

Endomorphism seminear-rings of Brandt semigroups

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Abstract

We consider the endomorphisms of a Brandt semigroup B_n , and the semigroup of mappings $E(B_n)$ that they generate under pointwise composition. We describe all the elements of this semigroup, determine Green's relations, consider certain special types of mapping which we can enumerate for each n , and give complete calculations for the size of $E(B_n)$ for small n .

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1 Introduction

For a group G , the set $M(G)$ of all functions $G \rightarrow G$ admits two natural binary operations: it is a semigroup under composition of functions (written multiplicatively) and a group under pointwise composition (written additively) using the group operation in G . If we write maps on the right, we find that function composition distributes on the left over pointwise composition, so that $f(g + h) = fg + fh$ for all $f, g, h \in M(G)$. This endows the set $M(G)$ with the structure of a *near-ring*, see Meldrum (1985). Now $M(G)$ contains the set $\text{End}(G)$ of endomorphisms of G (a semigroup under composition of functions), and it is easy to see that the endomorphisms are precisely the elements that always distribute on the right: $(f + g)h = fh + gh$ for all $f, g \in M(G)$ if and

only if $h \in \text{End}(G)$. We let $E(G)$ be the subnear-ring of $M(G)$ generated by the subset $\text{End}(G)$. The fact that $\text{End}(G)$ is a right distributive semigroup implies that $E(G)$ is generated by $\text{End}(G)$ as a group (that is, using only the pointwise composition). These ideas have their origin – as part of the more general theory of *distributively generated* near-rings – in Neumann (1956) and more particularly in Fröhlich (1958). The near-ring $E(G)$ is called the endomorphism near-ring of G , see Lyons and Malone (1970).

If the group G is replaced by a semigroup S , then the above ideas may be generalised. The set $M(S)$ of all functions $S \rightarrow S$ is now a *seminear-ring*: it is a semigroup under both composition of functions and pointwise composition, and left distributivity holds. We consider the subsemigroup $E^+(S)$ of $M(S)$ generated by $\text{End}(S)$ using pointwise composition: $E^+(S)$ will be a subseminear-ring, but we focus on its semigroup structure. Earlier work has been done in the second author's thesis (Samman, (1998)) and the case of a Clifford semigroup S has been considered in Gilbert and Samman (2009) where it is shown that for certain semilattices of groups S , the semigroup $E^+(S)$ is again a semilattice of groups with a precisely defined structure.

In the present paper, we turn to another class of inverse semigroups, and take S to be a finite Brandt semigroup B_n . The endomorphism semigroup of B_n is obtained by adjoining a zero to the symmetric group S_n of degree n , and so we have a rich but fully understood supply of endomorphisms. The key components in our approach to the structure of $E^+(B_n)$ are then: combinatorial information about the symmetric group S_n ; Green's relations; and a filtration by ideals determined by the support of mappings in $E^+(B_n)$, that is by the subsets not mapped to 0. In addition to some general structural results on $E^+(B_n)$, we also record the results of some calculations in $E^+(B_n)$ for $n \leq 6$ carried out by the computer algebra package GAP (The GAP group, 2007).

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2 Background

A (left) *seminear-ring* is a set L admitting two associative binary operations $+$ and \cdot that satisfy the left distributive law: for all $a, b, c \in L$ we have $a(b + c) = ab + ac$. An element $d \in L$ is called *distributive* if, for all $a, b \in S$, we have $(a + b)d = ad + bd$: the set of distributive elements is clearly a subsemigroup of (L, \cdot) . We say that L is a *distributively generated* seminear-ring if it contains a subsemigroup of distributive elements (K, \cdot) that generates $(L, +)$.

Let S be a semigroup and let $M(S)$ be the set of all functions $S \rightarrow S$. Then $M(S)$ is a seminear-ring, under the operations of composition of functions and

pointwise composition: for $s \in S$ and $f, g \in M(S)$ we have

$$s(fg) = (sf)g \quad \text{and} \quad s(f + g) = (sf)(sg).$$

Our first result identifies the distributive elements in $M(S)$, and is a straightforward generalisation of Lemma 9.6 of Meldrum (1985).

Lemma 2.1. *The set of distributive elements of $M(S)$ is precisely the set of endomorphisms $\text{End}(S)$.*

Proof. Suppose that $f, g \in M(S)$ and $\phi \in \text{End}(S)$. Then for each $s \in S$,

$$s(f + g)\phi = ((sf)(sg))\phi = (sf\phi)(sg\phi) = s(f\phi + g\phi),$$

and hence ϕ is a distributive element. Conversely, suppose that $d \in M(S)$ is distributive, and for any $s \in S$ let $c_s \in M(S)$ be the constant function at $s \in S$, defined by $xc_s = s$ for all $x \in S$. Then for any $s, t, x \in S$ we have

$$(st)d = ((xc_s)(xc_t))d = x(c_s + c_t)d = x(c_sd + c_td) = (sd)(td),$$

and d is an endomorphism. \square

The subsemigroup of $(M(S), +)$ generated by $\text{End}(S)$ is therefore a distributively generated seminear-ring that we denote by $E^+(S)$. We call $E^+(S)$ the *endomorphism seminear-ring* of S .

We now define the Brandt semigroups, and determine their endomorphisms. For any integer $n \geq 1$, we set $[n] = \{1, 2, \dots, n\}$. The *Brandt semigroup* B_n has underlying set $([n] \times [n]) \cup \{0\}$ with multiplication

$$(i, j)(k, l) = \begin{cases} (i, l) & \text{if } j = k \\ 0 & \text{if } j \neq k \end{cases}$$

with 0 acting as a (two-sided) zero element in B_n . The set of idempotents of B_n is $\{0, (1, 1), (2, 2), \dots, (n, n)\}$, and the product of distinct idempotents in B_n is always 0. We shall denote the set of idempotents of B_n by $\text{Idem}(B_n)$, to avoid a clash with the established use of E for the endomorphism seminear-ring. We now determine the endomorphisms of B_n : the following result is probably well-known, and is in any case a simple consequence of Munn's description (Munn (1955)) of all endomorphisms of Rees matrix semigroups (see also Houghton (1977)), but we give a proof for the sake of completeness.

Proposition 2.2. *The endomorphism monoid $\text{End}(B_n)$ is isomorphic to the monoid $(S_n)^0$ obtained by adjoining the zero map to the group S_n , where S_n is the symmetric group of degree n . A permutation $\sigma \in S_n$ induces the endomorphism of B_n mapping $(i, j) \mapsto (i\sigma, j\sigma)$ and $0 \mapsto 0$.*

Proof. Let $\theta \in \text{End}(B_n)$. Then $0\theta = 0$. Suppose that, for some $(i, j) \in B_n$, we have $(i, j)\theta = 0$. Then for any $p, q \in [n]$,

$$(p, q)\theta = ((p, i)(i, j)(j, q))\theta = (p, i)\theta(i, j)\theta(j, q)\theta = 0.$$

Therefore, if $\theta \neq 0$ we have $(i, j)\theta \neq 0$ for all $i, j \in [n]$.

A non-zero $\theta \in \text{End}(B_n)$ therefore determines two functions $\theta_1, \theta_2 : [n] \times [n] \rightarrow [n]$ such that

$$(i, j)\theta = ((i, j)\theta_1, (i, j)\theta_2). \quad (2.1)$$

Now for any $k \in [n]$ we have $(i, k)\theta(k, j)\theta \neq 0$ if and only if $(i, k)\theta_2 = (k, j)\theta_1$, and then

$$(i, k)\theta(k, j)\theta = ((i, k)\theta_1, (k, j)\theta_2). \quad (2.2)$$

Comparing (2.1) and (2.2) we deduce that

$$\begin{aligned} (i, j)\theta_1 &= (i, k)\theta_1 \\ (i, j)\theta_2 &= (k, j)\theta_2. \end{aligned}$$

It follows that θ_1 depends only on the first coordinate, θ_2 depends only on the second coordinate, and then the equality $(i, k)\theta_2 = (k, j)\theta_1$ implies that $\theta_1 = \theta_2$.

We write $\sigma = \theta_1 = \theta_2$, with σ now regarded as a function $[n] \rightarrow [n]$.

Now σ must be injective, for suppose that $j\sigma = k\sigma$. Then

$$\begin{aligned} ((i, j)(k, l))\theta &= (i\sigma, j\sigma)(k\sigma, l\sigma) \\ &= (i\sigma, l\sigma) \neq 0. \end{aligned}$$

Therefore $(i, j)(k, l) \neq 0$ and so $j = k$. Hence σ is a permutation of $[n]$.

Conversely, it is clear that for any permutation σ of $[n]$, the mapping $(i, j) \mapsto (i\sigma, j\sigma), 0 \mapsto 0$ is an endomorphism of $B_n(G)$. \square

3 Green's relations

If $\alpha \in E^+(B_n)$ we define its *support* to be the set

$$\text{supp}(\alpha) = \{(i, j) : (i, j)\alpha \neq 0\}.$$

Let $\alpha \in E^+(B_n)$ and suppose that

$$\alpha = \sigma_1 + \sigma_2 + \cdots + \sigma_m, \quad \sigma_r \in S_n, m \geq 2,$$

where each σ_r is regarded as an endomorphism of B_n as in Proposition 2.2, with $(i, j)\sigma_r = (i\sigma_r, j\sigma_r)$. Then

$$\begin{aligned} (i, j)\alpha &= (i\sigma_1, j\sigma_1)(i\sigma_2, j\sigma_2) \cdots (i\sigma_m, j\sigma_m) \\ &= \begin{cases} (i\sigma_1, j\sigma_m) & \text{if } (i, j) \in \text{supp}(\alpha) \\ 0 & \text{otherwise} \end{cases} \end{aligned}$$

and $(i, j) \in \text{supp}(\alpha)$ if and only if

$$j\sigma_1 = i\sigma_2, j\sigma_2 = i\sigma_3, \dots, j\sigma_{m-1} = i\sigma_m,$$

For $r \geq 2$ write $\varphi_r = \sigma_r \sigma_{r-1}^{-1}$ and set $\varphi_1 = \sigma_1$. Then

$$(i, j) \in \text{supp}(\alpha) \iff j = i\varphi_2 = i\varphi_3 = \cdots = i\varphi_m$$

and $\varphi_1, \varphi_2, \dots, \varphi_m$ determine the σ_r (and hence determine α) since we have $\sigma_r = \varphi_r \varphi_{r-1} \cdots \varphi_2 \varphi_1$. Moreover, for a given $\alpha \in E^+(B_n)$, if $(i, j) \in \text{supp}(\alpha)$ then j is determined by i : hence for a given $i \in [n]$ there exists at most one j with $(i, j) \in \text{supp}(\alpha)$, and hence $|\text{supp}(\alpha)| \leq n$. We record these observations, and some other useful facts about supports, in our next result.

Lemma 3.1. (a) If $\alpha, \beta \in E^+(B_n)$ then $\text{supp}(\alpha + \beta) \subseteq \text{supp}(\alpha) \cap \text{supp}(\beta)$.

(b) If $\sigma_1, \sigma_2 \in \text{End}(B_n)$ then $|\text{supp}(\sigma_1 + \sigma_2)| = n$.

(c) If $\alpha \notin \text{End}(B_n)$ then $|\text{supp}(\alpha)| \leq n$ and there exists $U \subseteq [n]$ and $\pi \in S_n$ such that $\text{supp}(\alpha) = \{(i, i\pi) : i \in U\}$.

(d) If $\alpha \in E^+(B_n)$ and $(i, i) \in \text{supp}(\alpha)$ then $(i, i)\alpha = (j, j)$ for some $j \in [n]$.

Proof. (a) is obvious. To prove (b), consider $\alpha = \sigma_1 + \sigma_2$ with $\sigma_1, \sigma_2 \in \text{End}(B_n)$. Then $(i, j) \in \text{supp}(\alpha)$ if and only if $j\sigma_1 = i\sigma_2$ and so $\text{supp}(\sigma_1 + \sigma_2) = \{(i, j) : j = i\sigma_2\sigma_1^{-1}\}$. It follows that $|\text{supp}(\sigma_1 + \sigma_2)| = n$. Part (c) was proved above, and for part (d) we note that since the idempotents in B_n commute, any $\alpha \in E^+(B_n)$ must map idempotents to idempotents. \square

Rephrasing part (c) of Lemma 3.1, the support of $\alpha = \sigma_1 + \cdots + \sigma_m \in E^+(B_n)$ (with $m \geq 2$) is determined by a partial bijection $U \rightarrow V$ of $[n]$, that is by an element π of the symmetric inverse monoid \mathcal{I}_n (see Howie (1995)). Then α is determined by its *support mapping* π and by two further partial bijections $\lambda = \sigma_1|_U$ and $\rho = \sigma_m|_V$. We call λ and ρ the *left action* and the *right action* of α . Then if $(i, j) \in \text{supp}(\alpha)$ we have $j = i\pi$ and $(i, j)\alpha = (i\lambda, j\rho)$. However, not every choice of π, λ, ρ gives rise to an element of $E^+(B_n)$, and we now characterize those choices that do. In what follows, for any subset $U \subseteq [n]$, we denote by $\text{stab}_{S_n}(U)$ the *pointwise stabiliser* of U in S_n .

Proposition 3.2. The triple $(\pi; \lambda, \rho)$ of partial bijections of $[n]$ represents an element $\alpha = \sigma_1 + \cdots + \sigma_m$ of $E^+(B_n)$ with $m \geq 2$ if and only if π, λ and ρ extend to permutations π_*, λ_* and ρ_* such that, if U is the domain of π and H is the subgroup of S_n generated by π_* and $\text{stab}_{S_n}(U)$, then $H\lambda_* = H\rho_*$.

Proof. Suppose that π, λ and ρ arise from an element $\alpha = \sigma_1 + \cdots + \sigma_m$ of $E^+(B_n)$. Set $\lambda_* = \sigma_1, \rho_* = \sigma_m$ and as above, let π be the partial bijection defined by $i\pi = j$ if and only if $(i, j) \in \text{supp}(\alpha)$. Choose π_* to be any permutation of $[n]$ extending π . Then $H = \langle \text{stab}_{S_n}(U), \pi_* \rangle$ does not depend on the choice of π_* . Now $\sigma_m = \varphi_m \varphi_{m-1} \cdots \varphi_2 \sigma_1$ where $\varphi_k = \sigma_k \sigma_{k-1}^{-1}$ and satisfies $i\varphi_k = j = i\pi$ for all $(i, j) \in \text{supp}(\alpha)$. Hence $\varphi_k \in H$ for all k and therefore $H\sigma_1 = H\sigma_m$.

Conversely, suppose that π, λ and ρ do extend to permutations π_*, λ_* and ρ_* such that $\rho_* = h\lambda_*$ for some $h \in H$. We may write $h = s_m \pi_* s_{m-1} \pi_* \cdots s_2 \pi_*$ for some $m \geq 2$, where $s_k \in \text{stab}_{S_n}(U)$ for all k . We may assume that at least one

s_k acts fixed-point-free on $[n] \setminus U$, for if no such s_k exists in the given expression for h , we may choose such an $s \in \text{stab}_{S_n}(U)$ and if $q = o(s\pi_*)$ in S_n , we consider the expression $h(s\pi_*)^q$ instead. Now set $\psi_k = s_k\pi_*$, and define $\sigma_1 = \lambda_*$ and $\sigma_k = \psi_k \cdots \psi_2 \lambda_*$ for $k \geq 2$. Let $\alpha = \sigma_1 + \cdots + \sigma_m$. Now $\sigma_k \sigma_{k-1}^{-1} = \psi_k = s_k\pi_*$ and so, if $i \in U$ and $j = i\pi$, then $i\sigma_k = is_k\pi_*\sigma_{k-1} = j\sigma_{k-1}$, and hence $\{(i, i\pi_*) : i \in U\} \subseteq \text{supp}(\alpha)$. But if $r \notin U$ and s_k acts fixed-point-free on $[n] \setminus U$ then $r\sigma_k = rs_k\pi_*\sigma_{k-1} \neq r\pi_*\sigma_{k-1}$ and so $(r, r\pi_*) \notin \text{supp}(\alpha)$. It follows that $\text{supp}(\alpha) = \{(i, i\pi) : i \in U\}$, and clearly α has left action equal to λ and right action equal to ρ . \square

The support and actions of an element in $E^+(B_n)$ also determines its Green's classes, as our next result explains.

Proposition 3.3. (a) *The \mathcal{R} -class and the \mathcal{L} -class of an endomorphism σ in $E^+(B_n)$ each consists only of σ .*

(b) *Two elements in $E^+(B_n)$ are \mathcal{R} -related if and only if they have the same support and the same left action, and are \mathcal{L} -related if and only if they have the same support and the same right action.*

(c) *For any $\alpha \in E^+(B_n)$, the \mathcal{R} -class of α and the \mathcal{L} -class of α have the same size.*

(d) *If α has support mapping $\pi : U \rightarrow V$ with $\pi_* \in S_n$ extending π , then $|R_\alpha|$ is equal to the size of the orbit of $H = \langle \text{stab}_{S_n}(U), \pi_* \rangle$ on the subset U .*

(e) *The \mathcal{H} -relation on $E^+(B_n)$ is trivial.*

(f) *Two elements α, β are \mathcal{D} -related if and only if they have the same support mapping $\pi : U \rightarrow V$ extending to $\pi_* \in S_n$ such that their left and right actions extend to permutations in the same coset of $H = \langle \text{stab}_{S_n}(U), \pi_* \rangle$ in S_n .*

Proof. For (a), we observe that if $\sigma \mathcal{R} \beta$ with $\sigma \neq \beta$ then $\sigma = \beta + \gamma$ for some $\gamma \in E^+(B_n)$, and then σ cannot be an endomorphism, by Proposition 3.1. The same reasoning applies to the \mathcal{L} -relation.

(b) Let $\alpha = \sigma_1 + \cdots + \sigma_m$ and suppose that α has support mapping $\pi_\alpha : U_\alpha \rightarrow V_\alpha$ with left and right actions $\lambda_\alpha, \rho_\alpha$. Similarly, let $\beta = \tau_1 + \cdots + \tau_t$ with support mapping $\pi_\beta : U_\beta \rightarrow V_\beta$ with left and right actions $\lambda_\beta, \rho_\beta$.

Suppose that $\alpha \mathcal{R} \beta$ so that, for some $\gamma, \delta \in E^+(B_n)$ we have $\alpha = \beta + \gamma$ and $\beta = \alpha + \gamma$. By part (a) of Lemma 3.1, $\text{supp}(\alpha) \subseteq \text{supp}(\beta)$ and $\text{supp}(\beta) \subseteq \text{supp}(\alpha)$: it follows that $\text{supp}(\alpha) = \text{supp}(\beta)$, and so $\pi_\alpha = \pi_\beta$. Then for all (i, j) in the support, we have

$$(i, j)\alpha = (i\lambda_\alpha, j\rho_\alpha) = (i, j)(\beta + \gamma) = (i, j)\beta(i, j)\gamma = (i\lambda_\beta, j\rho_\beta)(i, j)\gamma = (i\lambda_\beta, j\rho_\gamma)$$

and hence $i\lambda_\alpha = i\lambda_\beta$. Therefore α and β have the same support and the same left actions.

Conversely, suppose that α and β as above have the same support mapping $\pi : U \rightarrow V$ and the same left action λ . It suffices to show that there exists $\gamma \in E^+(B_n)$ such that $\alpha = \beta + \gamma$. Represent α by the triple $(\pi; \lambda, \rho)$ and β by the triple $(\pi; \lambda, \xi)$. Extend π to a permutation π_* and extend λ to a permutation λ_* . (Note that we could take $\lambda_* = \sigma_1$ or $\lambda_* = \tau_1$). Then if $H = \langle \text{stab}_{S_n}(U), \pi_* \rangle$, the coset $H\lambda_*$ does not depend on the choice of λ_* , and by Proposition 3.2 the right actions ρ and ξ extend to permutations ρ_*, ξ_* such that $H\rho_* = H\lambda_* = H\xi_*$. There exist $\varphi_k, 2 \leq k \leq m$ and $\psi_l, 2 \leq l \leq t$ such that, if $i \in U$ then $i\varphi_k = i\pi = i\psi_l$, and with $\rho_* = \varphi_m \cdots \varphi_2 \lambda_*, \xi_* = \psi_t \cdots \psi_2 \lambda_*$. Now let $p_j = o(\psi_j) - 1$ and consider the factors in the product

$$\omega = \varphi_m \varphi_{m-1} \cdots \varphi_2 \psi_2^{p_2} \cdots \psi_t^{p_t} \psi_t \psi_{t-1} \cdots \psi_2 \lambda_*,$$

regarding $\psi_j^{p_j}$ as the product of p_j separate factors each equal to ψ_j . Hence there are $(m-1) + p_2 + \cdots + p_t + t = q$ factors in all, which we rename in order as $\chi_i, (1 \leq i \leq q)$, starting with $\chi_1 = \lambda_*$ and concluding with $\chi_q = \varphi_m$. Then for $i \geq 2$, $\chi_i|_U = \pi$ and clearly $\omega = \varphi_m \varphi_{m-1} \cdots \varphi_2 \lambda_*$ in S_n so that $\omega|_V = \rho$. We define $v_i = \chi_i \chi_{i-1} \cdots \chi_1$. Then for $1 \leq k \leq t$, we have $v_i = \tau_i$ so that $\beta = v_1 + \cdots + v_t$. If we then define $\gamma = v_{t+1} + \cdots + v_q$ we find that $\beta + \gamma$ has support mapping π , left action λ and right action obtained by restricting $v_q = \chi_q \chi_{q-1} \cdots \chi_2 \chi_1 = \omega$ to V , so that the right action is $v_q|_V = \omega|_V = \rho$. It follows that $\beta + \gamma = \alpha$.

The proof of the characterisation of Green's \mathcal{L} -relation proceeds in the same way, and we omit the details.

Now part (c) follows for an endomorphism $\sigma \in E^+(B_n)$ by part (a). So consider $\alpha \in E^+(B_n) \setminus \text{End}(B_n)$, with support mapping π and actions λ, ρ . By part (b) the \mathcal{R} -class R_α of α consists of those mappings represented by triples $(\pi; \lambda, \xi)$ where ξ is a partial bijection with domain V that extends to some permutation ξ_* such that $H\lambda_* = H\xi_*$, and the \mathcal{L} -class L_α of α consists of those mappings represented by triples $(\pi; \eta, \rho)$ where η is a partial bijection with domain U that extends to some permutation η_* such that $H\eta_* = H\rho_*$. The mappings

$$R_\alpha \rightarrow L_\alpha, (\pi; \lambda, \xi) \mapsto (\pi; \pi\xi, \rho)$$

and

$$L_\alpha \rightarrow R_\alpha, (\pi; \eta, \rho) \mapsto (\pi; \lambda, \pi^{-1}\eta)$$

(where $\pi^{-1} : V \rightarrow U$ is a partial bijection on $[n]$) are then inverse bijections.

For part (d), let α have left action λ and extend λ to $\lambda_* \in S_n$. As above, the coset $H\lambda_*$ does not depend on the choice of λ_* , and we see by part (a) and Proposition 3.2 that $|R_\alpha|$ is the number of distinct actions on V by permutations ρ_* such that $H\lambda_* = H\rho_*$. There are $|H|$ choices for ρ_* , and the number of distinct actions on V is equal to the number of distinct actions of H on U .

Part (e) follows from part (b). Two mappings that are both \mathcal{R} and \mathcal{L} -related have the same support and the same left and right actions and so are equal.

Part (f) also follows from part (b) and Proposition 3.2. Suppose that $\alpha \mathcal{D} \beta$ and let $\gamma \in E^+(B_n)$ be such that $\alpha \mathcal{R} \gamma \mathcal{L} \beta$. By part (b), α, β and γ have the

same support mapping π , and if α is represented by the triple $(\pi; \lambda, \rho)$, β by $(\pi; \eta, \xi)$, then γ is represented by $(\pi; \lambda, \xi)$ and $H\rho_* = H\lambda_* = H\xi_* = H\eta_*$. Conversely, if α, β are represented by $(\pi; \lambda, \rho)$ and $(\pi; \eta, \xi)$ respectively, with $H\rho_* = H\lambda_* = H\xi_* = H\eta_*$, we take γ represented by $(\pi; \lambda, \xi)$ and then $\alpha\mathcal{R}\gamma\mathcal{L}\beta$. \square

4 Classification by support

4.1 Endomorphisms

$\sigma \in \text{End}(B_n)$ is induced by a permutation $\sigma \in S_n$, and such elements of $E^+(B_n)$ are characterised by their support:

$$\sigma \in \text{End}(B_n) \iff \text{supp}(\sigma) = B_n \setminus \{0\}.$$

As shown in Proposition 3.3, endomorphisms lie in singleton \mathcal{R} and \mathcal{L} -classes, and we further observe:

Proposition 4.1. *For any $\sigma \in \text{End}(B_n)$ we have $\sigma + \sigma = \sigma + \sigma + \sigma$ in $E^+(B_n)$, with support $\{(i, i) : 1 \leq i \leq n\}$. Hence σ generates a subsemigroup of order 2 in $E^+(B_n)$.*

4.2 Elements with full support

An element $\alpha \in E^+(B_n)$ is said to have *full support* if $|\text{supp}(\alpha)| = n$. Proposition 3.1 shows that the sum of any two endomorphisms in $E^+(B_n)$ has full support, and that for any $\alpha \in E^+(B_n)$ with full support we have $\text{supp}(\alpha) = \{(i, i\pi) : 1 \leq i \leq n\}$ for some permutation $\pi \in S_n$.

Let $\alpha = \sigma_1 + \sigma_2 + \cdots + \sigma_m$ have full support, where $\sigma_j \in \text{End}(B_n)$ and $m \geq 2$. Then as in section 3,

$$(i, j) \in \text{supp}(\alpha) \iff j = i\varphi_2 = i\varphi_3 = \cdots = i\varphi_m$$

and hence $\pi = \varphi_k$ for all $k, 2 \leq k \leq m$. Since $\sigma_k = \varphi_k \cdots \varphi_2 \sigma_1$ it follows that

$$\alpha = \sigma_1 + \pi\sigma_1 + \cdots + \pi^{m-1}\sigma_1.$$

Then $(i, j)\alpha = (i\sigma_1, j\pi^{m-1}\sigma_1) = (i, j\pi^{m-1})\sigma_1$. Hence α is determined by its support mapping π and its left action σ_1 . For fixed π, σ_1 we obtain a sequence of distinct mappings α for $m = 1, 2, \dots, o(\pi)$, where $o(\pi)$ is the order of the permutation π in the symmetric group S_n .

Proposition 4.2. *The number of elements of full support in $E^+(B_n)$ is given by*

$$n! \sum_{\pi \in S_n} o(\pi).$$

The sequence $(\sum_{\pi \in S_n} o(\pi))$ is sequence A060014 in Sloane (2007): its initial values are

n	1	2	3	4	5	6
	1	3	13	67	471	3271

As a corollary of part (d) of Proposition 3.3 we have:

Proposition 4.3. *The size of an \mathcal{R} or \mathcal{L} -class of an element α having full support and action permutation π is equal to the order of π in S_n .*

Proposition 4.4. *The set $\{\alpha \in E^+(B_n) : |\text{supp}(\alpha)| \leq n\}$ is an ideal of $E^+(B_n)$ and is generated, as a subsemigroup, by the subset of elements with full support.*

Proof. It is obvious from part (a) of Lemma 3.1 that $\{\alpha \in E^+(B_n) : |\text{supp}(\alpha)| \leq n\}$ is an ideal. In order to show that, as a subsemigroup, it is generated by the elements of full support, by part (b) of Lemma 3.1 it suffices to show that a sum $\alpha = \sigma_1 + \sigma_2 + \sigma_3$ of three endomorphisms may also be written as the sum of elements of full support.

To this end, let $\varphi_2 = \sigma_2\sigma_1^{-1}$ and $\varphi_3 = \sigma_3\sigma_2^{-1}$, and let r be the order of φ_2 in S_n . Consider the mappings

$$\beta = \sigma_1 + \varphi_2\sigma_1 + \cdots + \varphi_2^{r-1}\sigma_1 + \sigma_1$$

and $\gamma = \varphi_2\sigma_1 + \varphi_3\varphi_2\sigma_1 = \varphi_2\sigma_1 + \sigma_3$. Then β and γ are of full support, and $\beta + \gamma$ has the same left and right actions as α . Moreover, we have $(i, j) \in \text{supp}(\beta + \gamma)$ if and only if $i\varphi_2 = j = i\varphi_3$ and so $\text{supp}(\beta + \gamma) = \text{supp}(\alpha)$. It follows that $\alpha = \beta + \gamma$. \square

4.3 Elements with partial support

An element $\alpha \in E^+(B_n)$ with $|\text{supp}(\alpha)| < n$ is said to have *partial support*. In this case, by part (c) of Lemma 3.1, the support is given by $\text{supp}(\alpha) = \{(i, i\pi) : i \in U\}$ for some partial bijection $\pi \in \mathcal{I}_n$ with domain U .

Lemma 4.5. *If $0 \neq \alpha \in E^+(B_n)$ has partial support then $3 \leq n$ and $1 \leq |\text{supp}(\alpha)| \leq n - 2$. Moreover, given any k with $1 \leq k \leq n - 2$, and any partial bijection $\pi : U \rightarrow V$ between two subsets $U, V \subseteq [n]$ of size k , there exists $\alpha = \sigma_1 + \sigma_2 + \sigma_3 \in E^+(B_n)$ such that $\text{supp}(\alpha) = \{(i, i\pi) : 1 \leq i \leq n\}$.*

Proof. We know from part (b) of Proposition 4.3 that if $\sigma_1, \sigma_2 \in \text{End}(B_n)$ then $\alpha = \sigma_1 + \sigma_2$ has full support. So suppose that $\alpha = \sigma_1 + \sigma_2 + \sigma_3$ with $\sigma_j \in \text{End}(B_n)$. Regarding the σ_j as permutations in S_n , we set $\varphi_2 = \sigma_2\sigma_1^{-1}$ and $\varphi_3 = \sigma_3\sigma_2^{-1}$. Then

$$(i, j) \in \text{supp}(\alpha) \iff j = i\varphi_2 = i\varphi_3.$$

Hence φ_2 and φ_3 are permutations agreeing with the partial bijection π on its domain. If $|\text{supp}(\alpha)| > n - 2$ then φ_2 and φ_3 are permutations of degree n

agreeing on $n - 1$ elements: hence $\varphi_2 = \varphi_3$, and α has full support. So if α has partial support, then $|\text{supp}(\alpha)| \leq n - 2$. Since $\alpha \neq 0$ we must have $n \geq 3$. Now given two sets of distinct integers $U = \{a_1, \dots, a_k\}$ and $V = \{b_1, \dots, b_k\}$, each of size $k \leq n - 2$ and with $1 \leq a_p, b_q \leq n$ for all p, q , choose φ_2 to be any permutation in S_n such that $a_p \varphi_2 = b_p$ for all $p, 1 \leq p \leq k$, and let ϕ be any permutation in S_n whose set of fixed points is precisely $\{a_1, \dots, a_k\}$. (This is always possible if $k \leq n - 2$). Set $\varphi_3 = \phi \varphi_2$. Then $i \varphi_2 = i \varphi_3$ if and only if $i \in \{a_1, \dots, a_k\}$. Hence if we set

$$\alpha = \sigma_1 + \varphi_2 \sigma_1 + \varphi_3 \varphi_2 \sigma_1$$

for any $\sigma_1 \in S_n$, it follows that

$$\text{supp}(\alpha) = \{(a_1, b_1), \dots, (a_k, b_k)\}.$$

□

However, although we can construct each possible support for some α of the form $\alpha = \sigma_1 + \sigma_2 + \sigma_3$, it is not true that every mapping in $E^+(B_n)$ arises in this way.

Example 4.6. Take $n = 3$. Then the mapping $\alpha = (1\ 2) + (2\ 3) + (1\ 3\ 2) + \text{id}$ has partial support equal to the singleton set $\{(1, 2)\}$, with $(1, 2)\alpha = (2, 2)$. Suppose that $\alpha = \sigma_1 + \sigma_2 + \sigma_3$. Then $1\sigma_1 = 2, 2\sigma_3 = 2$ and for $\{a, b\} = \{1, 3\}$ we have $a = 2\sigma_1 = 1\sigma_2, b = 2\sigma_2 = 1\sigma_3$. Hence only two possibilities arise, and for each we find that $\sigma_1 + \sigma_2 + \sigma_3$ has full support, with support permutation $(1\ 2\ 3)$ in each case.

4.4 Elements with singleton support

By Lemma 4.5 there are no elements of $E^+(B_2)$ with singleton support. For $n \geq 3$, we shall now describe the subsemigroup of $E^+(B_n)$ consisting of the elements of singleton support, together with 0. For each $(i, j) \in B_n, (n \geq 3)$ we let

$$E_{(i,j)} = \{\alpha \in E^+(B_n) : \text{supp}(\alpha) = \{(i, j)\}\} \cup \{0\}.$$

Proposition 4.7. *Let $n \geq 3$.*

- (a) *The number of mappings in $E^+(B_n)$ with singleton support is equal to $n^2(n^2 - n + 1)$.*
- (b) *If $i \neq j$ then $E_{(i,j)}$ is a subsemigroup of $E^+(B_n)$ that is isomorphic to B_n .*
- (c) *If $i = j$ then $E_{(i,i)}$ is a subsemigroup of $E^+(B_n)$ that is isomorphic to $\text{Idem}(B_n)$.*
- (d) *The set of all mappings in $E^+(B_n)$ with singleton support, together with zero, forms a subsemigroup of $E^+(B_n)$ isomorphic to the zero direct union of $n(n - 1)$ copies of B_n and n copies of $\text{Idem}(B_n)$.*

Proof. (a) Suppose that $\alpha \in E^+(B_n)$ with $|\text{supp}(\alpha)| = 1$. If $\text{supp}(\alpha) = \{(i, i)\}$ then, by part (d) of Lemma 3.1, $(i, i)\alpha = (j, j)$. Given i , any value of j can occur: without loss of generality, suppose $i = 1$. Set $\varphi_2 = 1$ and $\varphi_3 = (2\ 3 \dots n)$. Then $(1, 1)(1 + 1 + (2\ 3 \dots n)) = (1, 1)$ and so if $\sigma : 1 \mapsto j$ then $(1, 1)(\sigma + \sigma + (2\ 3 \dots n)\sigma) = (j, j)$. There are n possibilities for i and n for j , and therefore $E^+(B_n)$ contains n^2 mappings α with $\text{supp}(\alpha) = \{(i, i)\}$.

Now suppose that $\text{supp}(\alpha) = \{(i, j)\}$ with $i \neq j$. We claim that for any $p, q \in [n]$ we can find α (with support $\{(i, j)\}$) such that $(i, j)\alpha = (p, q)$. Without loss of generality suppose that $(i, j) = (1, 2)$. Now take $\varphi_2 = (1\ 2)$ and $\varphi_3 = (2\ 3 \dots n)$. Then $(1, 2)(1 + \varphi_2 + \varphi_3) = (1, 2)(2, 1)(1, 3) = (1, 3)$. If $p \neq q$, choose σ with $1\sigma = p$ and $3\sigma = q$. Then $(1, 2)(\sigma + \varphi_2\sigma + \varphi_3\sigma) = (p, q)$. If $p = q$ we need a slightly different approach. Again assuming that $(i, j) = (1, 2)$, choose $\varphi_2 = (1\ 2)$, $\varphi_3 = (2\ 3 \dots n)$ and $\varphi_4 = (1\ 3\ 2)$. Then if $\beta = 1 + \varphi_2 + \varphi_3 + \varphi_4$ we have $\text{supp}(\beta) = \{(1, 2)\}$ with $(1, 2)\beta = (1, 1)$. Then for any $\sigma : 1 \mapsto p$ the map $\alpha = \sigma + \varphi_2\sigma + \varphi_3\sigma + \varphi_4\sigma$ has $\text{supp}(\alpha) = \{(1, 2)\}$ with $(1, 2)\alpha = (p, p)$. There are $n(n-1)$ possibilities for the support element (i, j) , and for each of these there are n^2 possibilities for $(i, j)\alpha$. Therefore $E^+(B_n)$ contains $n^3(n-1)$ mappings α with $\text{supp}(\alpha) = \{(i, j)\}$ with $i \neq j$.

The total number of mappings in $E^+(B_n)$ with singleton support is therefore $n^2 + n^3(n-1) = n^2(n^2 - n + 1)$.

For part (b), we observe that each $\alpha \in E_{(i, j)}$ is completely determined by the element $(i, j)\alpha$, and that if $\beta \in E_{(i, j)}$ then either $\alpha + \beta = 0$ or the element $(i, j)\alpha(i, j)\beta$ is non-zero and completely determines $\alpha + \beta$. It follows that $E_{(i, j)}$ is a subsemigroup of $E^+(B_n)$, and that the mapping defined by $\alpha \mapsto (i, j)\alpha$ and $0 \mapsto 0$ is an isomorphism $E_{(i, j)} \rightarrow B_n$. Similarly, for part (c), we observe that each $\alpha \in E_{(i, i)}$ is completely determined by the element $(i, i)\alpha \in \text{Idem}(B_n)$, and that the mapping defined by $\alpha \mapsto (i, i)\alpha$ and $0 \mapsto 0$ is then an isomorphism $E_{(i, i)} \rightarrow \text{Idem}(B_n)$. For part (d) it follows from part (a) of Lemma 3.1 that if $\alpha \in E_{(i, j)}$ and $\beta \in E_{(k, l)}$ with $(i, j) \neq (k, l)$ then $\alpha + \beta = 0$. \square

5 Enumerating elements of $E^+(B_n)$

5.1 $n = 2$

When $n = 2$, $\text{End}(B_2) = \{1, \tau, 0\}$, where τ is the transposition $(1\ 2)$. There are six other non-zero elements of $E^+(B_2)$, with full support. The Cayley table for the semigroup $(E^+(B_2), +)$ is

+	1	τ	γ	μ	ν	δ	η	ξ	0
1	δ	μ	0	0	η	δ	0	μ	0
τ	ν	γ	γ	ξ	0	0	ν	0	0
γ	0	γ	γ	0	0	0	0	0	0
μ	η	0	0	μ	0	0	η	0	0
ν	0	ξ	0	0	ν	0	0	ξ	0
δ	δ	0	0	0	0	δ	0	0	0
η	0	μ	0	0	η	0	0	μ	0
ξ	ν	0	0	ξ	0	0	ν	0	0
0	0	0	0	0	0	0	0	0	0

The actions of the elements of $E^+(B_2)$ on the non-zero elements of B_2 are shown in the following table:

α	$(1,1)\alpha$	$(1,2)\alpha$	$(2,1)\alpha$	$(2,2)\alpha$
1	(1,1)	(1,2)	(2,1)	(2,2)
τ	(2,2)	(2,1)	(1,2)	(1,1)
γ	(2,2)	0	0	(1,1)
μ	0	(1,1)	(2,2)	0
ν	0	(2,2)	(1,1)	0
δ	(1,1)	0	0	(2,2)
η	0	(1,2)	(2,1)	0
ξ	0	(2,1)	(1,2)	0
0	0	0	0	0

5.2 $n = 3$

For $n = 3$, Proposition 4.2 gives us $3! \times 13 = 78$ elements of full support. By Lemma 4.5, the only possible partial supports are singleton sets, and by Proposition 4.7 we find 63 such mappings. Hence $|E^+(B_3)| = 7 + 78 + 63 = 148$.

5.3 $n > 3$

We have investigated the size of the semigroup $E^+(B_n)$ for $n = 4, 5, 6$ using the computational discrete algebra system GAP (The GAP group, 2007). Propositions 4.3 and 4.4 give exact calculations for support sizes n and 1, but our calculations show that the bulk of the elements of $E^+(B_n)$ have support size $n-2$. Our GAP code, which is given in an appendix, counts elements of $E^+(B_n)$ by enumerating triples $(\pi; \lambda, \rho)$ as in Proposition 3.2. We summarize our findings (including those for $n = 2, 3$) in the following table, recalling that support size $n-1$ does not occur (by Lemma 4.5).

Enumeration of elements of $E^+(B_n)$ by support size

n	2	3	4	5	6
endomorphisms	3	7	25	121	721
full support	6	78	1,608	56,520	2,355,120
support size $n - 2$	—	63	5,112	1,005,000	142,533,000
support size $n - 3$	—	—	208	53,400	17,743,200
support size $n - 4$	—	—	—	525	289,350
support size $n - 5$	—	—	—	—	1,116
$ E^+(B_n) $	9	148	6,953	1,115,566	162,922,507

Appendix: GAP code for enumeration

The following GAP code produces a list `supplist` of all possible supports of size `suppsize` for elements of $E^+(B_n)$, and then constructs for each support U , a list `actionlist` of all triples $(\pi; \lambda, \rho)$ that represent elements of $E^+(B_n)$ with support U , as in Proposition 3.2. The number of elements found for each U is the summed by the counter `esize`.

```
#Enumeration of E(B_n) by triples

#Set required value of n here
n:=4;

#Set required support size here
suppsize:=2;

#Initialize counter
esize:=0;

sn:=SymmetricGroup(n);

#Define supplist as list of possible sets U of size suppsize in degree n

ulist:=[];
seed:=[1..suppsize];
for g in sn do
Add(ulist,AsSortedList(OnTuples(seed,g)));
od;
supplist:=DuplicateFreeList(ulist);

#List all possible U,V,pi in the list pilist

pilist:= [];
```

```

for uset in suppulist do
stabuset:=Stabilizer(sn,uset,OnTuples);
stabtrans:=RightTransversal(sn,stabuset);
for g in stabtrans do
newpi:=[uset,OnTuples(uset,g),g];
Add(pilist,newpi);
od;
od;

for pee in pilist do
actionlist:=[];
stabu:=Stabilizer(sn,pee[1],OnTuples);
transv:=RightTransversal(sn,stabu);
cosetgens:=AsList(RightCoset(stabu,pee[3]));
hgp:=Group(cosetgens);

#for each left action lam find all possible distinct right actions rho

for lam in transv do
lamaction:=OnTuples(pee[1],lam);
bigrholist:=[];
for rho in RightCoset(hgp,lam) do
Add(bigrholist,OnTuples(pee[2],rho));
rholist:=Unique(bigrholist);;
od;
for rhoaction in rholist do
newaction:=[pee[1],pee[2],lamaction,rhoaction];
Add(actionlist,newaction);
od;
od;

#add new actions to running total

esize:=esize+Length(actionlist);;
od;

#Reveal final total
esize;

```

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