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ENDS OF LEAVES OF CODIMENSION-ONE FOLIATIONS

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1. Introduction and statement of results. In Nishimori [2] the author began the study of the limit sets of ends of leaves of codimension-one foliations and obtained the necessary and sufficient condition under which the limit set of an isolated end becomes a compact leaf. The purpose of this paper is to strengthen and generalize the result mentioned above. The main theme is the study of the situation where the limit set of an end becomes the closure of a leaf and consists of a finite number of leaves.

In this paper M is always a closed orientable C^∞ manifold, and $\mathscr{F}$ is a transversely orientable codimension-one C^r foliation on M where $r \geqq 1$. Note that there is a C^∞ flow $\varphi: M \times \boldsymbol{R} \to M$ transverse to $\mathscr{F}$.

REMARK 1. The reason for the $C^r(r \geqq 1)$ assumption is that we use Thom's theorem guaranteeing the existence of a compact submanifold whose fundamental class coincides with a given codimension-one homology class of a compact manifold, see Thom [6].

REMARK 2. In the case of non-orientable manifolds or transversely non-orientable codimension-one foliations, we obtain similar results by considering, as usual, the two-fold or four-fold covering space and the induced foliation.

Now we recall the definition of ends of non-compact manifolds.

DEFINITION 1. Let F be a non-compact manifold. A family ε of non-empty connected open subsets of F is called an *end* of F if ε satisfies the following conditions (1)-(4).

(1) $\partial U = \mathrm{Cl}_F(U) - U$ is compact for all $U \in \varepsilon$ where $\mathrm{Cl}_F(\ )$ means the closure with respect to the topology of F.

(2) If $U, U' \in \varepsilon$, then there is $U'' \in \varepsilon$ with $U'' \subset U \cap U'$.

(3) $\bigcap \{U | U \in \varepsilon\} = \emptyset$.

(4) ε is a maximal family satisfying (1), (2) and (3).

REMARK 3. It is very useful to consider a Morse function $f: F \to [0, \infty)$ such that (1) if K is a compact subset of $[0, \infty)$ then $f^{-1}(K)$ is compact and (2) the singular points of f are contained in $f^{-1}(\{1/2, 1+1/2, 2+1/2, \cdots\})$. Each end ε of F contains uniquely a sequence $U_1 \supset U_2 \supset \cdots$ such that U_i is a connected component of $f^{-1}((i, \infty))$. We call the sequence $\{U_i\}_{i=1}^{\infty}$ the cofinal sequence of ε with respect to f. Conversely given such a sequence $\{U_i\}_{i=1}^{\infty}$, we see that $\varepsilon = \{U \mid U$ is a non-empty connected open subset of F, ∂U is compact and $U \supset U_i$ for some $i\}$ is an end of F.

DEFINITION 2. For a subset A of a non-compact leaf F of $\mathscr{F}$, let $L(A) = \{y \in M \mid$ There is a sequence $x_1, x_2, \cdots \in A$ such that (1) $\{x_i\}_{i=1}^{\infty}$ does not have any accumulation point in F with respect to the topology of F and (2) $\lim x_i = y$ with respect to the topology of $M\}$. We call $L(A)$ the *limit set* of A. Let ε be an end of a non-compact leaf F of $\mathscr{F}$. Let $L_\varepsilon(F) = \bigcap \{\mathrm{Cl}_M(U) \mid U \in \varepsilon\}$ and call it the ε *limit set* of F or the *limit set* of ε.

REMARK 4. By using the cofinal sequence $\{U_i\}_{i=1}^{\infty}$ of ε in Remark 3, we can write $L_\varepsilon(F) = \bigcap_{i=1}^{\infty} \mathrm{Cl}_M(U_i)$.

The fundamental properties of $L(F)$ and $L_\varepsilon(F)$ are as follows. See Nishimori [2], [4].

PROPOSITION 1. (1) *$L(F)$ and $L_\varepsilon(F)$ are non-empty, closed and saturated subsets of M.* (2) *$L(F) = \bigcup \{L_\varepsilon(F) \mid \varepsilon$ is an end of $F\}$.* (3) *$L_\varepsilon(F)$ is connected.*

REMARK 5. In the previous papers [2], [3] and [4], we used the term "invariant" instead of the term "saturated".

DEFINITION 3. Let F be a leaf of $\mathscr{F}$. Let $d(F) = \mathrm{Sup}\,\{k \mid$ There is a sequence $F_1, \cdots, F_k$ of leaves such that $F_i \subset \mathrm{Cl}_M(F_{i+1})$, $F_i \neq F_{i+1}$ and $F_k = F\}$. We call $d(F)$ or $d(\mathscr{F})$ the *depth* of F or $\mathscr{F}$, respectively.

DEFINITION 4. Let ε be an end of a non-compact manifold F. We define $d(\varepsilon)$ by induction as follows and call it the *depth* of ε.

(1) $d(\varepsilon) = 1$ if there is $U \in \varepsilon$ such that if an end ε' of F contains U then $\varepsilon' = \varepsilon$. If $d(\varepsilon) = 1$, ε is called *isolated*.

(2) $d(\varepsilon) = d > 1$ if (a) there is $U \in \varepsilon$ such that if an end $\varepsilon' \neq \varepsilon$ contains U then $d(\varepsilon') < d$ and (b) for all $U \in \varepsilon$ there is an end $\varepsilon' \neq \varepsilon$ containing U such that $d(\varepsilon') = d - 1$.

DEFINITION 5. An end ε of a leaf F is called *pseudotame* if $d(\varepsilon) < \infty$ and there is $U \in \varepsilon$ such that, for any end ε' containing U, if $F' \subset L_{\varepsilon'}(F)$ then $d(F') \leqq d(\varepsilon')$. If an end ε is not pseudotame, we call it *wild*.

DEFINITION 6. Let ε be an end of a leaf F of $\mathscr{F}$ with $d(\varepsilon) < \infty$.

Let $L_\varepsilon^*(F) = L_\varepsilon(F) - \bigcap_{U\in\varepsilon}(\bigcup_{U\in\varepsilon'\neq\varepsilon} L_{\varepsilon'}(F))$. We call $L_\varepsilon^*(F)$ the *principal part* of $L_\varepsilon(F)$.

We define tame ends with respect to $\mathscr{F}$ by induction on the depth. We fix a metric d on M throughout this paper.

DEFINITION 7. Let ε be an end of a leaf F of $\mathscr{F}$. (1) ε is a *tame end of depth* 1 if (a) $d(\varepsilon) = 1$, (b) $L_\varepsilon(F) \cap F = \emptyset$ and (c) ε *approaches* $L_\varepsilon(F)$ *from one side*, that is, for all $x \in L_\varepsilon(F)$ there are $\delta > 0$ and $U \in \varepsilon$ such that $\varphi(\{x\} \times (-\delta, 0)) \cap U = \emptyset$, or for all $x \in L_\varepsilon(F)$ there are $\delta > 0$ and $U \in \varepsilon$ such that $\varphi(\{x\} \times (0, \delta)) \cap U = \emptyset$. For a tame end ε of depth 1, let $a(\varepsilon) = \mathrm{Sup}\,\{d(\partial U, L_\varepsilon(F)) \mid U \in \varepsilon$ and if an end ε' contains U then $\varepsilon' = \varepsilon\}$.

(2) ε is a *tame end of depth* $d > 1$ if (a) $d(\varepsilon) = d$, (b) $L_\varepsilon(F) \cap F = \emptyset$, (c) ε approaches $L_\varepsilon^*(F)$ from one side, and (d) there are $U \in \varepsilon$ and $a > 0$ such that if an end $\varepsilon' \neq \varepsilon$ contains U then ε' is a tame end of depth $< d$ and $a(\varepsilon') > a$. For a tame end ε of depth $< d$, let $a(\varepsilon) = \mathrm{Sup}\,\{d(\partial U, L_\varepsilon(F)) \mid U \in \varepsilon$ and if $\varepsilon \neq \varepsilon' \ni U$ then ε' is a tame end of depth $< d\}$.

(3) ε is called *tame* if ε is a tame end of depth d for some positive integer d.

REMARK 6. The term "tame end", which depends on the foliation, is clearly different from that in Siebenmann [5].

Now we can state the results. Theorem 1 is stronger than Theorem B of Nishimori [2].

THEOREM 1. *Let ε be an end of a leaf F of $\mathscr{F}$. Then $L_\varepsilon(F)$ consists of just one leaf if and only if ε is a tame end of depth* 1.

COROLLARY 1. *Let ε be an isolated end. Then ε is tame if and only if ε is pseudotame.*

The limit sets of tame ends have the following property.

THEOREM 2. *Let ε be a tame end of a leaf F of $\mathscr{F}$. Then we have the following.*

(1) *The limit set $L_\varepsilon(F)$ of ε consists of a finite number of proper leaves of depth $\leqq d(\varepsilon)$.*

(2) *The principal part $L_\varepsilon^*(F)$ of $L_\varepsilon(F)$ is the unique leaf of depth $d(\varepsilon)$ in $L_\varepsilon(F)$ and $L_\varepsilon(F) = \mathrm{Cl}_M(L_\varepsilon^*(F))$.*

(3) *If a leaf F' is contained in $L_\varepsilon(F)$, then all ends of F' are tame ends of depth $< d(F')$ and F' has at least one and only a finite number of ends of depth $d(F') - 1$.*

COROLLARY 2. *A tame end is pseudotame.*

A pseudotame end is not necessarily tame, as the following theorem shows.

THEOREM 3. *There is a* C^0 *foliation* $\mathscr{F}$ *on* $S_2 \times [0, 1]$ *satisfying the following conditions, where* S_2 *is the closed surface of genus* 2.

(1) *All leaves of* $\mathscr{F}$ *are* C^∞ *submanifolds and transverse to* $\{x\} \times [0, 1]$ *for all* $x \in S_2$.

(2) $d(\mathscr{F}) = 3$ *and each leaf of depth* 3 *has a pseudotame end which is not tame.*

Theorem 2 and the following theorem may be considered as generalizations of Theorem 1.

THEOREM 4. (1) *Let* F_0 *be a leaf of* $\mathscr{F}$ *such that* $2 \leqq d = d(F_0) < \infty$ *and all ends of* F_0 *are tame. Then all ends of* F_0 *have depth* $< d$, *and* F_0 *has at least one and at most a finte number of ends of depth* $d - 1$. F_0 *is proper.*

(2) *Furthermore, suppose that* $\mathscr{F}$ *is of class* C^2 *and all leaves contained in* $\mathrm{Cl}_M(F_0) - F_0$ *have abelian holonomy and that* $\mathrm{Cl}_M(F_0) = L_\varepsilon(F_1)$ *for an end* ε *of some leaf* F_1 *of* $\mathscr{F}$. *Then* ε *is a tame end of depth* d *and* $L_\varepsilon^*(F_1) = F_0$.

2. **Preliminary.** In the proof of the theorems, we often consider foliations of the following type. Let X be a compact manifold with corner on the boundary such that the corner divides the boundary ∂X of X into two submanifolds Y_1, Y_2, that is, $\partial X = Y_1 \cup Y_2$ and $Y_1 \cap Y_2 = \partial Y_1 = \partial Y_2$ is the corner. Then we can consider a foliation $\mathscr{F}$ on X such that the connected components of Y_1 are leaves of $\mathscr{F}$ and $\mathscr{F}$ is transverse to Y_2. We introduce some notations. Let A be a compact manifold with or without boundary, and B a transversely oriented codimension-one compact submanifold of A such that $\partial B = B \cap \partial A$. We denote by $C(A, B)$ the compact manifold obtained from $A - B$ by attaching two copies B_1, B_2 of B, where the suffixes 1, 2 depend on the transverse orientation of B. Let $f\colon [0, \delta_1] \to [0, \delta_2]$ be a diffeomorphism such that $f(0) = 0$ and $\delta_1 > \delta_2$. We denote by $X(A, B, f)$ the quotient space of

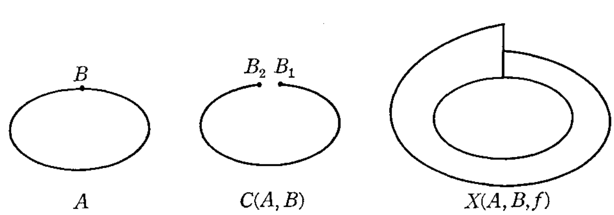


FIGURE 1

$C(A, B) \times [0, \delta_1]$ by the equivalence relation $\sim$ defined by $(x_1, t) \sim (x_2, f(t))$ for $t \in [0, \delta_1]$ and $x_1 \in B_1, x_2 \in B_2$ such that $x_1 = x_2$ as elements of B. Then we see that $X(A, B, f)$ is a compact manifold with corner on the boundary. See Figure 1. We denote by $\mathscr{F}(A, B, f)$ the foliation on $X(A, B, f)$ induced by that on $C(A, B) \times [0, \delta_1]$ with leaves $C(A, B) \times \{t\}, t \in [0, \delta_1]$.

DEFINITION 8. A codimension-zero compact submanifold S with corner on the boundary of M is called a *staircase* if there are a codimension-zero compact submanifold F of a leaf of $\mathscr{F}$, a codimension-one transversely oriented compact submanifold N of F with $\partial N = N \cap \partial F$ and with $F - N$ connected, a diffeomorphism $f: [0, \delta_1] \to [0, \delta_2]$ with $\delta_1 > \delta_2$, and an embedding $h: X(F, N, f) \to M$ satisfying the following conditions (1)-(4).

(1) $h(X(F, N, f)) = S$.

(2) $h(\{x\} \times [0, \delta_1]) \subset \varphi(\{x\} \times \boldsymbol{R})$ for all $x \in F$.

(3) $h(x, 0) = x$ for all $x \in F$.

(4) $h(C(F, N) \times \{\delta_1, f(\delta_1), f^2(\delta_1), \cdots\})$ is contained in a leaf.

We call F, $C = h(C(F, N) \times \{\delta_1\})$, $W = h(N_2 \times [\delta_2, \delta_1])$ and $D = h(\partial F \times [0, \delta_1])$ the *floor*, the *ceiling*, the *wall* and the *door* of the staircase S respectively, where N_2 is the copy of N with suffix 2. Note that $\partial S = F \cup C \cup W \cup D$. See Figure 2. The projection of S is the map $p: S \to F$ defined by $p(h(x, t)) = x$ for $x \in F, t \in [0, \delta_1]$. We say that S is *on the*

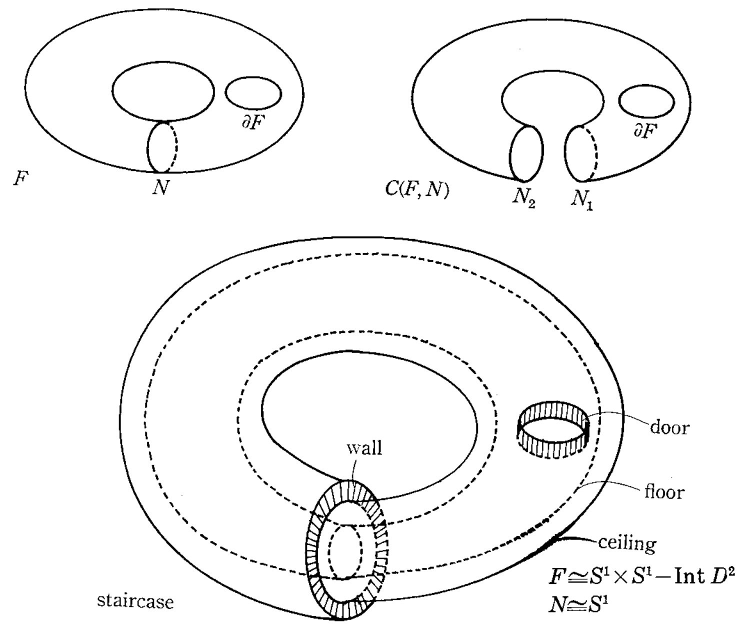


FIGURE 2

$d(F^*)$-th floor where F^* is the leaf of $\mathscr{F}$ containing F. If $h^*\mathscr{F} = \mathscr{F}(F, N, f)$, we call S *regular*.

The proof of the following two lemmas can be found in Nishimori [4].

LEMMA 2.1. *A leaf F is compact if and only if $d(F) = 1$.*

LEMMA 2.2. *Let Ω be a path-connected saturated subset of M containing no compact leaf. Then $\mathrm{Cl}_M(\Omega)$ contains at most a finite number of compact leaves.*

Now we recall the theory of ends. Let F be a non-compact manifold. We denote by $\hat{F}$ the topological space whose underlying set is

$$F \cup \{\varepsilon \mid \varepsilon \text{ is an end of } F\}$$

and whose topology is as follows: For $x \in F$, take the fundamental system of neighborhoods of x in F as that in $\hat{F}$. For $\varepsilon \in \hat{F} - F$, take $\{U \cup \{\varepsilon' \mid U \in \varepsilon' \in F\} \mid U \in \varepsilon\}$ as the fundamental system of neighborhoods of ε in $\hat{F}$. Here is the fundamental property of $\hat{F}$ with a proof.

PROPOSITION 2. (1) *$\hat{F}$ is a compact Hausdorff space and $\hat{F}$ satisfies the second countability axiom.* (2) *For any infinite sequence $\{\varepsilon_i\}_{i=1}^{\infty} \subset \hat{F} - F$, there is a subsequence $\{\varepsilon_{i_\nu}\}_{\nu=1}^{\infty}$ which converges to an end ε.*

PROOF. We use the notation of Remark 3. For $x, y \in F$ with $x \neq y$, there are open subsets U and V of F such that $x \in U$, $y \in V$, and $U \cap V = \varnothing$ since F is Hausdorff. For $x \in F$ and $\varepsilon \in \hat{F} - F$, choose $i > f(x)$ and let $U = f^{-1}((0, i))$ and let V be a connected component of $f^{-1}((i, \infty))$ such that $V \in \varepsilon$. Then U and $\tilde{V} = V \cup \{\varepsilon' \mid V \in \varepsilon'\}$ are open in F, and furthermore $x \in U$, $\varepsilon \in \tilde{V}$, and $U \cap \tilde{V} = \varnothing$. For $\varepsilon_1, \varepsilon_2 \in \hat{F} - F$ with $\varepsilon_1 \neq \varepsilon_2$, consider the cofinal sequence $U_1^i \supset U_2^i \supset \cdots$ of ε_i, $i = 1, 2$. Since $\varepsilon_1 \neq \varepsilon_2$, there is a positive integer j such that $U_j^1 \cap U_j^2 = \varnothing$. Then $\varepsilon_i \in \tilde{U}_j^i = U_j^i \cup \{\varepsilon' \mid U_j^i \in \varepsilon'\}$ and $\tilde{U}_j^1 \cap \tilde{U}_j^2 = \varnothing$. Thus $\hat{F}$ is Hausdorff.

Now we show that $\hat{F}$ satisfies the second countability axiom. Let $\mathscr{U} = \{U \cup \{\varepsilon \mid U \in \varepsilon\} \mid U$ is a connected component of $f^{-1}(i, \infty)$ for a positive integer $i\}$. Then $\mathscr{U}$ consists of a countable number of open sets of $\hat{F}$. Since F satisfies the second countability axiom, there is a countable base $\mathscr{V}$ of open sets of F. Since $\mathscr{U} \cup \mathscr{V}$ is a countable base of open sets of $\hat{F}$, it follows that $\hat{F}$ satisfies the second countability axiom.

Finally we show the compactness of $\hat{F}$. Let $\mathscr{U}$ be an open covering of $\hat{F}$. Since $\hat{F}$ satisfies the second countability axiom, there is a countable open covering $\mathscr{U}' = \{U_1, U_2, \cdots\}$. Let $V_i = \bigcup_{j=1}^{i} U_j$. Now suppose that $V_i \neq F$ for all i, from which we will bring out a contradiction. There is a sequence $\{x_i\}_{i=1}^{\infty} \subset \hat{F}$ such that $x_i \in \hat{F} - V_i$. We consider the following two cases.

(I) The case where $\{x_i\}_{i=1}^{\infty}$ contains an infinite subsequence $\{x_{i_\nu}\}_{\nu=1}^{\infty} \subset F$. Then the subset $\{x_{i_\nu} | \nu > 0\}$ has no accumulation point in F. Since $f^{-1}((t, \infty))$ consists of a finite number of connected components for all t, there is a connected component W_1 of $f^{-1}((1, \infty))$ containing an infinite number of x_{i_ν}'s. For the same reason, there is a connected component W_2 of $f^{-1}((2, \infty)) \cap W_1$ which contains an infinite number of x_{i_ν}'s. Clearly we can repeat this process infinitely often and get $W_1 \supset W_2 \supset \cdots$ of open sets. As in Remark 3, there is an end ε containing $\{W_j\}_{j=1}^{\infty}$. Since $\mathscr{U}'$ is an open covering of $\hat{F}$, there is $V_{i_0} \ni \varepsilon$. Since $\{W_j \cup \{\varepsilon' | W_j \in \varepsilon'\}\}_{j=1}^{\infty}$ is a fundamental system of neighborhoods of ε, there is j_0 such that $W_{j_0} \cup \{\varepsilon' | W_{j_0} \in \varepsilon'\} \subset V_{i_0}$. Since W_{j_0} contains an infinite number of x_{i_ν}'s, there is $i_* > i_0$ such that $x_{i_*} \in V_{i_0}$, a contradiction.

(II) The case where $\{x_i\}_{i=1}^{\infty} \cap F$ is a finite set. There is k such that if $i \geqq k$ then $x_i \in \hat{F} - F$. Let us write $\{\varepsilon_i\}_{i=k}^{\infty}$ instead of $\{x_i\}_{i=k}^{\infty}$. There is a connected component W_1 of $f^{-1}((1, \infty))$ belonging to an infinite number of ε_i's There is a connected component W_2 of $W_1 \cap f^{-1}((2, \infty))$ belonging to an infinite number of ε_i's. Thus we have a sequence $W_1 \supset W_2 \supset \cdots$. As before, there is an end ε containing $\{W_i\}_{i=1}^{\infty}$ and there are i_0, j_0 such that $W_{j_0} \cup \{\varepsilon' | W_{j_0} \in \varepsilon'\} \subset V_{i_0}$. Since W_{j_0} belongs to an infinite number of ε_i's, there is $i_1 > i_0$ such that $W_{j_0} \in \varepsilon_{i_1}$. Hence $\varepsilon_{i_1} \in V_{i_0}$, a contradiction. Note that this argument also shows (2).

By (I) and (II), there is an integer p such that $V_p = \hat{F}$. Therefore $\hat{F} = U_1 \cup \cdots \cup U_2$ and we obtain a finite subcovering of $\mathscr{U}$. Thus $\hat{F}$ is compact.

COROLLARY 3. *If F has an infinite number of ends of depth d, say $\{\varepsilon_i\}_{i=1}^{\infty}$, then F has an end ε of depth $> d$ and there is a subsequence $\{\varepsilon_{i_\nu}\}_{\nu=1}^{\infty}$ such that for all $U \in \varepsilon$ there is a positive integer N such that $U \in \varepsilon_i$ for all $\nu \geqq N$.*

PROOF. This follows from (2) of Proposition 2.

3. The proof of Theorem 1 and Corollary 1. First we prove Corollary 1 by assuming Theorem 1. Let ε be an isolated end of a leaf F. Then $d(\varepsilon) = 1$ and there is $U \in \varepsilon$ such that if an end ε' contains U then $\varepsilon' = \varepsilon$. Suppose that ε is a tame end and that $F_1 \subset L_\varepsilon(F)$. By Theorem 1, $F_1 = L_\varepsilon(F)$ and F_1 is compact. By Lemma 2.1, $d(F_1) = 1$. Therefore $d(F_1) \leqq d(\varepsilon)$ and ε is pseudotame. Conversely suppose that ε is pseudotame. If $F_1 \subset L_\varepsilon(F)$ then $d(F_1) \leqq d(\varepsilon) = 1$. Therefore $L_\varepsilon(F)$ consists of compact leaves. Since $L_\varepsilon(F)$ is connected by Proposition 1, it is easy to see that $L_\varepsilon(F)$ consists of just one leaf. Therefore ε is tame, which completes the proof of Corollary 1.

Now we prove Theorem 1. In Nishimori [2], we have already shown the following.

THEOREM 5. *Let* ε *be an isolated end of* F. *Then* $L_\varepsilon(F)$ *is a compact leaf if and only if* $L_\varepsilon(F) \cap F = \emptyset$ *and* ε *approaches* $L_\varepsilon(F)$ *from one side. In this case there is a staircase whose floor is the compact leaf* $L_\varepsilon(F)$ *and whose ceiling is contained in* $U \in \varepsilon$ *such that if an end* ε' *contains* U *then* $\varepsilon' = \varepsilon$.

REMARK 7. By Nakatsuka [1], we can take the staircase in Theorem 5 so that the wall is connected if $\dim F \geqq 3$. In this paper we suppose that the wall of any staircase is connected for simplicity. When $\dim F = 2$, we need a slight modification.

Therefore it is sufficient to show that if $L_\varepsilon(F)$ is a compact leaf then ε is isolated.

Suppose that $L_\varepsilon(F)$ is a compact leaf F_1. Clearly $F \neq F_1$. There are a closed tubular neighborhood T and a projection $p\colon T \to F_1$ such that $(M - T) \cap F \neq \emptyset$ and $p^{-1}(\{x\}) \subset \varphi(\{x\} \times \boldsymbol{R})$ for all $x \in F_1$.

FIRST STEP. We will prove that there is $U \in \varepsilon$ such that $L(U) = F_1$.

LEMMA 3.1. *Let* U *be a non-empty connected open subset of the leaf* F *such that* $\partial U = \mathrm{Cl}_F(U) - U$ *is compact, then* $L(U) = \bigcup \{L_{\varepsilon'}(F) \mid \varepsilon'$ *is an end containing* $U\}$.

PROOF. It is clear that $L(U) \supset \bigcup \{L_{\varepsilon'}(F) \mid \varepsilon'$ is an end containing $U\}$. Let $y \in L(U)$. Then there is a sequence $\{x_i\}_{i=1}^{\infty} \subset U$ without accumulation point with respect to the topology of F, and with $\lim x_i = y$ in M. We use the notations in Remark 3. Since ∂U is compact, $\partial U \subset f^{-1}((0, N))$ for an integer N. Clearly $f^{-1}((N, \infty))$ consists of a finite number of connected components. There is a connected component U_N of $f^{-1}((N, \infty))$ containing an infinite number of x_i's. We can find a connected component U_{N+1} of $U_N \cap f^{-1}((N+1, \infty))$ containing an infinite number of x_i's. Thus we obtain a sequence $U_N \supset U_{N+1} \supset \cdots$ such that U_j is a connected component of $f^{-1}((j, \infty))$ containing an infinite number of x_i's. Let ε' be the end corresponding to the sequence $U_N \supset U_{N+1} \supset \cdots$. Then $y \in L_{\varepsilon'}(F)$. Therefore $L(U) \subset \bigcup \{L_{\varepsilon'}(F) \mid \varepsilon'$ is an end containing $U\}$.

LEMMA 3.2. *There is* $U \in \varepsilon$ *such that* $U \subset \mathrm{Int}\, T$.

PROOF. Suppose that $U \cap (M - \mathrm{Int}\, T) \neq \emptyset$ for all $U \in \varepsilon$. In the notation of Remark 3, we can take $x_i \in U_i \cap (M - \mathrm{Int}\, T)$ where the sequence $U_1 \supset U_2 \supset \cdots$ corresponds to ε. Then $\bigcap_{i=1}^{\infty} \mathrm{Cl}_M(\{x_i, x_{i+1}, \cdots\})$ is non-empty and $\bigcap_{i=1}^{\infty} \mathrm{Cl}_M(\{x_i, x_{i+1}, \cdots\}) \subset (M - \mathrm{Int}\, T) \cap (\bigcap_{i=1}^{\infty} \mathrm{Cl}_M(U_i)) = (M - \mathrm{Int}\, T) \cap L_\varepsilon(F)$, a contradiction since $F_1 \cap (M - \mathrm{Int}\, T) = \emptyset$.

Since $U \in \varepsilon$ is connected, it follows that U in Lemma 3.2 is contained in one of the two connected components of $T - F_1$.

LEMMA 3.3. *There is $U \in \varepsilon$ such that if an end ε' contains U then $L_{\varepsilon'}(F) \cap F = \emptyset$.*

PROOF. Suppose that for all $U \in \varepsilon$ there is an end $\varepsilon' \ni U$ with $L_{\varepsilon'}(F) \supset F$. Then $\mathrm{Cl}_M(U) = \mathrm{Cl}_F(U) \cup L(U) \supset L_{\varepsilon'}(F) \supset F$. Therefore $L_\varepsilon(F) = \bigcap_{U \in \varepsilon} \mathrm{Cl}_M(U) \supset F$, a contradiction.

Let $U_2 \in \varepsilon$ satisfy Lemma 3.2 and $U_3 \in \varepsilon$ satisfy Lemma 3.3. By the definition of ends, there is $U \in \varepsilon$ such that $U \subset U_2 \cap U_3$. Then U satisfies Lemma 3.2 and Lemma 3.3. From now on we consider such U. Since $L(U) \cap F = (\bigcup_{\varepsilon' \ni U} L_{\varepsilon'}(F)) \cap F = \bigcup_{\varepsilon' \ni U} (L_{\varepsilon'}(F) \cap F) = \emptyset$, it follows that $L(U) = \mathrm{Cl}_M(U) - \mathrm{Cl}_F(U)$. Therefore $\mathrm{Cl}_M(U) - \mathrm{Cl}_F(U)$ is compact and saturated. Since $p^{-1}(x) \cap (\mathrm{Cl}_M(U) - \mathrm{Cl}_F(U))$ is compact, we can consider the farthest point $a(x) \in p^{-1}(x) \cap (\mathrm{Cl}_M(U) - \mathrm{Cl}_F(U))$ from $x \in F_1$. By using the fact that $\mathrm{Cl}_M(U) - \mathrm{Cl}_F(U)$ is saturated and that $\mathrm{Cl}_M(U) - \mathrm{Cl}_F(U) \subset T$, we see that the set $\{a(x) \mid x \in F_1\}$ is a leaf of $\mathscr{F}$. Let $F_2 = \{a(x) \mid x \in F_1\}$.

LEMMA 3.4. $F_2 = F_1$.

PROOF. Suppose that $F_2 \neq F_1$. Since $U \subset T$ and $\mathrm{Cl}_M(U) \supset F_1 \cup F_2$, U is contained in the connected component C of $T - F_1 - F_2$ which does not intersect ∂T. Since the component C is saturated, it follows that $F \subset C$, which contradicts the assumption $F \cap (M - T) \neq \emptyset$.

By Lemma 3.4, $L(U) = F_1$ which finished the first step.

SECOND STEP. The following arguments are analogous to those in Nishimori [2]. Let $x_0 \in F_1$. Consider $U \in \varepsilon$ in the first step. Since $L(U) = F_1$, the set $U \cap p^{-1}(x_0)$ has just one accumulation point x_0. We can number the points of $U \cap p^{-1}(x_0)$ as y_i in such a way that y_i is farther from x_0 than y_j if $i < j$. Let $\varphi(\{x_0\} \times [t_*, t^*]) = p^{-1}(x_0)$ and $\varphi(x_0, t_i) = y_i$, $t_* < t_i < t^*$.

LEMMA 3.5. *Let ω: $([0, 1], \{0, 1\}) \to (F_1, x_0)$ be a closed path. Let $\mathscr{F}_\omega$ be the foliation on $[0, 1] \times \boldsymbol{R}$ induced from $\mathscr{F}$ by the composed map $[0, 1] \times \boldsymbol{R} \xrightarrow{\omega \times \mathrm{id}} M \times \boldsymbol{R} \xrightarrow{\varphi} M$. There are an integer p and a positive integer ν such that the leaf of $\mathscr{F}_\omega$ containing the point $(0, t_i)$ contains $(1, t_{i+p})$ for $i > \nu$. The integer p depends only on the homotopy class $[\omega]$ of ω and the map u: $\pi_1(F_1, x_0) \to Z$ defined by $u([\omega]) = p$ is a non-trivial homomorphism. There is a positive integer a such that $u(\pi_1(F_1, x_0)) = \{ia \mid i \in Z\}$.*

We omit the proof of Lemma 3.5.

Since Z is abelian, there is a homomorphism $v: H_1(F_1, Z) \to Z$ such that $u = v \circ H$, where $H: \pi_1(F_1, x_0) \to H_1(F_1, Z)$ is the Hurewicz homomorphism. Let α: Hom $(H_1(F_1, Z), Z) \to H^1(F_1, Z)$ be the canonical isomorphism and we consider the homology class $d \circ \alpha(v) \in H_{n-2}(F_1^{n-1}, Z)$ where $d: H^1(F_1^{n-1}, Z) \to H_{n-2}(F_1^{n-1}, Z)$ is the Poincaré duality. By Thom's representation theorem, there is a closed submanifold N^{n-2} of F_1^{n-1} containing x_0 such that the fundamental class $[N]$ of N coincides with $d \circ \alpha(v)$. Then we see the following as in [2].

LEMMA 3.6. *In the above situation, there are a diffeomorphism* $f: [0, \delta_1] \to [0, \delta_2]$ *with* $\delta_1 > \delta_2$ *and an embedding* $h: X(F_1, N, f) \to T$ *satisfying the following conditions.*

(1) $S = h(X(F_1, N, f))$ *is a staircase whose floor is* F_1 *and whose ceiling is contained in* U.

(2) $h((x_0)_2, \delta_1) = y_{i_0}$ *and* $h((x_0)_1, \delta_1) = y_{i_0} + a$ *for some* i_0, *where* $(x_0)_j$ *belongs to the copy* N_j *of* N *in the notations of Preliminary.*

(3) *The subsets* $V_j = h(C(F_1, N) \times \{\gamma_j, f(\gamma_j), f^2(\gamma_j), \cdots\})$, $j = 1, \cdots, a$, *are the connected components of* $U \cap S$ *where* $\gamma_j \in (\delta_2, \delta_1]$ *is defined by* $h((x_0)_2, \gamma_j) = y_{i_0+j-1}$.

We omit the proof of Lemma 3.6.

Since $L_\varepsilon(F) = F_1$, there is $V \in \varepsilon$ such that $V \subset U \cap S$. Therefore $V \subset V_{j_0}$ for some $j_0 \in \{1, \cdots, a\}$. Since V_{j_0} belongs to just one end and $V_{j_0} \in \varepsilon$, it follows that ε is isolated. This completes the proof of Theorem 1.

4. The proof of Theorem 2 and Corollary 2. First we prove Corollary 2 by assuming Theorem 2. Let ε be a tame end of a leaf F. By definition, there is $U \in \varepsilon$ such that if an end ε' contains U then ε' is a tame end. If a leaf F' is contained in $L_{\varepsilon'}(F)$ for an end ε' containing U, then $d(F') \leqq d(\varepsilon)$ by (1) of Theorem 2. Therefore ε is pseudotame, which completes the proof of Corollary 2.

Now we prove Theorem 2. Let ε_0 be a tame end of a leaf F_0 of $\mathscr{F}$. We prove Theorem 2 for ε_0 and F_0. When $d_0 = d(\varepsilon_0) = 1$, Theorem 2 follows from Theorem 1. We consider the case $d_0 \geqq 2$. By the definition of tame ends, there are $U_0 \in \varepsilon_0$ and $a_0 > 0$ such that if an end $\varepsilon \neq \varepsilon_0$ contains U_0 then ε is a tame end of depth $< d_0$ and $a(\varepsilon) > a_0$. Since $L(U_0) = \bigcup \{L_\varepsilon(F_0) | U_0 \in \varepsilon\}$ by Lemma 3.1 and $L_\varepsilon(F_0) \cap F_0 = \varnothing$ for any tame end ε of F_0, we have $L(U_0) \cap F_0 = \varnothing$. Let $a_1 = d(\partial U_0, L(U_0))$. Since $L(U_0) \cap \partial U_0 = \varnothing$, we have $a_1 > 0$. Now we prove the following propositions by induction on d. The propositions for $d = d_0$ clarify the behavior of the end ε_0.

PROPOSITION 4.1 $[d](d \leqq d_0)$. *In the above situation, there are a number* $a(d)$ *with* $0 < a(d) < \mathrm{Min}\{a_0, a_1\}$ *and a finite set* $\mathscr{S}(d)$ *of staircases of the foliated manifold* $(M, \mathscr{F})$ *satisfying the following conditions.*

(1.1) *The interiors* $\mathrm{Int}_M S$ *for* $S \in \mathscr{S}(d)$ *are disjoint.*

(1.2) *Each staircase* $S \in \mathscr{S}(d)$ *is contained in the* $a(d)$*-neighborhood of the floor* $F(S)$ *of* S.

(1.3) *The leaf* $F^*(S)$ *of* $\mathscr{F}$ *containing the floor* $F(S)$ *of a staircase* $S \in \mathscr{S}(d)$ *has depth* $\leqq d$.

(1.4) *The ceiling* $C(S)$ *of a staircase* $S \in \mathscr{S}(d)$ *is contained in* U_0 *and the leaf* $\bar{C}(S)$ *of the restricted foliation* $\mathscr{F} | \bigcup \{S' \in \mathscr{S}(d) | S' \cap F^*(S) \neq \emptyset\}$ *containing* $C(S)$ *belongs to only one end, of leaf* F_0, *with depth* $d(F^*(S))$.

(1.5) *Each connected component of the door* $D(S)$ *of a staircase* $S \in \mathscr{S}(d)$ *is contained in the wall* $W(S')$ *of a staircase* $S' \in \mathscr{S}(d)$ *with* $d(F^*(S')) < d(F^*(S))$. *For each* $S \in \mathscr{S}(d)$ *the intersection* $W(S) \cap (\bigcup \{S' | S \neq S' \in \mathscr{S}(d)\})$ *is contained in* $\bigcup \{D(S') | S' \in \mathscr{S}(d)\}$.

(1.6) *For any end* ε *of depth* $\leqq d$ *containing* U_0, *the limit set* $L_\varepsilon(F_0)$ *is contained in* $\bigcup \{F^*(S) | S \in \mathscr{S}(d)\}$. *Furthermore, there is* $U \in \varepsilon$ *contained in* $\mathrm{Int}_M(\bigcup \{S | F^*(S) \subset L_\varepsilon(F_0)\})$.

PROPOSITION 4.2 $[d](d \leqq d_0)$. *Let* $\mathscr{S}(d)$ *be a finite set of staircases of the foliated manifold* $(M, \mathscr{F})$ *satisfying the conditions* (1.1)-(1.6) *of Proposition* 4.1$[d]$ *for a number* $a(d)$ *with* $0 < a(d) < \mathrm{Min}\{a_0, a_1\}$.

Then for any end ε *of depth* $< d$ *containing* U_0, *there is an element* $U_\varepsilon \in \varepsilon$ *satisfying the following condition* (2.1).

(2.1) $U_\varepsilon \subset U_0 \cap \mathrm{Int}_M(\bigcup \{S | F^*(S) \subset L_\varepsilon(F_0)\})$ *and* ∂U_ε *is a compact leaf of the restricted foliation* $\mathscr{F} | W(S_\varepsilon)$, *where* $W(S_\varepsilon)$ *is the wall of a staircase* $S_\varepsilon \in \mathscr{S}(d)$.

Now let $U_\varepsilon \in \varepsilon$ *satisfy the condition* (2.1). *Then*

(2.2) $d(\varepsilon) = d(F^*(S_\varepsilon))$, $L_\varepsilon(F_0) = \mathrm{Cl}_M(F^*(S_\varepsilon))$ *and* $L_\varepsilon^*(F_0) = F^*(S_\varepsilon)$.

(2.3) *If an end* $\varepsilon' \neq \varepsilon$ *contains* U_ε, *then* $d(\varepsilon') < d(\varepsilon)$.

(2.4) *For any staircase* $S \in \mathscr{S}(d)$ *intersecting* U_ε, *the intersection* $U_\varepsilon \cap W(S)$ *consists of compact leaves of the restricted foliation* $\mathscr{F} | W(S)$.

In the proof of these propositions, the following term "thinning" will be useful.

DEFINITION 9. Let $\mathscr{S}(d)$ be a finite set of staircases which satisfies the conditions (1.1)-(1.6). Let $0 < a < a(d)$. A finite set $\tilde{\mathscr{S}}(d)$ of staircases is called an *a-thinning* of $\mathscr{S}(d)$ if $\tilde{\mathscr{S}}(d)$ satisfies the corresponding conditions (1.1)-(1.6) with $a(d)$ replaced by a, and moreover the following condition $(*)$.

(*) For each $S \in \mathscr{S}(d)$, there is $\tilde{S} \in \tilde{\mathscr{S}}(d)$ such that the floor $F(\tilde{S})$ of $\tilde{S}$ contains the floor $F(S)$ of S and $\tilde{S}$ is contained in $S \cup (\bigcup \{S' \in \mathscr{S}(d) \mid F^*(S') \subset \mathrm{Cl}_M(F^*(S))\})$.

When we are not interested in the number a, we call $\tilde{\mathscr{S}}(d)$ a thinning of $\mathscr{S}(d)$.

We omit the proof of the following easy lemma.

LEMMA 4.0. *Let $\mathscr{S}(d)$ be a finite set of staircases. For any $a > 0$, there is an a-thinning of $\mathscr{S}(d)$.*

PROOF OF PROPOSITION 4.1[$d+1$] ($d + 1 \leqq d_0$). We suppose Proposition 4.1[d] and Proposition 4.2[d] as induction assumption. Therefore we have a finite set $\mathscr{S}(d)$ of staircases of $(M, \mathscr{F})$ satisfying the conditions (1.1)-(1.6) of Proposition 4.1[d].

LEMMA 4.1. *Let $M_d = M - \mathrm{Int}_M(\bigcup \{S \mid S \in \mathscr{S}(d)\})$. Then the intersection $\mathrm{Cl}_{F_0}(U_0) \cap M_d$ is not compact.*

PROOF. Let ε_1 be an end of F_0 such that $U_0 \in \varepsilon_1$ and $d(\varepsilon_1) = d + 1$. By the choice of $a(d)$, there is $U_1 \in \varepsilon_1$ such that if an end $\varepsilon \neq \varepsilon_1$ contains U_1 then $d(\varepsilon) < d(\varepsilon_1)$ and $d(\partial U_1, L_{\varepsilon_1}(F_0)) > a(d)$. By the definition of ends, there is $U' \in \varepsilon_1$ contained in $U_0 \cap U_1$. Let U_1' be the connected component of $F_0 - \partial U_0 - \partial U_1$ containing U'. Then $\partial U_1' \subset \partial U_0 \cup \partial U_1$ and $\partial U_1'$ is compact. Therefore $d(\partial U_1', L_{\varepsilon_1}(F_0)) > \mathrm{Min}\,\{d(\partial U_0, L(U_0)), d(\partial U_1, L_{\varepsilon_1}(F_0))\} > a(d)$. Furthermore, $U_1' \subset U_0 \cap U_1$. Clearly $U_1' \in \varepsilon_1$. Since $U_1' \subset U_0 \cap U_1$, it follows that, for any end $\varepsilon \neq \varepsilon_1$ of F_0 containing U_1', ε is a tame end of depth $<$ $d(\varepsilon_1) = d + 1$ containing U_0 and so there is $U_\varepsilon \in \varepsilon$ satisfying the condition (2.1) of Proposition 4.2[d]. Since there are an infinite number of ends ε of depth d containing U_1', there is a staircase $S \in \mathscr{S}(d)$ such that $d(F^*(S)) = d$ and the wall $W(S)$ of S contains an infinite number of ∂U_ε's. By the condition (2.1) of Proposition 4.2[d], the intersection $\mathrm{Cl}_{F_0}(U_0) \cap W(S)$ contains an infinite number of leaves of the restricted foliation $\mathscr{F} \mid W(S)$. By the condition (1.5) of Proposition 4.1[d], the wall $W(S)$ is contained in M_d because $d(F^*(S)) = d$.

Now suppose that $\mathrm{Cl}_{F_0}(U_0) \cap M_d$ is compact. Then $\mathrm{Cl}_{F_0}(U_0) \cap W(S)$ is compact. Choose a point $x_\varepsilon \in \partial U_\varepsilon \subset \mathrm{Cl}_{F_0}(U_0) \cap M_d$ for each ε. Then we can choose a converging infinite sequence $\{y_i\}$ from x_ε's. The limit $y = \lim y_i$ belongs to $L(U_0) \cap \mathrm{Cl}_{F_0}(U_0)$. Since $L(U_0) = \bigcup \{L_\varepsilon(U_0) \mid U_0 \in \varepsilon\}$ and since an end ε containing U_0 is tame, it follows that $L(U_0) \cap F_0 = \bigcup \{L_\varepsilon(U_0) \cap F_0 \mid U_0 \in \varepsilon\} = \varnothing$ because of the condition (1b) or (2b) of the definition of tame ends. Since $\mathrm{Cl}_{F_0}(U_0) \subset F_0$, this is a contradiction, which completes the proof of Lemma 4.1.

Now let Ω^d be the connected component of

$$M_d - \bigcup\{F \mid F \text{ is a compact leaf of } \mathscr{F} \mid M_d\}$$

containing $F_0 \cap M_d$. Let $\underline{\Omega}^d$ be the compact manifold obtained from Ω^d by attaching the boundary. This is rational by Lemma 2.2. Note that there is the canonical immersion $\iota\colon \underline{\Omega}^d \to M_d$. The connected components $G_1, \cdots, G_k$ of the attached part of the boundary $\partial\underline{\Omega}^d$ are the compact leaves of the induced foliation $\mathscr{F} \mid \underline{\Omega}^d = \iota^*\mathscr{F}$. Let ε_1 be an end of F_0 such that $U_0 \in \varepsilon_1$ and $d(\varepsilon_1) = d+1$. We take $U_1' \in \varepsilon_1$ as in the proof of Lemma 4.1. Let $\tilde{\mathscr{S}}(d)$ be a thinning of $\mathscr{S}(d)$ such that $\partial U_1' \cap (\bigcup\{S \mid S \in \tilde{\mathscr{S}}(d)\}) = \emptyset$. By taking $\tilde{\mathscr{S}}(d)$ instead of $\mathscr{S}(d)$ in the above construction, we have the compact manifold M_d, the canonical immersion $\tilde{\iota}\colon M_d \to M$ and the connected component $\tilde{\Omega}_d$.

LEMMA 4.2. *For any staircase $S \in \tilde{\mathscr{S}}(d)$, the intersection $U_1' \cap W(S) \cap \tilde{\Omega}^d$ is contained in the union $\bigcup\{\partial U_\varepsilon \mid U_1' \in \varepsilon \neq \varepsilon_1\}$ where $U_\varepsilon \in \varepsilon$ is the one in Proposition* 4.2[d]. (*See Figure* 3.)

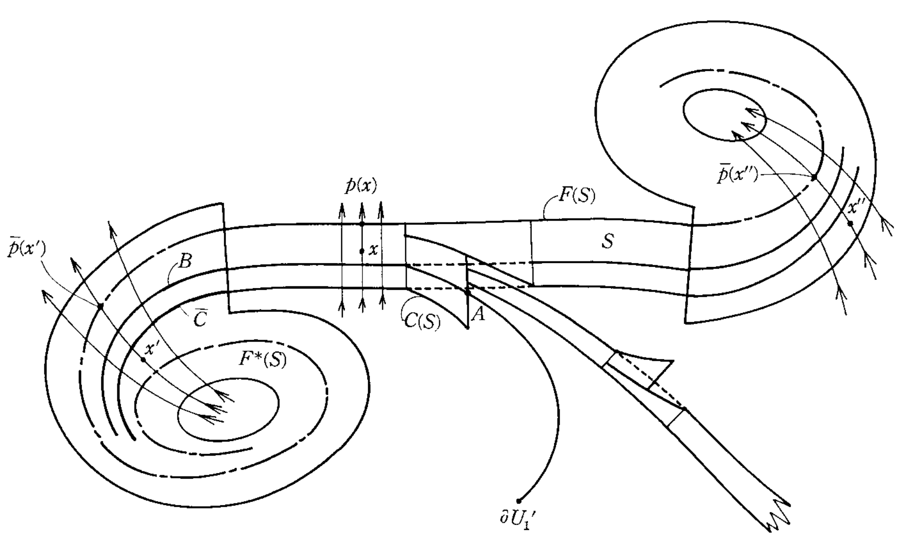


FIGURE 3

PROOF. Note that $L(U_1') = L_{\varepsilon_1}(F_0) \cup (\bigcup\{L_\varepsilon(F_0) \mid U_1' \in \varepsilon \neq \varepsilon_1\})$ by Lemma 3.1. Since $\bigcup\{L_\varepsilon(F_0) \mid U_1' \in \varepsilon \neq \varepsilon_1\} \subset \bigcup\{F^*(S') \mid S' \in \tilde{\mathscr{S}}(d)\}$ by the condition (1.6) and since clearly $W(S) \cap \tilde{\Omega}^d \cap (\bigcup\{F^*(S') \mid S' \in \tilde{\mathscr{S}}(d)\}) = \emptyset$, it follows that $L(U_1') \cap W(S) \cap \tilde{\Omega}^d \subset L^*_{\varepsilon_1}(F_0)$. Note further that $U_1' \cap W(S) \cap \tilde{\Omega}^d$ consists of leaves of $\mathscr{F} \mid W(S)$ since $\partial U_1' \cap S = \emptyset$.

(i) $U_1' \cap W(S) \cap \tilde{\Omega}^d$ does not contain a non-compact leaf of $\mathscr{F} \mid W(S)$.

Indeed if $U_1' \cap W(S) \cap \tilde{\Omega}^d$ contains a non-compact leaf, we have two points $x, y \in L(U_1') \cap W(S) \cap \mathrm{Cl}_M(\tilde{\Omega}^d)$ such that ε_1 approaches x and y from different sides. This contradict the assumption that ε_1 approaches the principal part $L_{\varepsilon_1}^*(F_0)$ from one side.

(ii) $U_1' \cap W(S) \cap \tilde{\Omega}^d$ does not contain two leaves E_1, E_2 of $\mathscr{F} | W(S)$ which belong to the same leaf E of $\mathscr{F} | \bigcup \{S' | S' \in \tilde{\mathscr{S}}(d)$ and $S' \cap F^*(S) \neq \emptyset\}$. Otherwise, there is a path $\omega: [0, 1] \to E$ such that $\omega(0) \in E_1$, $\omega(1) \in E_2$ and $p(\omega(0)) = p(\omega(1))$ where $p: S \to F(S)$ is the projection of the staircase S to the floor $F(S)$.

Now we show that we can extend p uniquely to a C^r map $\bar{p}: S \cup X \to F^*(S)$ where X will be defined below. By Theorem 2[d], we see that the leaf $F^*(S)$ is proper and the limit set of $F^*(S)$ consists of the floors of the staircases intersecting $F^*(S)$. Let $x \in \bigcup \{S' \in \tilde{\mathscr{S}}(d) | S' \cap F^*(S) \neq \emptyset\} - \mathrm{Cl}_M(F^*(S))$. Then $x \in X$ if and only if we have the first intersection point $\bar{p}(x)$ with $F^*(S)$ when we trace the orbit of the transversal flow φ starting from x in the same direction as the points of S going to $F(S)$ along φ.

Let $\Phi: p^{-1}(\omega(0)) \to p^{-1}(\omega(0))$ be the holonomy map with respect to the closed path $p \circ \omega$ in $F^*(S)$. Then it follows that $\Phi(W(S) \cap \tilde{\Omega}^d) = W(S) \cap \tilde{\Omega}^d$. Therefore there exist the limits $\alpha = \lim_{n \to \infty} \Phi^n(\omega(0))$ and $\beta = \lim_{n \to -\infty} \Phi^n(\omega(0))$. Since $\alpha, \beta \in L(U_1') \cap W(S) \cap \mathrm{Cl}_M(\tilde{\Omega}^d)$ and ε_1 approaches α and β from different sides, this is a contradiction. Then (ii) follows.

Let A be a connected component of $U_1' \cap W(S) \cap \tilde{\Omega}^d$. Clearly A is a leaf of $\mathscr{F} | W(S)$. Let B be the leaf of $\mathscr{F} | \bigcup \{S' \in \tilde{\mathscr{S}}(d) | S' \cap F^*(S) \neq \emptyset\}$ containing A. By (i) and (ii) we see that $B \cap W(S) \cap \tilde{\Omega}^d = A$ and $\partial B = \mathrm{Cl}_{F_0}(B) - B = A$ is a compact leaf of $\mathscr{F} | W(S)$. Let $\bar{C}$ be the leaf of $\mathscr{F} | \bigcup \{S' \in \tilde{\mathscr{S}}(d) | S' \cap F^*(S) \neq \emptyset\}$ containing the ceiling $C(S)$ of S. Then there exists uniquely a C^r diffeomorphism $f: \bar{C} \to B$ such that $\bar{p} \circ f = p$ where $\bar{p}: S \cup X \to F^*(S)$, $X \subset \bigcup \{S' \in \tilde{\mathscr{S}}(d) | S' \cap F^*(S) \neq \emptyset\} - \mathrm{Cl}_M(F^*(S'))$, is the map in the proof of (ii). Therefore B and $\bar{C}$ behave in the same way and $L(B) = L(\bar{C})$. By (1.5) of Proposition 4.1[d], $\bar{C}$ belongs to only one end ε of depth $d(F^*(S))$. Then B also belongs to only one end ε' of depth $d(F^*(S))$. Clearly $U_1' \in \varepsilon' \neq \varepsilon_1$, $\mathrm{Int}_{F_0}(B) = U_{\varepsilon'}$ and $A = \partial U_{\varepsilon'}$. Thus Lemma 4.2 is proved.

Let $V_1 = U_1' - \bigcup \{U_\varepsilon | U_1' \in \varepsilon \neq \varepsilon_1\}$. Then V_1 belongs to an isolated end $\bar{\varepsilon}_1$ of the leaf $F_0 \cap \tilde{\Omega}^d$ of the foliation $\mathscr{F} | \tilde{\Omega}^d$.

LEMMA 4.3. *Consider the leaf* $F_0 \cap \tilde{\Omega}^d$ *as a leaf of the induced foliation* $\mathscr{F} | \tilde{\Omega}^d = \tilde{\iota}^* \mathscr{F}$. *Then* $L_{\bar{\varepsilon}_1}(F_0 \cap \tilde{\Omega}^d) = L_{\varepsilon_1}(F_0) \cap \underline{\tilde{\Omega}}^d = L_{\varepsilon_1}^*(F_0) \cap \underline{\tilde{\Omega}}^d$.

PROOF. Note that $L(V_1) = L_{\bar{\varepsilon}_1}(F_0 \cap \tilde{\Omega}^d)$ and $L(U_1') = L_{\varepsilon_1}(F_0) \cup (\bigcup \{L_\varepsilon(F_0) | U_1' \in$

$\varepsilon \neq \varepsilon_1\}$) by Lemma 3.1. Since $\bigcup\{L_\varepsilon(F_0)\,|\,U_1' \in \varepsilon \neq \varepsilon_1\} \subset \bigcup\{F^*(S')\,|\,S' \in \tilde{\mathscr{S}}(d)\}$ and $(\bigcup\{F^*(S')\,|\,S' \in \tilde{\mathscr{S}}(d)\}) \cap \underline{\tilde{\Omega}}^d = \varnothing$, it follows that $L(U_1') \cap \underline{\tilde{\Omega}}^d = L_{\varepsilon_1}(F_0) \cap \underline{\tilde{\Omega}}^d$. Clearly $L(V_1) = L(U_1') \cap \underline{\tilde{\Omega}}^d$. Therefore we see that $L_{\bar{\varepsilon}_1}(F_0 \cap \tilde{\Omega}^d) = L_{\varepsilon_1}(F_0) \cap \underline{\tilde{\Omega}}^d$.

Lemma 4.2 implies that, for a staircase $S' \in \tilde{\mathscr{S}}(d)$, $F^*(S') \subset L_{\varepsilon_1}(F_0)$ if and only if, for all $U \in \varepsilon_1$, there is an end $\varepsilon \ni U$ such that $L_\varepsilon(F_0) = \mathrm{Cl}_M(F^*(S'))$. On the other hand, by Theorem 2[d'] (where $d' \leq d$) it is clear that $F^*(S') \subset \bigcap_{U \in \varepsilon_1}(\bigcup_{U \in \varepsilon \neq \varepsilon_1} L_\varepsilon(F_0))$ if and only if, for all $U \in \varepsilon_1$, there is an end $\varepsilon \ni U$ such that $L_\varepsilon(F_0) = \mathrm{Cl}_M(F^*(S'))$. Thus the condition $F^*(S') \subset L_{\varepsilon_1}(F_0)$ is equivalent to $F^*(S') \subset \bigcap_{U \in \varepsilon_1}(\bigcup_{U \in \varepsilon \neq \varepsilon_1} L_\varepsilon(F_0))$. Note that $\bigcap_{U \in \varepsilon_1}(\bigcup_{U \in \varepsilon \neq \varepsilon_1} L_\varepsilon(F_0)) \subset \bigcup\{F^*(S')\,|\,S' \in \tilde{\mathscr{S}}(d)\}$ by Proposition 4.1[d](1.6). Therefore $L_{\varepsilon_1}(F_0) - (\bigcup\{F^*(S')\,|\,S' \in \tilde{\mathscr{S}}(d)\}) = L^*_{\varepsilon_1}(F_0)$. Then it follows that $L_{\varepsilon_1}(F_0) \cap \underline{\tilde{\Omega}}^d = L^*_{\varepsilon_1}(F_0) \cap \underline{\tilde{\Omega}}^d$, which completes the proof of Lemma 4.3.

Since ε_1 approaches $L^*_{\varepsilon_1}(F_0)$ from one side, Lemma 4.3 implies that the isolated end $\bar{\varepsilon}_1$ of $F_0 \cap \tilde{\Omega}^d$ approaches $L_{\varepsilon_1}(F_0 \cap \tilde{\Omega}^d)$ from one side. Clearly $L_{\varepsilon_1}(F_0 \cap \tilde{\Omega}^d) \cap (F_0 \cap \tilde{\Omega}^d) = \varnothing$. Although the manifolds $F_0 \cap \tilde{\Omega}^d$ and $\underline{\tilde{\Omega}}^d$ have boundaries, Theorem 5 is still valid. Then $L_{\bar{\varepsilon}_1}(F_0 \cap \tilde{\Omega}^d)$ is a compact leaf of $\mathscr{F}\,|\,\underline{\tilde{\Omega}}^d$ and $L_{\bar{\varepsilon}_1}(F_0 \cap \tilde{\Omega}^d)$ coincides with one of the connected components $\tilde{G}_1, \cdots, \tilde{G}_k$ of the boundary of $\underline{\tilde{\Omega}}^d$. Since $\tilde{G}_i \supset G_i$ for any thinning $\tilde{\mathscr{S}}(d)$, $L_{\bar{\varepsilon}_1}(F_0 \cap \tilde{\Omega}^d)$ contains one of $G_1, \cdots, G_k$.

Now consider G_i's which are contained in $L_{\bar{\varepsilon}_1}(F_0 \cap \tilde{\Omega}^d)$'s for some ends ε_1 of F_0 such that $U_0 \in \varepsilon_1$ and $d(\varepsilon_1) = d + 1$ and for some thinnings $\tilde{\mathscr{S}}(d)$ of $\mathscr{S}(d)$. We rename them $F_1', \cdots, F_l'$. For each F_i', take one end ε_{1i} and one thinning $\tilde{\mathscr{S}}(d)_i$ of $\mathscr{S}(d)$ such that $F_i' \subset L_{\bar{\varepsilon}_{1i}}(F_0 \cap \tilde{\Omega}^d{}_i)$. Let $\tilde{\tilde{\mathscr{S}}}(d)$ be a common thinning of $\tilde{\mathscr{S}}(d)_1, \cdots, \tilde{\mathscr{S}}(d)_l$. By Theorem 5, we have disjoint staircases $S_1', \cdots, S_l'$ of $\mathscr{F}\,|\,\underline{\tilde{\tilde{\Omega}}}^d$ whose leaves are compact leaves of $\mathscr{F}\,|\,\underline{\tilde{\tilde{\Omega}}}^d$ containing $F_1', \cdots, F_l'$, respectively, and whose ceilings are contained in $U_{11}', \cdots, U_{1l}'$ of Lemma 4.2, respectively. Let $a(d+1)$ be a number with $0 < a(d+1) < \mathrm{Min}\,\{a(d), d(\partial U_{11}', L(U_{11}')), \cdots, d(\partial U_{1l}', L(U_{1l}'))\}$. We may suppose that the staircase S_i' is contained in the $a(d+1)$-neighborhood of the floor $F(S_i')$ of S_i' and that $\tilde{\tilde{\mathscr{S}}}(d)$ is an $a(d+1)$-thinning. Furthermore by choosing S_i' small enough, we may suppose that, if $F(S_i') \cap W(S) \neq \varnothing$ for some $S \in \tilde{\tilde{\mathscr{S}}}(d)$, then the connected components of $D(S_i')$ intersecting $W(S)$ are contained in $W(S)$. Let $\tilde{\tilde{\iota}}(S_i') = S_i$ and let $\mathscr{S}(d+1) = \tilde{\tilde{\mathscr{S}}}(d) \cup \{S_1, \cdots, S_l\}$, where $\tilde{\tilde{\iota}}\colon \tilde{\tilde{\Omega}}^d \to M$ is the canonical immersion.

Now we check that $\mathscr{S}(d+1)$ and $a(d+1)$ satisfy the conditions (1.1)-(1.6) of Proposition 4.1[$d+1$].

The conditions (1.1) and (1.2) are clear.

The condition (1.3). It is sufficient to consider the case $S = S_i \in \mathscr{S}(d+1) - \mathscr{S}(d)$. By the way of constructing S_i, we have the end ε_{1i} and the element U'_{1i} of Lemma 4.2. By Lemma 4.2, it follows that $U'_{1i} \cap W(S') \cap \tilde{\tilde{\Omega}}^d \subset \bigcup \{\partial U_\varepsilon \mid U'_{1i} \in \varepsilon \neq \varepsilon_{1i}\}$ for any $S' \in \tilde{\tilde{\mathscr{S}}}(d)$.

LEMMA 4.4. *Let $F^*(S_i)$ be the leaf of $\mathscr{F}$ containing the floor $F(S_i)$ of S_i as usual. Then* $\mathrm{Cl}_M(F^*(S_i)) \subset F^*(S_i) \cup (\bigcup \{F^*(S') \mid S' \in \tilde{\tilde{\mathscr{S}}}(d)\})$.

PROOF. Note first that $F^*(S_i) \subset L^*_{\varepsilon_{1i}}(F_0)$ by Lemma 4.3. Therefore ε_{1i} approaches $F^*(S_i)$ from one side. We may assume that for all $x \in F^*(S_i)$ there are $\delta_x > 0$ and $U_x \in \varepsilon_{1i}$ such that $\varphi(\{x\} \times (-\delta_x, 0)) \cap U_x = \emptyset$. Let G be a connected component of the boundary $\partial F(S_i)$ of the floor $F(S_i)$. Then G is contained in the wall $W(S')$ for some $S' \in \tilde{\tilde{\mathscr{S}}}(d)$. Furthermore by Lemma 4.2, there are a number $\alpha > 0$ and a neighborhood V of G in $W(S')$ such that $V \cap \varphi(G \times (-\alpha, 0]) \cap U'_{1i} = \emptyset$ and $A = V \cap \varphi(G \times (0, \alpha)) \cap U'_{1i}$ consists of a countable number of ∂U_ε's which are discrete. Note that if $\partial U_\varepsilon \subset A$ then $d(\varepsilon) = d(F^*(S'))$ by (2.2) of Proposition 4.2[d]. Now consider the map $\bar{p}: S' \cup X \to F^*(S')$, $X \subset \bigcup \{S'' \in \tilde{\tilde{\mathscr{S}}}(d) \mid S'' \cap F^*(S') \neq \emptyset\} - \mathrm{Cl}_M(F^*(S'))$ as in the proof of Lemma 4.2. Let $\bar{G}$ be the leaf of $\mathscr{F} \mid \bigcup \{S'' \mid S'' \in \tilde{\tilde{\mathscr{S}}}(d)\}$ containing G. For each pair $\partial U_{\varepsilon'}, \partial U_{\varepsilon''} \subset A$ we have a unique C^r diffeomorphism $f_{\varepsilon', \varepsilon''}: U_{\varepsilon'} \to U_{\varepsilon''}$ such that $p \circ f_{\varepsilon', \varepsilon''} = p$ and $f_{\varepsilon'', \varepsilon'''} \circ f_{\varepsilon', \varepsilon''} = f_{\varepsilon', \varepsilon'''}$. Since $L(U'_{1i}) \cap V = G$, we can number the ∂U_ε's in A as $\partial U_{\varepsilon(1)}, \partial U_{\varepsilon(2)}, \cdots$ in such a way that $\partial U_{\varepsilon(j)}$ is nearer to G than $\partial U_{\varepsilon(i)}$ if $i < j$. Then for each $x \in \partial U_{\varepsilon(1)}$ the sequence $\{f_{\varepsilon(1), \varepsilon(i)}(x)\}_{i=1}^\infty$ converges to a point in G. Furthermore for any $x \in U_{\varepsilon(1)}$, the sequence $\{f_{\varepsilon(1), \varepsilon(i)}(x)\}_{i=1}^\infty$ converges to a point $g(x)$ in $\varphi(\{x\} \times \boldsymbol{R})$. Then the limit point $g(x)$ belongs to $\bar{G}$. Thus we have a map $g: U_{\varepsilon(1)} \to \bar{G}$. It is easy to see that g is a C^r diffeomorphism. Therefore $U_{\varepsilon(1)}$ and $\bar{G}$ behave in the same way. The open subset $\bar{G} - G$ of $F^*(S)$ belongs to a tame end ε of $F^*(S)$. Then $d(\varepsilon) = d(\varepsilon(1)) \leqq d$. If an end $\varepsilon' \neq \varepsilon$ of $F^*(S)$ contains $\bar{G} - G$ then ε' is a tame end of depth $< d(\varepsilon)$.

Thus we see that all ends of $F^*(S)$ are tame ends of depth $\leqq d$. Therefore $d(F^*(S)) \leqq d + 1$.

The condition (1.4). Again it is sufficient to cosider the case $S = S_i$. By the way of construction, the ceiling $C(S_i)$ is contained in U_0. Clearly $\bar{C}(S_i)$ belongs to the end ε_{1i} and, furtheremore, if an end $\varepsilon \neq \varepsilon_{1i}$ contains $\bar{C}(S_i)$, then $d(\varepsilon) < d(\varepsilon_{1i}) = d + 1$. Now we prove that $d(F^*(S_i)) = d + 1$. Since $d(F^*(S_i)) \leqq d + 1$ by the condition (1.3), it is sufficient to show that $d(F^*(S_i)) \geqq d + 1$. Since $d(\varepsilon_{1i}) = d + 1$, there is a sequence $e_1, e_2, \cdots$ of ends of depth d converging to ε_{1i}. We may suppose that $U'_{1i} \in e_j$ for all j. Since the set $\tilde{\tilde{\mathscr{S}}}(d)_d = \{S \in \tilde{\tilde{\mathscr{S}}}(d) \mid d(F^*(S)) = d\}$ is finite, there is a staircase

$S_* \in \tilde{\tilde{\mathscr{S}}}(d)_d$ whose wall $W(S_*)$ contains an infinite number of $\partial U_{\varepsilon_j}$'s. Then we see that $L_{\bar{\varepsilon}_{1\iota}}(F_0 \cap \tilde{\tilde{\Omega}}^d) = F(S_\iota)$ intersects $W(S_*)$, hence $\mathrm{Cl}_M(F^*(S_\iota)) \supset F^*(S_*)$. Since $d(F^*(S_*)) = d$ by Proposition 4.2[d] it follows that $d(F^*(S_\iota)) \geqq d + 1$.

The condition (1.5). For the first part it is sufficient to consider the case $S = S_\iota$. Since any connected component K of the boundary $\partial F(S_\iota)$ of the floor $F(S_\iota)$ is contained in the wall $W(S')$ of some $S' \in \tilde{\tilde{\mathscr{S}}}(d)$, it follows the that the connected component of the door $D(S_\iota)$ containing K is contained in $W(S')$, by the choice of S_ι. Clearly $d(F^*(S')) \leqq d(F^*(S_\iota))$. For the second part note that if $W(S) \cap S' \neq \emptyset$ for $S, S' \in \mathscr{S}(d + 1)$, then $W(S) \cap S' = W(S) \cap D(S')$ by Proposition 4.1[d] or by the choice of the S_i's. Therefore $W(S) \cap (\bigcup \{S' \mid S \neq S' \in \mathscr{S}(d + 1)\}) \subset \bigcup \{D(S') \mid S' \in \mathscr{S}(d + 1)\}$.

The condition (1.6). It is sufficient to consider the case $d(\varepsilon) = d + 1$. Let $\varepsilon = \varepsilon_1$ as in the proof of Lemma 4.1. Then we have $U_1' \in \varepsilon_1$, $V_1 = U_1' - (\bigcup \{U_\varepsilon \mid U_1' \in \varepsilon \neq \varepsilon_1\})$ and $\bar{\varepsilon}_1 \ni V_1$. The limit set $L_{\bar{\varepsilon}_1}(F_0 \cap \tilde{\tilde{\Omega}}^d)$ of ε_1 is one of $F_1', \cdots, F_l'$, say F_ι'. In the case $\varepsilon_1 = \varepsilon_{1i}$, V_1 contains the ceiling $C(S_i')$ and the limit set $L_{\varepsilon_{1\iota}}(F_0)$ is contained in $F^*(S_\iota) \cup (\bigcup \{L_\varepsilon(F_0) \mid U_1' \in \varepsilon \neq \varepsilon_{1\iota}\})$. Since $\bigcup \{L_\varepsilon(F_0) \mid U_1' \in \varepsilon \neq \varepsilon_{1i}\} \subset \bigcup \{F^*(S) \mid S \in \tilde{\tilde{\mathscr{S}}}(d)\}$ by Proposition 4.1[d], it follows that $L_{\varepsilon_{1\iota}}(F_0) \subset \bigcup \{F^*(S) \mid S \in \mathscr{S}(d + 1)\}$. Clearly there is $U \in \varepsilon_{1i}$ contained in $\mathrm{Int}_M(\bigcup \{S \mid F^*(S) \subset L_\varepsilon(F_0)\})$.

Now we consider the case $\varepsilon_1 \neq \varepsilon_{1\iota}$. By the choice of V_1 and S_ι', it follows that $(\mathrm{Cl}_{F_0}(V_1) - V_1) \cap S_i' = \emptyset$. Therefore $V_1 \cap W(S_i)$ consists of leaves of $\mathscr{F} \mid W(S_\iota)$.

Lemma 4.5. *$V_1 \cap W(S_\iota)$ is a compact leaf of $\mathscr{F} \mid W(S_i)$. Furthermore, for the leaf $\bar{V}_1$ of $\mathscr{F} \mid S_\iota \cup (\bigcup \{S \mid S \in \tilde{\tilde{\mathscr{S}}}(d)\})$ containing $V_1 \cap W(S_\iota)$, we have $\bar{V}_1 \cap W(S_\iota) = V_1 \cap W(S_\iota)$.*

Proof. $V_1 \cap W(S_\iota)$ does not contain a non-compact leaf of $\mathscr{F} \mid W(S_i)$. Otherwise, we see that $L(V_1) \cap W(S_\iota) \neq \emptyset$, a contradiction since $L(V_1) = L_{\bar{\varepsilon}_1}(F_0 \cap \tilde{\tilde{\Omega}}^d) = F_\iota$.

$V_1 \cap W(S_\iota)$ is a compact leaf of $\mathscr{F} \mid W(S_\iota)$. Otherwise, we have two distinct compact leaves G_1, G_2 of $\mathscr{F} \mid W(S_i)$ in $V_1 \cap W(S_i)$. As in (ii) in the proof of Lemma 4.2, we see that $L(V_1) \cap W(S_\iota) \neq \emptyset$, a contradiction.

For the same reason, we see that $\bar{V}_1 \cup W(S_\iota) = V_1 \cap W(S_\iota)$. This completes the proof of Lemma 4.5.

As before let $\bar{C}(S_i)$ be the leaf of $\mathscr{F} \mid S_\iota \cup (\bigcup \{S \mid S \in \tilde{\tilde{\mathscr{S}}}(d)\})$ containing the ceiling $C(S_i)$ of S_i. Let $\bar{p}\colon S_\iota \cup X \to F^*(S_\iota)$ with $X \subset \bigcup \{S \in \tilde{\tilde{\mathscr{S}}}(d) \mid S \cap F^*(S_\iota) \neq \emptyset\} - \mathrm{Cl}_M(F^*(S_\iota))$ be the C^r map as in the Lemma 4.2. Then we have a unique C^r diffeomorphism $f\colon \bar{C}(S_\iota) \to V_1$ such that $\bar{p} \circ f = \bar{p}$. Therefore the ends ε_1 and ε_{1i} behave in the same way. This completes the proof of the condition (1.6).

PROOF OF PROPOSITION 4.2[$d+1$] ($d+1 \leqq d_0$). We suppose Theorem 2[d] and Proposition 4.2[d]. Let ε be an end of F_0 containing U_0. It is sufficient to consider the case $d(\varepsilon) = d+1$. Then ε is a tame end with $a(\varepsilon) > a_0$. Hence there is $U_1 \in \varepsilon$ such that if an end $\varepsilon' \neq \varepsilon$ contains U_1 then $d(\varepsilon') < d(\varepsilon)$ and $d(\partial U_1, L_\varepsilon(F_0)) > a_0 > a(d+1)$. By the proof of Proposition 4.1[$d+1$], we see that $L_\varepsilon(F_0) = \mathrm{Cl}_M(F^*)$, where F^* is a leaf of $\mathscr{F}$ with $d(F^*) = d+1$. The condition (1.6) implies that F^* coincides with $F^*(S_0)$ for some $S_0 \in \mathscr{S}(d+1)$. Since $U_1 \cap S_0 \neq \varnothing$, it follows that $U_1 \cap W(S_0)$ is non-empty and consists of leaves of $\mathscr{F} \mid W(S_0)$. The proof of the following lemma is similar to those of Lemma 4.5.

LEMMA 4.6. (a) *The intersection* $U_1 \cap W(S_0)$ *is a compact leaf of* $\mathscr{F} \mid W(S_0)$. (b) *Let* $\bar{U}_1$ *be the leaf of* $\mathscr{F} \mid \bigcup \{S \mid F^*(S) \subset L_\varepsilon(F_0)\}$ *containing* $U_1 \cap W(S_0)$. *Then* $\bar{U}_1 \cap W(S_0) = U_1 \cap W(S_0)$.

Now let $U_\varepsilon = \mathrm{Int}_{F_0}(\bar{U}_1)$. By (1.6), there is $U_2 \in \varepsilon$ contained in $U_1 \cap \mathrm{Int}_M(\bigcup \{S \mid F^*(S) \subset L_\varepsilon(F_0)\})$. Then $U_2 \subset U_\varepsilon$. Therefore $U_\varepsilon \in \varepsilon$ and U_ε satisfies (2.1).

For the second part of Proposition 4.2[$d+1$], let $U_\varepsilon \in \varepsilon$ satisfy the condition (2.1). Again it is sufficient to consider the case $d(\varepsilon) = d+1$. By the proof of the first part, (2.2) and (2.3) follow. For the proof of (2.4), let $S \in \mathscr{S}(d+1)$ intersect U_ε and suppose that the intersection $U_\varepsilon \cap W(S)$ contains a non-compact leaf of $\mathscr{F} \mid W(S)$. Let $\omega\colon [0,1] \to \mathrm{Cl}_{F_0}(U_\varepsilon)$ be a path such that $\omega(0) \in U_\varepsilon \cap W(S)$ and $\omega(1) \in \partial U_\varepsilon = \mathrm{Cl}_{F_0}(U_\varepsilon) - U_\varepsilon$. We consider the map $\bar{p}\colon S_\varepsilon \cup X \to F^*(S_\varepsilon)$ with $X \subset \bigcup \{S' \in \mathscr{S}(d+1) \mid S' \cap F^*(S_\varepsilon) \neq \varnothing\} - \mathrm{Cl}_M(F^*(S_\varepsilon))$, as in the proof of Lemma 4.2. Now we consider the path $\bar{p} \circ \omega$ in $F^*(S_\varepsilon)$. Note that $\bar{p}^{-1}(\bar{p} \circ \omega(0))$ contains an infinite number of points in $U_\varepsilon \cap W(S)$. Since the foliation $\mathscr{F}$ is transverse to the fibers of $\bar{p}$, we have the holonomy C^r diffeomorphism $h\colon \bar{p}^{-1}(\bar{p} \circ \omega(0)) \to \bar{p}^{-1}(\bar{p} \circ \omega(1))$ as usual. It is easy to see that $h(U_\varepsilon \cap W(S)) \subset W(S_\varepsilon)$. Then $U_\varepsilon \cap W(S_\varepsilon) \cap \bar{p}^{-1}(\bar{p} \circ \omega(1))$ contains an infinite number of points hence $L(U_\varepsilon) \cap W(S_\varepsilon) \neq \varnothing$. This is a contradiction, which completes the proof of the condition (2.4).

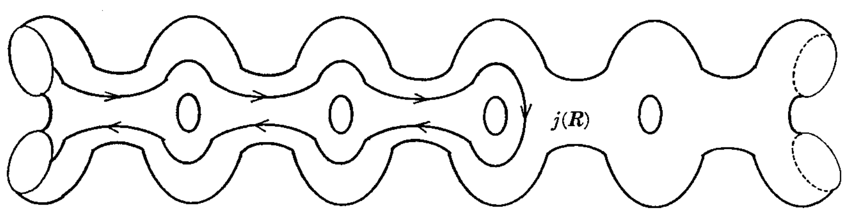
Now we can prove Theorem 2. We have a finite set $\mathscr{S}(d_0)$ of staircases of $(M, \mathscr{F})$ satisfying the conditions (1.1)-(1.6) of Proposition 4.1[d_0]. The condition (1.6) of Proposition 4.1[d_0] implies Theorem 2 (1). Proposition 4.2[d_0] implies Theorem 2(2).

For the proof of the condition (3), let $S' \in \mathscr{S}(d_0)$ satisfy $F^*(S') = F'$. Consider $S'' \in \mathscr{S}(d_0)$ with $W(S'') \cap D(S') \neq \varnothing$. Then the leaf $\bar{C}(S'')$ of

$\mathscr{F}\,|\,\bigcup\{S\in\mathscr{S}(d_0)\,|\,S\cap F^*(S'')\neq\varnothing\}$ containing the ceiling $C(S'')$ of S'' behaves in the same way as $F'\cap(\bigcup\{S\in\mathscr{S}(d_0)\,|\,S\cap F^*(S'')\neq\varnothing\})$. Then an end of F' containing $U(F', S'')=\mathrm{Int}_{F'}(F'\cap(\bigcup\{S\,|\,S\cap F^*(S'')\neq\varnothing\}))$ is tame. Since the floor $F(S')$ is compact, any end of F' contains $U(F', S'')$ for some S''. Therefore all ends of F' are tame. Then Theorem 4(1) implies Theorem 2(3). We will prove Theorem 4(1) later without assuming Theorem 2(3).

5. The proof of Theorem 3. Let C be a circle in S_2 with S_2-C connected. Let T be a tubular neighborhood of C. Then there are a C^∞ diffeomorphism $h: C\times[-1, 1]\to T$. Let $\alpha: [-1, 1]\to[0, 1]$ be a C^∞ map such that $\alpha(t)=0$ in a neighborhood of -1 and $\alpha(t)=1$ in a neighborhood of 1. Let $f: \boldsymbol{R}\to\boldsymbol{R}$ be a C^∞ diffeomorphism such that $f\,|\,(-\infty, 0]\cup[1, \infty)$ is the identity and $f(t)<t$ for all $t\in(0, 1)$. Let $\mathscr{F}_1$ be the foliation on $T\times[0, 1]$ whose leaves are $\{(h(x, s), t)\,|\,x\in C, s\in[-1, 1], t=(1-\alpha(s))t_0+\alpha(s)f(t_0)\}$, with t_0 running through $[0, 1]$. Let $\mathscr{F}_2$ be the foliation on $(S_2-\mathrm{Int}\, T)\times[0, 1]$ whose leaves are $(S_2-\mathrm{Int}\, T)\times\{t\}$ for $t\in[0, 1]$. By connecting $\mathscr{F}_1$ and $\mathscr{F}_2$, we have a foliation $\mathscr{F}_3$ on $S_2\times[0, 1]$. $\mathscr{F}_3$ has two compact leaves $S_2\times\{0\}$ and $S_2\times\{1\}$, and the other leaves are diffeomorphic to the connected sum of a countable number of copies of the torus $S^1\times S^1$. Note that $d(\mathscr{F}_3)=2$.

Now we are going to modify $\mathscr{F}_3$. Take a non-compact leaf F of $\mathscr{F}_3$ and a C^∞ embedding $j: \boldsymbol{R}\to F$ as in Figure 4.



$F\cong\cdots\#(S^1\times S^1)\#(S^1\times S^1)\#(S^1\times S^1)\#(S^1\times S^1)\#\cdots$

FIGURE 4

We extend j to a C^∞ embedding $j^*: \boldsymbol{R}\times[-1, 1]\to F$. Furthermore, we extend j^* to a C^∞ embedding $k: R\times[-1, 1]\times[0, 1]\to S_2\times(0, 1)$ satisfying the following conditions (1)–(3).

(1) $k\,|\,R\times[-1, 1]\times\{0\}=j^*$.

(2) $k(\{x\}\times\{s\}\times[0, 1])\subset j^*((x, s))\times(0, 1)$ for all $x\in R$ and $s\in[0, 1]$.

(3) $k(\boldsymbol{R}\times[-1, 1]\times\{t\})$ is contained in a leaf of $\mathscr{F}_3$ for all $t\in[0, 1]$.

Let $\mathscr{F}_4$ be a C^∞ foliation on $k(\boldsymbol{R}\times[-1, 1]\times[0, 1])$ whose leaves are $\{k(x, s, t)\,|\,x\in\boldsymbol{R}, s\in[-1, 1], t=(1-\alpha(s))t_0+\alpha(s)f(t_0)\}$ for $t_0\in[0, 1]$. By connecting $\mathscr{F}_4$ and $\mathscr{F}_3|(S_2 \times [0, 1] - k(\boldsymbol{R} \times (-1, 1) \times (0, 1)))$, we obtain a C^0 foliation $\mathscr{F}$ on $S_2 \times [0, 1]$. Then $\mathscr{F}$ satisfies the condition (1) of Theorem 3 and $d(\mathscr{F}) = 3$.

Let F_1 be the leaf of $\mathscr{F}$ containing $k(\boldsymbol{R} \times [-1, 1] \times \{1\})$ and let F_2 be a leaf of $\mathscr{F}$ intersecting $k(\boldsymbol{R} \times [-1, 1] \times (0, 1))$. Then F_2 has one end ε of depth 2 and a countable number of ends ε_i of depth 1. It is easy to check that $L_\varepsilon(F_2) = F \cup F_1 \cup S_2 \times \{0\} \cup S_2 \times \{1\}$ and $L_{\varepsilon_i}(F_2) = S_2 \times \{1\}$. Since $d(F) = d(F_1) = 2$, ε is not tame by Theorem 2(2). Clearly ε is pseudotame and we are done.

REMARK 8. ε satisfies the conditions (2)(a) and (2)(b) in Definition 7, but ε does not satisfy 2(c).

6. The proof of Theorem 4. (1) Since all ends of F_0 are tame and since $L(F_0) = \bigcup_\varepsilon L_\varepsilon(F_0)$ contains a leaf of depth $d - 1$, F_0 has an end of depth $\geqq d - 1$. The leaf F_0 has no end of depth $\geqq d$. Indeed if F_0 has an end of depth $\geqq d$ then $L(F_0)$ contains a leaf of depth $\geqq d$, a contradiction to the fact that $d = d(F_0)$. Now suppose that F_0 has an infinite number of ends of depth $d - 1$. By Corollary 3 in §2, F_0 has an end of depth $> d - 1$, a contradiction. Furthermore by the proof of Theorem 2, F_0 is proper. This completes the proof of (1). Note that we do not use Theorem 2(3).

(2) Let $\varepsilon_1, \cdots, \varepsilon_l$ be the ends of F_0 of depth $d - 1$. For each i, we have $U(\varepsilon_i) \in \varepsilon_i$ and $a_i > 0$ satisfying the condition (d) of Definition 7(2). By tracing the proof of Proposition 4.1 carefully, we see that there are a finite set $\mathscr{S}$ of staircases and a number a with $0 < a < \mathrm{Min}\,\{a_1, \cdots, a_l, d(\partial U(\varepsilon_1), L(U(\varepsilon_1)), \cdots, d(\partial U(\varepsilon_l), L(U(\varepsilon_l)))\}$ satisfying the conditions (1.1)-(1.6) of Proposition 4.1[$d - 1$], where we take $\mathscr{S}$, a and F_0 instead of $\mathscr{S}(d-1)$, $a(d-1)$ and U_0, respectively. Furthermore by using the result in Nishimori [3] we can take $\mathscr{S}$ in such a way that all the staircases in $\mathscr{S}$ are regular, since $\mathscr{F}$ is of class C^2 and all the leaves contained in $\mathrm{Cl}_M(F_0) - F_0$ have abelian holonomy. Let $K(\mathscr{S}) = \bigcup\,\{S \mid S \in \mathscr{S}\}$ and let $M_* = M - \mathrm{Int}_M(K(\mathscr{S}))$. Then $F_* = F_0 \cap M_*$ is a compact leaf of the restricted foliation $\mathscr{F}|M_*$ and $\mathrm{Cl}_M(F_0) \cap M_* = F_*$. We consider a small closed tubular neighborhood T of F_* in M_* and let $p\colon T \to F_*$ be a projection such that $p^{-1}(x) \subset \varphi(\{x\} \times \boldsymbol{R})$ for all $x \in F_*$. T is somewhat curious. Note that for $x \in \mathrm{Cl}_M(F_* \cap \mathrm{Int}\, M_*)$ the set $p^{-1}(x) - \{x\}$ has two connected components and for $x \in F_* - \mathrm{Cl}_M(F_* \cap \mathrm{Int}\, M_*)$ the set $p^{-1}(x) - \{x\}$ is connected. See Figure 5.

We may suppose that $F_1 \cap (M_* - T) \neq \varnothing$.

Since $L_\varepsilon(F_1) = \mathrm{Cl}_M(F_0)$, we see by the proof of Lemma 3.2 that there

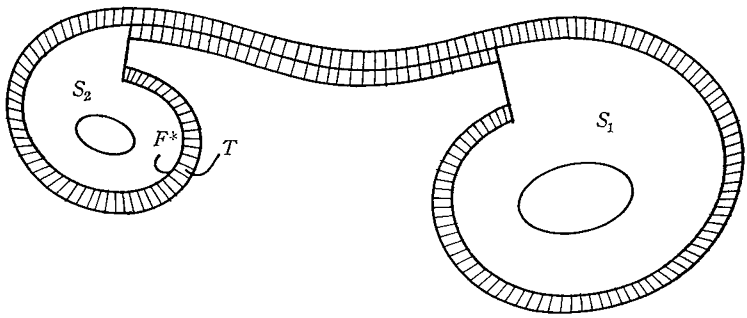


FIGURE 5

is $U \in \varepsilon$ such that (3.2)′ $U \subset T \cup K(\mathscr{S})$. Since Lemma 3.3 is still valid in this case, there is $U \in \varepsilon$ such that (3.3)′ if an end ε' contains U then $L_{\varepsilon'}(F_1) \cap F_1 = \emptyset$. As in the proof of Theorem 1, there is $U \in \varepsilon$ satisfying the conditions (3.2)′ and (3.3)′. From now on we fix one such U. Clearly $L(U) \subset T \cup K(\mathscr{S})$. Since $L(U) \cap F_1 = \bigcup_{\varepsilon' \ni U} (L_{\varepsilon'}(F_1) \cap F_1) = \emptyset$, it follows that $L(U) = \mathrm{Cl}_M(U) - \mathrm{Cl}_{F_1}(U)$. Therefore $L(U)$ is compact and saturated.

Now we prove that $L(U) \cap M_* = F_*$. Let $x_0 \in F_* \cap \mathrm{Int}\, M_*$ and $p^{-1}(x_0) = \varphi(\{x_0\} \times [\alpha, \beta])$ for $\alpha < 0 < \beta$. Suppose that $(L(U) \cap M_*) - F_* \neq \emptyset$, from which we will bring out a contradiction. It follows that $L(U) \cap (p^{-1}(x_0) - \{x_0\}) \neq \emptyset$. Consider the case where $L(U) \cap \varphi(\{x_0\} \times (0, \beta]) \neq \emptyset$. Let y_0 be the point in $L(U) \cap \varphi(\{x_0\} \times (0, \beta])$ farthest from x_0. Let F_2 be the leaf of $\mathscr{F}$ containing y_0. Since $F_2 \subset L(U)$, it follows that $F_2 \subset T \cup K(\mathscr{S})$. As in the proof of Lemma 4.2, we consider the map $\bar{p}$: $T_0 \cup X \to F_0$ with $X \subset K(\mathscr{S}) - \mathrm{Cl}_M(F_0)$, where $x \in X$ if and only if $\bar{p}(x)$ is the first intersection point with F_0 when we trace the orbit of φ starting from x in the direction opposite to that of φ, and where T_0 is the connected component of $T - F_*$ containing y_0. Since all the staircases in $\mathscr{S}$ are regular, F_0 and F_2 behave in the same way in $K(\mathscr{S})$ and $F_2 \cap \bar{p}^{-1}(x)$ consists of one point for all $x \in F_0$. Since U is connected and $L(U) \supset F_0 \cup F_2$, U must be contained in the connected component C of $T_0 \cup K(\mathscr{S}) - \mathrm{Cl}_M(F_0)$ surrounded by F_0 and F_2. Note that C is saturated. Therefore F_1 must also be contained in C, a contradiction to the fact that $F_1 \cap (M_* - T) \neq \emptyset$. When $L(U) \cap \varphi(\{x_0\} \times [\alpha, 0)) \neq \emptyset$, a similar argument brings out a contradiction. Thus we have $L(U) \cap M_* = F_*$.

Let $\mathscr{S}'$ be a thinning of $\mathscr{S}$ with $\partial U \cap K(\mathscr{S}') = \emptyset$, where $K(\mathscr{S}') = \bigcup \{S \mid S \in \mathscr{S}'\}$. As in the second step of the proof of Theorem 1, there is a staircase $S \subset M'_*$ such that $F'_* \cap \mathrm{Int}\, M'_* \subset F(S) \subset F'_*$, $C(S) \subset U$, $D(S) \subset \bigcup \{W(S') \mid S' \in \mathscr{S}'\}$, and $U \cap W(S)$ consists of a finite number of compact leaves of $\mathscr{F} \mid W(S)$, where $M'_* = M - \mathrm{Int}_M(K(\mathscr{S}'))$, $F'_* = F_0 \cap M'_*$ and

$F(S)$, $C(S)$, $D(S)$, $W(S)$ are the floor, ceiling, door, wall of S, respectively.

Let $G_1, \cdots, G_k$ be the connected components of $U \cap W(S)$. Let G_i' be the leaf of $\mathscr{F} | S \cup K(\mathscr{S}')$ containing G_i. Since all the staircases in $\mathscr{S}'$ are regular, it is easy to check that $V_i = \mathrm{Int}_{F_1}(G_i')$ belongs to an end e_i of F_1 of depth d and that if an end $e \neq e_i$ of F_1 contains V_i then e is a tame end of depth $< d$. The end e_i is tame. Indeed let $a_* = \mathrm{Min}\,\{d(W(S), F(S)) | S \in \mathscr{S}'\}$. Then e_i satisfies the condition (d) of Definition 7 (2) with U and a replaced by V_i and a^* respectively.

Since $L_\varepsilon(F_1) = \mathrm{Cl}_M(F_0)$, there is $V \in \varepsilon$ such that $V \subset \mathrm{Int}_M(S \cup K(\mathscr{S}'))$. By the definition of ends, we may suppose that $V \subset U$. Since $U \cap (S \cup K(\mathscr{S}')) = G_1' \cup \cdots \cup G_k'$, it follows that V is contained in V_i for some i. By the definition of ends, the open subset V_i of F_1 belongs to ε. By Theorem 2, ε is not a tame end of depth $< d$. Therefore $\varepsilon = e_i$ and ε is a tame end of depth d. Clearly $L_\varepsilon^*(F_1) = F_0$.

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