section 2, let x^* be the highest value of income that triggers default during the liquid regime. This threshold depends on the structure of the debt —the values of (b, η, K)— as well as the parameters defining the economic environment, $(\sigma, \upsilon, \mu, r)$, but to keep the notation simple we do not make this dependence explicit.

Price of Strategic Risk To study the properties of $S(x; b, \eta, K)$ we first need to establish some properties of the market value of the debt. As shown in the previous section, the default rule at maturity in the liquid regime is to default whenever $x < \bar{x}(K) = (r - \mu)K$, while the optimal rule before maturity is to default the first time that x drops below x^*. This last event is associated with the stopping time T_{x^*} given by

$$T_{x^*} = \inf\{t : t \geq T\upsilon \text{ and } x_t \leq x^*\}$$

The value of the bond once the firm is in the liquid regime, denoted $B(x; b, \eta, K)$, is given by

$$B(x; b, \eta, K) = E\left\{\int_0^{T_\eta \wedge T_{x^*}} e^{-rt} b dt + e^{-rT_\eta}[K\aleph_{[(T_\eta < T_{x^*}) \cap \left(x_{T_\eta} \geq \bar{x}(K)\right)]}] \mid x_0 = x\right\}.$$

It is immediate to show that $B(x)$ —again suppressing most of the arguments— is an increasing function of x (see the explicit solutions in Appendix C). The value of the debt in the illiquid regime —which is the relevant one to study the choice of maturity since it is at this stage that the firm selects the structure of the debt— is given by

$$L(x; b, \eta, K) = E\left\{\int_0^{T_\upsilon \wedge T_\eta} e^{-rt} b dt + e^{-rT_\upsilon}[B(x_{T_\upsilon})\aleph_{[T_\upsilon > T_\eta]}] \mid x_0 = x\right\}.$$

It follows that the market value of the bond in the illiquid state also increasing in x, since this corresponds to a lower probability of default.

Equation (13) then implies that $S(x; b, \eta, K)$ is *decreasing* in potential earnings (x). Thus, our model implies that firms with better prospects (higher x) face lower prices of strategic risk. $S(x; b, \eta, K)$ also depends on the structure of the debt and the features of the economic environment.

We can now use the results in Appendix C to analyze the implications of changing some of the parameters. Since the details depend on the particular value of x, as well as the relationship between x^* and $\bar{x}(K)$, here we report some general results. A full account of the details is in Appendix C.

We find that:

- **Uncertainty**. When the degree of uncertainty —as measured by σ— is arbitrarily low, the market price of strategic risk depends on the face value of the bond.

  – If the face value K is low (the precise value is given in Appendix C), we show that,

$$\lim_{\sigma\to 0} S = \begin{cases} 0 & \text{for } x \geq x_I^* \\ \frac{\upsilon}{r+\eta+\upsilon}\left(1-\left(\frac{x}{x_I^*}\right)\right) & \text{for } x \leq x_I^* \end{cases},$$

  where x_I^* is the level of income at which the firm decides to go bankrupt (see Appendix C).

  If the face value K is high, we get that

$$\lim_{\sigma\to 0} S = \begin{cases} 0 & \text{for } x \geq \bar{x}(K) \\ \frac{\upsilon}{r+\eta+\upsilon} - \hat{S}_M(x) & \text{for } \bar{x}(K) \geq x \geq x_{II}^* \\ \frac{\upsilon}{r+\eta+\upsilon} - \hat{S}_L(x) & \text{for } x_{II}^* \geq x \end{cases},$$

  where the nonnegative functions $\hat{S}_M(x)$ and $\hat{S}_L(x)$ are defined in Appendix C, and x_{II}^* is the level of income that triggers bankruptcy in this case.

  At the other end —when $\sigma \to \infty$— the price of strategic risk is higher and, independently of the state of the firm, satisfies

$$\lim_{\sigma\to\infty} S = \frac{\upsilon}{r+\eta+\upsilon}.$$

  In general our numerical results show that S is increasing in σ.

- **Asset Maturity**. Changes in the expected duration of the illiquid regime have the following impact on the price of strategic risk:

$$\lim_{\upsilon\to 0} S = \frac{\upsilon}{r+\eta+\upsilon} = 0$$

  and for $x_j^* \in \{x_I^*, x_{II}^*\}$ (depending, as before, on the value of K)

$$\lim_{\upsilon\to\infty} S = \begin{cases} > 0 & \text{for } x \geq \min(x_j^*, \bar{x}(K)) \\ 1 & \text{for } x \leq \min(x_j^*, \bar{x}(K)) \end{cases}.$$

  When the illiquid regime lasts a long time ($\upsilon \to 0$), the market value of the debt is governed by the illiquidity risk and, consequently, the price of strategic risk is zero. At the other end, when the duration of the illiquid regime is arbitrarily short, there is no illiquidity risk and the price of strategic risk is high if x is low because this implies immediate default. If x is above the level that triggers an instantaneous default, then the price is lower but still not zero since in this case the price of debt is given by the function $B(x)$.

- **Expected Maturity of Debt**. We find that

$$\lim_{\eta \to \infty} S = 0, \text{ and } \lim_{\eta \to 0} S > 0.$$

The market charges a positive price for the strategic risk in the case of long bonds ($\eta < \infty$). Our numerical results show that S is decreasing in η.

How can understanding the factors that move the prices of the two risks help us understand the forces that influence debt maturity? Here we present a heuristic argument that suggests a possible connection between changes in the economic environment and the choice of maturity. The next section discusses similar forces in the using our calibrated model.

Potential Earnings and Maturity Consider a firm that has optimally chosen the maturity of its bond and suppose that there is an increase in the value of its potential earnings, x. How should the firm adjust the maturity of its debt if it could reissue it? A higher x implies that the price of strategic risk, $S(x)$, is lower while the price of illiquidity risk, Q, has not changed. If the firm **decreases** the expected duration of the debt (higher η) this results in a lower price of illiquidity risk and, using the limiting results as good indicators of the direction of the change, a higher price of strategic risk. Thus, in a sense, lowering maturity allows the firm economize in the risk factor whose price has increased (strategic risk) at the cost of increasing the price of the other risk (illiquidity risk).

Uncertainty and Maturity Let us consider the impact of an increase in σ. According to our results (again using the limits as good indicators of movements over the whole range), this increases the price that the market changes for strategic risk, S, while the price of default risk, Q, is unchanged. What can the firm do to compensate for those changes? If the firm shortens the maturity of the debt (increases η) then it simultaneously increases the price of illiquidity risk and decreases the price of strategic risk. Thus, as before, such a change in maturity has the effect of economizing on the factor that has become more expensive (strategic risk) by increasing the amount paid for illiquidity risk. Changing the maturity of the debt provides one way of accomplishing this.

In the following section, we show the quantitative implications of a calibrated version of the model that allows for refinancing.

4 Quantitative Analysis

4.1 Calibration

In order to better characterize the choice of maturity, we present a quantitative illustration of the full model. The only simplification of the full model is that $b = 0$. We made this assumption because it does not affect the choice of maturity. In Section 4.3 we incorporate coupons and show how they are optimally chosen.

The calibration, shown in Table 3, is extremely simple since there are very few relevant parameters. We set the value of 3 parameters equal to zero: z and θ are set to zero because they are relevant only to the case in which $b > 0$. Similarly, the value of μ is also set to zero as a normalization because the relevant variable for discounting is $r - \mu$.

Table 3: Parameters

Parameters	Value	Basis
(μ, z, θ)	0	Normalization
Risk free rate, r	0.05	Standard
Refinancing cost, C	0.05	Issuance cost-to-debt ratio
Volatility, σ	0.2	Sales volatility
Arrival prod. phase, υ	1	Debt maturing under 5 years
Recovery, δ	0.6	Debt-to-equity

There are 5 remaining parameters. The value of the cost of issuing debt, C, is set to 0.05, which is slightly above 1 percent of the amount of debt issue. The interest rate, r, is set to a standard value, 5 percent. The value determining the value of volatility of the process for x, σ, is calibrated a priori at 0.2 because that is the median value for the standard deviation of the first difference of sales (see Table 4). The value of the funds that must be raised, K, is chosen such that it coincides with the maximum that can be raised for the lowest x, $K = 1.68$. The remaining 2 parameters—the Poisson parameter determining the probability of arrival of the productive phase, υ, and the share recovered of the value of the firm in case of default, δ—are calibrated such that the model provides reasonable predictions for the debt-to-asset ratio and debt maturity. Note, however, that to compare the model with data we need to pick the value of x that would represent the median firm. We pick the value of x which best represents the state of the median firm in the data in terms of the debt-to-equity ratio.

Table 4: Fit of Calibration Targets

	Model	Median Firm
Debt-to-Equity ratio	0.205	0.208
Debt-to-Asset ratio	0.176	0.195
Share of debt with maturity under 3 years	0.465	0.606
Share of debt with maturity under 5 years	0.647	0.870
Volatility (Sales)	0.200	0.209

Note: See Appendix A for an explanation of the construction of these variables.

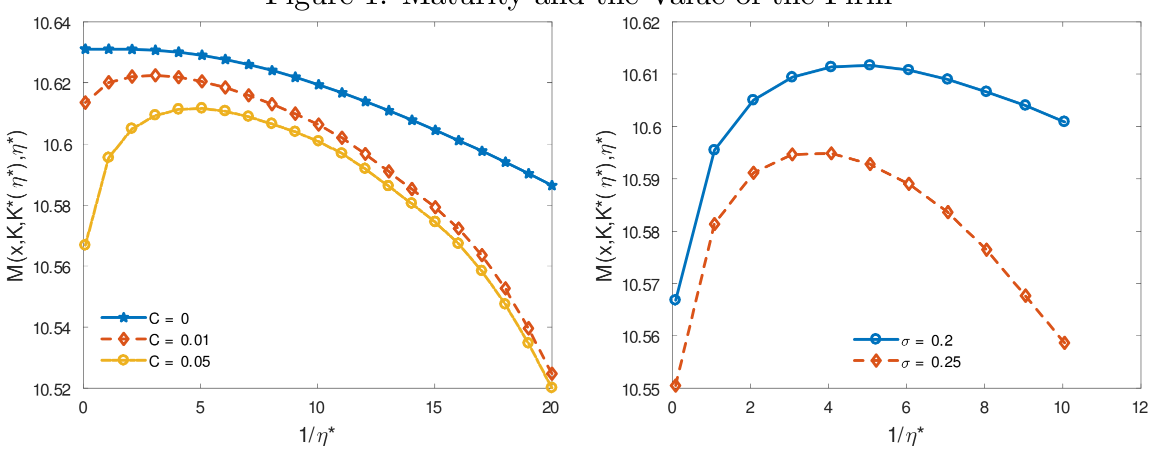
4.2 Results

4.2.1 Firm Value, Bond Characteristics and Maturity

It seems useful to first discuss how arbitrary choices of maturity impact the value of the project and the set of feasible bonds. In this section we fix the value of potential earnings (x) at the value used in the calibration to match the median of the statistics in the data and study how changes in maturity affect the value of a firm, denoted $M(x, K, K^*(\eta^*), \eta^*)$, that has to refinance a bond of a fixed value (K).

Figure 1 shows the value of the firm, denoted $M(x, K, K^*(\eta^*), \eta^*)$, as a function of exogenously chosen maturity, $1/\eta^*$, for alternative values of the cost of refinancing, C, and measures of volatility of the firm's earnings, σ.

Figure 1: Maturity and the Value of the Firm



Cost of Refinancing (C) | Uncertainty (σ)

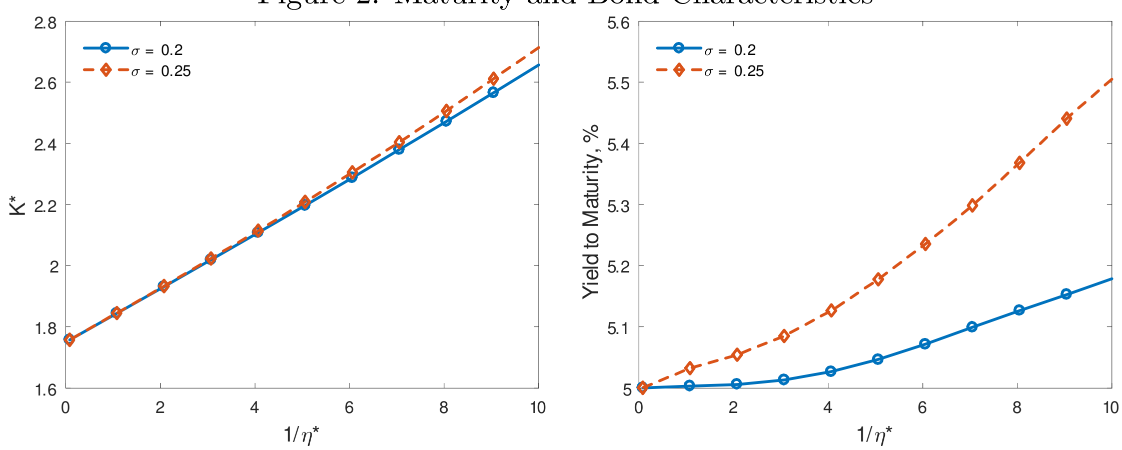
As expected, the function $M(x, K, K^*(\eta^*), \eta^*)$ is concave in maturity and, in all cases the maximum is interior. Thus, the model is consistent with a well defined optimal choice of maturity —the maximum of the M function— even though all market participants are risk neutral and use the same discount factor. In the base case —$C = 0.05$ and $\sigma = 0.20$— the optimal debt structure has an expected maturity close to five years.

The left panel of Figure 1 summarizes the impact of changes in the cost of refinancing. If this cost decreases from the base case ($C = 0.05$) to $C = 0.01$ (a charge equal to 0.25% of the initial debt), the optimal maturity decreases from the base case of 5 years to less than 3. As the cost gets close to zero the optimal maturity falls below 1 year. The lower costs of refinancing decrease the shadow cost of a bond maturing in the illiquid phase and, hence, decrease optimal maturity. We find that, under our parameterization, optimal maturity responds significantly to changes in the costs of refinancing.

The market value of a firm with a given stock of debt decreases as earnings volatility increases. The right panel of Figure 1 illustrates the impact of increasing volatility by 25% for a range of maturities from 1 month to 10 years. As in the case of the cost of refinancing, the effects are significant. The peak of the $M(x, K, K^*(\eta^*), \eta^*)$ function in the high volatility case is attained at a maturity slightly above 3.5 years which corresponds to approximately a 25% decrease in optimal maturity when compared to the low volatility case.

How do the characteristics of the bond change with maturity? Figure 2 reports the face value —holding the market value of the bond constant— and the yield to maturity as a function of the expected maturity of the bond. If the firm chooses to issue longer bonds then the face value must increase and the yield to maturity must increase as well. The market prices long maturity bonds as riskier investments.

Figure 2: Maturity and Bond Characteristics



Face Value | Yield to Maturity

The left panel of Figure 2 makes clear that the value maximizing debt structure does not pick the lowest face value, which would minimize the probability of strategic default since this is attained by the shortest possible bond. Similarly, the right panel of Figure 2 shows that the optimal bond does minimize the interest cost as measured by the yield to maturity. For example, in the base case, the firm chooses to pay a small spread over the risk-free rate to issue debt with a longer maturity even though it had available the option of issuing shorter term debt at a rate close to zero.

The negative impact of earnings volatility also shows up in the bond characteristics. A firm in the higher variance environment ($\sigma = 0.25$) issues shorter debt that nevertheless pays a higher risk premium with a very small difference in the face value. The higher financing cost reflects that higher uncertainty implies a higher probability of strategic default as well as lower likelihood that the firm will be able to refinance early maturing debt.

4.2.2 Maturity Choice and Potential Earnings

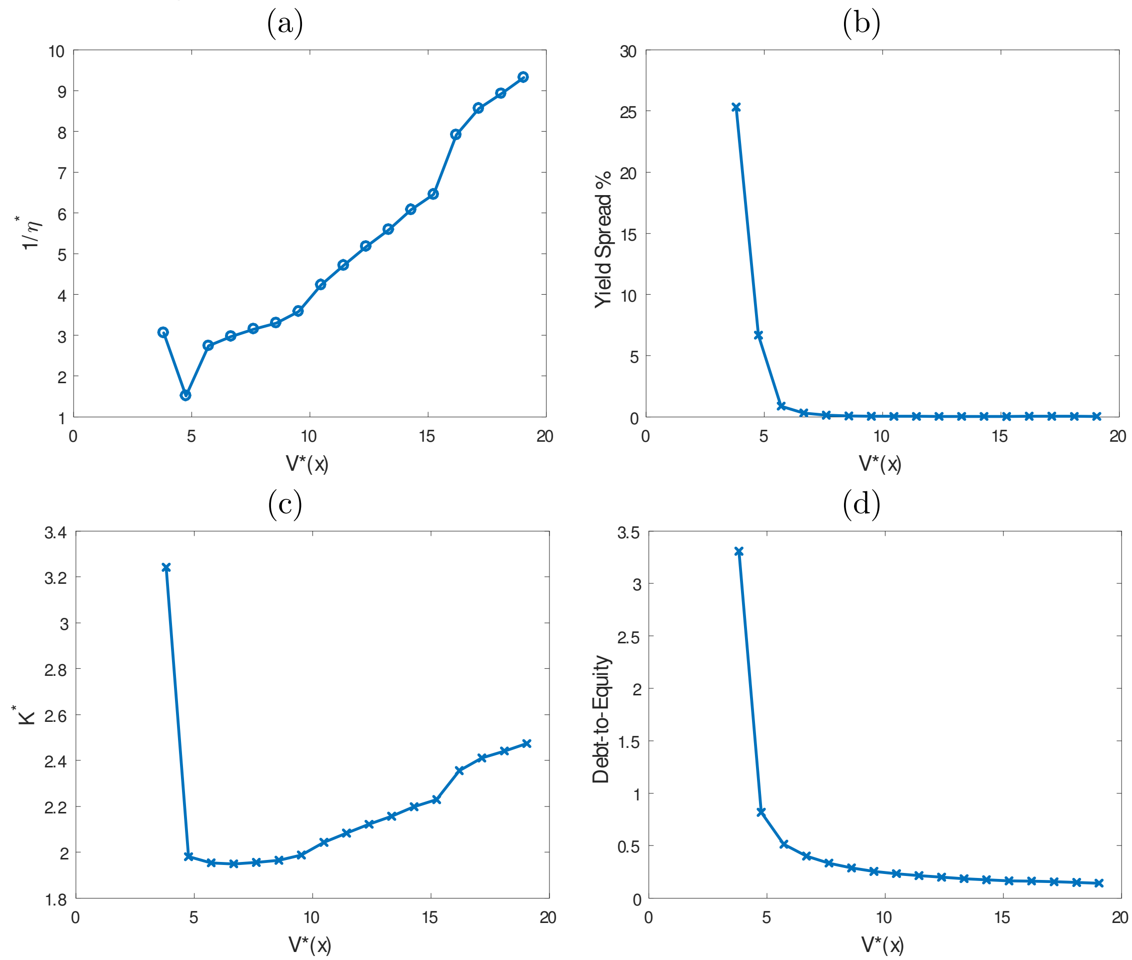
In this model the value of a 100% equity financed firm, which we denote $V^*(x)$, is proportional to x. In what follows we interpret lower values of $V^*(x)$ as indicating (temporarily) poorly performing firms, while higher values characterize (temporarily) highly productive firms.

We find that in all experiments, optimal maturity increases with the level of potential output. We view this as an evidence that firms that need to issue debt in "bad" times will **choose** to issue relatively short term debt. In all the cases that we study, the firms could have issued longer term debt —say with the expected duration that it chooses when $V^*(x)$ is slightly higher— but the tradeoff between cost and risk makes this unprofitable.

Figure 3 presents the values for key variables as potential output increases. First, Figure 3 (a) displays expected maturity. It is interesting to note occurs happen at the lowest $V^*(x)$ for which financing K would be profitable. Recall that we have chosen that value of K such that it coincides with the maximum that can be raised for that lowest x. In this case, the cost of refinancing the debt relative to the potential earnings of the project becomes too high. The firm chooses to issue a very high face value (3.2) that would never refinance and 3 years of maturity. The yield to maturity reaches 25% indicating that this firm would be in financial distress.

Beyond that minimum value of x, expected maturity increases steadily from about 1 year to about 9 years as potential earnings increase. The face value, which is displayed in Figures 3 (c), shows that it follows a similar pattern as expected maturity, although less steep. As potential earnings increases and the firms decide that debt will be refinanced in the event that it matures during the illiquid phase, yields to maturity decrease to a value close to 0%, as shown in Figure 3 (b). The debt-to-equity ratio shows similar pattern like yield with less curvature, where firms who do not refinance borrow as much as 3 times the amount of their equity, but the ratio decreases to less than half as potential earnings increases.

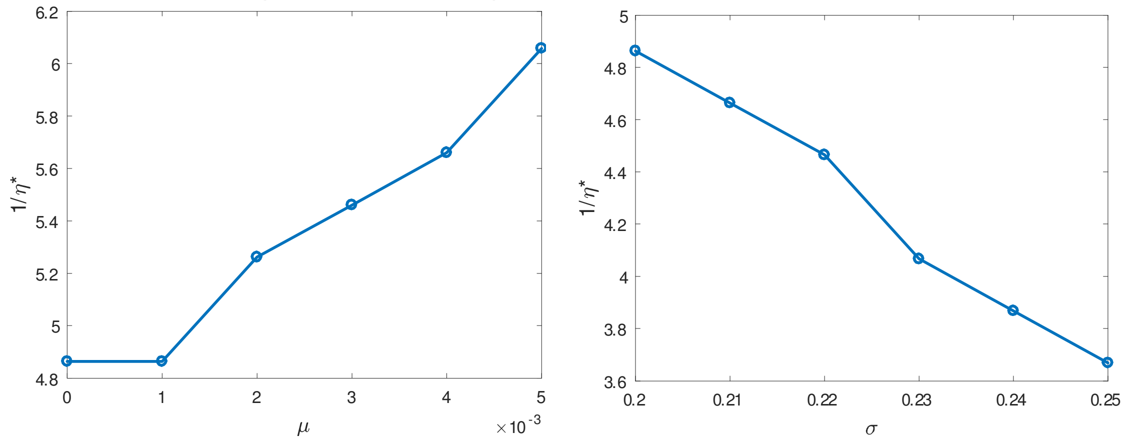
Figure 3: Optimal Debt as a Function of the State of the Firm



4.2.3 How Does Technology Influence Equilibrium Maturity?

The earnings process of the firm is determined by four parameters: μ, the expected growth rate, σ, a measure of the variability of the growth rate, υ, an indicator of liquidity ($1/\upsilon$ is a measure of the expected duration of the illiquid phase), and $1-\delta$, which measures the fraction of the value of the firm that debt holders can appropriate in case of default. Figure 4 shows the impact of changes in the first two parameters on the equilibrium maturity of debt for the median firm in our calibration data.

Figure 4: Technology and Equilibrium Maturity I



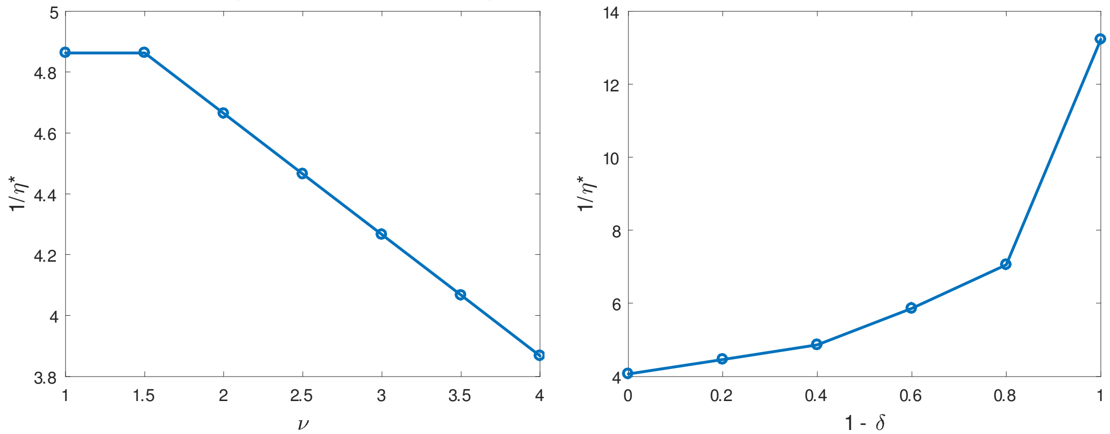
Expected Growth Rate — Variability Growth Rate

Firms with better growth prospects (higher μ) issue longer debt. The intuition —as discussed in the case of a single bond— is that higher expected growth reduces the likelihood of default if the firm is in the liquid regime. Thus, for a fixed maturity the market should lower the excess yield for longer maturities. The firm will find optimal in this case to issue longer debt.

The impact of uncertainty is almost exactly the opposite. For a bond with a given face value, increases in σ lower the expected value of the repayment in the liquid regime. This increases its price. The firm will then find it optimal to choose a shorter bond that can be refinanced at a time in which earnings are (potentially) higher.

Figure 5 summarizes how equilibrium maturity responds to changes in the other two technological parameters: υ and δ.

Figure 5: : Technology and Equilibrium Maturity II



(Inverse) Duration Illiquid Regime   Residual Value of Assets

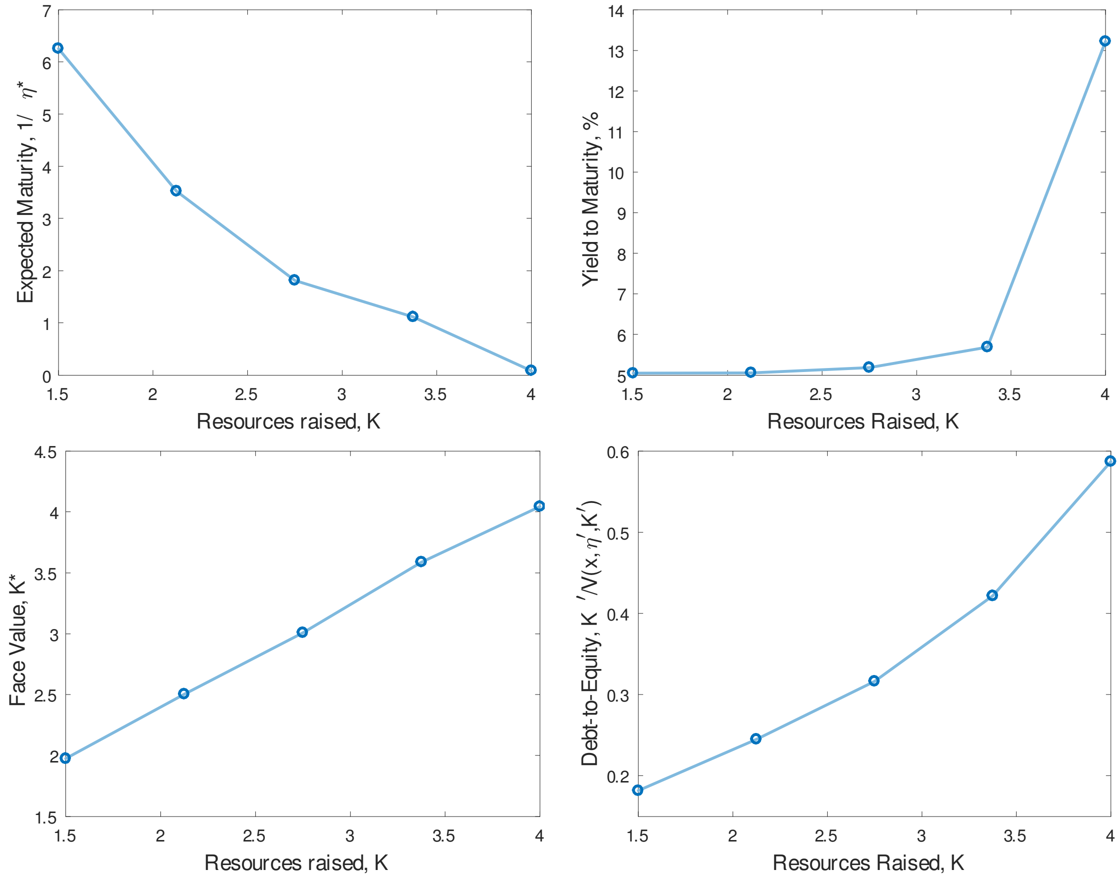
Consistent with the literature that emphasizes a connection between the maturity of the assets and the debt, our model implies that the shorter the duration of the illiquid regime, the shorter the expected duration of debt. The model also implies that the higher the resale value of the assets in case of default —that is, the higher $1 - \delta$— the longer the optimal maturity. A higher $1-\delta$ is associated with lower costs (for the debt holders) of default in the liquid regime and, hence, it is optimal for a firm to redesign its debt by lowering the implicit cost of illiquidity default. This is accomplished by increasing average maturity. Benmelech (2009) report a consistent result: American railroads that used rolling stock that could be more easily redeployed (higher resale value) tended to issue longer term bonds.

Appendix D shows how the other characteristics of the bond—its face value, yield spread, and debt-to-equity ratio—vary with these parameters. To the extent that differences in technology correspond to different sectors, the model implies that firms with similar values of expected earnings will issue debt with different characteristics.

4.2.4 Leverage

Figure 6 shows how the structure of the debt varies with the financing needs of the firm. We interpret the funds raised by the firm, K, as an indicator for its leverage given that all the other characteristics of the firm are constant. The model implies that firms with higher leverage issue shorter term bonds (panel (a)) with a higher face value (panel (c)). The impact of increases in leverage on yields appear highly nonlinear: As K increases such that the debt to equity ratio increases from 15% to 30%, the model implies small changes in the spread. When K increases beyond this threshold, increases in the debt-equity ratio from 30% to 40% are associated with large increases in the yield to maturity. Higher values of K cannot be financed in the market.

Figure 6: Debt Structure and Leverage



4.3 Bonds with Coupons

The previous analysis was simplified because it assumed that there was no revenue during the first phase ($z = 0$) and, as a consequence, bonds were issued with no coupon, $b = 0$. In this section, we make minimal modifications to the values of the parameters to consider the role of bond coupons. In particular, the revenue during the first phase is increased to $z = 0.1$. Thus, firms can now issue a bond with coupons of up to $b = 0.1$.

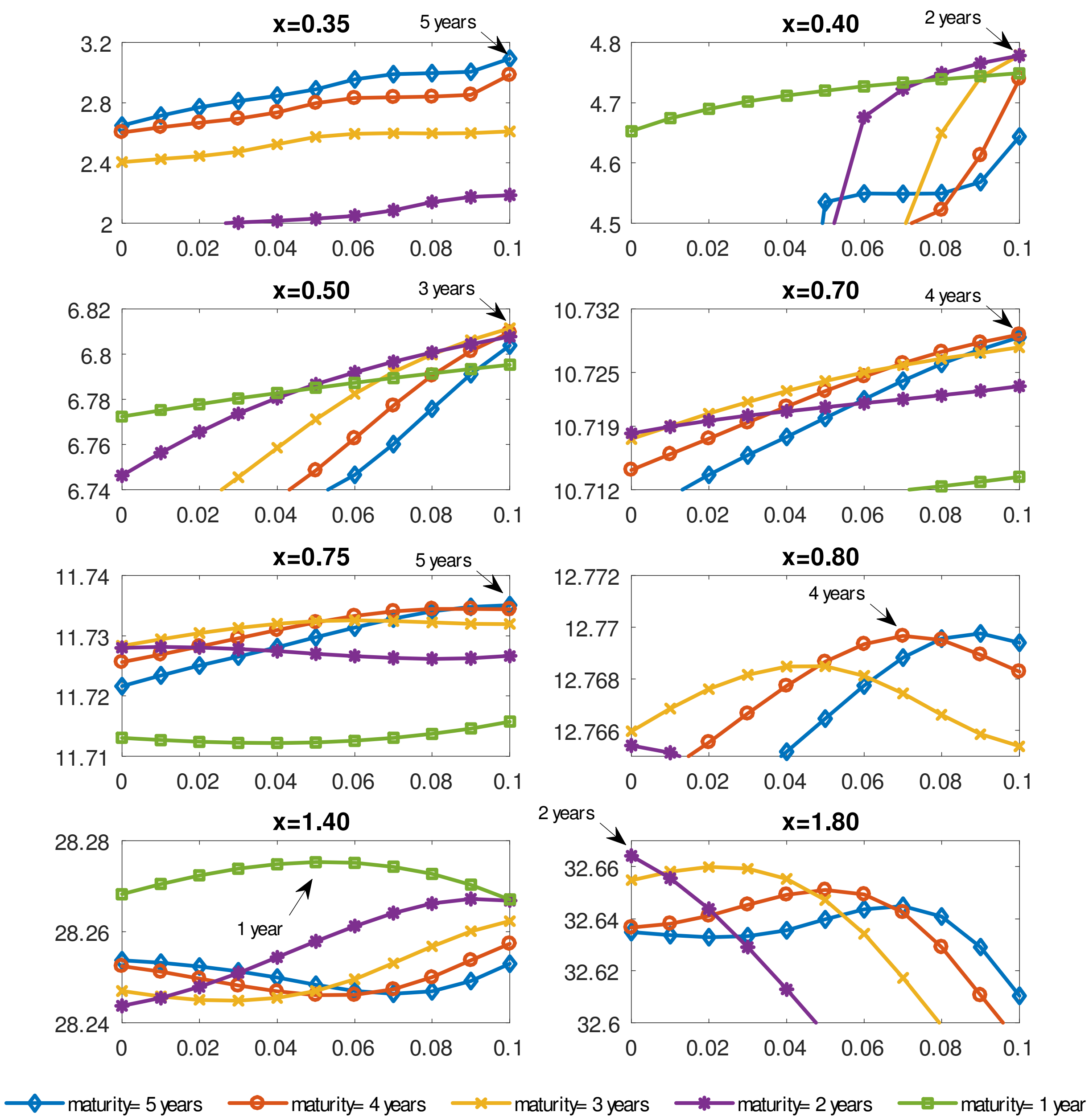
To show the role of the coupon, it is useful to show how arbitrary choices of the coupon impact the value of the project for a set of feasible bonds. Note now that there are three choices, K, η, and b. For each combination of (η, b) considered we select the value of K such that the bond raises the desired resources. In particular, we analyze how changes in the bond coupon impact the value of a firm, denoted $M(x, K, K^*(\eta, b), \eta, b)$, that has to refinance a bond of a fixed value.[13]

The value of potential earnings (x) is set at many values from $x = 0.35$ to $x = 1.8$. Figure 7 shows the value of the firm for alternative values of maturity, $1/\eta$, exogenously chosen, as a function of the bond coupon. We find that for values of x between 0.35 to 0.75 the largest possible coupon $(b = 0.1)$ would be preferred. Actually, for values of x between 0.35 to 0.70, reducing the coupon is worse for the value of the firm for any value of maturity (between 1 and 5) of the newly issued debt. For $x = 0.75$ we note that for some value of maturity other values of the coupon are preferred. For instance, for a 2-year maturity bond, no coupon would be preferred. In terms of maturity, in this range of values of x we see a pattern similar to Figure 3: maturity initially decreases and then increases steadily.

Here, however, optimal maturity reaches 5 years at $x = 0.75$ and it starts to decrease. Note that given that in this example we are not allowing for bonds with maturity higher than 5 years, the alternative would be to keep maturity at 5 years and $b = 0.1$ and decrease K'. This is an option that we allow but as Figure 7 shows for $x > 0.75$ there are other better alternatives. For $x = 0.8$, maturity decreases to 4 years and the optimal coupon 0.7. For $x = 1.4$, optimal maturity is 1 year and the $b = 0.1$. As $b = 0$, maturity continues to increase with x as in the case with no coupon. For instance, for $x = 1.8$ the optimal maturity is 2 years.

[13] We set $K = 2.8$.

Figure 7: Bond Coupon and the Value of the Firm



Even though we are not maximizing over the set of all possible bonds, our results suggest that:

- For firms that have projects with low value, $V^*(x)$, it is optimal to issue bonds with a high coupon ($b = 1$ is the upper bound) and of short-to-moderate maturity . These firms face the highest price of strategic risk (driven by the low x) and they can economize on this factor by both frontloading the debt and choosing a relatively short maturity.

- For firms with intermediate value, the best bond still frontloads payments because it chooses the highest possible coupon. However, since the higher x implies a lower price of strategic risk, it is optimal to issue longer term debt (5 years).

- Finally, firms with very high values of x face low prices of strategic risk and issue zero coupon bonds with an intermediate maturity (2 years). This implies that the only risk that bondholders face is the risk that, when the bond matures, the earnings will fall below $\bar{x}(K)$.

5 Conclusion

We study the optimal choice of the structure of the debt by a risk-neutral agent–which we interpret as a firm–borrowing from a risk-neutral lender. The optimal maturity strikes a balance between two risks: the risk of default in the low growth (illiquid) regime and the risk of strategic default (in the high growth regime).

Several results are obtained relating the characteristics of the project (the firm or its sector) and the choice of maturity. As the level of (potential) output increases, the shadow price of the risk of strategic default goes down and the cost of default in the low growth regime goes up. Hence, it is optimal to lower the risk of default in the low growth (initial) regime by extending the maturity of the debt. The degree of uncertainty in the economic environment also influences the choice of financial structure. Higher uncertainty about growth rates is associated with shorter term debt. Finally, investment projects with a higher value (to the lenders) upon default (e.g. higher resale value or lower cost of fire sales) and projects with lower leverage are financed with longer-term debt.

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Appendices

A Maturity and Key Firms Characteristics

In this appendix we reproduce some previous empirical findings in the existent literature. We use the non-financial and non-regulated firms in Standard & Poor's Compustat from 1988 to 2006 to draw our sample of US firms.[14] The final sample size is 17,169 firm-year observations, with 3,023 different firms.

The construction of Compustat variables is the following:[15]

- Debt Ratio under 3 years*: A ratio of short-term debt (DD2 + DD3 + DLC) to total debt (DLTT + DLC).
- Debt Ratio under 5 years: A ratio of short-term debt (DD2 + DD3 + DD4 + DD5 + DLC) to total debt (DLTT + DLC).
- R&D Investment: A ratio of R&D expenditure (XRD) to sales (SALE). This variable captures "growth opportunities" following Guedes and Opler (1996).
- Volatility (Sales): A ratio of the standard deviation of the first difference in sales (SALE) to the mean of total assets (AT) across time weighted by the average total asset of the sample period, following Stohs and Mauer (1996) and Okzan (2002).
- Volatility (EBITDA): A ratio of the standard deviation of the first difference of EBITDA (SALE - COGS - XSGA) to the mean of total assets (AT) across time.
- Return Volatility: Tge tandard deviation of stock return, where stock return is measured by the growth rate of adjusted closing price of stock (PRCCD/AJEXDI). This is a measure of return volatility.

[14] We have excluded financial firms from the data because the capital structure of the financial firms are influenced by different factors from other sectors. Similarly, we have excluded utilities because those firms are heavily regulated, and different debt maturity structure predictions may apply to such firms. We have also trimmed the extreme 1% of all independent variables to exclude outliers. We have discarded observations that have erroneous data for any of the variables used in the regression. To be specific, we have discarded any observation with short-term debt that is higher than total debt, and with current asset higher than total asset. See Stohs and Mauer (1996) and Johnson (2003) for similar practices.

[15] (*) indicates variables that are included in the calibration target (Table 2) but not in the regression (Table 2A).

- Asset Maturity: The weighted average of short-term assets (ACT/COGS) and long-term assets (PPEGT/DP), where weights are defined, respectively, as (ACT/AT) and (PPEGT/AT), following Stohs and Mauer (1996).

- Asset Salability: A ratio of total net property, plant and equipment (PPENT) to total asset (AT), following De Jong et al. (2008).

- Kaplan-Zingales index: Following equation (14), where CF = (IB + DP), DIV = (DVC + DVP), C = (CHE), LEV = (DLTT + DLC)/(DLTT + DLC + SEQ), and Q = (LSE + CSHO*PRCC - CEQ -TXDB)/(LSE).[16] We interpret a higher value of this index as more reliance on external financing.

- Size: Logarithm of sales (SALE).

- Debt-to-Equity*: A ratio of total debt (DLTT + DLC) to equity (AT - DLC).

- Debt-to-Asset*: A ratio of total debt (DLTT + DLC) to assets (AT).

- Earning Growth (Sales)*: A growth rate of sales (SALE)

- Std. Dev. of Earning Growth (Sales)*: The standard deviation of the growth rate of sales (SALE).

We use the short-term debt ratio calculated as the ratio of debt that matures within 5 years period to the total debt as the endogenous variable. Table 1A reports the descriptive statistics for the constructed variables. Importantly, short-term debt is prevalent among these firms: on average, 25 percent of the debt has maturity longer than 5 years. There is also large variation in debt maturity, as shown by the fact that the standard deviation of the short-term debt ratio is 30 percent.

[16] The index is defined as following:

$$KZ_{it} = -1.002\frac{CF_{it}}{AT_{it-1}} - 39.368\frac{Div_{it}}{AT_{it-1}} - 1.315\frac{C_{it}}{AT_{it-1}} + 3.139Lev_{it} + 0.283Q_{it} \tag{14}$$

where CF is the cash flow, Div is the dividend, C is the cash and short-term investment, Lev is the leverage, Q is the Tobin's Q and AT is the lagged total asset, following the paper by Lamont et al. (2001). The paper argues that these variables obtained from the restricted version of the ordered logit of the central regression by Kaplan and Zingales (1997) provides a good explanation for the wedge that firms face between internal and external costs of funds.

Table 1A: Descriptive Statistics

	Mean	Median	Std. Dev.
Debt Ratio under 5 years	0.750	0.892	0.300
R&D Investment	0.103	0.040	0.335
Volatility (Sales)	0.280	0.213	0.220
Volatility (EBITDA)	0.113	0.075	0.118
Return Volatility	4.023	3.894	1.604
Asset Maturity	6.490	5.398	4.650
Asset Salability	0.221	0.201	0.129
Kaplan-Zingales Index	-2.948	-0.787	8.678
Size	5.134	4.984	2.288

In Table 2A, we report panel-data regression on the short-term debt ratio. We also show the "expected" sign given findings in previous empirical literature. In terms of the regressions, we show several columns to argue that considering alternative measures volatility does not alter the result in any significant way. We also show that we can control by sector or firm fixed effects. For some of the intuitions that will be derived later variations across sectors are important, so we also show the results controlling only by year fixed effects.

We find that firms/sectors with:

- Higher growth opportunity → longer debt maturity.
- Higher volatility → shorter debt maturity.
- Longer asset maturity → longer debt maturity (although not significant).
- Higher asset salability → longer debt maturity.
- More reliance on external financing → shorter debt maturity (although significant only with variation across industries).

These findings are quite similar to the articles discussed in subsection 1.1. However, our new theory offer an alternative explanation to the one usually provided in previous literature.

Table 2A: Panel Data Fixed Effects Regression on 5 Years Short-Term Debt Ratio

	Expected Sign	(1) Coef./SE	(2) Coef./SE	(3) Coef./SE	(4) Coef./SE	(5) Coef./SE	(6) Coef./SE	(7) Coef./SE
R&D Investment	$-,+$	-0.0123**	-0.0102*	-0.0165***	-0.0139**	-0.0149**	-0.0125**	-0.0045
		(0.0060)	(0.0062)	(0.0061)	(0.0062)	(0.0061)	(0.0062)	(0.0097)
Volatility (Sales)	$+$	0.0838***	0.0945***	-	-	-	-	-
		(0.0145)	(0.0152)					
Volatility (EBITDA)	$+$	-	-	0.1191***	0.1249***	-	-	-
				(0.0249)	(0.0270)			
Return Volatility	$+$	-	-	-	-	0.0085***	0.0097***	-
						(0.0031)	(0.0034)	
Asset Maturity	$-$	-0.0011	-0.0009	-0.0012	-0.0010	-0.0011	-0.0009	-0.0005
		(0.0008)	(0.0009)	(0.0008)	(0.0009)	(0.0008)	(0.0009)	(0.0013)
Asset Salability	$-$	-0.1997***	-0.1596***	-0.1989***	-0.1601***	-0.2019***	-0.1631***	-0.1042*
		(0.0359)	(0.0403)	(0.0360)	(0.0403)	(0.0360)	(0.0404)	(0.0613)
Kaplan-Zingales Index	$+$	0.0003	0.0003	0.0003	0.0003	0.0003	0.0003	0.0007*
		(0.0003)	(0.0003)	(0.0003)	(0.0003)	(0.0003)	(0.0003)	(0.0004)
Size	$-$	-0.0424***	-0.0404***	-0.0420***	-0.0405***	-0.0411***	-0.0394***	-0.0346***
		(0.0019)	(0.0022)	(0.0020)	(0.0023)	(0.0025)	(0.0028)	(0.0082)
Year Fixed Effect		Yes	Yes	Yes	Yes	Yes	Yes	Yes
Industry Fixed Effect		No	Yes	No	Yes	No	Yes	No
Firm Fixed Effect		No	No	No	No	No	No	Yes
No. of Obs		17169	17169	17169	17169	17169	17169	17169
R^2		0.15	0.19	0.15	0.19	0.15	0.19	0.57

Note: Standard errors robust to heteroscedasticity are reported in parenthesis. ***, **, and * coefficient is significant at 1, 5, and 10 percent level, respectively. Regression 1 and 2 uses volatility of sales as a measure of risk, while regression 3 and 4 uses volatility of EBITDA as a measure of risk, and regression 5 and 6 uses return volatility as a measure of risk. Regression 7 does not contain any measure of risk because each firm corresponds to only one value of risk throughout the sample period.

B Additional Results: Valuations in the High Growth/Liquid Regime

Preliminaries Let λ_i be a root of

$$r + \eta - \mu\lambda = \frac{\sigma^2}{2}\lambda(\lambda - 1).$$

Given our assumptions it follows that $\lambda_1 < 0$ and λ_2. Moreover,

$$\lim_{\sigma\to\infty} \lambda_2 = 1 \text{ and } \lim_{\sigma\to\infty} \lambda_1 = 0,$$
$$\lim_{\sigma\to 0} \lambda_2 = \infty \text{ and } \lim_{\sigma\to 0} \lambda_1 = -\infty$$
$$\frac{\partial\lambda_2}{\partial\sigma} < 0 \text{ and } \frac{\partial\lambda_1}{\partial\sigma} > 0.$$

Similarly, let π_i be a root of

$$r + \eta + \upsilon - \mu\pi = \frac{\sigma^2}{2}\pi(\pi - 1),$$

and, as before, denote by π_1 to be the negative root and π_2 to be the positive root. Simple calculations show that the impact of changes in σ upon the π_i mimic their effect on the λ_i. It also follows that

$$\pi_1 < \lambda_1 \text{ and } \pi_2 > \lambda_2,$$

and that

$$\lim_{\upsilon\to\infty} \pi_2 = \infty \text{ and } \lim_{\upsilon\to\infty} \pi_1 = -\infty.$$

Recall that default at maturity happens if

$$x < \bar{x}(K, \theta z) = (r - \mu)\left(K - \frac{\theta z}{r}\right).$$

We assume that, upon default, bondholders get $(1 - \delta)$ of the value of the assets.

Valuations There are four possible cases depending on the parameters:

1. Case I: Perfectly safe bond: $b \leq \theta z$ and $K \leq \theta\frac{z}{r}$.

2. Case II: No default in the liquid regime except possibly at maturity: : $b \leq \theta z$ and $K \leq \theta\frac{z}{r}$.

3. Case III: Default is possible in the liquid regime both before the debt matures and at maturity. However, it is never optimal for the firm to default at maturity: $b > \theta z$ and $K < \frac{r+\eta}{r+\eta-\lambda_1 r}(\frac{\theta z}{r} - \lambda_1 \frac{b}{r+\eta})$. This corresponds to the case $x^* > \bar{x}(K, \theta z)$.

4. Case IV: Default is possible in the liquid regime both before the debt matures and at maturity. This corresponds to $b > \theta z$ and $K \geq \frac{r+\eta}{r+\eta-\lambda_1 r}(\frac{\theta z}{r} - \lambda_1 \frac{b}{r+\eta})$ or, $x^* \leq \bar{x}(K, \theta z)$.

Case I. In this case there is no risk and the values of the equity and debt are given by

$$\begin{aligned} B(x; I) &= \frac{b + \eta K}{r + \eta} = B^*, \\ T(x; I) &= \frac{x}{r - \mu} + \frac{\theta z}{r} - \frac{b + \eta K}{r + \eta}. \end{aligned}$$

Case II. The value of the bond is given by

$$B(x; II) = \begin{cases} B^* + \bar{B}^1 II)(\frac{x}{\bar{x}})^{\lambda_1}, & x \geq \bar{x}(K, \theta z), \\ \frac{b}{r+\eta} + \frac{\eta(1-\delta)}{r+\eta}(\frac{\theta z}{r} + \frac{x}{r-\mu}\frac{r+\eta}{r+\eta-\mu}) + \bar{B}^2(II)(\frac{x}{\bar{x}})^{\lambda_2}, & x < \bar{x}(K, \theta z) \end{cases},$$

where,

$$\begin{aligned} \bar{B}^1(II) &= \frac{\eta}{r + \eta}\frac{1}{\lambda_1 - \lambda_2}\left[\frac{(1-\delta)\bar{x}}{r - \mu}\frac{(r+\eta)(1-\lambda_2)}{r + \eta - \mu} + \lambda_2(K - (1-\delta)\frac{\theta z}{r})\right], \\ \bar{B}^2(II) &= \frac{\eta}{r + \eta}\frac{1}{\lambda_1 - \lambda_2}\left[\frac{(1-\delta)\bar{x}}{r - \mu}\frac{(r+\eta)(1-\lambda_1)}{r + \eta - \mu} + \lambda_1(K - (1-\delta)\frac{\theta z}{r})\right]. \end{aligned}$$

In this case the value of the firm is

$$T(x; II) = \begin{cases} \frac{x}{r-\mu} + \frac{\theta z - b}{r+\eta} + \frac{\eta}{r+\eta}(\frac{\theta z}{r} - K) + \bar{T}^1(II)(\frac{x}{\bar{x}})^{\lambda_1}, & x \geq \bar{x}(K, \theta z), \\ \frac{x}{r+\eta-\mu} + \frac{\theta z - b}{r+\eta} + \bar{T}^2(II)(\frac{x}{\bar{x}})^{\lambda_2} & x < \bar{x}(K, \theta z) \end{cases},$$

where

$$\begin{aligned} \bar{T}^1(II) &= \frac{1}{\lambda_1 - \lambda_2}\left[(\frac{\bar{x}}{r + \eta - \mu} - \frac{\bar{x}}{r - \mu})(1 - \lambda_2) + \lambda_2\frac{\eta}{r + \eta}(\frac{\theta z}{r} - K)\right], \\ \bar{T}^2(II) &= \frac{1}{\lambda_1 - \lambda_2}\left[(\frac{\bar{x}}{r + \eta - \mu} - \frac{\bar{x}}{r - \mu})(1 - \lambda_1) + \lambda_1\frac{\eta}{r + \eta}(\frac{\theta z}{r} - K)\right]. \end{aligned}$$

Case III. In this case we conjecture that the optimal default is such that x^* —the level of x that triggers default— is greater than $\bar{x}(K, \theta z)$. This implies that if, for some t, $x_t < x^*$, then the firm will default. However, if the stopping time T_η precedes such an event then the bond will be repaid in full. Thus, in this case the strategic risk is concentrated on the period before the bond matures.

The relevant HJB equation for the value of equity solves the following

$$rT(x;III) = x + \theta z - b + \eta[\frac{x}{r-\mu} + \frac{\theta z}{r} - K - T(x;III)]$$
$$+T'(x,III)\mu x + T''(x;III)\frac{\sigma^2}{2}x, \quad x \geq x^*,$$

whith standard boundary conditions

$$T(x^*;III) = 0 \text{ and } \lim_{x\to\infty} \frac{T(x;III)}{x} < \infty$$

and the smooth pasting condition given by

$$T'(x^*;III) = 0.$$

The value of the firm is zero for $x < x^*$, and for $x \geq x^*$,

$$T(x;III) = \frac{x}{r-\mu} + \frac{\theta z}{r} - B^* + \underbrace{\left[\frac{1}{1-\lambda_1}(B^* - \frac{\theta z}{r})\right]}_{+}(\frac{x}{x^*})^{\lambda_1}.$$

The optimal bankruptcy decision rule is given by

$$x^* = \frac{(r-\mu)\lambda_1}{\lambda_1 - 1}(B^* - \frac{\theta z}{r}).$$

Since this case requires that $x^* \geq \bar{x}(K, \theta z)$, this implies that the face value of the bond must satisfy

$$K < \frac{r+\eta}{r+\eta-\lambda_1 r}(\frac{\theta z}{r} - \lambda_1 \frac{b}{r+\eta}) \equiv K_L$$

for the solution to be in this region. The value of the debt is

$$(1-\delta)\left[\frac{x}{r-\mu} + \frac{\theta z}{r}\right], \text{ for } x < x^*,$$

and

$$B(x;III) = B^* + \left[(1-\delta)(\frac{\theta z}{r} + \frac{x^*}{r-\mu}) - B^*\right](\frac{x}{x^*})^{\lambda_1}.$$

Case IV. If the face value of the bond is sufficiently high then it is possible that the pre-maturity default threshold, x^*, is lower than the default threshold at maturity, $\bar{x}(K, \theta z)$. In this case it is possible that default may occur before maturity (if x drops below x^* and the bond has not matured) or at maturity. The values of the firm and the bond are given by

$$B(x; IV) = \begin{cases} B^* + \bar{B}_H^1(IV)(\frac{x}{\bar{x}})^{\lambda_1}, & x \geq \bar{x}(K, \theta z). \\ \frac{b}{r+\eta} + \frac{\eta(1-\delta)}{r+\eta}(\frac{\theta z}{r} + \frac{x}{r-\mu}\frac{r+\eta}{r+\eta-\mu}) + \bar{B}_M^1(IV)(\frac{x}{\bar{x}})^{\lambda_1} + \bar{B}_M^2(IV)(\frac{x}{\bar{x}})^{\lambda_2}, & x^* < x < \bar{x}(K, \theta z) \\ (1-\delta)(\frac{x}{r-\mu} + \frac{\theta z}{r}), & x < x^*. \end{cases}$$

The constants solve the following system of equations,

$$\begin{aligned} \bar{B}_M^2(IV) &= \frac{1}{\lambda_1 - \lambda_2}[(1-\lambda_1)\frac{\eta(1-\delta)}{r+\eta-\mu}\frac{\bar{x}}{r-\mu} - \lambda_1\frac{\eta}{r+\eta}((1-\delta)\frac{\theta z}{r} - K)], \\ \bar{B}_M^1(IV) &= (\frac{x^*}{\bar{x}})^{-\lambda_1}[-\bar{B}_M^2(IV)(\frac{x^*}{\bar{x}})^{\lambda_2} \\ &\quad + (1-\delta)(\frac{x^*}{r-\mu} + \frac{\theta z}{r}) - \frac{b}{r+\eta} - \frac{\eta(1-\delta)}{r+\eta}(\frac{\theta z}{r} + \frac{x^*}{r-\mu}\frac{r+\eta}{r+\eta-\mu})], \\ \bar{B}_H^1(IV) &= \bar{B}_M^1(IV) + \bar{B}_M^2(IV) + \frac{\eta(1-\delta)}{r+\eta}(\frac{\theta z}{r} + K\frac{r+\eta}{r+\eta-\mu}) - \frac{\eta K}{r+\eta}. \end{aligned}$$

The value of the firm is

$$T(x; IV) = \begin{cases} \frac{x}{r-\mu} + \frac{\theta z}{r} - B^* + \bar{T}_H^1(IV)(\frac{x}{\bar{x}})^{\lambda_1}, & x \geq \bar{x}(K, \theta z) \\ \frac{x}{r+\eta-\mu} + \frac{\theta z - b}{r+\eta} + \bar{T}_M^1(IV)(\frac{x}{\bar{x}})^{\lambda_1} + \bar{T}_M^2(IV)(\frac{x}{\bar{x}})^{\lambda_2}, & x^* < x < \bar{x}(K, \theta z) \\ 0, & x < x^* \end{cases}$$

with the constants being given by the solution of the following system (imposing boundary and smooth pasting conditions)

$$\begin{aligned} \bar{T}_M^2(IV) &= \frac{1}{\lambda_1 - \lambda_2}[(1-\lambda_1)(\frac{\bar{x}}{r+\eta-\mu} - \frac{\bar{x}}{r-\mu}) + \lambda_1\frac{\eta}{r+\eta}(\frac{\theta z}{r} - K)], \\ \bar{T}_M^1(IV) &= -(\frac{x^*}{\bar{x}})^{-\lambda_1}[\frac{x^*}{r+\eta-\mu} + \frac{\theta z - b}{r+\eta} + \bar{T}_M^2(IV)(\frac{x^*}{\bar{x}})^{\lambda_2}], \\ \bar{T}_H^1(IV) &= \frac{1}{\lambda_1}[\frac{\bar{x}}{r+\eta-\mu} - \frac{\bar{x}}{r-\mu} + \lambda_1\bar{T}_M^1(IV) + \lambda_2\bar{T}_M^2(IV)]. \end{aligned}$$

The optimal default boundary solves

$$(x^*)^{\lambda_2}[\frac{\eta(r+\eta-\lambda_1\mu)}{(r-\mu)^{\lambda_2}(r+\eta-\mu)(r+\eta)}(K - \frac{\theta z}{r})^{1-\lambda_2}] = \frac{(1-\lambda_1)}{r+\eta-\mu}x^* + \lambda_1\frac{b-\theta z}{r+\eta},$$

and it is immediate to verify that, the solution is greater that $\bar{x}(K,\theta z)$ iff

$$K \geq \frac{r+\eta}{r+\eta-\lambda_1 r}\left(\frac{\theta z}{r}-\lambda_1\frac{b}{r+\eta}\right) \equiv K_L$$

C The One Bond Case

In this section we illustrate how the two prices of risk depend on properties of the economic environment. To simplify, we take a special version of cases III and IV. We assume that $\theta = 0$ —which corresponds to the case of no riskless income in the liquid regime— and $\delta = 1$, which implies that debt holders get nothing in the case of default. Allowing for a more general specification does not change the basic properties of the results but makes the algebra more cumbersome.

Preliminaries In this case the value of K_L is given by

$$K_L = \frac{-\lambda_1 b}{(1-\lambda_1)\, r+\eta},$$

and default at maturity occurs if

$$x < \bar{x}(K) = K(r-\mu).$$

For $K \leq K_L$ —we label this case I—the bankruptcy threshold is

$$x_I^* = \frac{-\lambda_1}{1-\lambda_1}B^*(r-\mu),$$

while for $K \geq K_L$ —we label this case II— the bankruptcy threshold solves

$$\underbrace{\left[\lambda_1\left(\frac{r-\mu}{r+\eta-\mu}-\frac{r}{r+\eta}\right)+\frac{\eta}{r+\eta-\mu}\right]}_{+} K\left(\frac{x_{II}^*}{\bar{x}}\right)^{\lambda_2} = \underbrace{-\lambda_1\frac{b}{r+\eta}}_{+} + \underbrace{\frac{(\lambda_1-1)}{r+\eta-\mu}}_{-} x_{II}^*.$$

Properties of K

$$\lim_{\eta\to\infty} K_L = 0, \text{ and } \lim_{\eta\to 0} K_L = \frac{-\lambda_1(0)}{1-\lambda_1(0)}\frac{b}{r},$$

where $\lambda_i(0)$ is the value of the corresponding root when $\eta = 0$. Similarly we use $\pi_i(0)$ as value of the root when $\eta = 0$ as well.

Given the properties of K_L, there are some configurations that are not possible. For example, since $\lim_{\eta\to\infty} K_L = 0$, there is very short duration debt in case I, since for any finite K it must be the case that —for a sufficiently short duration $1/\eta$— $K > K_L$. This case is also ruled out in environments with very high variance as $\lim_{\sigma\to\infty} K_L = 0$.

Similar considerations apply to x^*. For example, x_I^* and x_{II}^* both converge to zero as $\sigma \to \infty$ since the high option value of not defaulting dominates. Moreover, x_{II}^* converges to zero when expected duration goes to zero. Finally, for consols (i.e. $\eta \to 0$) we have that $\lim_{\eta\to 0} x_I^* = \lim_{\eta\to 0} x_{II}^* > 0$.

Risk Prices The price of illiquidity risk, which we label Q is independent of the state and given by

$$Q = \frac{\eta K}{b + \eta K} \frac{r+\eta}{r+\eta+\upsilon}. \tag{15}$$

The price of strategic risk during the illiquid regime —which is the relevant one when it comes to analyzing refinancing decisions— depends on whether the firm is in case I or II and on the particular value of potential earnings, x.

Case I In this case the prices of risk differ depending on whether x is above —which we label L^H— or below —denoted L^L— the threshold that would trigger default in the liquid regime. Standard calculations show that

$$\begin{aligned}
L_I^H(x) &= B^* - \frac{\eta K}{r+\eta+\upsilon} - B^* \left(\frac{x}{x_I^*}\right)^{\lambda_1} + \bar{L}_I^H \left(\frac{x}{x_I^*}\right)^{\pi_1}, \ x \geq x_I^* \\
L_I^L(x) &= \frac{b}{r+\eta+\upsilon} + \bar{L}_I^L \left(\frac{x}{x_I^*}\right)^{\pi_2}, \ x \leq x_I^*.
\end{aligned}$$

where

$$\begin{aligned}
\bar{L}_I^H &= \frac{B^*}{\pi_2 - \pi_1} \left[-\lambda_1 + \pi_2 \frac{r+\eta}{r+\eta+\upsilon}\right], \\
\bar{L}_I^L &= \frac{B^*}{\pi_2 - \pi_1} \left[-\lambda_1 + \pi_1 \frac{r+\eta}{r+\eta+\upsilon}\right].
\end{aligned}$$

Case II In this case the optimal default threshold before the debt matures, x_{II}^*, lies below the default threshold at maturity, $\bar{x}(K)$. This defines three regions in terms of the value of potential earnings that require different treatment. We define the market value of the debt $L_{II}^H(x)$ when $x \geq \bar{x}(K)$. For intermediate values of x (that is, $\bar{x}(K) \geq x \geq x_{II}^*$) we denote the value of the debt in the illiquid region by $L_{II}^M(x)$. Finally, for low values of x

($x^*_{II} \geq x$), the value of the debt is $L^L_{II}(x)$. Standard pricing arguments can be used to show that the market value of the debt in each of these regions is given by

$$
\begin{aligned}
L^H_{II}(x) &= B^* - \frac{\eta K}{r+\eta+\upsilon} + \bar{B}^H_1 \left(\frac{x}{\bar{x}(K)}\right)^{\lambda_1} + \bar{L}^H_{II} \left(\frac{x}{\bar{x}(K)}\right)^{\pi_1}, \\
L^M_{II}(x) &= \frac{b}{r+\eta} + \bar{B}^L_1 \left(\frac{x}{\bar{x}(K)}\right)^{\lambda_1} + \bar{B}^L_2 \left(\frac{x}{\bar{x}(K)}\right)^{\lambda_2} + \bar{L}^M_1 \left(\frac{x}{\bar{x}(K)}\right)^{\pi_1} + \bar{L}^M_2 \left(\frac{x}{\bar{x}(K)}\right)^{\pi_2}, \\
L^L_{II}(x) &= \frac{b}{r+\eta+\upsilon} + \bar{L}^L_{II} \left(\frac{x}{\bar{x}(K)}\right)^{\pi_2},
\end{aligned}
$$

where the constants satisfy

$$
\begin{aligned}
\bar{B}^H_1 &= \frac{\lambda_1}{\lambda_2-\lambda_1}\frac{\eta K}{r+\eta}\left(\frac{x^*_{II}}{\bar{x}(K)}\right)^{\lambda_2-\lambda_1} - \frac{b}{r+\eta}\left(\frac{x^*_{II}}{\bar{x}(K)}\right)^{-\lambda_1} - \frac{\lambda_2}{\lambda_2-\lambda_1}\frac{\eta K}{r+\eta}, \\
\bar{B}^L_1 &= \frac{\lambda_1}{\lambda_2-\lambda_1}\frac{\eta K}{r+\eta}\left(\frac{x^*_{II}}{\bar{x}(K)}\right)^{\lambda_2-\lambda_1} - \frac{b}{r+\eta}\left(\frac{x^*_{II}}{\bar{x}(K)}\right)^{-\lambda_1}, \\
\bar{B}^L_2 &= \frac{-\lambda_1}{\lambda_2-\lambda_1}\frac{\eta K}{r+\eta}, \\
\bar{L}^H_{II} &= \bar{L}^M_1 + \frac{\pi_2}{\pi_2-\pi_1}\frac{\eta K}{r+\eta+\upsilon}, \\
\bar{L}^M_2 &= \frac{\pi_1}{\pi_2-\pi_1}\frac{\eta K}{r+\eta+\upsilon}, \\
\bar{L}^M_1 &= \left(\frac{x^*_{II}}{\bar{x}(K)}\right)^{-\pi_1}\left[\frac{-\lambda_1}{\pi_2-\pi_1}\left(\frac{b}{r+\eta} + \frac{\eta K}{r+\eta}\left(\frac{x^*_{II}}{\bar{x}(K)}\right)^{\lambda_2}\right) + \frac{\pi_2}{\pi_2-\pi_1}\frac{b}{r+\eta+\upsilon}\right], \\
\bar{L}^L_{II} &= \left(\frac{x^*_{II}}{\bar{x}(K)}\right)^{-\pi_2}\left[\frac{\pi_1}{\pi_2-\pi_1}\left(\frac{b}{r+\eta+\upsilon} + \frac{\eta K}{r+\eta+\upsilon}\left(\frac{x^*_{II}}{\bar{x}(K)}\right)^{\pi_2}\right)\right. \\
&\quad \left. + \frac{-\lambda_1}{\pi_2-\pi_1}\left(\frac{b}{r+\eta} + \frac{\eta K}{r+\eta}\left(\frac{x^*_{II}}{\bar{x}(K)}\right)^{\lambda_2}\right)\right].
\end{aligned}
$$

Given the market value of the bond, $L(x)$, the price of strategic risk is (see equation (6)

$$
S(x; b, \eta, K) = \frac{b}{b+\eta K} + \frac{\eta K}{b+\eta K}\frac{\upsilon}{r+\eta+\upsilon} - \frac{r+\eta}{b+\eta K} L(x; b, \eta, K)
$$

Maturity and Risk Prices In this section we report the limiting values of the two risk prices in the case of a short bond ($1/\eta \to 0$) and a consol ($1/\eta \to \infty$). We find that

Table C.1: Maturity and Risk Prices

		$1/\eta \to 0$	$1/\eta \to \infty$
$x \geq x^*$	Q	1	0
$x \geq x^*$	S	0	$\left(\frac{x}{x_I^*}\right)^{\lambda_1(0)} - \frac{\left(\frac{x}{x_I^*}\right)^{\pi_1(0)}}{\pi_2(0)-\pi_1(0)}\left[-\lambda_1(0) + \pi_2(0)\frac{r}{r+\upsilon}\right]$
$x \leq x^*$	Q	1	0
$x \leq x^*$	S	0	$\frac{\upsilon}{r+\upsilon} - \frac{\left(\frac{x}{x_I^*}\right)^{\pi_2(0)}}{\pi_2(0)-\pi_1(0)}\left[-\lambda_1(0) + \pi_1(0)\frac{r}{r+\upsilon}\right]$

.

Uncertainty and Risk Prices Table C.2 reports the behavior of the risk prices as a function of σ. Since Q —the price of illiquidity risk— is independent of the degree of uncertainty, we only report the values for the price of strategic risk.

Table C.2: Uncertainty and the Price of Strategic Risk

		$\sigma \to 0$	$\sigma \to \infty$
$x \geq x_I^*$	S	0	$\frac{\upsilon}{r+\eta+\upsilon}$
$x_I^* \geq x$	S	$\frac{\upsilon}{r+\eta+\upsilon}\left(1-\left(\frac{x}{x_I^*}\right)^{\pi_2^+}\right)$	$\frac{\upsilon}{r+\eta+\upsilon}$
$x \geq \bar{x}(K)$	S	0	$\frac{\upsilon}{r+\eta+\upsilon}$
$\bar{x}(K) \geq x \geq x_{II}^*$	S	$\frac{\upsilon}{r+\eta+\upsilon} - \hat{S}_M(x)$	$\frac{\upsilon}{r+\eta+\upsilon}$
$x_{II}^* \geq x$	S	$\frac{\upsilon}{r+\eta+\upsilon} - \hat{S}_L(x)$	$\frac{\upsilon}{r+\eta+\upsilon}$

where

$$\hat{S}_M(x) = -\frac{b}{r+\eta+\upsilon}\frac{1}{B^*} + \frac{\eta K}{B^*}\left[\frac{1}{r+\eta}\left(\frac{x}{\bar{x}(K)}\right)^{\lambda_2^+} - \frac{1}{r+\eta+\upsilon}\left(\frac{x}{\bar{x}(K)}\right)^{\pi_2^+}\right],$$

$$\hat{S}_L(x) = \frac{1-(r+\eta)}{r+\eta+\upsilon} + \frac{1}{B^*}\left(\frac{x}{\bar{x}(K)}\right)^{\pi_2^+} \times$$

$$\left[-\frac{\eta K}{r+\eta+\upsilon} + \frac{b}{r+\eta}\left(\frac{x_{II}^*}{\bar{x}(K)}\right)^{-\pi_2^+} - \frac{b}{r+\eta+\upsilon}\left(\frac{x_{II}^*}{\bar{x}(K)}\right)^{\pi_2^+} + \frac{\eta K}{r+\eta}\left(\frac{x_{II}^*}{\bar{x}(K)}\right)^{-\frac{\upsilon}{\mu}}\right].$$

In these formulas we use λ_2^+ and π_2^+ as the values of the roots when $\sigma = 0$. They are given by

$$\lambda_2^+ = \frac{r+\eta}{\mu} \text{ and } \pi_2^+ = \frac{r+\eta+\upsilon}{\mu}.$$

Illiquidity and Risk Prices In the model $1/\upsilon$ is the expected duration of the illiquid regime. The price of illiquidity risk responds to changes in υ in a very intuitive way

$$\lim_{\upsilon\to 0} Q = \frac{\eta K}{b+\eta K}, \quad \lim_{\upsilon\to\infty} Q = 0, \text{ and } \frac{\partial Q}{\partial \upsilon} < 0.$$

Table C.3 reports the limiting values of the price of strategic risk

Table C.3: Illiquidity and the Price of Strategic Risk

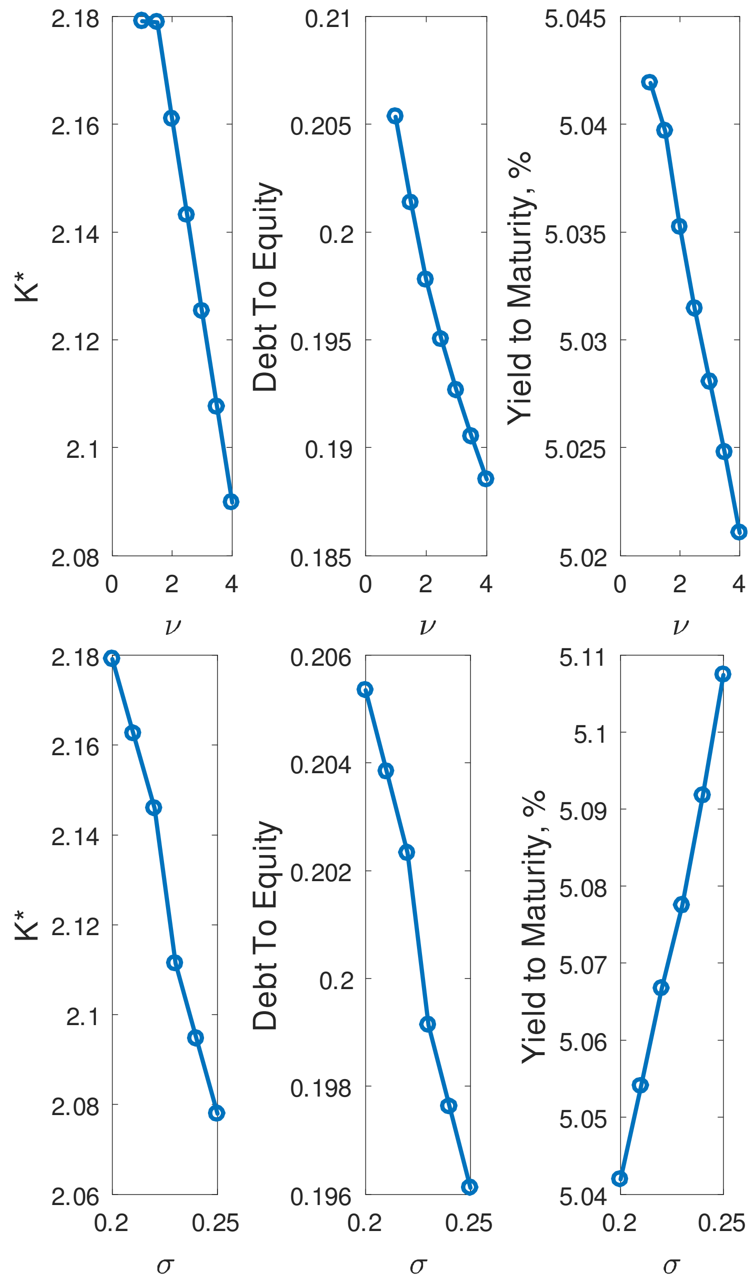
		$1/\upsilon \to \infty$	$1/\upsilon \to 0$
$x \geq x_I^*$	S	0	$\left(\frac{x}{x_I^*}\right)^{\lambda_1}$
$x_I^* \geq x$	S	0	1
$x \geq \bar{x}(K)$	S	0	$\tilde{S}_M(x)$
$\bar{x}(K) \geq x \geq x_{II}^*$	S	0	$\frac{\eta K}{b+\eta K} - \tilde{S}_L(x)$
$x_{II}^* \geq x$	S	0	1

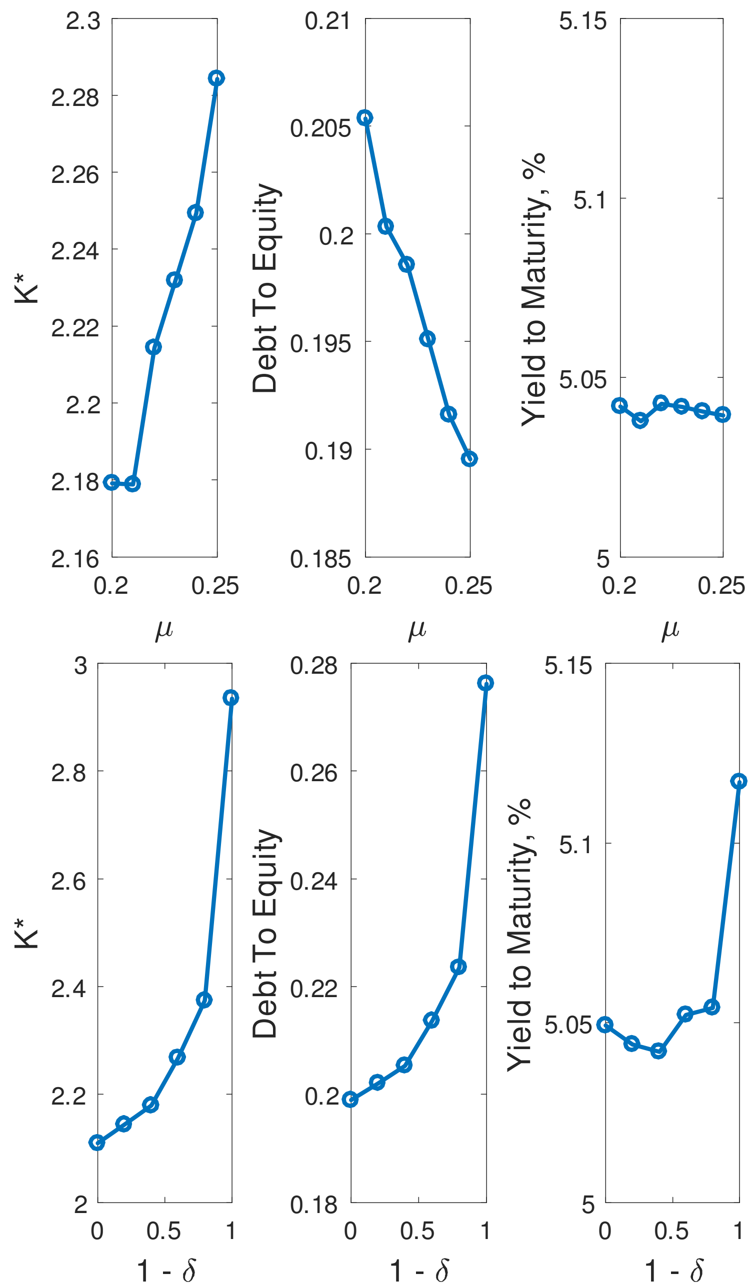
where

$$\tilde{S}_M(x) = \frac{-\lambda_1}{\lambda_2-\lambda_1}\frac{\eta K}{r+\eta}\left(\frac{x_{II}^*}{\bar{x}(K)}\right)^{\lambda_2}\left(\frac{x}{x_{II}^*}\right)^{\lambda_1} + \frac{b}{b+\eta K}\left(\frac{x}{x_{II}^*}\right)^{\lambda_1} + \frac{\lambda_2}{\lambda_2-\lambda_1}\frac{\eta K}{b+\eta K}\left(\frac{x}{\bar{x}(K)}\right)^{\lambda_1},$$

$$\tilde{S}_L(x) = \left[\frac{\lambda_1}{\lambda_2-\lambda_1}\frac{\eta K}{b+\eta K}\left(\frac{x_{II}^*}{\bar{x}(K)}\right)^{\lambda_2}\left(\frac{x}{x_{II}^*}\right)^{\lambda_1} - \frac{b}{b+\eta K}\left(\frac{x}{x_{II}^*}\right)^{\lambda_1}\right] - \frac{\lambda_1}{\lambda_2-\lambda_1}\frac{\eta K}{b+\eta K}\left(\frac{x}{\bar{x}(K)}\right)^{\lambda_2}$$

D Additional Graphs





E Algorithm for Numerical Solution

The algorithm is based on value function interation.

1. Solve for the valuation of the firm $T(x, \eta, K)$ and bond $B(x, \eta, K)$ in the liquid regime using the solution given in the Appendix C. Note that the valuation of the firm and bond in the liquid regime does not change

2. Solve for the valuation of the firm $V^1(x, \eta, K)$ and bond $L^1(x, \eta, K)$ in the illiquid regime using the solution given in the Appendix C. This will served as an initial guess.

3. Find the optimal value of the firm that needs to be refinanced such that $M^1(x; K) = \max_{(b', \eta', K')} V^1(x, b', \eta', K')$ such that $L^1(x; b', \eta', K') \geq K$.

4. Solve for $V^2(x, \eta, K)$ and $L^2(x, \eta, K)$ using $M^1(x; K)$, following the equations given in (3) and (4).

5. If $|V^2(x, \eta, K) - V^1(x, \eta, K)| + |L^2(x, \eta, K) - L^1(x, \eta, K)| < \epsilon$, finish. Otherwise, set $V^1(x, \eta, K) = V^2(x, \eta, K)$ and $L^1(x, \eta, K) = L^2(x, \eta, K)$, then back to step (3).

F Optimal Maturity with Zero Cost

In this appendix we consider the case in which there is no cost in refinancing a loan and the safe income, z, is equal to zero. To simplify we study only the case of a pure discount bond, that is, $b = 0$.

The conjecture is that in this case the optimal maturity will be zero (i.e. $\eta = \infty$). To check this conjecturem, we set the value of refinancing to the firm, $M(x)$, equal to what would be the value of a firm that issues zero maturity bonds. Thus, it keeps its principal constant and default only occurs in Phase II (high growth) whenever $x < \bar{x}(K)$. In particular, we conjecture that if the firm takes this continuation payoff as given and bonds are priced as if there is zero probability of default in Phase I (because of refinancing) then the optimal choice of a bond, the optimal (η', K'), equals (∞, K), where K is the initial level of investment and, moreover, the resulting value of the firm under this choice is indeed equal to the conjectured $M(x)$.

First, we display the final equation (so that the problem can be computed) and then we describe the logic.

The value of the firm, $V(x; \eta, K)$, has two branches, one corresponding to $x \geq \bar{x}(K)$ —which we label H— and the other when $x < \bar{x}(K)$, which we denote by L.

$$V^H(x) = \frac{x}{r-\mu} - \frac{\eta\upsilon}{r+\eta+\upsilon}\left(\frac{1}{r+\upsilon} + \frac{1}{r+\eta}\right)K$$
$$+ \bar{M}_1^H K\left(\frac{x}{\bar{x}(K)}\right)^{\tau_1} + \bar{T}_1^H K\left(\frac{x}{\bar{x}(K)}\right)^{\lambda_1} + \bar{V}_1^H K\left(\frac{x}{\bar{x}(K)}\right)^{\pi_1}$$

where,

$$\bar{M}_1^H = \frac{1-\tau_2}{\tau_2-\tau_1}\frac{\upsilon}{r+\upsilon-\mu} + \frac{\tau_2}{\tau_2-\tau_1}\frac{\upsilon}{r+\upsilon}.$$
$$\bar{T}_1^H = \frac{1-\lambda_2}{\lambda_2-\lambda_1}\frac{\eta}{r+\eta-\mu} + \frac{\lambda_2}{\lambda_2-\lambda_1}\frac{\eta}{r+\eta}.$$
$$\bar{V}_1^H = -\frac{1-\pi_2}{\pi_2-\pi_1}\frac{\eta+\upsilon}{r+\eta+\upsilon-\mu} - \frac{\pi_2}{\pi_2-\pi_1}\frac{r+\eta}{r+\eta+\upsilon}.$$

The lower branch is

$$V^L(x) = \frac{x}{r+\eta-\mu} + \frac{x\eta}{r+\eta+\upsilon-\mu}\frac{1}{r+\upsilon-\mu}$$
$$+ \bar{M}_2^L K\left(\frac{x}{\bar{x}(K)}\right)^{\tau_2} + \bar{T}_2^L K\left(\frac{x}{\bar{x}(K)}\right)^{\lambda_2} + \bar{V}_2^L K\left(\frac{x}{\bar{x}(K)}\right)^{\pi_2},$$

where

$$\bar{M}_2^L = \frac{1-\tau_1}{\tau_2-\tau_1}\frac{\upsilon}{r+\upsilon-\mu} + \frac{\tau_1}{\tau_2-\tau_1}\frac{\upsilon}{r+\upsilon}.$$
$$\bar{T}_2^L = \frac{1-\lambda_1}{\lambda_2-\lambda_1}\frac{\eta}{r+\eta-\mu} + \frac{\lambda_1}{\lambda_2-\lambda_1}\frac{\eta}{r+\eta}.$$
$$\bar{V}_2^L = -\frac{1-\pi_1}{\pi_2-\pi_1}\frac{\eta+\upsilon}{r+\eta+\upsilon-\mu} - \frac{\pi_1}{\pi_2-\pi_1}\frac{r+\eta}{r+\eta+\upsilon}.$$

The value of the bond is given by the $L(x)$ function. As in the previous case, it has two branches

$$L^H(x) = \frac{\eta}{r+\eta}K + \bar{B}_1^H(K-D_G)\left(\frac{x}{\bar{x}(K)}\right)^{\lambda_1} + \bar{L}_1^H(K-D_G)\left(\frac{x}{\bar{x}(K)}\right)^{\pi_1},$$

where

$$\bar{B}_1^H = -\frac{\lambda_2}{\lambda_2-\lambda_1}\frac{\eta}{r+\eta}.$$

$$\bar{L}_1^H = \frac{\pi_2}{\pi_2 - \pi_1} \frac{\eta}{r + \eta + \upsilon}.$$

The lower branch is

$$L^H(x) = \frac{\eta}{r+\eta} D_G + \frac{\eta}{r+\eta+\upsilon}(K - D_G) + \bar{B}_2^L (K - D_G) \left(\frac{x}{\bar{x}(K)}\right)^{\lambda_2} + \bar{L}_2^L (K - D_G) \left(\frac{x}{\bar{x}(K)}\right)^{\pi_2},$$

where

$$\bar{B}_2^L = -\frac{\lambda_1}{\lambda_2 - \lambda_1} \frac{\eta}{r+\eta}.$$

$$\bar{L}_2^L = \frac{\pi_1}{\pi_2 - \pi_1} \frac{\eta}{r+\eta+\upsilon}.$$

G Proof of Proposition 1

Recall that the operator H is given by

$$H(M)(x, K) = \sup_{(b', \eta', K') \in \Sigma(x, K; M)} V(x; b', \eta', K'; M(x; K')),$$

where

$$\Sigma(x, K; M) \equiv \{(b', \eta', K') : L(x; b', \eta', K') \geq K + C_F$$
$$\text{and } \frac{x}{r-\mu} + \frac{r + \upsilon\theta}{r+\upsilon} \frac{z}{r} \geq \frac{b'}{r+\upsilon} + \frac{\upsilon}{r+\upsilon} \frac{b' + \eta K'}{r+\eta}\}.$$

Let $\geq$ be the natural order in the space of functions. Thus $F \geq G$ iff for all (x, K) $F(x, K) \geq G(x, K)$. Since $M(x; K')$ is a continuation value of the firm $V(x; b', \eta', K'; M(x; K'))$ increasing in $M(x; K')$. Moreover, higher continuation values increase the likelihood that the debt will be repaid at maturity. Thus the set $\Sigma(x, K, M)$ is also increasing (in the set inclusion sense) in $M(x; K')$.

Let $\mathcal{M}$ be the set of functions that satisfy

$$0 \leq M(x, K) \leq x/(r - \mu).$$

Then, it follows that H maps $\mathcal{M}$ into itself. Moreover, $\mathcal{M}$ is a complete lattice and hence the Knaster-Tarski theorem implies that the set of fixed points is nonempty. This completes the argument.