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ENDOMORPHISM RINGS OF COMPLETELY PURE-INJECTIVE MODULES

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ABSTRACT. Let R be a ring, $E = E(R_R)$ its injective envelope, $S = \mathrm{End}(E_R)$ and J the Jacobson radical of S. It is shown that if every finitely generated submodule of E embeds in a finitely presented module of projective dimension ≤ 1, then every finitley generated right S/J-module X is canonically isomorphic to $\mathrm{Hom}_R(E, X \otimes_S E)$. This fact, together with a well-known theorem of Osofsky, allows us to prove that if, moreover, E/JE is completely pure-injective (a property that holds, for example, when the right pure global dimension of R is ≤ 1 and hence when R is a countable ring), then S is semiperfect and R_R is finite-dimensional. We obtain several applications and a characterization of right hereditary right noetherian rings.

INTRODUCTION

Let R be a ring, M_R a right R-module, and $S = \mathrm{End}(M_R)$. Then there exists an adjoint pair:

$$\mathrm{Hom}_R(M, -) : \mathrm{Mod}\text{-}R \leftrightarrows \mathrm{Mod}\text{-}S : - \otimes_S M$$

which induces a functorial morphism $\alpha : 1_{\mathrm{Mod}\text{-}S} \to \mathrm{Hom}_R(M, - \otimes_S M)$. If X is a right S-module such that α_X is an isomorphism, we will say that X_S is M-invariant. It is well known that when every right S-module X is M-invariant, useful information can be passed from M_R to S. This is what happens, for example, when M_R is a finitely generated projective module, which makes it possible to characterize properties of the endomorphism ring S in terms of M_R. This property also holds when M_R is finitely presented and S is a (von Neumann) regular ring and this, coupled with Osofsky's theorem [8, 9] that asserts that a ring whose cyclic right modules are all injective is semisimple, has been exploited in [3] to obtain an easy proof of the result of Damiano that shows that a right PCI ring (i.e., a ring with each proper cyclic right module injective) is right noetherian.

This technique was also (implicitly) applied in [1] to a right hereditary ring R whose injective envelope $E(R_R)$ is projective, showing that R is, in this case, a (two-sided) hereditary artinian QF-3 ring. An extension in [3, Corollary 6] shows that if $E(R_R)$ is just finitely presented (instead of projective), then R is a right

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artinian ring with Morita duality. The key point of this proof is to show that R is right finite-dimensional. But, as the endomorphism ring S of $E = E(R_R)$ is regular, all the cyclic right S-modules are E-invariant. This makes it possible to transfer the injectivity property and then to use Osofsky's theorem to show that S is semisimple.

In this paper we consider the rather more general situation that arises when the injective envelope $E_R = E(R_R)$ of a ring R has the property that every finitely generated submodule embeds in a finitely presented module whose projective dimension is ≤ 1 (this includes the right hereditary rings with finitely presented injective envelope, but also the rings R such that every finitely generated submodule of E_R embeds in a free module). If $S = \mathrm{End}(E_R)$ and J is the radical of S, we prove in Theorem 1.6 that each finitely generated right S/J-module is E-invariant—a result that will be our main tool in the rest of the paper. This allows us to apply the transfer techniques sketched above to the ring S/J and hence substantially broaden the scope of these methods. In this setting, we usually cannot expect that the endomorphism ring S is semisimple. In general, it is not even regular. However, we show that when certain quotients of E_R are pure-injective, then S is semiperfect and hence R_R is finite-dimensional. More specifically, we assume that E/JE is a *completely pure-injective* R-module, i.e., a module such that each pure quotient of itself is pure-injective. We give several applications and we extend [3, Corollary 6] by proving that if R is right hereditary and every finitely generated submodule of E_R is finitely presented, then R is right noetherian.

In the last part of the paper we consider rings R whose right pure global dimension (cf. [6, 7]) is ≤ 1. This includes all countable rings. If every finitely generated submodule of E_R embeds in a finitely presented module of projective dimension ≤ 1, then we show that E/JE is pure-injective (Theorem 2.1), so that E/JE is completely pure-injective in this case and hence R is, again, finite-dimensional. As an application we show that, for these rings, the property that R is right nonsingular and every finitely generated right R-module embeds in a free module is right-left symmetric.

We refer to [5] and [11] for all undefined notions used in the text.

1. M-invariant modules

Let ${}_SM_R$ be a bimodule. We have a pair or adjoint functors $\mathrm{Hom}_R(M,-):$ $\text{Mod-}R \leftrightarrows \text{Mod-}S : - \otimes_S M$ and the corresponding adjunction morphisms α_X, for every $X \in \text{Mod-}S$. The right S-modules X such that α_X is an isomorphism will, again, be called *M-invariant*. The following result is well known (cf. [12], [11]).

Proposition 1.1. *Let ${}_SM_R$ be a bimodule. Then the following assertions hold:*

(i) *If L_R is pure-injective, then* $\mathrm{Hom}_R(M, L)$ *is a pure-injective right S-module.*
(ii) *If ${}_SM$ is flat and L_R is M-injective, then* $\mathrm{Hom}_R(M, L)$ *is injective.*

Our interest in M-invariant modules is motivated by the fact that certain injectivity properties are easily transferred to these modules. From Proposition 1.1 we have:

Proposition 1.2. *Let ${}_SM_R$ be a bimodule and X an M-invariant right S-module. Then the following assertions hold:*

(i) *If $X \otimes_S M$ is pure-injective, then X is pure-injective.*
(ii) *If ${}_SM$ is flat and $X \otimes_S M$ is M-injective, then X is injective.*

In order to exploit Proposition 1.2 we need to have M-invariant S-modules. Recall that if E_R is (quasi-)injective (or pure-injective), then S/J (where $S = \mathrm{End}(E_R)$ and $J = J(S)$) is a regular ring and idempotents lift modulo J. We want to apply Osofsky's theorem to S/J and for this we need to prove that the cyclic right S/J-modules are E-invariant. We start by giving a useful sufficient condition for α_X to be a monomorphism.

Proposition 1.3. *Let P_R be a finitely generated projective module, $E = E(P_R)$ and $S = \mathrm{End}(E_R)$. Then α_X is a monomorphism for each finitely generated right S/J-module X.*

Proof. Since X is an S/J-module and $XJ = 0$, we have a free presentation of X in Mod-S, say $S^{(I)} \xrightarrow{h} S^n \xrightarrow{p} X \to 0$, where $J^n = J(S^n) \subseteq \mathrm{Ker}\, p = \mathrm{Im}\, h$. Applying $- \otimes_S E$ we obtain an exact sequence in Mod-R

$$E^{(I)} \xrightarrow{h_*} E^n \xrightarrow{p_*} X \otimes_S E \to 0.$$

Let $Z := \mathrm{Im}\, h_* = \mathrm{Ker}\, p_*$, with canonical projection $v : E^{(I)} \to Z$ and canonical injection $u : Z \to E^n$. Then each $f \in \mathrm{Hom}_R(E, E^n)$ such that $p_* \circ f = 0$ factors in the form $f = u \circ f'$, where $f' \in \mathrm{Hom}_R(E, Z)$. Since P is projective, we obtain a morphism $g : P \to E^{(I)}$ that makes the diagram

$$\begin{array}{ccc} P & \xrightarrow{j} & E \\ \Big\downarrow g & & \Big\downarrow f' \\ E^{(I)} & \xrightarrow{v} & Z \end{array}$$

commute, where j is the canonical inclusion. Since P is finitely generated, $g(P) \subseteq E^{(F)}$ for some finite subset F of I. As E is injective, there exists a homomorphism $t : E \to E^{(I)}$ such that $t \circ j = g$. Hence $h_* \circ t \circ j = h_* \circ g = f \circ j$, so that $(h_* \circ t - f) \circ j = 0$. Since j is an essential monomorphism by hypothesis, $\mathrm{Ker}(h_* \circ t - f)$ is essential in E. Consider the following commutative diagram of right S-modules:

$$\begin{array}{ccccccc} S^{(I)} & \xrightarrow{h} & S^n & \xrightarrow{p} & X & \longrightarrow & 0 \\ \Big\downarrow \alpha_{S^{(I)}} & & \Big\downarrow \alpha_{S^n} & & \Big\downarrow \alpha_X & & \\ \mathrm{Hom}_R(E, E^{(I)}) & \xrightarrow{h_{**}} & \mathrm{Hom}_R(E, E^n) & \xrightarrow{p_{**}} & \mathrm{Hom}_R(E, X \otimes_S E) & & \end{array}$$

Then $f \in \mathrm{Hom}_R(E, E^n)$ and $f \in \mathrm{Ker}\ p_{**}$, so there exists $t \in \mathrm{Hom}_R(E, E^{(I)})$ such that $h_{**}(t) - f$ has essential kernel and, hence, belongs to $J(S)^n$. Thus $h_{**}(t) - f \in \alpha_{S^n}(\mathrm{Ker}\ p)$. On the other hand, since $\mathrm{Im}\, t \subseteq E^{(F)}$ for F finite, there exists $q \in S^{(I)}$ such that $t = \alpha_{S^{(I)}}(q)$ and so $h_{**}(t) = (\alpha_{S^n} \circ h)(q) \in \alpha_{S^n}(\mathrm{Ker}\ p)$. Thus we have that $f \in \alpha_{S^n}(\mathrm{Ker}\ p)$ and this implies that α_X is a monomorphism. □

Recall that R is called a right Kasch ring whenever $E(R_R)$ is a cogenerator of Mod-R. From the preceding result we immediately obtain:

Corollary 1.4. *Let R be a right Kasch ring. Then $\mathrm{End}(E(R_R))$ is also a right Kasch ring.*

Proof. Let $E = E(R_R)$, $S = \mathrm{End}(E_R)$ and $J = J(S)$. If C is a simple right S-module, then $CJ = 0$ and so C is an S/J-module. Thus α_C is a monomorphism by

Proposition 1.3 and, as $C \otimes_S E$ is cogenerated by E, we obtain a monomorphism $C \xrightarrow{\alpha_C} \operatorname{Hom}_R(E, C \otimes_S E) \to \operatorname{Hom}_R(E, E^I) \cong S^I$, for some set I. Hence C embeds in S_S. $\square$

Now, in order to obtain E-invariant modules from Proposition 1.3, we need to give conditions for α_X to be an epimorphism. The following lemma will be crucial for this purpose.

Lemma 1.5. *Let P_R be a finitely generated projective right R-module, $E = E(P_R)$ its injective hull, and $S = \operatorname{End}(E_R)$. Assume that each finitely generated submodule of E embeds in a finitely presented module of projective dimension ≤ 1. Then, for each finitely generated right S/J-module X, $\operatorname{Hom}_R(E/P, X \otimes_S E) = 0$.*

Proof. Let $f \in \operatorname{Hom}_R(E/P, X \otimes_S E)$ and $\pi : E \to E/P$ the canonical projection. We want to prove that $g = f \circ \pi = 0$. Since P is finitely generated, E is the direct limit of all its finitely generated submodules that contain P. Thus it will be enough to show that if $P \subseteq Z \subseteq E$ and Z is finitely generated, then $g(Z) = 0$. By hypothesis, there exists a finitely presented right R-module F such that $\operatorname{pd}(F) \leq 1$, and a monomorphism $\varphi : Z \to F$. Then, regarding P as a submodule of F, we get the following commutative diagram:

$$\begin{array}{ccccc}
Z & & \xrightarrow{u} & & E \\
\downarrow{\scriptstyle \pi'} & \searrow^{\varphi} & & \nearrow_{\gamma} & \downarrow{\scriptstyle \pi} \\
& & F & & \\
Z/P & & \xrightarrow{v} & & E/P \\
& \searrow^{\beta} & \downarrow & \nearrow_{\delta} & \\
& & F/P & &
\end{array}$$

where β is the monomorphism induced by φ, γ is obtained by the injectivity of E, and δ is induced by γ. We have that F/P is a finitely presented module. Consider the functorial exact sequence

$$0 = \operatorname{Ext}^1_R(P, -) \to \operatorname{Ext}^2_R(F/P, -) \to \operatorname{Ext}^2_R(F, -) = 0.$$

Since $\operatorname{pd}(F) \leq 1$, the last term is zero, and so $\operatorname{pd}(F/P) \leq 1$. Next let $S^{(I)} \to S^n \xrightarrow{p} X \to 0$ be a free presentation of X in Mod-S and consider the induced exact sequence in Mod-R, $E^{(I)} \to E^n \xrightarrow{p \otimes E} X \otimes_S E \to 0$. Set $Y = \operatorname{Ker}(p \otimes_S E)$. From the short exact sequence $0 \to K \to E^{(I)} \to Y \to 0$ we obtain the natural exact sequence

$$\operatorname{Ext}^1_R(F/P, E^{(I)}) \to \operatorname{Ext}^1_R(F/P, Y) \to \operatorname{Ext}^2_R(F/P, K).$$

Since $\operatorname{pd}(F/P) \leq 1$, we have that $\operatorname{Ext}^2_R(F/P, K) = 0$ and, as F/P is finitely presented and E is injective, $\operatorname{Ext}^1_R(F/P, E^{(I)}) \cong \operatorname{Ext}^1_R(F/P, E)^{(I)} = 0$. Thus $\operatorname{Ext}^1_R(F/P, Y) = 0$ and so we have an exact sequence

$$\operatorname{Hom}_R(F/P, E^n) \xrightarrow{(p \otimes E)_*} \operatorname{Hom}_R(F/P, X \otimes E) \to \operatorname{Ext}^1_R(F/P, Y) = 0$$

which shows that $(p \otimes E)_* = \operatorname{Hom}_R(F/P, p \otimes E)$ is an epimorphism. Hence, there exists a morphism $\epsilon : F/P \to E^n$ such that $f \circ \delta = (p \otimes E) \circ \epsilon$. But, as E^n is injective and v is a monomorphism, $\epsilon \circ \beta : Z/P \to E^n$ can be extended to a map $\mu : E/P \to E^n$ such that $\mu \circ v = \epsilon \circ \beta$. This gives $(p \otimes E) \circ \mu \circ v = (p \otimes E) \circ \epsilon \circ \beta = f \circ \delta \circ \beta =$

$f \circ v$. Thus we have that $g|_Z = g \circ u = f \circ \pi \circ u = f \circ v \circ \pi' = (p \otimes E) \circ \mu \circ v \circ \pi' = (p \otimes E) \circ \mu \circ \pi \circ u$, so that it remains to prove that $(p \otimes E) \circ \mu \circ \pi \circ u = 0$.

If $p_i : E^n \to E$ are the canonical projections for $i = 1, \dots, n$, then each $p_i \circ \mu \circ \pi$ is an element of S whose kernel contains P. Therefore $p_i \circ \mu \circ \pi \in J(S)$. Now, let x be an element of E and set $e_i = (\delta_{ij})_{j=1,\dots,n} \in S$. Since $XJ = 0$ and $p_i \circ \mu \circ \pi \in J$, $((p \otimes E) \circ \mu \circ \pi \circ u)(x) = (p \otimes E)((\mu \circ \pi)(x)) = \sum_{i=1}^n p(e_i) \otimes (p_i \circ \mu \circ \pi)(x) = \sum_{i=1}^n p(e_i) \cdot (p_i \circ \mu \circ \pi) \otimes x = 0$. This completes the proof. □

Theorem 1.6. *Let P_R be a finitely generated projective module, $E = E(P_R)$ and $S = \mathrm{End}(E_R)$. Assume that each finitely generated submodule of E embeds in a finitely presented module of projective dimension ≤ 1. Then each finitely generated right S/J-module is E-invariant.*

Proof. Let X be a finitely generated right S/J-module. By Proposition 1.3 α_X is a monomorphism. It remains to prove that α_X is an epimorphism. Consider a free presentation $S^{(I)} \to S^n \xrightarrow{p} X \to 0$ of X in Mod-S. Tensoring with ${}_SE$ yields an exact sequence in Mod-R, $E^{(I)} \to E^n \xrightarrow{p \otimes E} X \otimes_S E \to 0$. Now, if $\varphi \in \mathrm{Hom}_R(E, X \otimes_S E)$ and $j : P \to E$ is the canonical inclusion, there is by the projectivity of P a morphism $t : P \to E^n$ such that $\varphi \circ j = (p \otimes E) \circ t$. Then, as E is injective, there exists $h : E \to E^n$ such that $h \circ j = t$. Thus we have $(p \otimes E) \circ h \circ j = (p \otimes E) \circ t = \varphi \circ j$, so that $(\varphi - (p \otimes E) \circ h) \circ j = 0$. Hence $g := \varphi - (p \otimes E) \circ h$ factors through the projection $\pi : E \to E/P$, say as $g = f \circ \pi$. By Lemma 1.5 we have that $f = 0$, and so $g = 0$ and $\varphi = (p \otimes E) \circ h$. Thus we see that $(p \otimes E)_*$ is an epimorphism and the commutative diagram:

$$\begin{array}{ccccc} S^n & \xrightarrow{p} & X & \longrightarrow & 0 \\ \downarrow{\scriptstyle \alpha_{S^n}} & & \downarrow{\scriptstyle \alpha_X} & & \\ \mathrm{Hom}_R(E,\, E^n) & \xrightarrow{(p\otimes E)_*} & \mathrm{Hom}_R(E,\, X \otimes_s E) & \longrightarrow & 0 \end{array}$$

shows that α_X is indeed an epimorphism. □

If E_R is quasi-injective and $S = \mathrm{End}(E_R)$, then S/J is a regular right self-injective ring. If we set $\bar{E} := (S/J) \otimes_S E = E/JE$, then we have a bimodule ${}_{S/J}\bar{E}_R$ and, if $X \in$ Mod-S/J, we have that

$$X \otimes_S E \cong (X \otimes_{S/J} S/J) \otimes_S E \cong X \otimes_{S/J} ((S/J) \otimes_S E) \cong X \otimes_{S/J} \bar{E}.$$

Thus, if we identify $X \otimes_S E$ with $X \otimes_{S/J} \bar{E}$, and if $\bar{\alpha}_X : X \to \mathrm{Hom}_R(\bar{E}, X \otimes_{S/J} \bar{E})$ is the canonical morphism and $p : E \to \bar{E}$ the canonical projection, we see that $\mathrm{Hom}_R(p, X \otimes_S E) \circ \bar{\alpha}_X = \alpha_X$. Since $\mathrm{Hom}_R(p, X \otimes_S E)$ is a monomorphism, if X_S is E-invariant, then $X_{S/J}$ is $\bar{E}$-invariant.

Specifically, if $X = S/J$, then we have proved

Corollary 1.7. *Let P_R be a finitely generated projective module, $E = E(P_R)$, $S = \mathrm{End}(E_R)$ and $J = J(S)$. If every finitely generated submodule of E embeds in a finitely presented module of projective dimension ≤ 1, there is a canonical isomorphism $S/J = \mathrm{End}(E/JE)$.*

Proposition 1.8. *Let E_R be quasi-injective (or pure-injective) and let X be a right S/J-module which is E-invariant. If $X \otimes_S E$ is either E-injective or pure-injective, then $X_{S/J}$ is injective.*

Proof. Let $\bar{E} = E/JE$. Since X is E-invariant, it is also $\bar{E}$-invariant. On the other hand, as S/J is regular, ${}_{S/J}\bar{E}$ is flat. By Proposition 1.2 applied to the adjunction defined by ${}_{S/J}\bar{E}_R$, if we assume that $X \otimes_S E \cong X \otimes_{S/J} \bar{E}$ is E-injective, we get that $X_{S/J}$ is injective. Similarly, if $X \otimes_{S/J} \bar{E}$ is pure-injective, then $X_{S/J}$ is pure-injective and hence, since S/J is regular, injective. □

We will say that a module M is *completely pure-injective* when every *pure* quotient of M is pure-injective. (Note the change of terminology with respect to [3].)

Corollary 1.9. *Let P_R be a finitely generated projective module, $E = E(P_R)$, $S = \mathrm{End}(E_R)$, and $J = J(S)$. Assume that every finitely generated submodule of E_R embeds in a finitely presented right R-module of projective dimension ≤ 1 and that E/JE is completely pure-injective. Then S is semiperfect and P_R is finite-dimensional.*

Proof. By Theorem 1.6, each finitely generated right S/J-module X is E-invariant. Since the canonical projection $S/J \to X$ is a pure epimorphism (since S/J is regular), we have that the induced R-epimorphism $E/JE \to X \otimes_S E$ is also pure. Thus $X \otimes_S E$ is a pure-injective right R-module by hypothesis, and by Proposition 1.8, $X_{S/J}$ is injective. Then, by Osofsky's theorem [8, 9], S/J is semisimple and hence S is semiperfect. This is equivalent to E_R (and hence to P_R) being finite-dimensional. □

The preceding corollary can be regarded as a generalization of [3, Corollary 6]. A more specific extension of this result is the following:

Corollary 1.10. *Let R be a right hereditary ring. Then R is right noetherian if and only if every finitely generated submodule of $E(R_R)$ is finitely presented.*

Proof. If every finitely generated submodule of $E(R_R)$ is finitely presented, then R_R is right finite-dimensional by Corollary 1.9. Thus, using [5, Corollary 5.20], we see that R is right noetherian. The converse is clear. □

2. Rings of pure global dimension less than or equal to one

Recall that the pure-injective dimension of a right R-module M is defined as the smallest nonnegative integer (or ∞) such that there exists an exact sequence $0 \to M \to E_0 \to E_1 \to \cdots \to E_n \to 0$, where the E_i, $i = 0, \ldots, n$, are pure-injective modules and the associated short exact sequences are pure exact. The supremum of the pure-injective dimensions of the right R-modules is called the right pure global dimension of R [7, 6], and is denoted by $\mathrm{r.\,pgldim}(R)$. Thus the rings R such that $\mathrm{r.\,pgldim}(R) \leq 1$ provide a natural source of completely pure-injective modules. The following theorem will be useful in order to apply our results to these rings.

Theorem 2.1. *Let R be a ring, $E = E(R_R)$, $S = \mathrm{End}(E_R)$ and $J = J(S)$. If every finitely generated submodule of E_R embeds in a finitely presented module of projective dimension ≤ 1, then E/JE is a pure-injective R-module.*

Proof. Let $\bar{E} = E/JE$. Consider the exact sequence in Mod-R, $0 \to R \xrightarrow{j} E \to E/R \to 0$, and let $g \in \mathrm{Hom}_R(R, \bar{E}) \cong \bar{E}$. Then g induces a homomorphism $h : R_R \to E$ such that if $q : E \to \bar{E}$ is the canonical projection, then

$q \circ h = g$. By the injectivity of E, h extends to $t : E \to E$, so g extends to a morphism $q \circ t : E \to \bar{E}$. Thus, in the exact sequence

$$\mathrm{Hom}_R(E/R, \bar{E}) \to \mathrm{Hom}_R(E, \bar{E}) \xrightarrow{j_*} \mathrm{Hom}_R(R, \bar{E}),$$

j_* is an epimorphism and hence an isomorphism since $\mathrm{Hom}_R(E/R, \bar{E}) = 0$ by Lemma 1.5. Since S/J is E-invariant by Theorem 1.6, we have isomorphisms of left S/J-modules:

$$\bar{E} \cong \mathrm{Hom}_R(E, \bar{E}) \cong \mathrm{Hom}_R(E, (S/J) \otimes_S E) \cong S/J.$$

Let $\bar{E}^* = \mathrm{Hom}_{S/J}(\bar{E}, S/J)$. Since $\bar{E}$ is reflexive as a S/J-module,

$$\bar{E} \cong \mathrm{Hom}_{S/J}(\bar{E}^*, S/J).$$

Since S/J is right self-injective, applying Proposition 1.1 to the bimodule ${}_R\bar{E}^*_{S/J}$ we obtain that $\bar{E}$ is a pure-injective right R-module. □

Remark. As a consequence of Theorem 2.1 we see that, in Corollary 1.9, it is enough to assume that every *proper* pure quotient of E/JE is pure-injective, instead of requiring that E/JE be completely pure-injective.

Corollary 2.2. *Let R be a ring such that* $\mathrm{r.pgldim}(R) \leq 1$. *Assume, further, that every finitely generated submodule of $E(R_R)$ embeds in a finitely presented module of projective dimension ≤ 1. Then R is right finite-dimensional.*

Proof. If $E = E(R_R)$ we have, by Theorem 2.1, that E/JE is pure-injective and hence completely pure-injective. Then R is right finite-dimensional by Corollary 1.9. □

An interesting class of rings of right pure global dimension ≤ 1 is the class of countable rings [6, 7]. For instance, it follows from the preceding results that every countable ring R such that every finitely generated submodule of $E(R_R)$ embeds in a finitely presented module of projective dimesion ≤ 1 is finite-dimensional.

The following result is a partial generalization of [1, Theorem 3.2], and shows that the rings such that $\mathrm{r.pgldim}(R) \leq 1$ and $E(R_R)$ is projective are not far from being right QF-3 rings (but they need not be, as the ring $R = \left(\begin{smallmatrix} \mathbb{Q} & \mathbb{Q} \\ 0 & \mathbb{Z} \end{smallmatrix}\right)$ shows).

Corollary 2.3. *Let R be a ring such that* $\mathrm{r.pgldim}(R) \leq 1$ *and $E(R_R)$ is projective. Then R has a faithful injective right ideal.*

Proof. By Corollary 2.2 R is right finite-dimensional and, using [10, Lemma 2], we obtain the result. □

The rings R such that every finitely generated right R-module embeds in a free module have been called right FGF by Faith [2]. It is still an open problem whether a right FGF ring must be QF.

Corollary 2.4. *Let R be a right FGF ring such that* $\mathrm{r.pgldim}(R) \leq 1$ *and R has essential right socle. Then R is QF.*

Proof. R is right finite-dimensional by Corollary 2.2. Thus $\mathrm{Soc}(R_R)$ is finitely generated and, as R_R has essential socle, we see that R_R has finite essential socle. Since each finitely generated right module embeds in a (finitely generated) free right R-module, we see that every finitely generated right module has finite essential socle, so that R is right artinian. Then R is QF by [2]. □

Recall that a ring homomorphism $\varphi : R \to Q$ is a *right flat epimorphism of rings* (or a perfect right localization of R) precisely when ${}_RQ$ is flat and the canonical morphism $Q \otimes_R Q \to Q$ is an isomorphism. Goodearl proved that if Q is the right maximal quotient ring of a right nonsingular ring R, then the canonical morphism $R \to Q$ is a *left* flat epimorphism if and only if every finitely generated nonsingular right R-module embeds in a free module [4, Theorem 7]. In general, this condition is not right-left symmetric, as is shown by the endomorphism ring of an infinite-dimensional vector space over a field. However, if $\operatorname{r.pgldim}(R) \leq 1$, then we have symmetry.

Corollary 2.5. *Let R be a ring such that* $\operatorname{r.pgldim}(R) \leq 1$. *Then the following conditions are equivalent:*

(i) *R is right nonsingular and every finitely generated nonsingular right R-module embeds in a free module.*
(ii) *R is left nonsingular and every finitely generated nonsingular left R-module embeds in a free module.*
(iii) *R has a semisimple two-sided maximal quotient ring.*

Proof. (i)⇒(iii) Let $Q = Q^r_{\max}(R)$ be the maximal right quotient ring of R. By Corollary 2.2, R is right finite-dimensional and so Q is semisimple [11, Theorem XII.2.5]. Further, Q_R is flat by the result of Goodearl mentioned above (cf. also [5, Theorem 5.17] and [11, Theorem XII.7.1]). But then it follows from [11, Corollary XII.7.3] that Q is also the maximal left quotient ring of R.

(iii)⇒(i) Since Q is semisimple, R is right nonsingular by [11, Proposition XII.2.2]. Also, since the left maximal quotient ring Q of R is semisimple, the canonical homomorphism $R \to Q$ is a left flat epimorphism. Then, using again [5, Theorem 5.17], we see that every finitely generated nonsingular right R-module embeds in a free module.

Finally, observe that the proof can be completed by symmetry, bearing in mind that condition (iii) is left-right symmetric. □

An entirely similar argument can be applied to the characterization given by Cateforis and Goodearl of the right nonsingular rings such that every finitely generated nonsingular right R-module is projective [5, Theorem 5.18]. This class of rings is not right-left symmetric in general [5] but, from the preceding corollary and [5, Theorem 5.18], we have:

Corollary 2.6. *Let R be a ring such that* $\operatorname{r.pgldim}(R) \leq 1$ *and Q its maximal right quotient ring. Then the following conditions are equivalent:*

(i) *R is right nonsingular and every finitely generated nonsingular right R-module is projective.*
(ii) *R is left nonsingular and every finitely generated nonsingular left R-module is projective.*
(iii) *R is left and right semihereditary, and Q is a semisimple two-sided maximal quotient ring of R.*

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