

ENDOMORPHISM RINGS OF ABELIAN SURFACES AND PROJECTIVE MODELS OF THEIR MODULI SPACES

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Abstract. We construct projective models for Humbert surfaces and QCM-curves, i.e., Shimura curves together with their natural embedding into the coarse moduli space for principally polarized abelian surfaces. The points of a QCM-curve correspond to an abelian surface, such that its algebra of complex multiplications is an order in an indefinite rational quaternion algebra. Moreover, we determine the structure of such orders.

1. Introduction. In this paper we are mainly concerned with constructing, in a concrete way, projective models for arbitrary QCM-curves. For maximal orders of the indefinite quaternion division algebras with discriminant 6 and 10, Hashimoto and Murabayashi constructed in a recent paper ([HM]) projective models of the corresponding QCM-curves. These examples inspired the author to study this question. In [HM] the authors furthermore construct a preimage under the Torelli map (which is an immersion).

Let A be a simple principally polarized complex abelian surface, $\text{End}(A)$ its ring of endomorphisms and $L = \text{End}(A) \otimes \mathbf{Q}$ the algebra of endomorphisms (= the algebra of complex multiplications). Then, as is well-known, the ring $\text{End}(A)$ is of one of the following types: an order in a CM-field of degree four, an order in an indefinite rational quaternion algebra, an order in a real quadratic field, or \mathbf{Z} . The dimensions of the corresponding moduli spaces—named Shimura varieties of PEL-type—is 0, 1, 2, 3, respectively. In the first three cases we refer to these Shimura varieties, together with their embeddings into the Satake compactification $\mathcal{A}_2 = \text{Proj}(A(\Gamma_2))$ of $\Gamma_2 \backslash \mathbf{H}_2$, as CM-points, QCM-curves (quaternionic complex multiplication), and Humbert surfaces. As a projective variety the Satake compactification \mathcal{A}_2 is a quotient of \mathbf{P}^3 by a finite group G of order 46080, see [R1]. We are mainly concerned with QCM-curves, i.e., Shimura curves with their natural embedding into \mathcal{A}_2 . CM-points are special points on Humbert surfaces, which are easy to compute.

Let us define a QCM-order to be any order in an indefinite rational quaternion algebra which occurs as an endomorphism ring of an abelian surface. The first result is to determine the structure of QCM-orders. We prove that a QCM-order may be written as $R = \mathbf{Z} \oplus \mathbf{Z}\alpha \oplus \mathbf{Z}\beta \oplus \mathbf{Z}\alpha\beta$, where α and β are Rosati invariant elements of (positive) discriminant $\Delta(\alpha)$, $\Delta(\beta)$, such that the discriminant matrix

$$S_\Delta = \begin{pmatrix} \Delta(\alpha) & \Delta(\alpha, \beta) \\ \Delta(\alpha, \beta) & \Delta(\beta) \end{pmatrix}$$

is positive definite. The discriminant of R is $d(R) = \det(S_\Delta)/4$. Changing the basis gives a similar matrix $S_\Delta[g] = {}^t g S_\Delta g$ for some $g \in Gl(2, \mathbf{Z})$. This implies that QCM-orders are parametrized by certain classes of binary quadratic forms.

The main result of this paper is to prove that the QCM-order uniquely determines the QCM-curve, if the QCM-order has a primitive discriminant matrix (i.e., $\text{g.c.d.}(\Delta(\alpha), \Delta(\alpha, \beta), \Delta(\beta)) = 1$). This is no longer true in the non-primitive case, where the embedding of the QCM-curve depends not only on the (isomorphism class of the) QCM-order R , but also on the (class of the) embedding $R \hookrightarrow M_4(\mathbf{Z})$. In that case it remains as an open problem to determine the QCM-curves. Finally, a projective model for QCM-curves (both in the primitive and in the non-primitive cases) can be given by determining the component in the intersection of two Humbert surfaces with discriminants $\Delta(\alpha), \Delta(\beta)$. This new method generalizes what was done in [HM]. For making this procedure a general method, it is indispensable to have an algorithm for computing a projective model for arbitrary Humbert surfaces.

Humbert surfaces H_Δ are classified by their discriminant Δ . For fundamental discriminants they are isomorphic to symmetric Hilbert modular surfaces, which were studied extensively by Hirzebruch, Zagier, van der Geer, and many others (see [HG], [vdG2]). For square discriminants they were studied by Hermann [He]. However, projective models are known only in rare cases. In this paper we present a new geometric argument which is crucial to get a model for any discriminant. The covering of \mathcal{A}_2 of level $\Gamma_g^*(2, 4)$ is isomorphic to \mathbf{P}^3 ([R1]). Hence, by Krull's Hauptidealsatz, any irreducible component of the covering of H_Δ of level $\Gamma_g^*(2, 4)$ is given as the zero set of a single irreducible homogeneous polynomial. This allows to get an algorithm for determining a projective model for any discriminant.

In the late 70's Ihara computed the equation of the Shimura curve of discriminant 6. In Kurihara's paper [K] one can find models of Shimura curves, when the discriminants are 6, 10, 14, 22 and 46, hence the corresponding orders are maximal. Ron Livne and Bruce Jordan have done more equations. In his Harvard thesis in 1981 Jordan uses the terminology "QM abelian surface" for an abelian surface with multiplication by a QCM-order. Such a surface produces a point on the corresponding QCM-curve in our terminology. Ron Livne kindly informed me that there are related results (universal families of Kummer surfaces) by Besser in his Tel-Aviv University thesis. For maximal orders R the discriminant is square-free, and hence the discriminant form is primitive. The approach in [K] is to view the group Γ of units in R of reduced norm 1 as a subgroup of $Sl(2, \mathbf{R})$ on fixing an isomorphism $R \otimes \mathbf{R} \cong M_2(\mathbf{R})$ (cf. [Sh1]). Then Γ is a Fuchsian group of the first kind and $\Gamma \backslash \mathbf{H}$ is compact if and only if $R \otimes \mathbf{Q}$ is a division algebra. This was already observed by Poincaré. These Shimura curves for orders in division algebras together with their natural embedding into \mathcal{A}_2 are just the simple QCM-curves in our terminology. They do not intersect the boundary in \mathcal{A}_2 .

2. Notations and first results. Throughout the paper we will use the same notation as in [R1], [R2]. For general facts we refer to [I] and [F]. So let (the traditional model of the

Siegel upper half space)

$$\begin{aligned} \mathbf{H}_g &= \{ \tau \in \text{Mat}_{g \times g}(\mathbf{C}); \tau \text{ is symmetric, } \text{Im}(\tau) > 0 \}, \\ \Gamma_g &= Sp(2g, \mathbf{Z}). \end{aligned}$$

Let Γ be a subgroup of finite index of Γ_g . Denote by $A(\Gamma) = \bigoplus_k [\Gamma, k]$ the ring of modular forms for Γ . Let $\mathcal{A}_g(\Gamma) = \text{Proj}(A(\Gamma))$ be the corresponding Satake compactification; it contains $\Gamma \backslash \mathbf{H}_g$ as an open dense subset. The open part $\Gamma \backslash \mathbf{H}_g$ is the coarse moduli space for principally polarized abelian varieties with level- Γ structure.

The important new ingredient is to consider a fixed order contained in the endomorphism ring. One may consider more generally an arbitrary level Γ and an arbitrary polarization. However, polarization and level have nothing to do with the endomorphism algebra. Changing the polarization just changes the Rosati anti-involution. Hence, to keep things simple, our general policy is to study the case of level 1 and principal polarization. Any other case is only notationally more difficult.

We recall some standard facts in an explicit form. For proofs we refer to [M]. For any τ in \mathbf{H}_g we have the lattice $\Lambda_\tau = \mathbf{Z}^g + \tau \mathbf{Z}^g$ in \mathbf{C}^g and the abelian variety $A_\tau = \mathbf{C}^g / \Lambda_\tau$. On A_τ we choose the principal polarization L_τ , which maps under the Chern class map to the standard alternating form (Riemann form)

$$E_\tau(\lambda, \mu) = \langle x_1, y_2 \rangle - \langle x_2, y_1 \rangle \quad \text{for } \lambda = \tau x_1 + x_2 \text{ and } \mu = \tau y_1 + y_2.$$

One extends the Riemann form to an \mathbf{R} -bilinear form on $\mathbf{C}^g \times \mathbf{C}^g \rightarrow \mathbf{R}$. This form satisfies $E(i\lambda, i\mu) = E(\lambda, \mu)$ and defines the (hermitian) Riemann form $H(\lambda, \mu) = E(i\lambda, \mu) + iE(\lambda, \mu)$. One easily computes, for $\lambda = \tau x_1 + x_2$,

$$H_\tau(\lambda, \lambda) = E_\tau(i\lambda, \lambda) = \langle \Im(\tau)^{-1}(\Re(\tau)x_1 + x_2), (\Re(\tau)x_1 + x_2) \rangle + \langle \Im(\tau)x_1, x_1 \rangle.$$

Hence $H_\tau(\cdot, \cdot)$ is positive definite. An endomorphism $\phi \in \text{End}(A_\tau)$ is given by

$$\phi = A + \tau C \quad \text{with } \phi\tau = B + \tau D,$$

where $A, B, C, D \in M_g(\mathbf{Z})$, and one easily checks that the rational representation

$$\rho_Q : \text{End}(A_\tau) \ni \phi \mapsto M_\phi = \begin{pmatrix} A & -B \\ -C & D \end{pmatrix} \in \text{Mat}_{2g}(\mathbf{Z})$$

is a \mathbf{Z} -algebra embedding. On $M_{2g}(\mathbf{Z})$ we have the Rosati anti-involution, defined by

$$M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \mapsto \hat{M} = J^{-1} {}^t M J = \begin{pmatrix} {}^t D & -{}^t B \\ -{}^t C & {}^t A \end{pmatrix}.$$

This map satisfies $\hat{\hat{M}} = M$ and $(\widehat{M_1 M_2}) = \widehat{M_2} \widehat{M_1}$, and hence is an anti-involution. The endomorphism $\hat{\phi}$ is defined by $M_{\hat{\phi}} = \widehat{M_\phi}$. It is easily verified that

$$E_\tau(\hat{\phi}-, -) = E_\tau(-, \phi-) \quad \text{and} \quad H_\tau(\hat{\phi}-, -) = H_\tau(-, \phi-),$$

and hence $\hat{\phi}$ is adjoint to ϕ with respect to E_τ and H_τ . Because of

$${}^t(C\tau + D)^{-1} = A - \sigma(\tau)C \quad \text{for } \sigma = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in Sp(2g, \mathbf{R}),$$

the action of the endomorphism ring is in accordance with the action of $Sp(2g, \mathbf{R})$ on $\mathbf{H}_g \times \mathbf{C}^g$ by

$$\sigma(\tau, z) = (\sigma(\tau), {}^t(C\tau + D)^{-1}z), \quad \sigma(\tau) = (A\tau + B)(C\tau + D)^{-1}.$$

We have $\deg(\phi) = \det(M_\phi)$ for the degree of an endomorphism. The positivity of the Riemann form implies that $\text{Tr}(\hat{\phi}\phi) = (1/2) \text{Tr}(M_\phi M_\phi) = \text{Tr}(A {}^t D - C {}^t B)$ is positive for $\phi \neq 0$. The positivity of the Rosati involution is essential for the study of the algebra of complex multiplications $\text{End}^0(A_\tau) = \text{End}(A_\tau) \otimes_{\mathbf{Z}} \mathbf{Q}$ of A_τ . This is a semi-simple algebra and was classified by Albert. We fix the following notation for $\tau \in \mathbf{H}_g$:

$$\begin{array}{ccccc} \mathbf{Q} \subset & F = K^{\text{Rosati}} & \subset & S = L^{\text{Rosati}} & \\ & \cap & & \cap & \\ & K & \subset & L = \text{End}^0(A_\tau) & \subset & M_{2g}(\mathbf{Q}). \end{array}$$

In the above diagram K is the center of L , and F is the Rosati invariant part of K . Hence $F = S \cap K$. For simple $\tau \in \mathbf{H}_g$ the algebra L is a division algebra. An algebra $L \subset M_{2g}(\mathbf{Q})$ is called an *Albert algebra* if L is Rosati invariant ($l \in L \Rightarrow \hat{l} \in L$) and the restriction of the Rosati anti-involution is positive. An algebra $L \subset M_{2g}(\mathbf{Q})$ is called *admissible* if there exists a $\tau \in \mathbf{H}_g$ with $L = \text{End}^0(A_\tau)$. An admissible algebra is an Albert algebra. The following theorem (the classification of Albert algebras) is well-known ([M, p. 201]):

THEOREM 1. *Let L be an Albert division algebra of dimension d^2 over its center K , then $F = K^{\text{Rosati}}$ is a totally real number field. We have one of the following cases:*

- (i) $F = K = S = L$ is a number field.
- (ii) $F = K \subset S \subset L$, $\dim_F S = 3$, $\dim_F L = 4$. In this case for any embedding $F \hookrightarrow \mathbf{R}$ it holds that $L \otimes_F \mathbf{R} \cong M_2(\mathbf{R})$ and the isomorphism may be chosen such that the F -vector space S is just the set of symmetric matrices.
- (iii) $F = K = S \subset L$, $\dim_F L = 4$. Then $L \otimes_F \mathbf{R}$ is the skew field of Hamilton quaternions for any embedding $F \hookrightarrow \mathbf{R}$.
- (iv) $[K : F] = 2 = \dim_{\mathbf{Q}}(L) : \dim_{\mathbf{Q}}(S)$, K is a totally imaginary number field. In this case for any embedding $F \hookrightarrow \mathbf{R}$ it holds that $L \otimes_F \mathbf{R} \cong L \otimes_K (K \otimes_F \mathbf{R}) \cong L \otimes_K \mathbf{C} \cong M_d(\mathbf{C})$ and the isomorphism may be chosen such that the F -vector space S maps into the set of hermitian matrices.

3. Algebraic families of principally polarized abelian varieties. For our purpose it turns out to be convenient to consider another model of the Siegel upper half space. A period matrix τ induces by

$$\phi_\tau \begin{pmatrix} x \\ y \end{pmatrix} = x - \tau y$$

an isomorphism $\phi_\tau : \mathbf{R}^{2g} \rightarrow \mathbf{C}^g$, and $M_\tau = \phi_\tau^{-1} i \phi_\tau$ defines the corresponding complex structure on \mathbf{R}^{2g} . As a matrix we have

$$\mathbf{H}_g \ni \tau = p + iq \mapsto M_\tau = \begin{pmatrix} -pq^{-1} & q + pq^{-1}p \\ -q^{-1} & q^{-1}p \end{pmatrix},$$

because for $i = \tau q^{-1} - pq^{-1}$ we get $i(x - \tau y) = (pq^{-1}py + qy - pq^{-1}x) - \tau(q^{-1}py - q^{-1}x)$. The matrices M_τ are elements in $Sp(2g, \mathbf{R})$ and satisfy $\widehat{M}_\tau = -M_\tau = M_\tau^{-1}$. We furthermore define $S_\tau = -M_\tau J$ and get an isomorphism of complex manifolds

$$\mathbf{H}_g \cong \{S_\tau \in Sp(2g, \mathbf{R}); S_\tau \text{ is symmetric and positive definite}\}.$$

In terms of matrices the bijection is given by

$$\mathbf{H}_g \ni \tau = p + iq \mapsto S_\tau = \begin{pmatrix} q + pq^{-1}p & pq^{-1} \\ q^{-1}p & q^{-1} \end{pmatrix} = \begin{pmatrix} q & 0 \\ 0 & q^{-1} \end{pmatrix} \left[\begin{pmatrix} 1 & 0 \\ p & 1 \end{pmatrix} \right].$$

We call this the real (or algebraic) model of the Siegel upper half space. The Rosati anti-involution restricts to an involution on \mathbf{H}_g (in the standard model $\tau \mapsto -\tau^{-1}$). The action of $Sp(2g, \mathbf{R})$ on \mathbf{H}_g corresponds to the action

$$\sigma \bullet M_\tau = \sigma M_\tau \sigma^{-1}$$

on matrices of type M_τ and

$$\sigma \circ S_\tau = \sigma S_\tau \sigma^t$$

on the real model. Remark that

$$\sigma \bullet M_\tau = \sigma S_\tau J \sigma^{-1} = \sigma S_\tau \sigma^t J = (\sigma \circ S_\tau) J,$$

hence the actions are equivariant. We will freely use τ , M_τ or S_τ to denote an element of the Siegel upper half space in the standard model or in the real model. For our purpose the algebraic model is more appropriate. One easily checks that for

$$M = \begin{pmatrix} A & -B \\ -C & D \end{pmatrix} \in \text{Mat}_{2g}(\mathbf{Z}) \quad \text{and} \quad M_\tau = \begin{pmatrix} -pq^{-1} & q + pq^{-1}p \\ -q^{-1} & q^{-1}p \end{pmatrix}$$

we have

$$\begin{aligned} M \in \text{End}(A_\tau) &\iff (A + \tau C)\tau = B + \tau D \\ &\iff MM_\tau = M_\tau M. \end{aligned}$$

This leads to the following definitions for any admissible algebra $L \subset M_{2g}(\mathbf{Q})$

$$\begin{aligned} \mathbf{H}(L) &= \{\tau \in \mathbf{H}_g; lM_\tau = M_\tau l \text{ for all } l \in L\}, \\ \Gamma(L) &= \{\sigma \in \Gamma_g; \sigma L = L\sigma\}. \end{aligned}$$

Moreover we consider the diagram

$$\begin{array}{ccc} \Gamma(L) \backslash \mathbf{H}(L) & \rightarrow & \Gamma_g \backslash \mathbf{H}_g \\ & \downarrow & \cap \\ \mathcal{A}(L) & \rightarrow & C(L) \hookrightarrow \mathcal{A}_g, \end{array}$$

where $C(L)$ denotes the closure of the image of $\Gamma(L) \backslash \mathbf{H}(L)$ in the Satake compactification \mathcal{A}_g and $\mathcal{A}(L)$ its normalisation. We call $\mathcal{A}(L)$ the Shimura variety of type L and $C(L)$ the cycle of type L . It is obvious that $\Gamma(L)$ is acting on $\mathbf{H}(L)$, and the induced map $\Gamma(L) \backslash \mathbf{H}(L) \rightarrow \Gamma_g \backslash \mathbf{H}_g$ is injective for points with $\text{End}^0(A_\tau) = L$. Therefore it is generically injective. The group $\Gamma(L)$ is the largest subgroup of Γ_g acting on $\mathbf{H}(L)$. It is proved in [R6]

that $C(L)$ and $\mathcal{A}(L)$ are projective varieties and that the map $L \mapsto \Gamma(L) \backslash \mathbf{H}(L)$ is an equivalence of the category of admissible algebras with Rosati-equivariant embeddings in $M_{2g}(\mathcal{Q})$ and that of irreducible varieties with closure $C(L)$ parametrizing principally polarized abelian varieties A_τ with $L \subset \text{End}^0(A_\tau)$. Similar embeddings are studied in [Sa].

For small genus the following division algebras occur as endomorphism algebras ([Sh2], [R6]): In genus $g = 1$ there occur only $\mathcal{A}(\mathcal{Q}) = \mathcal{A}_1$ and the CM-points $\mathcal{A}(K)$. In genus $g = 2$, the 3-fold $\mathcal{A}(\mathcal{Q}) = \mathcal{A}_2$, the surfaces $\mathcal{A}(F)$ for the real-quadratic fields F/\mathcal{Q} , the curves $\mathcal{A}(L)$ for the indefinite quaternion algebras L/\mathcal{Q} and the CM-points. In genus $g = 3$, the 6-fold $\mathcal{A}(\mathcal{Q}) = \mathcal{A}_3$, the 3-folds $\mathcal{A}(F)$ for the real-cubic fields F/\mathcal{Q} , the surfaces $\mathcal{A}(K)$ for the indefinite imaginary-quadratic extensions K/\mathcal{Q} and the CM-points.

4. Humbert surfaces. We start by reviewing the classical terminology. As remarked in the last chapter, the equation $MM_\tau = M_\tau M$ for a matrix $M = \begin{pmatrix} A & -B \\ -C & D \end{pmatrix}$ is equivalent to $(A + \tau C)\tau = B + \tau D$. The disadvantage of the classical terminology is that one gets a quadratic equation for the entries of the period matrix $\tau = \begin{pmatrix} \tau_1 & \tau_2 \\ \tau_2 & \tau_3 \end{pmatrix}$. A Rosati invariant matrix is of type

$$M = \hat{M} = \begin{pmatrix} a_1 & a_2 & 0 & b \\ a_3 & a_4 & -b & 0 \\ 0 & c & a_1 & a_3 \\ -c & 0 & a_2 & a_4 \end{pmatrix}$$

and the above equation is equivalent to

$$a_3\tau_1 + (a_4 - a_1)\tau_2 - a_2\tau_3 + c(\tau_2^2 - \tau_1\tau_3) + b = 0,$$

which is the type of equations studied by Humbert in the last century [Hu]. Humbert called such a relation a *singular relation*.

Any Rosati invariant matrix M satisfies $M^2 - \text{Tr}(A)M + \det(A) + bc = 0$, and hence has the reduced Trace $t(M) = \text{Tr}(A)$ and the discriminant

$$\Delta(M) = \text{Tr}(A)^2 - 4(\det(A) + bc) = (a_1 - a_4)^2 + 4(a_2a_3 - bc).$$

We call M primitive if $\mathbf{Z}[M] = \mathcal{Q}(M) \cap M_4(\mathbf{Z})$ and normalized if $\text{Tr}(A) \in \{0, 1\}$. The first aim is to prove the following theorem similar to the main result in [Hu]:

THEOREM 2. *Let $M_1 = \hat{M}_1$ and $M_2 = \hat{M}_2$ be Rosati invariant elements in $M_4(\mathbf{Z})$. Then there exists an element $\sigma \in \Gamma_2$ such that $\sigma \bullet M_1 = \sigma M_1 \sigma^{-1} = M_2$ if and only if $t(M_1) = t(M_2)$, $\text{g.c.d}(M_1) = \text{g.c.d}(M_2)$ and $\Delta(M_1) = \Delta(M_2)$.*

PROOF. Conjugation preserves the minimal equation, and hence preserves $t(M)$ and $\Delta(M)$. So let assume $t(M_1) = t(M_2)$ and $\Delta(M_1) = \Delta(M_2)$. Moreover, we may assume that both matrices are primitive and normalized.

First step: Let $M = \hat{M}$ be as above. We have

$$\begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & \alpha & 0 & \beta \\ -1 & 0 & 0 & 0 \\ 0 & \gamma & 0 & \delta \end{pmatrix} \cdot \begin{pmatrix} a_1 & a_2 & 0 & b \\ a_3 & a_4 & -b & 0 \\ 0 & c & a_1 & a_3 \\ -c & 0 & a_2 & a_4 \end{pmatrix} = \begin{pmatrix} * & * & * & * \\ * & * & * & * \\ 0 & b\gamma - a_2\delta & * & * \\ a_2\delta - b\gamma & 0 & * & * \end{pmatrix}.$$

Writing $-a_2/b$ as a reduced fraction γ/δ , $\text{g.c.d.}(\gamma, \delta) = 1$, it is easy to find $\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in$

$Sl(2, \mathbf{Z})$ such that $a_2\delta - b\gamma = 0$. Hence we may assume $c = 0$.

Second step:

$$\begin{pmatrix} 1 & 0 & \alpha & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} a_1 & a_2 & 0 & b \\ a_3 & a_4 & -b & 0 \\ 0 & a_1 & a_3 \\ a_2 & a_4 \end{pmatrix} = \begin{pmatrix} a_1 & a_2 & 0 & b + \alpha a_3 \\ a_3 & a_4 & * & 0 \\ 0 & a_1 & a_3 \\ a_2 & a_4 \end{pmatrix}.$$

Let $g_1 = \text{g.c.d.}(a_2, a_3, b)$ and $g_1 g_2 = \text{g.c.d.}(a_3, b)$, then $b/g_1 g_2$ and $a_3/g_1 g_2$ are coprime. Choose α such that $b/g_1 g_2 + \alpha a_3/g_1 g_2$ is prime to a_2 (e.g., a big prime number, using Dirichlet's theorem on primes in arithmetic progressions). Then the above equality shows that we may assume $(a_2, b) | a_3$.

Third step: Let

$$\begin{aligned} a_3 &= px, \\ a_2 &= py, \\ b &= pz \text{ with } (y, z) = 1. \end{aligned}$$

Then the equality

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \alpha & 0 & \beta \\ s & 0 & 1 & 0 \\ 0 & \gamma & 0 & \delta \end{pmatrix} \cdot \begin{pmatrix} a_1 & a_2 & 0 & b \\ a_3 & a_4 & -b & 0 \\ 0 & a_1 & a_3 \\ a_2 & a_4 \end{pmatrix} = \begin{pmatrix} * & * & 0 & \alpha b - a_2 \beta \\ * & * & * & 0 \\ 0 & s(a_2\delta - b\gamma) - a_3\gamma & * & * \\ * & 0 & * & * \end{pmatrix}$$

implies that for $\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} = \begin{pmatrix} y & z \\ \gamma & \delta \end{pmatrix} \in Sl(2, \mathbf{Z})$ and $s = x\gamma$ we get a matrix M with $b = c = 0$.

Fourth step: Using

$$\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a_1 & a_2 \\ a_3 & a_4 \end{pmatrix} \begin{pmatrix} 1 & -x \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} a_1 + xa_3 & a_2 + (a_4 - a_1)x - x^2 a_3 \\ a_3 & a_4 - xa_3 \end{pmatrix},$$

we may assume that the $\text{g.c.d.}(a_2, a_3) = 1$.

Fifth step: Let $\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in Sl(2, \mathbf{Z})$ with $\alpha = a_2, \beta = a_3$. Then

$$\begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & \alpha & 0 & \beta \\ \beta\gamma & 0 & 1 + \beta\gamma & 0 \\ 0 & \gamma & 0 & \delta \end{pmatrix} \cdot \begin{pmatrix} a_1 & a_2 & 0 \\ a_3 & a_4 & 0 \\ 0 & a_1 & a_3 \\ a_2 & a_4 \end{pmatrix} = \begin{pmatrix} a_1 & 1 & 0 \\ a_2 a_3 & a_4 & 0 \\ 0 & * & * \\ * & * & * \end{pmatrix}.$$

By using a step similar to Step 4, we may assume $a_1 = 0$. Therefore we finally obtain a normal form for primitive normalized matrices

$$M_{\text{standard}} = \begin{pmatrix} 0 & 1 & 0 \\ k & l & 0 \\ 0 & 0 & k \end{pmatrix}, \quad t(M) = l, \quad \Delta(M) = 4k + l$$

with $k \in \mathbf{Z}$ and $l \in \{0, 1\}$. □

COROLLARY 3. *Let M_1 and M_2 be Rosati invariant elements. Then $\mathcal{Q}(M_1) \cap M_4(\mathbf{Z})$ and $\mathcal{Q}(M_2) \cap M_4(\mathbf{Z})$ are isomorphic as \mathbf{Z} -algebras if and only if the orders are conjugate by an element of Γ_2 .*

In classical terminology Humbert proved that up to equivalence any relation may be written as $k\tau_1 + l\tau_2 = \tau_3$ (see [HM, 2.7]).

The theorem allows us to construct the following standard model for Humbert surfaces. Let $F = \mathcal{Q}[\omega]$ be a quadratic \mathcal{Q} -algebra with positive discriminant Δ_F , \mathcal{O}_F its ring of integers and $\mathcal{O} = \mathbf{Z}[\omega]$ the order of discriminant $\Delta = \Delta_F f^2$ in \mathcal{O}_F . Hence F is a real quadratic number field or is isomorphic to $\mathcal{Q} \times \mathcal{Q}$. Let σ_1, σ_2 be the projections $F \otimes_{\mathcal{Q}} \mathbf{R} \cong \mathbf{R} \oplus \mathbf{R} \rightarrow \mathbf{R}$. Let $\sigma : F \ni x \mapsto \text{diag}(\sigma_1(x), \sigma_2(x))$, which induces a map

$$\sigma : Sl(2, F) \ni \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \begin{pmatrix} \sigma(a) & \sigma(b) \\ \sigma(c) & \sigma(d) \end{pmatrix} \in Sp(4, \mathbf{R}).$$

We fix a \mathbf{Z} -basis $1 = \omega_1, \omega = \omega_2$ for \mathcal{O} and denote by $R = (\sigma_i(\omega_j))$ the Gram matrix of F . Then tRR is the symmetric positive definite rational matrix ${}^tRR = (T_{F/\mathcal{Q}}(\omega_i\omega_j))_{i,j}$, where $T_{F/\mathcal{Q}}$ denotes the trace map. Let $x \in F$ be arbitrary. Then $x\omega_i = \sum_j A_{ij}\omega_j$ for some matrix $A(x) = (A_{ij}) \in M_n(\mathcal{Q})$. It is easy to check that

$$A(x) = {}^tR\sigma(x){}^tR^{-1}$$

and

$$x \in \mathcal{O} \iff A(x) \in M_2(\mathbf{Z}).$$

We fix the embedding

$$(\mathbf{H}_1)^2 \ni (\tau_1, \tau_2) \mapsto \pi = \begin{pmatrix} \tau_1 \\ \tau_2 \end{pmatrix} \in \mathbf{H}_2.$$

Because of

$$(A)(\pi[R]) = ({}^tR\sigma(\omega){}^tR^{-1})({}^tR\pi R) = ({}^tR\pi R)(R^{-1}\sigma(\omega)R) = (\pi[R])({}^tA),$$

we get a diagram (twice the regular representation)

$$\begin{array}{ccccc} \mathcal{O} & \subset & \text{End}(A_{\pi[R]}) & \subset & M_4(\mathbf{Z}) \\ \cap & & \cap & & \cap \\ F & \subset & \text{End}(A_{\pi[R]}) & \subset & M_4(\mathcal{Q}), \end{array}$$

where the inclusion is given by

$$F \ni \omega \mapsto \begin{pmatrix} {}^tR & 0 \\ 0 & R^{-1} \end{pmatrix} \begin{pmatrix} \sigma(\omega) & 0 \\ 0 & \sigma(\omega) \end{pmatrix} \begin{pmatrix} {}^tR & 0 \\ 0 & R^{-1} \end{pmatrix}^{-1} \in M_4(\mathbf{Q}).$$

By the last theorem any period $\tau = \pi[R]$ with $\mathcal{O} \subset \text{End}(A_\tau)$ lies in a manifold $\mathbf{H}(F)$ as constructed above. By taking the quotient $\Gamma(F)\backslash\mathbf{H}(F)$, we get the standard model for Humbert surfaces. The Humbert modular group $\Gamma(F) = \Gamma(\Delta)$ is described in the following lemma.

LEMMA 4. *For any order \mathcal{O} in a real quadratic algebra F there exists a Rosati equivariant embedding $\psi_{\mathcal{O}} : M_2(F) \rightarrow M_4(\mathbf{Q})$ and an element $\sigma_{\text{Gal}} \in \Gamma_2 = \text{Sp}(4, \mathbf{Z})$ of order two, such that $\psi_{\mathcal{O}}^{-1}(\text{Im}(\psi_{\mathcal{O}}) \cap \Gamma_2) = \text{Sl}(2, \mathcal{O})$ and*

$$\sigma_{\text{Gal}} \psi_{\mathcal{O}} \left(\begin{pmatrix} a & b \\ c & d \end{pmatrix} \right) \sigma_{\text{Gal}} = \psi_{\mathcal{O}} \left(\begin{pmatrix} \bar{a} & -\bar{b} \\ -\bar{c} & \bar{d} \end{pmatrix} \right),$$

where $x \mapsto \bar{x}$ generates the Galois group $\text{Gal}(F/\mathbf{Q})$. The Humbert modular group $\Gamma(F) = \Gamma(\Delta)$ is generated by σ_{Gal} and $\psi_{\mathcal{O}}(\text{Sl}(2, \mathcal{O}))$.

PROOF. Let $\omega^2 - l\omega - k = 0$ be the minimal equation of ω . Hence $\bar{\omega} = l - \omega$ and $\Delta(\omega) = 4k + l$. Then define

$$\psi_{\mathcal{O}} \left(\begin{pmatrix} a_1 + a_2\omega & b_1 + b_2\omega \\ c_1 + c_2\omega & d_1 + d_2\omega \end{pmatrix} \right) = \begin{pmatrix} a_1 & a_2 & b_2 & b_1 + lb_2 \\ ka_2 & a_1 + la_2 & b_1 + lb_2 & kb_2 + l(b_1 + b_2) \\ kc_2 - lc_1 & c_1 & d_1 & kd_2 \\ c_1 & c_2 & d_2 & d_1 + ld_2 \end{pmatrix}.$$

This embedding is obtained by extending the above embedding of F by

$$M_2(F) \ni \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \begin{pmatrix} {}^tR & 0 \\ 0 & R^{-1} \end{pmatrix} \begin{pmatrix} \sigma(a) & \sigma(b/\mathcal{D}) \\ \sigma(c\mathcal{D}) & \sigma(d) \end{pmatrix} \begin{pmatrix} {}^tR & 0 \\ 0 & R^{-1} \end{pmatrix}^{-1},$$

where $\mathcal{D} = 2\omega - 1$ is the different of the order \mathcal{O} . The Galois element is given by

$$\sigma_{\text{Gal}} = \begin{pmatrix} 1 & 0 & 0 \\ l & -1 & 0 \\ 0 & 1 & l \\ 0 & 0 & -1 \end{pmatrix}.$$

We omit further details, which are similar to computations in [R3]. □

We remark that the Humbert modular group $\Gamma(\Delta_F)$ is just the symmetric Hilbert modular group if \mathcal{O} is the full ring of integers \mathcal{O}_F in a real quadratic number field.

5. Modular forms and projective models of Humbert surfaces. The theta constants (of second kind) are given by (we use Mumford’s notation f_a).

$$f_a(\tau) = \theta \begin{bmatrix} a \\ 0 \end{bmatrix} (2\tau) = \sum_{x \in \mathbf{Z}^g} \exp 2\pi i \left(\tau \left[x + \frac{1}{2}a \right] \right)$$

for $a \in \mathbf{Z}^g$. The functions $f_a(\tau)$ depend only on $a \bmod 2$, and hence a is regarded as an element in \mathbf{F}_2^g . The action of $Sp(2g, \mathbf{Z})$ on \mathbf{H}_g induces for any $k \in \mathbf{Z}$ a (right) group action on the algebra of holomorphic functions $\{f : \mathbf{H}_g \rightarrow \mathbf{C}\}$ by

$$f|_k\sigma(\tau) = \det(C\tau + D)^{-k} f(\sigma(\tau)).$$

A holomorphic function f on \mathbf{H}_g is a modular form of weight k , or in short $f \in [\Gamma_g, k]$, if and only if $f|_k\sigma = f$ for all $\sigma \in \Gamma_g$. In genus $g = 1$ one has to add a condition for the cusps. We recall from [R1] that the ring of modular forms of even weight is given by

$$A(\Gamma_g)_{(2)} = \bigoplus_{2|k} [\Gamma_g, k] = (\mathbf{C}[f_a(\tau)]^{\mathbf{H}_g})^N.$$

Here N denotes the normalization (in its field of fractions). H_g is a finite group obtained as the image of the theta representation $\rho_{\text{theta}} : \Gamma_g \rightarrow H_g/(\pm 1)$ (see [R3]). The kernel of ρ_{theta} is denoted by $\Gamma_g^*(2, 4)$.

DEFINITION 5. A holomorphic function f on $\mathbf{H}(L)$ is a modular form of weight k and type L , or in short $f \in [\Gamma(L), k]$, if $f|_k\sigma = f$ for all $\sigma \in \Gamma(L)$.

(In genus $g = 1$ one has to add a cusp condition, but this case is not interesting for our purpose. The varieties $\mathcal{A}(L)$ are just points for $L \neq \mathbf{Q}$.) Denote by $A(L) = \bigoplus_k [\Gamma(L), k]$ the ring of modular forms of type L and by $\mathcal{A}(L) = \text{Proj}(A(L))$ the corresponding Satake compactification, which is the normalization of $C(L)$.

Strictly speaking, it is very easy to compute the ring $A(L)$ of modular forms. The first step is to find all relations for the restricted functions $\widetilde{f_a(\tau)}$ on $\mathbf{H}(L)$. Let $G(L) = \{\sigma \in H_g; \sigma = (\pm)\rho_{\text{theta}}(g) \text{ for some } g \in \Gamma(L)\}$ be the image of $\Gamma(L)$ in the finite group H_g . The element i is always contained in $G(L)$, therefore we always get modular forms of even weight. The following theorem is a consequence of the corresponding result for the group $\Gamma_g^*(2, 4)$.

THEOREM 6. *The ring of modular forms of even weight is given by*

$$A(L)_{(2)} = \bigoplus_{2|k} [\Gamma(L), k] = (\mathbf{C}[\widetilde{f_a(\tau)}]^{G(L)})^N.$$

The problem is to find all relations and to compute the normalization. This is usually a difficult problem. However, in some low-dimensional cases it is possible to finish the computation ([I], [R2], [R4]). For small genus we have $A(\Gamma_1) = \mathbf{C}[f_a(\tau)]^{\mathbf{H}_1}$, $A(\Gamma_2)_{(2)} = \mathbf{C}[f_a(\tau)]^{\mathbf{H}_2}$ and $A(\Gamma_3) = \mathbf{C}[f_a(\tau)]^{\mathbf{H}_3}$. We use binary numbers to index the theta constants, i.e., (in genus $g = 2$) $f_0 = f_0$, $f_1 = f_1$, $f_2 = f_1$ and $f_3 = f_1$. In particular, $\text{Proj}(A(\Gamma_2))$ is isomorphic to the quotient of \mathbf{P}^3 by the finite group $G = H_2$ of order 46080.

By a well-known geometric argument (Krull's Hauptidealsatz, [H, Ex. I 2.8]) any Humbert surface H_Δ is given as the zero set of a single irreducible homogeneous polynomial $F_{\Delta,i}(f_0, f_1, f_2, f_3)$ of positive degree in \mathbf{P}^3 divided out by the finite group $G(\Delta, i) = \pm\rho_{\text{theta}}(\Gamma(H_{\Delta,i}))$, where i labels the components of the covering of H_Δ in level $\Gamma_2^*(2, 4)$. All the groups $G(\Delta, i)$ are conjugate in G . The equation is not unique. However, all these

equations are in one orbit under the action of the finite group G . It is a famous observation by van der Geer ([vdG1]) how to find the degree. Consider the modular form of weight $5/2$ on $\Gamma_{1,1}(4)$ (in the $+$ -space) with q -expansion ($q = \exp(2\pi i \tau)$)

$$f_0 \left(f_0^4 - \frac{5}{4} f_1^4 \right) (\tau) = 1 - \sum_{\Delta \geq 1} a_{\Delta} q^{\Delta} = 1 - (10q + 70q^4 + 48q^5 + 120q^8 + 250q^9 + 240q^{12} \dots)$$

Let

$$m(\Delta) = [\Gamma_2 : \Gamma_2^*(2, 4)\Gamma(H_{\Delta}, i)] = \begin{cases} 10 & \Delta \equiv 1(8) \\ 60 & \Delta \equiv 0(4) \\ 6 & \Delta \equiv 5(8) \end{cases}$$

be the number of components of the covering of H_{Δ} in level $\Gamma_2^*(2, 4)$ (this computation is straightforward, see also [B3, 7.13] for a similar computation in level 2, where the number of components is 10, 15 and 6, respectively for the cases as above) and let

$$v(x) = \begin{cases} 1/2 & x = 1, \\ 1 & x \geq 2, x \in \mathbf{N}, x \equiv 0, 1(4), \\ 0 & \text{else.} \end{cases}$$

Then we have

$$a_{\Delta} = \sum_x v \left(\frac{\Delta}{x^2} \right) m \left(\frac{\Delta}{x^2} \right) \deg(F_{(\Delta/x^2), i}).$$

Using a formula of Siegel, the coefficients of the modular form may be computed alternatively by

$$a_{\Delta} - 24 \sum_{x \in \mathbf{Z}} \sigma_1 \left(\frac{\Delta - x^2}{4} \right) = \begin{cases} 12\Delta - 2 & \text{if } \Delta \text{ is a square,} \\ 0 & \text{else.} \end{cases}$$

Therefore we get the following table:

Δ	1	4	5	8	9	12	13	16	17	20	21	24	25	28	29	32
a_{Δ}	10	70	48	120	250	240	240	550	480	528	480	720	1210	960	720	1080
$\deg(F_{\Delta})$	2	1	8	2	24	4	40	8	48	8	80	12	120	16	120	16

We remark that [R5] answers the question (posed in [vdG1, p. 333]) to give a direct way to compute the Hodge diamond of $\mathcal{A}_2(2)$. The Hodge numbers are given by $h_{i,i} = 1$ and $h_{i,j} = 0$ for $i \neq j$. After blowing up the singularities, one gets $h_{1,1} = h_{2,2} = 16$, $h_{0,0} = h_{3,3} = 1$ and $h_{i,j} = 0$ for $i \neq j$. This information is sufficient to compute $\deg(F_{\Delta, i})$, which is explained in [vdG1, Chapter 8]. In principle this allows us to compute equations $F_{\Delta, i}$ for H_{Δ} for any discriminant $\Delta = 4k + l$. One takes one of the components in level $\Gamma_2^*(2, 4)$ (we call this component the *standard component* and put $F_{\Delta} = F_{\Delta, \text{standard}}$, because it is associated to σ_{standard} with $t(\alpha_{\text{standard}}) = l \in \{0, 1\}$ and $\Delta(\alpha_{\text{standard}}) = 4k + l$) containing period matrices of type $\begin{pmatrix} \tau & z \\ z & k\tau + lz \end{pmatrix}$ and takes the $(p-q)$ -expansion with $pq = \exp 2\pi i \tau/4$ and $q = \exp 2\pi i z/4$ of the theta constants

$$f_b^a \left(\begin{pmatrix} \tau & z \\ z & k\tau + lz \end{pmatrix} \right) = \sum_{x, y \in \mathbf{Z}} pq^{(2x+a)^2 + k(2y+b)^2} q^{l(2y+b)^2 + 2(2x+a)(2y+b)},$$

which is a power series in $\mathbf{Z}[[p, q]]$. The equation $F_\Delta(f_0, f_1, f_2, f_3) \equiv 0$ in $\mathbf{Z}[p, q]/(q^n, p^n)$ for $n \gg 1$ has a unique non-trivial solution of the correct degree, which can be computed by solving a linear problem for the coefficients of F_Δ . For the following examples we use the notation $(a, b, c, d) = f_0^a f_1^b f_2^c f_3^d$. To get the equation of the other components in level $\Gamma_2^*(2, 4)$ one uses the theta representation. If $\sigma(\alpha) = \alpha_{\text{standard}}$, then $\sigma \bullet \mathbf{H}(\alpha) = \mathbf{H}(\alpha_{\text{standard}})$ and $\rho_{\text{theta}}(\sigma)(F_\Delta)$ is the equation of the component containing $\Gamma(\Delta, \alpha) \setminus \mathbf{H}(\alpha)$.

The easiest example is H_1 with $F_1 = (1, 0, 0, 1) - (0, 1, 1, 0) = f_0 f_3 - f_1 f_2$. This Humbert surface is a quotient of $\mathbf{P}^1 \times \mathbf{P}^1$. We refer to [R5] for more detailed information. For discriminant 4 we get the linear equation $F_4 = (0, 1, 0, 0) - (0, 0, 1, 0) = f_1 - f_2$. Therefore H_4 is a quotient of \mathbf{P}^2 . For discriminant 5 we get $F_5 = (4, 4, 0, 0) + (4, 0, 4, 0) + (4, 0, 0, 4) + (0, 4, 4, 0) + (0, 4, 0, 4) + (0, 0, 4, 4) + 2(2, 2, 2, 2) - 2((5, 1, 1, 1) + (1, 5, 1, 1) + (1, 1, 5, 1) + (1, 1, 1, 5))$. For discriminant 8 we get $F_8 = 2(1, 1, 0, 0) - (0, 0, 2, 0) - (0, 0, 0, 2)$, and for discriminant 12 we get $F_{12} = 4(2, 1, 1, 0) + 4(0, 1, 1, 2) - (0, 4, 0, 0) - (0, 0, 4, 0) - 2(0, 2, 2, 0) - 4(2, 0, 0, 2)$. In Appendix we give equations for the discriminants 16, 20, 24 and 28 of degree 8, 8, 12 and 16, respectively. Using the old-fashioned theta constants of Riemann, instead of the f_a 's, Humbert calculated an equation for discriminant 1, 4, 5 and 8 in a much more complicated way ([Hu]). This was reestablished by Hashimoto and Murabayashi in [HM] (see also [B1–3]).

6. QCM-orders and QCM-curves. We recall that an order R in an indefinite rational quaternion algebra $A = R \otimes \mathbf{Q}$ is called a QCM-order if $R = \text{End}(X)$ for some abelian surface X . This is equivalent to that $L = R \otimes \mathbf{Q}$ is admissible and $R = L \cap M_4(\mathbf{Z})$ for some Rosati equivariant embedding $L \subset M_4(\mathbf{Q})$. In a rational quaternion algebra A any element satisfies an equation $x^2 - t(x)x + n(x) = 0$, where $t(x)$ and $n(x)$ are called the reduced trace and norm (see [E]). The map $n : A \rightarrow \mathbf{Q}$ is multiplicative and $t : A \rightarrow \mathbf{Q}$ is additive. The main anti-involution is defined by $\bar{x} = t(x) - x$ and the “Zwischennorm” is defined by

$$n(x, y) = n(x + y) - n(x) - n(y) = n(y, x) = t(x)t(y) - t(xy) = x\bar{y} + y\bar{x} = y\bar{x} + x\bar{y}.$$

The discriminant form is defined by

$$\Delta(x, y) = \frac{1}{2}(\Delta(x + y) - \Delta(x) - \Delta(y)) = 2t(xy) - t(x)t(y) = t(x)t(y) - 2n(x, y),$$

and the discriminant $d(x_1, x_2, x_3, x_4)$ of a module generated by x_1, \dots, x_4 is defined by

$$d(x_1, x_2, x_3, x_4)^2 = -\det(t(x_i x_j)).$$

Obviously, $n(x, x) = 2n(x) = 2x\bar{x}$, $\Delta(x) = \Delta(x, x) = t(x)^2 - 4n(x)$ and $n(x, 1) = t(x)$. While the sign of the discriminant varies in the literature, we always choose the positive sign. The discriminant of A is defined to be the discriminant of some maximal order R in A . The discriminant form is identically zero on \mathbf{Q} , and hence determines a ternary quadratic form on A/\mathbf{Q} .

THEOREM 7. *Any QCM-order can be written as $R = \mathbf{Z} \oplus \mathbf{Z}\alpha \oplus \mathbf{Z}\beta \oplus \mathbf{Z}\alpha\beta$, where α and β are primitive Rosati invariant elements of positive discriminant $\Delta(\alpha), \Delta(\beta)$, such that*

the discriminant matrix

$$S_\Delta = \begin{pmatrix} \Delta(\alpha) & \Delta(\alpha, \beta) \\ \Delta(\alpha, \beta) & \Delta(\beta) \end{pmatrix}$$

is positive definite. The discriminant of R is $d(R) = \det(S_\Delta)/4$.

PROOF. It follows from Theorem 1 that $R \otimes \mathbf{Q} = \mathbf{Q} \oplus \mathbf{Q}\alpha \oplus \mathbf{Q}\beta \oplus \mathbf{Q}\alpha\beta$, where α and β are Rosati invariant elements of positive discriminant $\Delta(\alpha)$, $\Delta(\beta)$. Moreover, up to multiplication by a non-zero rational number, the element $\gamma = \alpha\beta - \beta\alpha$ is a unique element with $\hat{\gamma} = -\gamma$. The algebra $R \otimes \mathbf{Q}$ is admissible if and only if $\gamma^2 = -n(\gamma) = (\Delta(\alpha, \beta) - \Delta(\alpha)\Delta(\beta))/4$ is a negative number. Since we have $R = R \otimes \mathbf{Q} \cap M_4(\mathbf{Z})$, we may assume that α and β are primitive normalized elements of R . By Theorem 2 we may assume that (α, β) are as follows:

$$\alpha = \begin{pmatrix} 0 & 1 & & \\ k & l & & \\ & & 0 & k \\ & & 1 & l \end{pmatrix}, \quad \beta = \begin{pmatrix} a_1 & a_2 & 0 & b \\ a_3 & a_4 & -b & 0 \\ 0 & c & a_1 & a_3 \\ -c & 0 & a_2 & a_4 \end{pmatrix}.$$

We have $t(\alpha) = l$, $n(\alpha) = -k$, $\Delta(\alpha) = 4k + l^2$, $t(\beta) = a_1 + a_4$, $n(\beta) = bc + a_1a_4 - a_2a_3$, $\Delta(\beta) = (a_1 - a_4)^2 + 4(a_2a_3 - bc)$, $\Delta(\alpha, \beta) = t(\alpha)t(\beta) + 2(ka_2 + a_3 - la_1)$, $n(\alpha, \beta) = -(ka_2 + a_3 - la_1)$. We may assume furthermore that $\beta - a_2\alpha - a_1$ is primitive, and hence $(a_3 - ka_2, a_4 - a_1 - la_2, b, c) = 1$. We have

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ ky - lx & x & 1 & 0 \\ x & y & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} a_1 & a_2 & 0 & b \\ a_3 & a_4 & -b & 0 \\ 0 & c & a_1 & a_3 \\ -c & 0 & a_2 & a_4 \end{pmatrix} = \begin{pmatrix} * & * & 0 & b \\ * & * & -b & 0 \\ 0 & \tilde{c} & * & * \\ -\tilde{c} & 0 & * & * \end{pmatrix},$$

where $\tilde{c} = c + x(a_4 - a_1 - la_2) - y(ka_2 - a_3) + b(x^2 - ky^2 + lxy)$. We may choose x such that $c + x(a_4 - a_1 - la_2) = (c, a_4 - a_1 - la_2)p_1$ for some prime p_1 not dividing $ka_2 - a_3$, and choose y such that $c + x(a_4 - a_1 - la_2) - y(ka_2 - a_3) = (c, a_4 - a_1 - la_2, ka_2 - a_3)p_2$ for some big prime p_2 not dividing b . Hence we may assume that the $\text{g.c.d.}(b, c) = 1$ for a basis (α, β) . Now it is easy to check that $\text{g.c.d.}(b, c) = 1$ for a basis (α, β) implies that $\mathbf{Q}(\alpha, \beta) \cap M_4(\mathbf{Z}) = \mathbf{Z} \oplus \mathbf{Z}\alpha \oplus \mathbf{Z}\beta \oplus \mathbf{Z}\alpha\beta$. The computation of the discriminant is omitted. \square

LEMMA 8. Let α and β be Rosati invariant elements of $M_4(\mathbf{Q})$. Then it holds that

$$\alpha\beta + \beta\alpha = t(\alpha)\beta + t(\beta)\alpha - n(\alpha, \beta).$$

In particular, if $\mathbf{Q}(\alpha)$ and $\mathbf{Q}(\beta)$ are real quadratic fields such that $\mathbf{Q}(\alpha) \neq \mathbf{Q}(\beta)$, then $\mathbf{Q}(\alpha, \beta)$ is a quaternion algebra.

PROOF. This can be shown by routine calculation, which we omit here. \square

The following results generalize those in [Ha].

COROLLARY 9. For an order $\mathcal{O} = \mathbf{Z}[\omega]$ of positive discriminant Δ with $\Delta \equiv 0, 1(4)$ and a discriminant matrix S_Δ of a QCM-order R the following conditions are equivalent:

- (i) Δ is primitively represented by the quadratic form S_Δ .

- (ii) There exists an embedding $\mathcal{O} \hookrightarrow R$ such that $R \cap \mathcal{Q}(\omega) = \mathcal{O}$.
- (iii) A QCM-curve with QCM-order R is contained in the Humbert surface H_Δ .

Moreover, for $\Delta(\alpha) \neq \Delta(\beta)$, a QCM-curve with QCM-order R is a component in the intersection $H_{\Delta(\alpha)} \cap H_{\Delta(\beta)}$ if and only if

$${}^t g S_\Delta g = \begin{pmatrix} \Delta(\alpha) & * \\ * & \Delta(\beta) \end{pmatrix}$$

for some $g \in Gl(2, \mathbf{Z})$.

A QCM-curve with QCM-order R is a component in the quotient $X_1 \cap X_2 / G(\Delta, 1) \cap G(\Delta, 2)$ of two non-identical components X_1, X_2 of the covering of H_Δ in level $\Gamma_2^*(2, 4)$ if and only if

$${}^t g S_\Delta g = \begin{pmatrix} \Delta & * \\ * & \Delta \end{pmatrix}$$

for some $g \in Gl(2, \mathbf{Z})$.

PROOF. For any closed point τ of the QCM-curve $C(R)$ outside a set of measure zero (i.e., outside a countable set) we have $\text{End}(A_\tau) = R$. If the curve is a component in the intersection $H_{\Delta(\alpha)} \cap H_{\Delta(\beta)}$, then there exist Rosati invariant elements α, β in R with the given discriminants. By Lemma 8 the elements α and β generate an order contained in R , which generates the same quaternion algebra $\mathcal{Q}(\alpha, \beta) = R \otimes \mathcal{Q}$. Therefore $\mathbf{Z}[\alpha, \beta]$ is a QCM-order. This induces an embedding of algebraic curves $C(\mathbf{Z}[\alpha, \beta]) \subset C(R)$, which must be an isomorphism on an open set. Hence $\mathbf{Z}[\alpha, \beta] = R$. □

If Δ is a non-zero square, the Humbert surface H_Δ is the closure of a set of period points corresponding to abelian surfaces which are isogenous to a self product of an elliptic curve (i.e., $\sigma(\tau) = \pi \times \pi$ for some $\sigma \in M_4(\mathbf{Z})$ with $\sigma \hat{\sigma} = n \in \mathbf{N}$). Therefore, in this case $\text{End}(A_\tau)$ contains an order in $M_2(\mathcal{Q})$. More generally, if the discriminant form of an QCM-order R represents a non-zero square, all period points correspond to non-simple abelian surfaces. We call the corresponding QCM-curves *non-simple*. Otherwise we call it *simple*. A CM-point is *simple* if it corresponds to a simple abelian surface. There cannot be any simple CM-point on a QCM-curve. The only CM-points on a QCM-curve which can occur correspond to abelian surfaces with multiplication by some order in $M_2(K)$ for an imaginary quadratic number field K . However, up to countable many exceptions any period point on a simple QCM-curve corresponds to a simple abelian surface with complex multiplication (actually quaternionic complex multiplication) given by the QCM-order.

Our next problem is to characterize bases up to conjugation by a symplectic matrix. So consider two bases (α_1, β_1) and (α_2, β_2) of abstractly isomorphic orders R_1 and R_2 . By Theorem 2 we may assume that $\alpha = \alpha_1 = \alpha_2$ is normalized and primitive of discriminant $\Delta(\alpha) = 4k + l$. Moreover, we may assume that β_1 and β_2 are primitive and normalized. We will compare (α, β_i) with a certain *standard basis* (α, β) with the same discriminant matrix.

We have that

$$\begin{aligned}
 M \bullet \beta_1 &= \begin{pmatrix} a_1 & a_2 & b_2 & b_1 + lb_2 \\ ka_2 & a_1 + la_2 & b_1 + lb_2 & kb_2 + l(b_1 + b_2) \\ kc_2 - lc_1 & c_1 & d_1 & kd_2 \\ c_1 & c_2 & d_2 & d_1 + ld_2 \end{pmatrix} \bullet \begin{pmatrix} A_1 & A_2 & 0 & b \\ A_3 & t - A_1 & -b & 0 \\ 0 & c & A_1 & A_3 \\ -c & 0 & A_2 & t - A_1 \end{pmatrix} \\
 &= \beta = \begin{pmatrix} 0 & 0 & 0 & 1 \\ a & t & -1 & 0 \\ 0 & \kappa & 0 & a \\ -\kappa & 0 & 0 & t \end{pmatrix}
 \end{aligned}$$

with $a = A_3 + kA_2 - lA_1 = -n(\alpha, \beta_1)$ and $\kappa = bc + A_1t - A_1^2 - A_2A_3 = n(\beta_1)$, is equivalent to

$$\begin{aligned}
 c_1 &= A_1a_1 + A_3a_2 - c(b_1 + lb_2), \\
 c_2 &= A_2a_1 + (t - A_1)a_2 + cb_2, \\
 d_2 &= -ba_2 + A_1b_2 + A_2(b_1 + lb_2), \\
 d_1 &= ba_1 + A_3b_2 + (t - A_1)(b_1 + lb_2) - ld_2.
 \end{aligned}$$

The matrix M with these (c_1, c_2, d_1, d_2) is symplectic if and only if

$$1 = a_1d_1 + ka_2d_2 - kb_2c_2 - b_1c_1,$$

which is equivalent to

$$\begin{aligned}
 1 &= (ba_1^2 + (t - 2A_1 - lA_2)a_1b_1 + cb_1^2) + l(ba_1a_2 + (t - 2A_1 - lA_2)a_1b_2 + cb_1b_2) \\
 &\quad + (A_3 - kA_2)(a_1b_2 - b_1a_2) - k(ba_2^2 + (t - 2A_1 - lA_2)a_2b_2 + cb_2^2).
 \end{aligned}$$

Therefore we have reduced the question of the equivalence of a basis to the standard basis (α, β) to the question if 1 is represented by the quadratic form $q(x) = 1/2(xBx)$ with the symmetric integral matrix

$$B = \begin{pmatrix} 2b & t - 2A_1 - lA_2 & lb & A_3 - kA_2 + l(t - 2A_1 - A_2) \\ * & 2c & kA_2 - A_3 & lc \\ * & * & -2kb & -k(t - 2A_1 - lA_2) \\ * & * & * & -2kc \end{pmatrix}$$

of determinant $\det(B) = ((\Delta(\alpha)\Delta(\beta) - \Delta(\alpha, \beta)^2)/4)^2 = d(R)^2$. Considered as a quadratic form over the real numbers, the form has signature $(2, 2)$. The discriminant matrix

$$S_\Delta = \begin{pmatrix} \Delta(\alpha) & \Delta(\alpha, \beta) \\ \Delta(\alpha, \beta) & \Delta(\beta) \end{pmatrix}$$

of a QCM-order R is called primitive if $\text{g.c.d.}(\Delta(\alpha), \Delta(\alpha, \beta), \Delta(\beta)) = 1$. The main result of this paper is the following theorem, which is similar to Corollary 3:

THEOREM 10. *Let R_1 and R_2 be two QCM-orders with primitive discriminant matrix S_Δ . If they are isomorphic as \mathbf{Z} -algebras, then they are conjugate by an element of Γ_2 .*

To prove the theorem we recall some facts about quadratic forms. For proofs we refer to [Kn], [Ki] and [MH]. Let A be a commutative ring and E a free A -module together with

a quadratic form $q : E \rightarrow A$ ($q(ax) = a^2q(x)$ and $b(x, y) = q(x + y) - q(x) - q(y)$ is bilinear). Let $B = b(e_i, e_j)$ be the matrix of b with respect to some basis $\{e_i\}$. The quadratic module (E, q) is called regular if $d(E, q) = \det(B)$ is a unit in A . In our applications, A is \mathbf{Z} or the complete discrete valuation ring \mathbf{Z}_p with quotient field \mathbf{Q}_p for a prime p or $\mathbf{Z}_\infty = \mathbf{R}$. For a quadratic \mathbf{Z} -module E we put $E_p = E \otimes \mathbf{Z}_p$. Two quadratic \mathbf{Z} -modules are in the same class if and only if they are isomorphic as quadratic \mathbf{Z} -modules and belong to the same genus if and only if the E_p 's are isomorphic as quadratic \mathbf{Z}_p -modules for all p including $p = \infty$. If (E, q) is a quadratic \mathbf{Z} -module with $d(E, q) \neq 0$, and the rational number $t \in \mathbf{Q}^*$ is represented by all E_p including $p = \infty$, then there exists a quadratic \mathbf{Z} -module in the genus of E representing t ([Kn, 21.1]). If (E, q) is a regular quadratic \mathbf{Z}_p -module for a prime p of rank ≥ 2 , then every unit $t \in \mathbf{Z}_p^*$ is represented by (E, q) ([Kn, 14.6]). There is an intermediate notion of spinor genus between class and genus. Putting [Kn, 24.2] and [Kn, 24.4] together we have:

THEOREM 11. *Let (E, q) be a quadratic \mathbf{Z} -module with $d(E, q) \neq 0$ of rank ≥ 3 . Suppose that E_∞ is indefinite. For any (finite) prime p , suppose that $E_p = (M_p, c_p q_p) \perp N_p$ for some regular quadratic \mathbf{Z}_p -module (M_p, q_p) of $\text{rank}(M_p) \geq 2$. Then the genus of E contains only one class.*

Now we can prove Theorem 10.

PROOF. Let $R = \mathbf{Z} \oplus \mathbf{Z}\alpha \oplus \mathbf{Z}\beta \oplus \mathbf{Z}\alpha\beta$ be a QCM-order with α and β primitive Rosati invariant elements of positive discriminant $\Delta(\alpha)$, $\Delta(\beta)$ and discriminant matrix S_Δ . Using the notation as above, we see that some of the sub-determinants of B are given as follows:

$$\det \begin{pmatrix} 2b & t - 2A_1 - lA_2 \\ t - 2A_1 - lA_2 & 2c \end{pmatrix} = -\Delta(\beta) + 2A_2\Delta(\alpha, \beta) - \Delta(\alpha)A_2^2,$$

$$\det \begin{pmatrix} 2c & lc \\ lc & -2kc \end{pmatrix} = -c\Delta(\alpha), \quad \det \begin{pmatrix} 2b & lb \\ lb & -2kb \end{pmatrix} = -b\Delta(\alpha), \quad \det B = d(R)^2.$$

For any (finite) prime we have to study the conditions of Theorem 11. If the prime p is prime to $d(R)$ or $b\Delta(\alpha)$ or $c\Delta(\alpha)$, then the conditions are obviously satisfied and 1 is represented by q . So suppose that p divides $d(R)$, $b\Delta(\alpha)$ and $c\Delta(\alpha)$. If p divides $\Delta(\alpha)$, then it divides also $\Delta(\alpha, \beta)$, and hence by the primitivity of S_Δ we get that $\Delta(\beta)$ is prime to p and the conditions are satisfied. So suppose that p is prime to $\Delta(\alpha)$, and hence p divides b and c . But we may assume that $(b, c) = 1$ (see the proof of Theorem 7). Hence all assumptions of Theorem 11 can be satisfied. Therefore 1 is represented by the quadratic form B . Hence there is a standard basis like

$$\alpha = \begin{pmatrix} 0 & 1 & & 0 \\ k & l & & \\ & & 0 & k \\ 0 & 1 & l & \end{pmatrix}, \quad \beta = \begin{pmatrix} 0 & 0 & 0 & 1 \\ a & t & -1 & 0 \\ 0 & c & 0 & a \\ -c & 0 & 0 & t \end{pmatrix}$$

with the same discriminant matrix S_Δ . This standard basis has $\Delta(\alpha) = 4k + l^2$, $\Delta(\beta) = t^2 - 4c$, $\Delta(\alpha, \beta) = t(\alpha)t(\beta) + 2a$, which provides a basis for arbitrary discriminant matrix S_Δ . □

COROLLARY 12. A basis

$$\alpha = \begin{pmatrix} 0 & 1 & 0 \\ k & l & \\ 0 & 0 & k \end{pmatrix}, \quad \beta = \begin{pmatrix} 0 & 0 & 0 & b \\ a & t & -b & 0 \\ 0 & c & 0 & a \\ -c & 0 & 0 & t \end{pmatrix}$$

is equivalent to a standard basis, if the class number $h_{++}(\Delta_\beta)$ is 1, where

$$h_{++}(\Delta_\beta) = \# \left\{ \begin{pmatrix} b & t/2 \\ t/2 & c \end{pmatrix}; \begin{matrix} (b, t, c) = 1 \\ t^2 - 4bc = \Delta_\beta \end{matrix} \right\} \text{ mod } Gl(2, \mathbf{Z}).$$

EXAMPLE 13. Consider the elements

$$\alpha = \begin{pmatrix} 0 & 1 & 0 \\ 5 & 0 & \\ 0 & 1 & 0 \end{pmatrix}, \quad \beta_1 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 5 & 1 & -1 & 0 \\ 0 & 6 & 0 & 5 \\ -6 & 0 & 0 & 1 \end{pmatrix}, \quad \beta_2 = \begin{pmatrix} 0 & 0 & 0 & 2 \\ 5 & 1 & -2 & 0 \\ 0 & 3 & 0 & 5 \\ -3 & 0 & 0 & 1 \end{pmatrix}.$$

Then we have $\Delta(\alpha) = 20$, $\Delta(\beta_i) = 25$, $d(\mathbf{Z} \oplus \mathbf{Z}\alpha \oplus \mathbf{Z}\beta_i \oplus \mathbf{Z}\alpha\beta_i) = 100$, and the quadratic form q and the discriminant matrix S_Δ are given by

$$B = \begin{pmatrix} 4 & 1 & 0 & 5 \\ 1 & -6 & -5 & 0 \\ 0 & -5 & -20 & -5 \\ 5 & 0 & -5 & 30 \end{pmatrix}, \quad S_\Delta = \begin{pmatrix} 20 & 10 \\ 10 & 25 \end{pmatrix}.$$

It is easy to check, for example by computations mod 5, that 1 is *not* represented by q . (The Corollary 12 cannot be applied, because $h_{++}(25) \neq 1$.) Therefore the orders $\mathbf{Z} \oplus \mathbf{Z}\alpha \oplus \mathbf{Z}\beta_1 \oplus \mathbf{Z}\alpha\beta_1$ and $\mathbf{Z} \oplus \mathbf{Z}\alpha \oplus \mathbf{Z}\beta_2 \oplus \mathbf{Z}\alpha\beta_2$ are abstractly isomorphic. However, they are not conjugate mod I_2 . Therefore the primitivity of S_Δ is an important assumption in Theorem 10.

REMARK 14. It would be interesting to have a formula for the number of discriminant forms of QCM-orders

$$h_{\text{QCM}}(d) = \# \left\{ S_\Delta = \begin{pmatrix} \Delta_1 & \Delta \\ \Delta & \Delta_2 \end{pmatrix}; \begin{matrix} S_\Delta > 0, \det S_\Delta = 4d \\ \Delta_1, \Delta_2 \equiv 0, 1(4) \end{matrix} \right\} \text{ mod } Gl(2, \mathbf{Z})$$

for the cardinality $h_{\text{QCM,primitive}}$ of the subset of primitive classes and $h_{\text{QCM, simple}}$ for the cardinality of the subset of simple classes. (A class is simple if it does not represent non-zero squares, and a class is primitive if the g.c.d. is 1.) The first few values are given by

d	1	2	3	4	5	6	7	8	9	10	11
$h_{\text{QCM,primitive}}(d)$	1	1	1	2	2	2	1	2	3	2	2
$h_{\text{QCM, simple}}(d)$	0	0	0	0	0	1	0	0	0	1	0
$h_{\text{QCM}}(d)$	1	1	2	3	2	2	2	3	3	2	3

The computation is done by considering the *reduced* S_Δ , which we define by $0 \leq \Delta \leq \Delta_1/2 < \Delta_1 \leq \Delta_2 = (\Delta^2 + 4d)/\Delta_1$. Using the action of $Gl(2, \mathbf{Z})$, any S_Δ is similar to a reduced matrix. There are only finitely many such reduced S_Δ , which follows from $3\Delta_1^2 \leq 16d$.

We will denote by C_S the QCM-curves for QCM-orders R with primitive discriminant matrix S . This notation is justified by Theorem 10. With this notation we get:

COROLLARY 15. *For coprime discriminants Δ_1 and Δ_2 , the intersection $H_{\Delta_1} \cap H_{\Delta_2}$ contains all the QCM-curves $C \begin{pmatrix} \Delta_1 & a \\ a & \Delta_2 \end{pmatrix}$ with $0 \leq a, a^2 \leq \Delta_1 \Delta_2$ as irreducible components.*

EXAMPLE 16. The intersection $H_5 \cap H_8$ contains $C \begin{pmatrix} 5 & 0 \\ 0 & 8 \end{pmatrix}, C \begin{pmatrix} 5 & 2 \\ 2 & 8 \end{pmatrix}, C \begin{pmatrix} 5 & 1 \\ 1 & 5 \end{pmatrix}, C \begin{pmatrix} 1 & 0 \\ 0 & 4 \end{pmatrix}$ as irreducible components, because $\begin{pmatrix} 5 & 6 \\ 6 & 8 \end{pmatrix}$ is similar to $\begin{pmatrix} 1 & 0 \\ 0 & 4 \end{pmatrix}$ and $\begin{pmatrix} 5 & 4 \\ 4 & 8 \end{pmatrix}$ is similar to $\begin{pmatrix} 5 & 1 \\ 1 & 5 \end{pmatrix}$.

EXAMPLE 17. The intersection $H_1 \cap H_4 = C \begin{pmatrix} 1 & 0 \\ 0 & 4 \end{pmatrix}$ is a quotient of a plane quadric. It is a non-simple curve. If we put $\Gamma = Sl(2, \mathbf{R}) \cap R$ for the corresponding QCM-order R , we get a non-standard model of the compactification of the modular curve $\Gamma \backslash \mathbf{H}$ parametrizing abelian surfaces with quaternionic multiplication by a maximal order R (of discriminant 1) in $M_2(\mathbf{Z})$. There are two different QCM-curves for discriminant 6, namely $C \begin{pmatrix} 1 & 0 \\ 0 & 24 \end{pmatrix}$ (non-simple, which is a component of $H_1 \cap H_{24}$) and $C \begin{pmatrix} 5 & 1 \\ 1 & 5 \end{pmatrix}$ (simple, which is a component of $H_5 \cap H_8$). There are two different QCM-curves for discriminant 10, namely $C \begin{pmatrix} 1 & 0 \\ 0 & 40 \end{pmatrix}$ (non-simple) and $C \begin{pmatrix} 5 & 0 \\ 0 & 8 \end{pmatrix}$ (simple, which is another component of $H_5 \cap H_8$). For discriminant 26 the classes for discriminant matrices are given by

$$\begin{pmatrix} 1 & 0 \\ 0 & 104 \end{pmatrix}, \begin{pmatrix} 5 & 1 \\ 1 & 21 \end{pmatrix}, \begin{pmatrix} 8 & 0 \\ 0 & 13 \end{pmatrix}, \begin{pmatrix} 9 & 2 \\ 2 & 12 \end{pmatrix},$$

which are all primitive. Therefore there are 4 different QCM-curves with discriminant 26. For discriminant 15 the classes for discriminant matrices are given by

$$\begin{pmatrix} 1 & 0 \\ 0 & 60 \end{pmatrix}, \begin{pmatrix} 4 & 2 \\ 2 & 16 \end{pmatrix}, \begin{pmatrix} 5 & 0 \\ 0 & 12 \end{pmatrix}, \begin{pmatrix} 8 & 2 \\ 2 & 8 \end{pmatrix},$$

which are not all primitive. Therefore there are at least 4 different QCM-curves of discriminant 15. This covers the examples in [Ha].

These examples illustrate that in principle it is always possible to compute projective models for QCM-curves. One has to identify various components between intersections of

two Humbert surfaces. For this purpose it is useful to consider other Humbert surfaces containing the corresponding QCM-curve. This produces equations which help to decide which component is the right one.

APPENDIX. In this appendix we give formulas for Humbert surfaces of discriminant 16, 20, 24 and 28 of degree 8, 8, 12 and 16, respectively. For discriminant 16 we get

$$\begin{aligned} F_{16} = & (0, 0, 8, 0) + (0, 0, 0, 8) - 4((0, 0, 6, 2) + (0, 0, 2, 6)) + 6(0, 0, 4, 4) \\ & + 16((4, 0, 2, 2) + (2, 2, 4, 0) + (2, 2, 0, 4) + (0, 4, 2, 2)) \\ & - 8((3, 1, 4, 0) + (3, 1, 0, 4) + (1, 3, 4, 0) + (1, 3, 0, 4)) \\ & - 48((3, 1, 2, 2) + (1, 3, 2, 2)) + 64(2, 2, 2, 2) \end{aligned}$$

For discriminant 20 we get

$$\begin{aligned} F_{20} = & 64((6, 0, 0, 2) + (2, 0, 0, 6)) + 70(0, 4, 4, 0) \\ & - 16((4, 3, 1, 0) + (4, 1, 3, 0) + (0, 3, 1, 4) + (0, 1, 3, 4)) \\ & - 32((4, 2, 2, 0) + (2, 4, 0, 2) + (2, 0, 4, 2) + (0, 2, 2, 4)) \\ & + 128(4, 0, 0, 4) - 128(2, 2, 2, 2) - 96((2, 3, 1, 2) + (2, 1, 3, 2)) \\ & + ((0, 8, 0, 0) + (0, 0, 8, 0)) + 8((0, 7, 1, 0) + (0, 1, 7, 0)) \\ & + 28((0, 6, 2, 0) + (0, 2, 6, 0)) + 56((0, 5, 3, 0) + (0, 3, 5, 0)) \end{aligned}$$

For discriminant 24 we get

$$\begin{aligned} F_{24} = & ((0, 0, 12, 0) + (0, 0, 0, 12)) - 128((7, 3, 2, 0) + (7, 3, 0, 2)) + 64(6, 6, 0, 0) \\ & - 128((6, 2, 4, 0) + (6, 2, 0, 4) + (2, 6, 4, 0) + (2, 6, 0, 4)) \\ & + 64((5, 5, 2, 0) + (5, 5, 0, 2)) + 15((0, 0, 8, 4) + (0, 0, 4, 8)) + 20(0, 0, 6, 6) \\ & - 32((5, 1, 6, 0) + (5, 1, 0, 6) + (1, 5, 6, 0) + (1, 5, 0, 6)) \\ & - 352((5, 1, 4, 2) + (5, 1, 2, 4) + (1, 5, 4, 2) + (1, 5, 2, 4)) \\ & + 240((4, 4, 4, 0) + (4, 4, 0, 4)) + 480(4, 4, 2, 2) \\ & - 192((4, 0, 6, 2) + (4, 0, 2, 6) + (0, 4, 6, 2) + (0, 4, 2, 6)) \\ & + 640((4, 0, 4, 4) + (0, 4, 4, 4)) - 128((3, 7, 2, 0) + (3, 7, 0, 2)) \\ & + 288((3, 3, 6, 0) + (3, 3, 0, 6)) - 160((3, 3, 4, 2) + (3, 3, 2, 4)) \\ & + 124((2, 2, 8, 0) + (2, 2, 0, 8)) - 400((2, 2, 6, 2) + (2, 2, 2, 6)) \\ & + 1000(2, 2, 4, 4) + 20((1, 1, 10, 0) + (1, 1, 0, 10)) \\ & - 28((1, 1, 8, 2) + (1, 1, 2, 8)) - 184((1, 1, 6, 4) + (1, 1, 4, 6)) \\ & + 256((0, 8, 2, 2) + (8, 0, 2, 2)) + 6((0, 0, 10, 2) + (0, 0, 2, 10)) \end{aligned}$$

For discriminant 28 we get

$$\begin{aligned} F_{28} = & 1024((10, 4, 0, 2) + (10, 0, 4, 2) + (2, 4, 0, 10) + (2, 0, 4, 10)) \\ & - 1024((10, 3, 3, 0) + (0, 3, 3, 10)) - 1024((10, 1, 1, 4) + (4, 1, 1, 10)) \end{aligned}$$

$$\begin{aligned}
& - 512((8, 6, 2, 0) + (8, 2, 6, 0) + (0, 6, 2, 8) + (0, 2, 6, 8)) \\
& - 512((8, 5, 1, 2) + (8, 1, 5, 2) + (2, 5, 1, 8) + (2, 1, 5, 8)) \\
& + 1280((8, 4, 4, 0) + (0, 4, 4, 8)) + 1024((8, 3, 3, 2) + (2, 3, 3, 8)) \\
& - 512((8, 2, 2, 4) + (4, 2, 2, 8)) + 256(8, 0, 0, 8) \\
& - 64((6, 9, 1, 0) + (6, 1, 9, 0) + (6, 9, 1, 6) + (0, 1, 9, 6)) \\
& - 1024((6, 8, 0, 2) + (6, 0, 8, 2) + (2, 8, 0, 6) + (2, 0, 8, 6)) \\
& + 1536((6, 7, 3, 0) + (6, 3, 7, 0) + (0, 7, 3, 6) + (0, 3, 7, 6)) \\
& - 256((6, 6, 2, 2) + (6, 2, 6, 2) + (2, 6, 2, 6) + (2, 2, 6, 6)) \\
& - 896((6, 5, 5, 0) + (0, 5, 5, 6)) \\
& - 3328((6, 5, 1, 4) + (6, 1, 5, 4) + (4, 5, 1, 6) + (4, 1, 5, 6)) \\
& + 2560((6, 4, 4, 2) + (2, 4, 4, 6)) + 256((6, 4, 0, 6) + (6, 0, 4, 6)) \\
& + 512((6, 3, 3, 4) + (4, 3, 3, 6)) + 3584((6, 2, 2, 6)) \\
& + 480((4, 10, 2, 0) + (4, 2, 10, 0) + (0, 10, 2, 4) + (0, 2, 10, 4)) \\
& - 640((4, 9, 1, 2) + (4, 1, 9, 2) + (2, 9, 1, 4) + (2, 1, 9, 4)) \\
& - 128((4, 8, 4, 0) + (4, 4, 8, 0) + (0, 8, 4, 4) + (0, 4, 8, 4)) \\
& + 3168((4, 8, 0, 4) + (4, 0, 8, 4)) \\
& + 768((4, 7, 3, 2) + (4, 3, 7, 2) + (2, 7, 3, 4) + (2, 3, 7, 4)) \\
& - 1216((4, 6, 6, 0) + (0, 6, 6, 4)) + 4992((4, 6, 2, 4) + (4, 2, 6, 4)) \\
& - 9472((4, 5, 5, 2) + (2, 5, 5, 4)) + 1088(4, 4, 4, 4) \\
& + 48((2, 13, 1, 0) + (2, 1, 13, 0) + (0, 13, 1, 2) + (0, 1, 13, 2)) \\
& + 144((2, 12, 0, 2) + (2, 0, 12, 2)) \\
& + 32((2, 11, 3, 0) + (2, 3, 11, 0) + (0, 11, 3, 2) + (0, 3, 11, 2)) \\
& - 2272((2, 10, 2, 2) + (2, 2, 10, 2)) \\
& - 304((2, 9, 5, 0) + (2, 5, 9, 0) + (0, 9, 5, 2) + (0, 5, 9, 2)) \\
& + 1904((2, 8, 4, 2) + (2, 4, 8, 2)) - 576((2, 7, 7, 0) + (0, 7, 7, 2)) \\
& + 8640(2, 6, 6, 2) + ((0, 16, 0, 0) + (0, 0, 16, 0)) \\
& + 8((0, 14, 2, 0) + (0, 2, 14, 0)) + 28((0, 12, 4, 0) + (0, 4, 12, 0)) \\
& + 56((0, 10, 6, 0) + (0, 6, 10, 0)) + 70((0, 8, 8, 0))
\end{aligned}$$

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