

# ENDS OF THE MODULI SPACE OF HIGGS BUNDLES

RAFE MAZZEO, JAN SWOBODA, HARTMUT WEISS, AND FREDERIK WITT

ABSTRACT. We associate to each stable Higgs pair  $(A_0, \Phi_0)$  on a compact Riemann surface  $X$  a singular limiting configuration  $(A_\infty, \Phi_\infty)$ , assuming that  $\det \Phi$  has only simple zeroes. We then prove a desingularization theorem by constructing a family of solutions  $(A_t, t\Phi_t)$  to Hitchin's equations which converge to this limiting configuration as  $t \rightarrow \infty$ . This provides a new proof, via gluing methods, for elements in the ends of the Higgs bundle moduli space and identifies a dense open subset of the boundary of the compactification of this moduli space.

## 1. INTRODUCTION

The moduli space of solutions to Hitchin's equations on a compact Riemann surface occupies a privileged position at the cross-roads of gauge-theoretic geometric analysis, geometric topology and the emerging field of higher Teichmüller theory. These are equations for a pair  $(A, \Phi)$ , where  $A$  is a unitary connection on a Hermitian vector bundle  $E$  over a Riemann surface  $X$ , and  $\Phi$  an  $\text{End}(E)$ -valued  $(1, 0)$ -form (the 'Higgs field'). We will mostly be concerned with the fixed determinant case, i.e. we consider only connections which induce a fixed connection on the determinant line bundle of  $E$  and trace-free Higgs fields. Then the equations read

$$(1) \quad \begin{aligned} F_A^\perp + [\Phi \wedge \Phi^*] &= 0 \\ \bar{\partial}_A \Phi &= 0. \end{aligned}$$

Here  $F_A^\perp$  is the trace-free part of the curvature of  $A$ , which is a 2-form with values in the skew-Hermitian endomorphisms of  $E$ , and  $\Phi^*$  is computed with respect to the Hermitian metric on  $E$ . We always assume below that  $X$  is compact, and that its genus is bigger than 1.

When  $E$  is the trivial rank 2 bundle, (1) is the two-dimensional reduction of the standard self-dual Yang-Mills system on  $X \times \mathbb{R}^2$  to  $X$ . However, these equations also makes sense for higher rank nontrivial bundles, and have in addition been studied when  $X$  is a higher dimensional Kähler manifold [Si88, Si92]. There is an alternate presentation where this system can be studied in more purely algebraic and topological terms using representations of (a central extension of) the fundamental group of  $X$  into the Lie group  $\text{SL}(r, \mathbb{C})$ ,  $r = \text{rk}(E)$  (see [Go12] and references therein).

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A pair  $(A_0, \Phi_0)$  is called a *Higgs pair* if  $\bar{\partial}_{A_0} \Phi_0 = 0$ . Thus  $\Phi_0$  is holomorphic with respect to the  $(0, 1)$  component of the unitary connection  $A_0$ . The  $(1, 0)$  component of  $A_0$  is determined by  $A_0^{0,1}$  and its unitarity with respect to a fixed Hermitian metric  $H$  on  $E$ . In his initial paper on these equations [Hi87], Hitchin established the existence of a unique solution to (1) in the complex gauge orbit of any given Higgs pair which satisfies an additional stability condition slightly weaker than the standard slope stability condition for  $E$  alone. He went on to prove that the moduli space  $\mathcal{M}$  of solutions modulo the gauge group enjoys many nice properties. In particular, if  $\text{rk}(E) = 2$  and the degree of  $E$  is odd, then  $\mathcal{M}$  is a smooth manifold of dimension  $12\gamma - 12$ , where  $\gamma$  is the genus of  $X$ . (In other cases it is at least a quasi-projective variety, but we focus in this paper on this simplest setting.) Furthermore, it has a natural hyperkähler metric  $g$  of Weil-Petersson type, with respect to which it is complete. In the intervening years, much has been learned about its topology and many other features. However, surprisingly little is known about the metric structure at infinity.

In the past few years, a very intriguing conceptual picture has emerged through the work of Gaiotto, Moore and Neitzke [GMN10]. As part of a much broader picture concerning hyperkähler metrics on holomorphic integrable systems, they describe a decomposition of the natural metric  $g$  on  $\mathcal{M}$  as a leading term (the semiflat metric in the language of [Fr99]) plus an asymptotic series of non-perturbative corrections, which decay exponentially in the distance from some fixed point. The coefficients of these correction terms are given there in terms of a priori divergent expressions coming from a wall-crossing formalism.

A further motivation is Hausel's result about the vanishing of the image of compactly supported cohomology in the ordinary cohomology [Ha99]. In analogy with Sen's conjecture about the  $L^2$ -cohomology of the monopole moduli spaces [Se94], he conjectured further that the  $L^2$ -cohomology of the Higgs bundle moduli space must vanish. Partial confirmation of this conjecture was established shortly afterwards by Hitchin [Hi00] who showed that the  $L^2$ -cohomology is concentrated in the middle degree. Closely related results about  $L^2$ -cohomology of gravitational instantons, and partial confirmation of Sen's conjecture, were obtained by Hausel, Hunsicker and the first author [HHM05]. These papers make clear that any hope of obtaining better results about  $L^2$ -cohomology will require a better understanding of the metric asymptotics on  $\mathcal{M}$ .

One other recent development appears in the pair of papers by Taubes [Ta13.1, Ta13.2]. His setting is for a closely related gauge theory on three-manifolds with gauge group  $\text{SL}(2, \mathbb{C})$ , but he notes there that his results transfer simply (and presumably with fewer technicalities) to the case of surfaces. He proves a compactness theorem for those equations focusing on the specific problems caused by the noncompactness of the underlying group ( $\text{SL}(2, \mathbb{C})$  rather than  $\text{SU}(2)$ ). More specifically, he is able to deduce information about the limiting behavior of solutions which diverge in a specific

way in the moduli space. While the results in our paper are partly subsumed by those of Taubes, we hope that the constructive methods here will be of value in the various types of questions described above.

We can now describe our work and the results in this paper. Our initial motivation was to reach a more detailed understanding of the structure of the space  $\mathcal{M}$  near its asymptotic boundary, with the hope of using this to obtain information about the structure of the metric  $g$  there. In essence, we do this by reproving Hitchin's result for solutions which lie sufficiently far out in  $\mathcal{M}$ . We make here the simplifying assumption that the Higgs field  $\Phi_0$  is *simple*, in the sense that its determinant has simple zeroes. This implies, in particular, the stability of the Higgs pair  $(A_0, \Phi_0)$  in the sense of Hitchin, but has further technical implications too. We first consider a family of 'limiting configurations', consisting of certain singular pairs  $(A_\infty, \Phi_\infty)$  which satisfy a decoupled version of Hitchin's equations, namely

$$F_{A_\infty}^\perp = 0, \quad [\Phi_\infty \wedge \Phi_\infty^*] = 0, \quad \bar{\partial}_{A_\infty} \Phi_\infty = 0.$$

Thus each  $A_\infty$  is projectively flat with simple poles, while the limiting Higgs fields are holomorphic with respect to these connections and have a specified behavior near these poles.

**Theorem 1.1** (Existence and deformation theory of limiting configurations). *Let  $(A_0, \Phi_0)$  be a Higgs pair such that  $q := \det \Phi_0$  has only simple zeroes. Then there exists a complex gauge transformation  $g_\infty$  on  $X^\times = X \setminus q^{-1}(0)$  which transforms  $(A_0, \Phi_0)$  into a limiting configuration. Furthermore, the space of limiting configurations with fixed determinant  $q \in H^0(K_X^2)$  having simple zeroes is a torus of dimension  $6\gamma - 6$ , where  $\gamma$  is the genus of  $X$ .*

In a later paper [MSWW15], following a suggestion of Hitchin, we interpret limiting configurations in terms of more familiar parabolic Higgs pair data.

We also consider the family of desingularizing 'fiducial solutions' which will be used to 'round off' the singularities in these limiting configurations. These fiducial solutions are an explicit family of rotationally symmetric solutions on  $\mathbb{C}$ , the existence of which was pointed out to us by Neitzke, but since there does not seem to be an easily available reference for them in the literature, we provide a fairly complete derivation of their properties here.

With these two types of components, we now pursue a standard strategy to construct exact solutions. Namely, we construct families of approximate solutions, which lie in the gauge orbit of some  $(A, t\Phi)$  for  $t$  large, and then use the linearization of a relevant elliptic operator to correct these approximate solutions to exact solutions. This yields the

**Theorem 1.2** (Desingularization theorem). *If  $(A_\infty, \Phi_\infty)$  is a limiting configuration, then there exists a family  $(A_t, \Phi_t)$  of solutions of the rescaled Hitchin equation*

$$F_A^\perp + t^2[\Phi \wedge \Phi^*] = 0, \quad \bar{\partial}_A \Phi = 0$$

provided  $t$  is sufficiently large, where

$$(A_t, \Phi_t) \longrightarrow (A_\infty, \Phi_\infty)$$

as  $t \nearrow \infty$ , locally uniformly on  $X^\times$  along with all derivatives, at an exponential rate in  $t$ . Furthermore,  $(A_t, \Phi_t)$  is complex gauge equivalent to  $(A_0, \Phi_0)$  if  $(A_\infty, \Phi_\infty)$  is the limiting configuration associated with  $(A_0, \Phi_0)$ .

In particular, we obtain solutions to Hitchin's equation for Higgs pairs  $(A_0, t\Phi_0)$  when  $t$  is large and  $\det \Phi_0$  has simple zeroes. The advantage of this method is that we obtain precise estimates on the shape of these solutions. This mirrors what is obtained in [Ta13.1, Ta13.2], and it is not hard to deduce from this that the Weil-Petersson metric  $g$  does indeed decompose as a principal term (essentially given by the deformation theory of the limiting configurations) and an exponentially decreasing error. There is some work to identify the various parts of this metric, and this will appear in a subsequent paper. While we are able to capture the correct exponential rate, our method at present includes an extra polynomial factor, so in particular we are not yet able to say anything about the leading coefficient of the first decaying term.

To understand the entire end of the moduli space  $\mathcal{M}$  (when the degree of  $E$  is odd), one must also consider non-simple Higgs fields. When the simplicity condition fails, the desingularizing fiducial solutions must be replaced by some more complicated special solutions. These new fiducial solutions are being studied in the ongoing thesis work of Laura Fredrickson, and it is expected that these gluing methods will adapt readily to incorporate her 'multi-pole' fiducial solutions.

Following [Hi87], we have chosen to focus on pairs  $(A, \Phi)$  in a complex gauge orbit of a given Higgs pair as the objects of interest; a completely equivalent perspective is to regard the Hermitian metric on  $E$  as the object to be varied, which is the point of view taken by Simpson and many others.

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## 2. PRELIMINARIES ON GAUGE THEORY AND HIGGS BUNDLES

**2.1. Holomorphic vector bundles.** To fix notation, we briefly recall some classical facts about gauge theory and holomorphic vector bundles. Good general references are [Ko87, Chapter I and VII] or [WGP08, Chapter III and Appendix].

Let  $X$  be a Riemann surface of genus  $\gamma \geq 2$  with canonical line bundle  $K_X$ , which we usually denote just as  $K$ . We also fix a metric on  $X$  in the designated conformal class, normalised so that the corresponding Kähler form  $\omega$  satisfies  $\int_X \omega = 2\pi$ . Consider a complex vector bundle  $E \rightarrow X$  of rank  $r = \text{rk}(E)$  and degree  $d = \text{deg}(E)$ , where by definition,  $d$  is the degree of the complex line bundle  $\det E = \Lambda^r E$ . The *slope* of  $E$  is  $\mu(E) := \text{deg}(E)/\text{rk}(E)$ . We write  $\text{GL}(E)$  and  $\text{SL}(E)$  for the bundles of automorphisms, and automorphisms with determinant one, of  $E$ , and set  $\mathfrak{gl}(E) = E \otimes E^*$ , with  $\mathfrak{sl}(E)$  the subbundle of trace-free endomorphisms. The sections  $\Gamma(\text{GL}(E))$  and  $\Gamma(\text{SL}(E))$  are the *complex gauge transformations* in this theory; these are infinite-dimensional Lie groups in the sense of [Mi84], with Lie algebras  $\Omega^0(\mathfrak{gl}(E))$  and  $\Omega^0(\mathfrak{sl}(E))$ , respectively. A Hermitian metric  $H$  on the fibres of  $E$  determines the bundles  $\text{U}(E, H)$  and  $\text{SU}(E, H)$  of unitary and special unitary automorphisms of  $(E, H)$ ; the corresponding Lie algebra bundles are  $\mathfrak{u}(E, H)$  and  $\mathfrak{su}(E, H)$ . The sections  $\Gamma(\text{U}(E, H))$  are the *unitary gauge transformations*. For simplicity we usually omit mention of the metric  $H$  in this notation.

The affine space  $\mathcal{U}(E)$  of unitary connections (with respect to  $H$ ) has  $\Omega^1(\mathfrak{u}(E))$  as its group of translations. The action of the unitary gauge group  $\text{U}(E)$  on  $\mathcal{U}(E)$  is the familiar one:

$$(2) \quad d_A \mapsto d_{Ag} := g^{-1} \circ d_A \circ g = d_A + g^{-1} d_A g.$$

In the sequel, we tacitly fix a base connection  $A_0$  and hence identify an arbitrary unitary connection  $A$  with an element in  $\Omega^1(\mathfrak{u}(E))$ . The covariant derivative  $d_A: \Omega^0(E) \rightarrow \Omega^1(E)$  satisfies  $dH(s_1, s_2) = H(d_A s_1, s_2) + H(s_1, d_A s_2)$ ; in a local trivialization,  $d_A s = ds + As$ , where  $d$  is the usual differential and the *connection matrix*  $A$  is a matrix-valued 1-form. These three perspectives, regarding  $A$  as a point in  $\mathcal{U}(E)$ , a covariant derivative  $d_A$  or as a connection matrix, are used interchangeably below. In particular,  $A = 0$  can mean that  $A$  is the base connection, that  $d_A$  is given locally as  $d$ , or that the connection matrix vanishes. Finally, we obtain the *curvature* of  $A$ ,  $F_A = d_A \circ d_A \in \Omega^2(\mathfrak{u}(E))$  from the natural extension  $d_A: \Omega^p(E) \rightarrow \Omega^{p+1}(E)$ . Here, we have the familiar transformation rule

$$F_{Ag} = g^{-1} F_A g$$

for a unitary gauge transformation  $g$ .

We now explain the action of the complex gauge group on connections. Over a Riemann surface, holomorphic structures on  $E$  are in one-to-one correspondence with so-called  $\bar{\partial}$ -operators  $\bar{\partial}_E: \Omega^0(E) \rightarrow \Omega^{0,1}(E)$ , which satisfy

$\bar{\partial}_E(f\sigma) = \bar{\partial}f \cdot \sigma + f\bar{\partial}_E\sigma$  for any  $f \in C^\infty(X)$  and  $\sigma \in \Omega^0(E)$ . As for connections, the space of  $\bar{\partial}$ -operators  $\mathcal{C}(E)$  is affine, but modelled on  $\Omega^{0,1}(\mathfrak{gl}(E))$ . Further, the complex gauge group  $\Gamma(\mathrm{GL}(E))$  acts on  $\mathcal{C}(E)$  as in Eq. (2).

If  $A$  is a unitary connection (for some fixed Hermitian metric  $H$ ), then the projection of  $d_A$  onto  $(0, 1)$  forms,

$$\bar{\partial}_A := \mathrm{pr}^{0,1} \circ d_A,$$

is a  $\bar{\partial}$ -operator and hence determines a holomorphic structure; we also define  $\partial_A = \mathrm{pr}^{1,0} \circ d_A$ . Conversely, given the Hermitian metric  $H$ , then to any  $\bar{\partial}$ -operator  $\bar{\partial}_E$  we can uniquely associate a unitary connection  $A = A(H, \bar{\partial}_E)$ ; this is the so-called *Chern connection*, which has  $\bar{\partial}_A = \bar{\partial}_E$ . If  $\alpha$  denotes the local connection matrix of  $\bar{\partial}_E$ , then

$$A(H, \bar{\partial}_E) = \alpha - \alpha^*.$$

The natural action of  $\Gamma(\mathrm{U}(E, H))$  on  $\mathcal{U}(E, H)$  thus extends to an action by elements of  $\Gamma(\mathrm{GL}(E))$  by

$$A(H, \bar{\partial}_E)^g := A(H, \bar{\partial}_E^g)$$

or equivalently,

$$(3) \quad d_{A^g} = \bar{\partial}_{A^g} + \partial_{A^g} := g^{-1} \circ \bar{\partial}_A \circ g + g^* \circ \partial_A \circ g^{*-1}.$$

Note that this reduces to the action of (2) when  $g \in \Gamma(\mathrm{U}(E, H))$ . The curvature transforms as

$$(4) \quad F_{A^g} = g^{-1}(F_A + \bar{\partial}_A(G \cdot \partial_A G^{-1}))g$$

where  $G = gg^*$ .

**2.2. Hitchin's equations.** Fix a Hermitian vector bundle  $(E, H) \rightarrow X$  of rank 2. The background metric  $H$  will be used in an auxiliary manner; since any two Hermitian metrics are complex gauge equivalent the precise choice is immaterial. We shall be studying solutions  $(A, \Phi)$  of Hitchin's self-duality equations [Hi87]

$$(5) \quad \begin{aligned} F_A + [\Phi \wedge \Phi^*] &= -i\mu(E)\mathrm{Id}_E\omega, \\ \bar{\partial}_A\Phi &= 0. \end{aligned}$$

Here  $A \in \mathcal{U}(E)$  and  $\Phi \in \Omega^{1,0}(\mathfrak{gl}(E))$  is called a *Higgs field*.

The unitary gauge group  $\Gamma(\mathrm{U}(E))$  acts on unitary connections as above and on Higgs fields by conjugation  $\Phi^g := g^{-1}\Phi g$  and thus on the solution space of (5). Moreover, any solution  $(A, \Phi)$  determines a *Higgs bundle*  $(E, \bar{\partial}, \Phi)$ , i.e. a holomorphic structure  $\bar{\partial} = \bar{\partial}_A$  on  $E$  for which  $\Phi$  is holomorphic:  $\Phi \in H^0(X, \mathrm{End}(E) \otimes K)$ . To do so we simply forget the first equation in (5). Conversely, given a Higgs bundle  $(E, \bar{\partial}, \Phi)$ ,  $\bar{\partial}$  can be augmented to a unitary connection  $A$  such that the first Hitchin equation holds provided  $(E, \bar{\partial}, \Phi)$  is *stable*. This means that  $\mu(F) < \mu(E)$  for any nontrivial proper  $\Phi$ -invariant holomorphic subbundle  $F$ , i.e.  $\Phi(F) \subset F \otimes K$ .

**Example.** The *determinant* of a Higgs field  $\Phi$  is the holomorphic quadratic differential  $\det \Phi \in H^0(X, K^2)$ . Let  $\mathfrak{p}_\Phi$  be the zero locus of  $\det \Phi$ , and write  $X_\Phi^\times = X \setminus \mathfrak{p}_\Phi \not\subseteq X$  for its complement. (When there is no risk of confusion, we simply write  $\mathfrak{p}$  and  $X^\times$ .) We call  $\Phi$  *simple* if the zeroes of  $\det \Phi$  are simple; in this case,  $\mathfrak{p}_\Phi$  consists of precisely  $4(\gamma - 1)$  distinct points. For instance, the so-called fiducial Higgs field  $\Phi_t^{\text{fid}}$ ,  $t < \infty$ , which will be constructed in Section 3 is simple in this sense. Furthermore, if  $\Phi$  is a simple Higgs field, then the Higgs bundle  $(E, \bar{\partial}, \Phi)$  is necessarily stable (see [Hi87, Theorem 8.1 (iv)]). We are grateful to Richard Wentworth for pointing out to us this simple but important fact.

In this paper we will always work with a *fixed* connection on the determinant line bundle of  $E$ . According to the splitting  $\mathfrak{u}(2) = \mathfrak{su}(2) \oplus \mathfrak{u}(1)$ , where  $\mathfrak{su}(2)$  is the set of trace-free elements of the Lie algebra  $\mathfrak{u}(2)$  and  $\mathfrak{u}(1) = i\mathbb{R}$ , the bundle  $\mathfrak{u}(E)$  splits as  $\mathfrak{su}(E) \oplus i\mathbb{R}$ . If  $A$  is a unitary connection, then its curvature  $F_A$  decomposes as

$$F_A = F_A^\perp + \frac{1}{2} \text{Tr}(F_A) \otimes \text{Id}_E,$$

where  $F_A^\perp \in \Omega^2(\mathfrak{su}(E))$  is the *trace-free* part of the curvature and  $\frac{1}{2} \text{Tr}(F_A) \otimes \text{Id}_E$  is the *pure trace* or *central* part, see e.g. [LeP92]. Note that  $\text{Tr}(F_A) \in \Omega^2(i\mathbb{R})$  is precisely the curvature of the induced connection on  $\det E$ . Let us fix a background connection  $A_0$  from now on and consider only those connections  $A$  which induce the same connection on  $\det E$  as  $A_0$  does, i.e.  $A = A_0 + \alpha$  where  $\alpha \in \Omega^1(\mathfrak{su}(E))$ ; in other words, any such  $A$  is trace-free “relative” to  $A_0$ . We may now consider the pair of equations

$$(6) \quad \begin{aligned} F_A^\perp + [\Phi \wedge \Phi^*] &= 0, \\ \bar{\partial}_A \Phi &= 0, \end{aligned}$$

for  $A$  trace-free relative to  $A_0$ . Since the trace of a holomorphic Higgs field is a holomorphic  $(1, 0)$ -form which can be chosen at will, we may as well restrict to trace-free Higgs fields  $\Phi \in \Omega^{1,0}(\mathfrak{sl}(E))$ . There always exists a unitary connection  $A_0$  on  $E$  such that  $\text{Tr} F_{A_0} = -i \deg(E)\omega$ , and with this as background connection, a solution of (6) provides a solution to (5), even though the latter system is a priori more stringent. The relevant gauge groups are now  $\Gamma(\text{SL}(E))$  and  $\Gamma(\text{SU}(E))$  which we denote by  $\mathcal{G}^c$  and  $\mathcal{G}$  respectively.

### 3. THE FIDUCIAL SOLUTION

Our first goal is to determine the model ‘fiducial’ solutions of Hitchin’s equations for Higgs fields with simple zeroes. These are the elements of a one-parameter radial family of ‘radial’ global solutions on  $\mathbb{R}^2$ , and are a key ingredient in the gluing construction below. The limiting element of this family is a pair  $(A_\infty^{\text{fid}}, \Phi_\infty^{\text{fid}})$  which is singular at 0 and satisfies a decoupled

version of Hitchin's equations:

$$(7) \quad F_{A_\infty^{\text{fid}}} = 0, \quad [\Phi_\infty^{\text{fid}} \wedge (\Phi_\infty^{\text{fid}})^*] = 0, \quad \text{and} \quad \bar{\partial}_{A_\infty^{\text{fid}}} \Phi_\infty^{\text{fid}} = 0.$$

The other elements of the family,  $(A_t^{\text{fid}}, t\Phi_t^{\text{fid}})$ ,  $0 < t < \infty$ , are smooth across 0, satisfy (6) and desingularize the limiting element. Further, they give rise to solutions of the self-dual Yang-Mills equation which are translation invariant in two directions and are also rotationally invariant. Symmetric solutions of this type (as well as others) have been studied in connection with integrable systems, and Mason and Woodhouse show that the resulting reduced equation is essentially Painlevé III [MaWo93], see also Eq. (24). On the other hand, some version of this family appears at least as far back as the paper of Ceccotti and Vafa [CeVa93], but see also the recent more paper of Gaiotto, Moore and Neitzke [GMN13]. Its existence can also be deduced from the work of Biquard and Boalch [BiBo04], although their method does not give the explicit formula for it. In any case, we present an explicit derivation of this family of solutions since this does not seem to appear in the literature. We are very grateful to Andy Neitzke for bringing this family of fiducial solutions to our attention and for explaining its main properties to us. We note that similar fiducial solutions in more general settings, e.g. Higgs fields with determinants having non-simple zeroes, or for higher rank groups, are being constructed in the forthcoming thesis of Laura Fredrickson [F] at UT Austin.

We begin with a useful lemma.

**Lemma 3.1.** *Let  $\Phi$  and  $\Phi'$  be two Higgs fields on  $X^\times$  with  $\det \Phi = \det \Phi'$  such that both  $\Phi$  and  $\Phi'$  are normal on  $X^\times$ . Then there exists a unitary gauge transformation  $g$  on  $X^\times$  such that  $\Phi^g = \Phi'$ .*

*Proof.* Since  $X^\times$  is homotopy equivalent to a bouquet of circles, any complex vector bundle over  $X^\times$  is topologically trivial. More generally, any fibre bundle with connected fibre admits a global section over  $X^\times$ . In particular we may identify  $\Phi$  and  $\Phi'$  with functions  $\varphi, \varphi': X^\times \rightarrow \mathfrak{sl}(2, \mathbb{C})$ . Since  $\varphi$  and  $\varphi'$  are pointwise normal and have the same determinant, then locally on  $X^\times$  we can find unitary gauge transformations  $g$  such that  $g^{-1}\varphi g = \varphi'$ . Hence

$$\mathcal{C}_{\varphi, \varphi'} = \{(p, g_p) \in X^\times \times \text{SU}(2) \mid g_p^{-1}\varphi(p)g_p = \varphi'(p)\} \rightarrow X^\times$$

is a smooth fibre bundle. The fiber is diffeomorphic to the pointwise stabilizer

$$\text{Stab}_{\text{SU}(2)} \begin{pmatrix} \lambda & 0 \\ 0 & -\lambda \end{pmatrix} = \left\{ \begin{pmatrix} \tau & 0 \\ 0 & \bar{\tau} \end{pmatrix} \mid \tau \in S^1 \right\}$$

which is a maximal torus  $S^1 \subset \text{SU}(2)$ . Since this is connected, there exists a global section over  $X^\times$ .  $\square$

**3.1. The limiting fiducial connection.** We first determine the *limiting fiducial solution*  $(A_\infty^{\text{fid}}, \Phi_\infty^{\text{fid}})$ , where  $A_\infty^{\text{fid}}$  is flat and  $\Phi_\infty^{\text{fid}}$  is normal. More precisely, we show that any Higgs pair  $(A, \Phi)$  on the punctured complex



plane  $\mathbb{C}^\times$ , where  $A$  is a flat unitary connection and  $\Phi$  is a normal Higgs field whose determinant has one simple zero at 0, can be brought into this particular form.

The construction below can be carried out either on all of  $\mathbb{C}$  or else over an open disk  $D$  centered at 0. To be specific, we suppose the latter. As usual,  $D^\times = D \setminus \{0\}$ . Let  $\Phi$  be a normal Higgs field over  $D^\times$  and assume that  $\det \Phi = -z dz^2$  on  $D$ . Fix a Hermitian metric  $H$  on  $E$  and corresponding unitary frame so that  $E|_{D^\times} \cong D^\times \times \mathbb{C}^2$ . Define the *limiting fiducial Higgs field* with respect to this frame by

$$(8) \quad \Phi_\infty^{\text{fid}} = \varphi_\infty^{\text{fid}} dz := \begin{pmatrix} 0 & \sqrt{|z|} \\ \frac{z}{\sqrt{|z|}} & 0 \end{pmatrix} dz.$$

This is continuous on  $D$  and smooth on  $D^\times$ . By Lemma 3.1, since  $\det \Phi_\infty^{\text{fid}} = \det \Phi$ , there is a unitary gauge transformation  $g$  on  $D^\times$ , unique up to the unitary stabilizer of  $\Phi_\infty^{\text{fid}}$ , which brings  $\Phi$  into this fiducial form, that is,  $g^{-1}\Phi g = \Phi_\infty^{\text{fid}}$  over  $D^\times$ . The *infinitesimal complex stabilizer* of  $\Phi_\infty^{\text{fid}}$  is the bundle

$$L_{\Phi_\infty^{\text{fid}}}^{\mathbb{C}} := \{\gamma \in \mathfrak{sl}(E) \mid [\gamma, \Phi_\infty^{\text{fid}}] = 0\}.$$

In this fixed unitary frame,  $\gamma \in \Omega^0(D^\times, L_{\Phi_\infty^{\text{fid}}}^{\mathbb{C}})$  if and only if

$$(9) \quad \gamma_\mu = \mu \begin{pmatrix} 0 & 1 \\ \frac{z}{|z|} & 0 \end{pmatrix}, \quad \mu: D^\times \rightarrow \mathbb{C}.$$

Note that  $\gamma_\mu$  is skew-Hermitian if and only if  $e^{i\theta}\mu + \bar{\mu} = 0$  (where  $z = re^{i\theta}$ ); this reflects the fact that this bundle of unitary stabilizers is a nontrivial  $S^1$ -bundle over  $D^\times$  (cf. also the end of the proof of Lemma 4.5).

**Proposition 3.2.** *Let  $A$  be a flat unitary connection over  $D^\times$  with respect to which  $\Phi_\infty^{\text{fid}}$  is holomorphic. Then there exists a unique unitary gauge transformation  $g = \exp(\gamma)$  with  $\gamma \in \Gamma(D^\times, \mathfrak{su}(E))$ , which stabilizes  $\Phi_\infty^{\text{fid}}$  and such that*

$$(10) \quad A^g = A_\infty^{\text{fid}} := \frac{1}{8} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} dz & -d\bar{z} \\ z & \bar{z} \end{pmatrix}.$$

*Note that this limiting fiducial solution  $(A_\infty^{\text{fid}}, \Phi_\infty^{\text{fid}})$  is defined with respect to a fixed unitary fiducial frame.*

*Proof.* Write  $A = A_r dr + A_\theta d\theta$  and name the components of these coefficient matrices with respect to the chosen fiducial frame as

$$A_r = \begin{pmatrix} i\beta & w \\ -\bar{w} & -i\beta \end{pmatrix}, \quad A_\theta = \begin{pmatrix} i\alpha & v \\ -\bar{v} & -i\alpha \end{pmatrix}$$

where  $\alpha, \beta: D^\times \rightarrow \mathbb{R}$  and  $v, w: D^\times \rightarrow \mathbb{C}$  are all smooth, and  $z = re^{i\theta}$ .

We now show how the fact that  $\Phi$  is holomorphic and  $A$  is flat restricts these coefficients, and then use this information to gauge away the off-diagonal terms.

**$\Phi$  holomorphic:** We compute the terms in the equality

$$\bar{\partial}_A \Phi_\infty^{\text{fid}} := \bar{\partial} \Phi_\infty^{\text{fid}} + [A^{0,1} \wedge \Phi_\infty^{\text{fid}}] = 0.$$

First,

$$(11) \quad \bar{\partial} \Phi_\infty^{\text{fid}} = \frac{1}{4} r^{-\frac{1}{2}} e^{i\theta} \begin{pmatrix} 0 & 1 \\ -e^{i\theta} & 0 \end{pmatrix} d\bar{z} \wedge dz.$$

Next, using  $dr = \frac{1}{2}(e^{-i\theta} dz + e^{i\theta} d\bar{z})$  and  $d\theta = \frac{1}{2ir}(e^{-i\theta} dz - e^{i\theta} d\bar{z})$ , we have

$$A^{0,1} = \frac{1}{2} e^{i\theta} (A_r + \frac{i}{r} A_\theta) d\bar{z} = \frac{1}{2} e^{i\theta} \begin{pmatrix} -\frac{\alpha}{r} + i\beta & w + \frac{i}{r} v \\ -\bar{w} - \frac{i}{r} \bar{v} & \frac{\alpha}{r} - i\beta \end{pmatrix} d\bar{z},$$

so that

$$(12) \quad [A^{0,1} \wedge \Phi_\infty^{\text{fid}}] \\ = \frac{1}{2} r^{1/2} e^{i\theta} \begin{pmatrix} e^{i\theta} w + \bar{w} + \frac{i}{r} (e^{i\theta} v + \bar{v}) & 2(-\frac{\alpha}{r} + i\beta) \\ 2e^{i\theta} (\frac{\alpha}{r} - i\beta) & -(e^{i\theta} w + \bar{w} + \frac{i}{r} (e^{i\theta} v + \bar{v})) \end{pmatrix} d\bar{z} \wedge dz.$$

Adding (11) to (12) and equating coefficients to zero gives  $\alpha = \frac{1}{4}$ ,  $\beta = 0$ , and

$$(13) \quad e^{i\theta} v + \bar{v} = e^{i\theta} w + \bar{w} = 0.$$

We have used here the identity  $e^{i\theta} u + \bar{u} = 2e^{i\theta/2} \text{Re}(e^{i\theta/2} u)$  (for any  $u$ ) to separate into real and imaginary parts. Altogether, we have now obtained that

$$(14) \quad A = \begin{pmatrix} 0 & w \\ -\bar{w} & 0 \end{pmatrix} dr + \begin{pmatrix} i/4 & v \\ -\bar{v} & -i/4 \end{pmatrix} d\theta \quad \text{and} \\ A^{0,1} = \frac{1}{2} e^{i\theta} \begin{pmatrix} -\frac{1}{4r} & w + \frac{i}{r} v \\ -\bar{w} - \frac{i}{r} \bar{v} & \frac{1}{4r} \end{pmatrix} d\bar{z}$$

with  $v, w$  subject to (13).

**Flatness:** The equation  $F_A = 0$  expands as

$$\partial_r A_\theta - \partial_\theta A_r + [A_r, A_\theta] = 0.$$

Substituting the expressions for  $A_r$  and  $A_\theta$  above now gives that  $\text{Im}(\bar{w}v) = 0$ , which is in fact the same as (13), and more significantly,

$$(15) \quad \partial_r v = iPw, \quad \text{where } P = \frac{1}{i} \partial_\theta + \frac{1}{2}.$$

We now wish to find a gauge transformation  $g_\mu$  in the stabilizer of  $\Phi_\infty^{\text{fid}}$  which simplifies  $A$  even further. We assume that  $g_\mu$  is the exponential of some section  $\gamma_\mu$  of the infinitesimal stabilizer bundle, so using the earlier expression for  $\gamma_\mu$  we have that

$$(16) \quad g_\mu = \begin{pmatrix} \cosh(e^{i\theta/2} \mu) & e^{-i\theta/2} \sinh(e^{i\theta/2} \mu) \\ e^{i\theta/2} \sinh(e^{i\theta/2} \mu) & \cosh(e^{i\theta/2} \mu) \end{pmatrix} =: \begin{pmatrix} \eta_1 & \eta_2 \\ e^{i\theta} \eta_2 & \eta_1 \end{pmatrix},$$

where the final equality defines  $\eta_1$  and  $\eta_2$ . Note that although  $e^{\pm i\theta/2}$  is only defined on the slit domain  $D_-^\times = D^\times \setminus (-1, 0)$ , both  $\eta_1$  and  $\eta_2$  make sense on all of  $D^\times$ .

Now,  $(A^{0,1})^{g_\mu} = g_\mu^{-1} A^{0,1} g_\mu + g_\mu^{-1} \bar{\partial} g_\mu$ , so we compute

$$g_\mu^{-1} \bar{\partial} g_\mu = \begin{pmatrix} e^{2i\theta} \eta_2^2 / 4r & e^{-i\theta} D\mu + \frac{1}{4r} e^{i\theta} \eta_1 \eta_2 \\ D\mu - \frac{1}{4r} e^{2i\theta} \eta_1 \eta_2 & -e^{2i\theta} \eta_2^2 / 4r \end{pmatrix} d\bar{z},$$

where we have written

$$D = e^{i\theta/2} \partial_{\bar{z}} e^{i\theta/2}$$

and are using the identity  $\eta_1^2 - e^{i\theta} \eta_2^2 = 1$ . Setting  $U = w + (i/r)v$ , and recalling from (13) that  $\bar{w} + \frac{i}{r}\bar{v} = -e^{i\theta}U$ , then further computation gives

$$g_\mu^{-1} A^{0,1} g_\mu = \frac{1}{2} e^{i\theta} \begin{pmatrix} -(1/4r)(\eta_1^2 + e^{i\theta} \eta_2^2) & U - (1/2r)\eta_1 \eta_2 \\ e^{i\theta}((1/2r)\eta_1 \eta_2 + U) & (1/4r)(\eta_1^2 + e^{i\theta} \eta_2^2) \end{pmatrix} d\bar{z}.$$

Adding these terms together yields

$$(17) \quad (A^{0,1})^{g_\mu} = \begin{pmatrix} -\frac{1}{8r} e^{i\theta} & e^{-i\theta} D\mu + \frac{1}{2} e^{i\theta} U \\ D\mu + \frac{1}{2} e^{2i\theta} U & \frac{1}{8r} e^{i\theta} \end{pmatrix} d\bar{z}.$$

Recall that our goal is to gauge away the off-diagonal components. To do this, we must choose  $\mu$  so that  $D\mu + \frac{1}{2} e^{2i\theta} U = 0$ . Using that

$$D = e^{i\theta} \left( \partial_{\bar{z}} - \frac{e^{i\theta}}{4r} \right), \quad \text{and} \quad \partial_{\bar{z}} = \frac{1}{2} e^{i\theta} \left( \partial_r + \frac{i}{r} \partial_\theta \right),$$

we write this equation, in terms of the operator  $P$  in (15), as

$$(18) \quad \left( \partial_r - \frac{1}{r} P \right) \mu = -U := -w - \frac{i}{r} v.$$

We solve this in a slightly unexpected way, by showing that the individual equations  $\partial_r \mu = -w$ ,  $P\mu = iv$  are compatible. Indeed,  $\partial_r P\mu = P\partial_r \mu$  is the same as  $\partial_r(iv) = P(-w)$ , which follows precisely from the flatness of  $A$  (as must be the case!). Noting that  $P$  is invertible, we can now simply take  $\mu = P^{-1}(iv)$ , and this satisfies both equations.

The final point is that if we write  $\bar{P} = -Q$ , where  $Q = P - 1$ , then  $Q(e^{i\theta} \mu) = e^{i\theta} P\mu$ , so that

$$Q(e^{i\theta} \mu + \bar{\mu}) = e^{i\theta} P\mu - \bar{P}\bar{\mu} = e^{i\theta} iv - \overline{(iv)} = i(e^{i\theta} v + \bar{v}) = 0,$$

by (13) again. Since  $Q$  is also invertible,  $e^{i\theta} \mu + \bar{\mu} = 0$ , hence  $\gamma_\mu$  is skew-Hermitian and  $g_\mu$  is a unitary gauge transformation, so that  $A^g$  is still flat.  $\square$

**Corollary 3.3.** *Let  $(A, \Phi)$  be a pair on  $D^\times$  with*

$$F_A = 0, \quad [\Phi \wedge \Phi^*] = 0, \quad \bar{\partial}_A \Phi = 0$$

*and  $\det \Phi = \det \Phi_\infty^{\text{fid}}$ . Then there exists a unitary gauge transformation  $g$  on  $D^\times$  such that  $(A, \Phi)^g = (A_\infty^{\text{fid}}, \Phi_\infty^{\text{fid}})$ .*

**3.2. The desingularized fiducial solutions.** We now find a family of solutions  $(A_t^{\text{fid}}, \Phi_t^{\text{fid}})$  of Hitchin's rescaled equation

$$(19) \quad \mathcal{H}_t(A, \Phi) := (F_A + t^2[\Phi \wedge \Phi^*], \bar{\partial}_A \Phi) = 0, \quad t > 0,$$

which are smooth across  $z = 0$  and which converge to  $(A_\infty^{\text{fid}}, \Phi_\infty^{\text{fid}})$  as  $t \nearrow \infty$ . Since this limiting pair is purely diagonal and purely off-diagonal, respectively, in the fiducial frame, it is natural to impose that  $A_t^{\text{fid}}$  and  $\Phi_t^{\text{fid}}$  have the same form. Thus we make the ansatz that in the same fiducial frame,

$$(20) \quad \begin{aligned} A_t^{\text{fid}} &= f_t(r) \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \left( \frac{dz}{z} - \frac{d\bar{z}}{\bar{z}} \right), \\ \Phi_t^{\text{fid}} &= \varphi_t^{\text{fid}} dz = \begin{pmatrix} 0 & r^{1/2} e^{h_t(r)} \\ r^{1/2} e^{i\theta} e^{-h_t(r)} & 0 \end{pmatrix} dz \end{aligned}$$

(according to [MaWo93] this ansatz essentially captures all possible solutions). We calculate that,

$$F_{A_t^{\text{fid}}} + t^2[\varphi_t^{\text{fid}} \wedge (\varphi_t^{\text{fid}})^*] = \left( \frac{1}{r} \partial_r f_t - 2rt^2 \sinh(2h_t) \right) \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} d\bar{z} \wedge dz,$$

and in addition,

$$\bar{\partial}_{A_t^{\text{fid}}} \Phi_t^{\text{fid}} = \left( \partial_{\bar{z}} \varphi_t^{\text{fid}} - \frac{f_t}{\bar{z}} [a_1, \varphi_t^{\text{fid}}] \right) d\bar{z} \wedge dz = 0.$$

After some computation, we are led to the pair of equations

$$(21) \quad \partial_r f_t(r) = 2t^2 r^2 \sinh 2h_t,$$

$$(22) \quad f_t(r) = \frac{1}{8} + \frac{1}{4} r \partial_r h_t(r).$$

Now apply  $r\partial_r$  to (22) and insert into (21) to get

$$(23) \quad (r\partial_r)^2 h = 8t^2 r^3 \sinh 2h.$$

To simplify this, set  $\rho = \frac{8}{3} tr^{3/2}$ , so that  $r\partial_r = \frac{3}{2} \rho \partial_\rho$ . Writing  $h_t(r) = \psi(\rho)$  for some function  $\psi$ , we obtain

$$(24) \quad (\rho \partial_\rho)^2 \psi = \frac{1}{2} \rho^2 \sinh 2\psi$$

which is  $t$ -independent. Once we identify a suitable solution of this equation, we will have the solutions

$$(25) \quad h_t(r) = \psi\left(\frac{8}{3} tr^{3/2}\right), \quad f_t(r) = \frac{1}{8} + \frac{1}{4} r \partial_r h_t$$

of the original system. The equation (24) is of Painlevé III type. It is known [MTW77], [Wi01] that there exists a unique solution which decays exponentially and has the correct behavior as  $\rho \rightarrow 0$ , namely

- $\psi(\rho) \sim -\log(\rho^{1/3} (\sum_{j=0}^{\infty} a_j \rho^{4j/3}))$ ,  $\rho \searrow 0$ ;
- $\psi(\rho) \sim K_0(\rho) \sim \rho^{-1/2} e^{-\rho}$ ,  $\rho \nearrow \infty$ ;
- $\psi(\rho)$  is monotonically decreasing (and hence strictly positive).

The notation  $A \sim B$  indicates a complete asymptotic expansion. In the first case, for example, for each  $N \in \mathbb{N}$ ,

$$\left| \rho^{-1/3} e^{-\psi(\rho)} - \sum_{j=0}^N a_j \rho^{4j/3} \right| \leq C \rho^{4(N+1)/3},$$

with a corresponding expansion for any derivative. The function  $K_0(\rho)$  is the Macdonald function (or Bessel function of imaginary argument) of order 0; it has a complete asymptotic expansion involving terms of the form  $e^{-\rho} \rho^{-1/2-j}$ ,  $j \geq 0$ , as  $\rho \rightarrow \infty$ .

All of these calculations were sketched to us in a personal communication by Andy Neitzke, and we gratefully acknowledge his assistance.

From (26) we can now compute the asymptotics of  $f_t(r)$  and  $h_t(r)$ .

**Lemma 3.4.** *The functions  $f_t(r)$  and  $h_t(r)$  have the following properties:*

- (i) *As a function of  $r$ ,  $f_t$  has a double zero at  $r = 0$  and increases monotonically from  $f_t(0) = 0$  to the limiting value  $1/8$  as  $r \nearrow \infty$ . In particular,  $0 \leq f_t \leq \frac{1}{8}$ .*
- (ii) *As a function of  $t$ ,  $f_t$  is also monotone increasing. Further,  $\lim_{t \nearrow \infty} f_t = f_\infty \equiv \frac{1}{8}$  uniformly in  $\mathcal{C}^\infty$  on any half-line  $[r_0, \infty)$ , for  $r_0 > 0$ .*
- (iii) *There are uniform estimates*

$$\sup_{r>0} r^{-1} f_t(r) \leq C t^{2/3} \quad \text{and} \quad \sup_{r>0} r^{-2} f_t(r) \leq C t^{4/3},$$

where  $C$  is independent of  $t$ .

- (iv) *When  $t$  is fixed and  $r \searrow 0$ , then  $h_t(r) \sim -\frac{1}{2} \log r + b_0 + \dots$ , where  $b_0$  is an explicit constant. On the other hand,  $|h_t(r)| \leq C \exp(-\frac{8}{3} t r^{3/2}) / (t r^{3/2})^{1/2}$  uniformly for  $t \geq t_0 > 0$ ,  $r \geq r_0 > 0$ .*
- (v) *There is a uniform estimate*

$$\sup_{r \in (0,1)} r^{1/2} e^{\pm h_t(r)} \leq C,$$

where  $C$  is independent of  $t$ .

*Proof.* Define  $\eta(\rho) = \frac{1}{8} + \frac{3}{8} \rho \psi'(\rho)$ , where  $\rho = \frac{8t}{3} r^{3/2}$ , so that  $f_t(r) = \eta(\rho)$ . By (24),

$$\eta'(\rho) = \frac{3}{8} \rho (\psi''(\rho) + \rho^{-1} \psi'(\rho)) = \frac{3}{16} \rho \sinh(2\psi(\rho)),$$

which implies that  $\eta'(\rho) \geq 0$  since  $\psi \geq 0$ . In fact, (26) also implies that  $\lim_{\rho \rightarrow \infty} \eta(\rho) = \frac{1}{8}$  and

$$(27) \quad \eta(\rho) \sim \frac{1}{8} + \frac{3}{8} \rho \left( -\frac{1}{3\rho} - \frac{4a_1}{3a_0} \rho^{1/3} + O(\rho^{4/3}) \right) = -\frac{a_1}{2a_0} \rho^{4/3} + O(\rho^{7/3}),$$

when  $\rho$  is small, so  $f_t$  has a double zero at 0 as a function of  $r$ . This proves (i) and (ii). Substituting  $r = (\frac{3\rho}{8t})^{2/3}$  now gives

$$\frac{f_t}{r} = \left( \frac{8t}{3} \right)^{2/3} \frac{\eta(\rho)}{\rho^{2/3}} \quad \text{and} \quad \frac{f_t}{r^2} = \left( \frac{8t}{3} \right)^{4/3} \frac{\eta(\rho)}{\rho^{4/3}}.$$

The estimates in (iii) thus follow from (27), which implies that  $\eta(\rho)/\rho^{2/3}$  and  $\eta(\rho)/\rho^{4/3}$  are bounded for  $\rho > 0$ . Property (iv) also follows directly from (26). Finally, uniform boundedness of  $r^{1/2}e^{\pm h_t(r)}$  on  $(0, 1)$  is via the substitution  $\rho = \frac{8}{3}tr^{3/2}$  equivalent to uniform boundedness of the function

$$\rho \mapsto \left(\frac{3}{8}t^{-1}\rho\right)^{1/3}e^{\pm\psi(\rho)}$$

on the interval  $(0, \frac{8}{3}t)$ . This again follows from (26).  $\square$

**Corollary 3.5.** *The solutions  $(A_t^{\text{fid}}, \Phi_t^{\text{fid}})$  of the rescaled Hitchin equation are smooth at  $z = 0$ . Further, they converge exponentially in  $t$ , uniformly in  $\mathcal{C}^\infty$  on any exterior region  $r \geq r_0 > 0$  to  $(A_\infty^{\text{fid}}, \Phi_\infty^{\text{fid}})$ .*

*Proof.* The preceding Lemma gives that  $r^{1/2}e^{h_t(r)} \sim 1 + \dots$  and  $r^{1/2}e^{i\theta}e^{-h_t(r)} \sim z + \dots$  for fixed  $t$  as  $r \rightarrow 0$ , and similarly,  $f_t \sim c_1|z|^2 + \dots$ , while if  $r$  is fixed, then

$$(A_t^{\text{fid}}, \Phi_t^{\text{fid}}) \longrightarrow (A_\infty^{\text{fid}}, \Phi_\infty^{\text{fid}})$$

exponentially in  $t$ , uniformly in  $\mathcal{C}^\infty$  on any exterior region  $r \geq r_0 > 0$ .  $\square$

**3.3. The complex gauge orbit of the fiducial solutions.** We first determine the holomorphic normal form of a Higgs bundle  $(E, \bar{\partial}, \Phi)$  near a simple zero of  $\det \Phi$ .

**Lemma 3.6.** *Near any simple zero of  $\det \Phi$ , there is a complex coordinate  $z$  and a holomorphic frame of  $E$  such that*

$$\Phi = \begin{pmatrix} 0 & 1 \\ z & 0 \end{pmatrix} dz, \quad \det \Phi = -z dz^2.$$

*Proof.* Choose a complex coordinate  $z$  near  $p \in \mathfrak{p}_\Phi$  such that  $\det \Phi = -z dz^2$ . Writing  $\Phi = \varphi dz$  with respect to a holomorphic frame, then since  $p$  is a simple zero,  $\varphi(0)$  must be nilpotent, but not the zero matrix (else  $\det \varphi$  would vanish like  $z^2$ ). Applying a constant gauge transformation, we can thus take

$$\varphi(z) = \begin{pmatrix} a(z) & b(z) \\ c(z) & -a(z) \end{pmatrix}, \quad \text{with } \varphi(0) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

Noting that  $\sqrt{b(z)}$  is holomorphic near 0, define the complex unimodular gauge transformation

$$g(z) = \frac{1}{\sqrt{b(z)}} \begin{pmatrix} b(z) & 0 \\ -a(z) & 1 \end{pmatrix};$$

it is straightforward to check that  $g^{-1}\Phi g$  has the correct form.  $\square$

**Remark.** Describing the Higgs bundle  $(E, \bar{\partial}, \Phi)$  using spectral curves as in [Hi87, Section 8], then Lemma 3.6 also follows from the pushforward-pullback formula for vector bundles (see [Hi99, Chapter 2, Proposition 4.2]).

In this holomorphic frame,  $\bar{\partial}_E$  is the standard  $\bar{\partial}$  operator, so the connection  $A$  is then completely determined by the fact that  $A^{0,1} = 0$  in this frame and its unitarity with respect to the Hermitian metric  $H$ . As a consequence, if  $(A_1, \Phi_1)$  and  $(A_2, \Phi_2)$  are two Higgs pairs with  $\det \Phi_1 = \det \Phi_2 := q$ , then for any simple zero  $p$  of  $q$  there exists a complex gauge transformation  $g$  such that  $(A_1, \Phi_1)^g = (A_2, \Phi_2)$  near  $p$ . (We note that the corresponding statement is false in neighborhoods of higher order zeroes of  $q$ .) In particular, since  $\det \Phi_t^{\text{fid}} = -z dz^2$ , all of the fiducial solutions  $(A_t^{\text{fid}}, \Phi_t^{\text{fid}})$  are equivalent under the complex gauge action near 0 for  $0 < t < \infty$ . More precisely, if we define the Higgs pair

$$A_0 = 0, \quad \Phi_0 = \begin{pmatrix} 0 & 1 \\ z & 0 \end{pmatrix} dz,$$

we get the following:

**Lemma 3.7.** (i) *Over  $D$ , the fiducial solution  $(A_t^{\text{fid}}, \Phi_t^{\text{fid}})$  is complex gauge equivalent to  $(A_0, \Phi_0)$  for all  $0 < t < \infty$ . An explicit gauge transformation achieving  $(A_0, \Phi_0)^{g_t} = (A_t^{\text{fid}}, \Phi_t^{\text{fid}})$  is given by*

$$g_t = \begin{pmatrix} r^{-\frac{1}{4}} e^{-\frac{1}{2} h_t(r)} & 0 \\ 0 & r^{\frac{1}{4}} e^{-\frac{1}{2} h_t(r)} \end{pmatrix}.$$

(ii) *Over  $D^\times$ , the limiting fiducial solution  $(A_\infty^{\text{fid}}, \Phi_\infty^{\text{fid}})$  is complex gauge equivalent to  $(A_0, \Phi_0)$  by the singular gauge transformation*

$$g_\infty = \begin{pmatrix} r^{-\frac{1}{4}} & 0 \\ 0 & r^{\frac{1}{4}} \end{pmatrix},$$

*i.e.,  $(A_0, \Phi_0)^{g_\infty} = (A_\infty^{\text{fid}}, \Phi_\infty^{\text{fid}})$ .*

*Proof.* Both assertions are checked by direct computation.  $\square$

#### 4. LIMITING CONFIGURATIONS

We now start on the global aspects of this problem. As explained in the introduction, our existence theorem for solutions of Eq. (5) involves gluing copies of the fiducial solution into what we call limiting configurations. These fiducial solutions have already been discussed, and our next goal is to describe the other building block, the limiting configurations.

**Definition 4.1.** Let  $(E, \bar{\partial}, \Phi)$  be a Higgs bundle, where  $\Phi$  is simple, and set  $q = \det \Phi$ . Suppose too that  $H$  is a Hermitian metric on  $E$ . A *limiting configuration* associated to  $q$  is a pair  $(A_\infty, \Phi_\infty)$  over  $X^\times$ , where  $\det \Phi_\infty = q$ , and which satisfies the decoupled Hitchin equations

$$(28) \quad F_{A_\infty}^\perp = 0, \quad [\Phi_\infty \wedge \Phi_\infty^*] = 0, \quad \bar{\partial}_{A_\infty} \Phi_\infty = 0.$$

If, in addition,  $(A_\infty, \Phi_\infty)$  agrees with  $(A_\infty^{\text{fid}}, \Phi_\infty^{\text{fid}})$  with respect to some fixed choice of holomorphic coordinate system and unitary frame near every point of  $\mathfrak{p}_\Phi$ , then  $(A_\infty, \Phi_\infty)$  is called a *framed limiting configuration*. Because the

determinant is fixed, we require  $A_\infty$  and  $\Phi_\infty$  to be trace-free, the former relative to some fixed background connection.

**Remark.** By Corollary 3.3, any limiting configuration  $\det \Phi_\infty = \det \Phi$  is gauge equivalent to a framed limiting configuration: simply extend the gauge transformation  $\gamma$  obtained there using cutoff functions.

The main objective in this section is to prove the following

**Theorem 4.1.** *Let  $(E, \bar{\partial}, \Phi)$  be a Higgs bundle with simple Higgs field. Then there is a Hermitian metric  $H_0$  on  $E$  so that if  $A = A(H_0, \bar{\partial})$  is the associated Chern connection then the Higgs pair  $(A, \Phi)$  is complex gauge equivalent via some transformation  $g_\infty \in \Gamma(X^\times, \text{SL}(E))$  to a framed limiting configuration  $(A_\infty, \Phi_\infty)$ , i.e.,  $(A_\infty, \Phi_\infty) := (A, \Phi)^{g_\infty}$ .*

**Remark.** As we shall see below in Theorem 6.6, every framed limiting configuration arises in this way. Further, any two framed limiting configurations arising from a Higgs pair  $(A, \Phi)$  are related by a complex gauge transformation which is the identity near the puncture. As in [Hi87, Theorem 2.7] we deduce that a limiting configuration is uniquely determined by the initial Higgs pair up to unitary gauge transformations.

**Remark.** The proof of this theorem given here is analytic. As pointed out by Nigel Hitchin, the same conclusion may be reached by interpreting limiting configurations as solutions corresponding to a certain parabolic structure on  $E$ . This is explained in [MSWW15]. The more explicit construction presented here is useful for other purposes however; in particular, it will be used in the application of the results of this paper to the study of asymptotics of the hyperkähler metric on  $\mathcal{M}$ .

The proof of Theorem 4.1 proceeds by first bringing the simple Higgs field  $\Phi$  into a normal form on all of  $X$ . Using some of the remaining gauge freedom (i.e., gauge transformations preserving this normal form), we then transform the connection to one with vanishing trace-free curvature, thus producing a limiting configuration. This requires a brief foray into the theory of conic operators. The final subsection considers the local deformation theory of the space of limiting configurations.

**4.1. Normal form for the Higgs field.** Fix the Higgs bundle  $(E, \bar{\partial}, \Phi)$  and assume that  $\Phi$  is a simple Higgs field. We now bring  $\Phi$  into a simple normal form by a complex gauge transformation. Specifically, we smoothly “off-diagonalize”  $\Phi$  near each of its zeroes, and make it normal elsewhere with respect to a hermitian metric.

Fix a coordinate chart  $(U_i, z_i)$  and holomorphic frame near each  $p_i$  so that  $\Phi|_{U_i}$  equals the expression in Lemma 3.6. Define a Hermitian metric  $H_0$  on  $U_i$  by declaring this frame to be unitary, and then extend  $H_0$  arbitrarily on the rest of  $X$ . Associated to  $H_0$  is its Chern connection  $A$ . The existence of a unitary holomorphic frame near each  $U_i$  implies that the connection matrix of  $A$  in this frame vanishes. Finally, using Lemma 3.7 we can choose



a complex gauge transformation  $g \in \Gamma(\cup U_i^\times, \mathrm{SL}(E))$  so that  $(A, \Phi)^g$  agrees with the fiducial solution in these neighbourhoods.

We now wish to extend this  $g$  to all of  $X$  so that  $\Phi^g$  is normal outside the  $U_i$ . To motivate this, recall first that any invertible matrix  $\varphi \in \mathfrak{sl}(2, \mathbb{C})$  is (pointwise) conjugate to a trace-free diagonal matrix. It is impossible to carry out this diagonalization consistently on  $X^\times$  since the eigenspaces are interchanged when traversing a loop surrounding any one of the  $p_i$ , so we settle on the less ambitious goal of conjugating to a normal matrix.

Given  $\varphi$ , there are subsets  $D_\varphi, N_\varphi \subset \mathrm{SL}(2, \mathbb{C})$  of matrices which diagonalize and normalize  $\varphi$ , respectively, at any point. Fixing a basepoint  $g_\varphi \in D_\varphi$ ,

$$D_\varphi = \left\{ g_\varphi \begin{pmatrix} \mu & 0 \\ 0 & \mu^{-1} \end{pmatrix} \mid \mu \in \mathbb{C}^* \right\} \cup \left\{ g_\varphi \begin{pmatrix} 0 & i\mu \\ i\mu^{-1} & 0 \end{pmatrix} \mid \mu \in \mathbb{C}^* \right\}$$

and

$$N_\varphi = D_\varphi \cdot \mathrm{SU}(2) = \left\{ g_\varphi \begin{pmatrix} \mu & 0 \\ 0 & \mu^{-1} \end{pmatrix} M \mid \mu \in \mathbb{C}^*, M \in \mathrm{SU}(2) \right\}.$$

The Hermitian metric  $H_0$  gives meaning to Hermitian adjoints and normal endomorphisms. Since any complex vector bundle is trivial over  $X^\times$ , we can write  $\Phi = \varphi \otimes \kappa$  on this punctured surface, where  $\kappa$  trivializes  $K$  over  $X^\times$  and  $\varphi \in \mathcal{C}^\infty(X^\times, \mathfrak{sl}(2, \mathbb{C}))$ . There is a smooth fibration  $\mathcal{N}_\varphi \rightarrow X^\times$ , where each fibre  $N_{\varphi(x)}$  is diffeomorphic to  $\mathcal{N} := N_{\mathrm{Id}}$ . If  $g: U \rightarrow \mathrm{SL}(2, \mathbb{C})$  diagonalizes  $\varphi$  over  $U$ , then  $g(x)N$  is a local trivialization of  $\mathcal{N}_\varphi$  over  $U$ . Since the complex square root is well-defined over simply-connected sets, such a section  $g$  always exists locally. However, the fibres  $\mathcal{N}$  are homotopy-equivalent to  $\mathrm{SU}(2) \cong S^3$  and  $X^\times$  retracts onto a bouquet of circles, so there are no obstructions to extending sections. This proves the

**Lemma 4.2.** *Any normalizing local section  $g: U \rightarrow \mathcal{N}_\varphi$  on an open set  $U \subset X^\times$  extends to a global section  $X^\times \rightarrow \mathcal{N}_\varphi$ . In particular, there exists a complex frame of  $E|_{X^\times}$  with respect to which  $\Phi$  is a normal matrix.*

**4.2. Gauging away  $F_A^\perp$ .** We now start the proof of Theorem 4.1. At this point we have a Hermitian metric  $H_0$  and a complex gauge transformation  $g_0 \in \Gamma(X^\times, \mathrm{SL}(E))$  such that  $(A, \Phi)^{g_0}$  is fiducial near each  $p_i$  in some unitary frame, and  $\Phi^{g_0}$  is normal on the entire surface. To simplify notation, replace  $(A, \Phi)^{g_0}$  by  $(A, \Phi)$  until further notice. We now show how to use the remaining gauge freedom to transform the connection  $A$  to one for which  $F_A^\perp = 0$ . In this subsection we set up the equations for doing so.

Recall from Section 3.1 that the infinitesimal complex stabilizer of  $\Phi$  is a holomorphic line bundle  $L_\Phi^\mathbb{C} = \{\gamma \in \mathfrak{sl}(E) \mid [\gamma, \Phi] = 0\}$ ; the skew-Hermitian and Hermitian elements are denoted  $L_\Phi := L_\Phi^\mathbb{C} \cap \mathfrak{su}(E)$  and  $iL_\Phi$ ; these are real line bundles over  $X^\times$ . The Jacobi identity shows that the bracket of sections of  $L_\Phi$  is again a section of  $L_\Phi$ .

**Lemma 4.3.** *If  $\Phi$  is normal,  $A$  is unitary and  $\bar{\partial}_A \Phi = 0$ , then  $F_A^\perp \in \Omega^2(L_\Phi)$ .*

*Proof.* This is a purely local statement. Choose a local complex coordinate and unitary eigenframe for  $\Phi$  so that

$$\Phi = \begin{pmatrix} \lambda & 0 \\ 0 & -\lambda \end{pmatrix} dz.$$

The connection form  $\alpha = \alpha^{0,1} - (\alpha^{0,1})^*$  is determined by its  $(0,1)$ -part

$$\alpha^{0,1} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} d\bar{z}.$$

Now

$$\bar{\partial}_A \Phi = \begin{pmatrix} \partial_{\bar{z}} \lambda & -2b\lambda \\ 2c\lambda & -\partial_{\bar{z}} \lambda \end{pmatrix} d\bar{z} \wedge dz = 0$$

implies  $b = c = 0$ , so

$$(29) \quad \alpha^{0,1} = \begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix} d\bar{z}.$$

In particular,  $[\alpha \wedge \alpha] = 0$ , so  $F_A = d\alpha$  and hence

$$F_A^\perp = \begin{pmatrix} \operatorname{Re} \partial_z(a-d) & 0 \\ 0 & \operatorname{Re} \partial_z(d-a) \end{pmatrix} dz \wedge d\bar{z}.$$

□

The bundles  $L_\Phi$  and  $iL_\Phi$  are parallel with respect to the induced unitary connection on  $\mathfrak{gl}(E)$ . Indeed,  $d_A \Phi = 0$  (the  $(1,0)$  part of the derivative automatically vanishes in this dimension), so  $[d_A \gamma \wedge \Phi] = d_A[\gamma, \Phi] = 0$ . In particular, the connection Laplacian

$$\Delta_A := d_A^* d_A: \Omega^0(\mathfrak{su}(E)) \rightarrow \Omega^0(\mathfrak{su}(E))$$

restricts to a map  $\Omega^0(iL_\Phi) \rightarrow \Omega^0(iL_\Phi)$ .

**Proposition 4.4.** *If  $A$  is a unitary connection and  $\gamma \in \Omega^0(iL_\Phi)$ , then*

$$(30) \quad F_{A^{\exp(\gamma)}}^\perp = 0 \Leftrightarrow \Delta_A \gamma = i * F_A^\perp.$$

*Proof.* Recalling Eq. (4), and since  $\gamma$  is Hermitian,  $g = \exp(\gamma) = g^*$ ,

$$(31) \quad F_{A^g}^\perp = 0 \Leftrightarrow F_A^\perp + \bar{\partial}_A(\exp(2\gamma)\partial_A \exp(-2\gamma)) = 0.$$

Computing in a local unitary eigenframe for  $\Phi$  gives

$$\begin{aligned} \partial_A \exp(-2\gamma) &= -2 \exp(-2\gamma) \partial_A \gamma \\ &\implies \bar{\partial}_A(\exp(2\gamma)\partial_A \exp(-2\gamma)) = -2 \bar{\partial}_A \partial_A \gamma. \end{aligned}$$

If  $\Lambda$  denotes contraction with the Kähler form  $\omega$ , then, by [Ni00, Prop. 1.4.21 and 1.4.22],

$$2i\Lambda \bar{\partial}_A \partial_A \gamma = \Delta_A \gamma - 2i\Lambda[F_A, \gamma].$$

We use here that the induced connection  $\operatorname{End}(A)$  on  $\operatorname{End}(E)$  has curvature satisfying  $F_{\operatorname{End}(A)} \gamma = [F_A, \gamma]$ . However, by Lemma 4.3,  $\Lambda[F_A, \gamma] = *[F_A, \gamma] = *[F_A^\perp, \gamma] = 0$ , so (31) becomes  $\Delta_A \gamma = i * F_A^\perp$ . □

**4.3. Gauging away  $F_A^\perp$ , continued.** We now show how to solve (30). In polar coordinates on each punctured disk  $U^\times$ ,

$$\Delta_A = \nabla_A^* \nabla_A = -\frac{1}{r^2} (\nabla_{r\partial_r}^2 + \nabla_{\partial_\theta}^2).$$

This is an elliptic conic operator; we shall appeal to the theory of such operators, referring to [MaMo11] and references therein for more details.

Fix a trivialization of  $iL_\Phi$  in  $U^\times$  to identify sections with functions  $\gamma: U^\times \rightarrow i\mathfrak{su}(2)$ . There is unitary frame in  $U^\times$  so that

$$(32) \quad A = \alpha d\theta = \frac{1}{4} \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} d\theta;$$

in this frame  $\nabla_{r\partial_r} = r\partial_r$  and  $\nabla_{\partial_\theta} = \partial_\theta + \alpha$ , hence

$$\Delta_A \gamma = -\frac{1}{r^2} ((r\partial_r)^2 \gamma + \partial_\theta^2 \gamma + 2[\alpha, \partial_\theta \gamma] + [\alpha, [\alpha, \gamma]]) = -(\partial_r^2 + \frac{1}{r} \partial_r + \frac{1}{r^2} T) \gamma.$$

Here  $T$  is the  $r$ -independent *tangential operator*, acting on sections of the restriction of  $\mathfrak{su}(E)$  over the  $S^1$  link. The coefficients of  $\Delta_A$  are smooth away from  $\mathfrak{p}_\Phi$ , and are *polyhomogeneous* at these points, i.e., near each such point, any coefficient  $a$  has a complete asymptotic expansion

$$(33) \quad a \sim \sum_j \sum_{k=0}^{N_j} r^{\nu_j} (\log r)^k a_{j,k}(\theta),$$

with a corresponding expansion for each of its derivatives. We encode the exponents which appear in this expansion as an index set  $\{\nu_j, N_j\} \subset \mathbb{C} \times \mathbb{N}$ , which has the property that  $\operatorname{Re} \nu_j \rightarrow \infty$  as  $j \rightarrow \infty$ .

**Definition 4.2.** A number  $\nu \in \mathbb{C}$  is called an *indicial root* for  $\Delta_A$  if there exists some  $\zeta = \zeta(\theta)$  such that  $\Delta_A(r^\nu \zeta(\theta)) = \mathcal{O}(r^{\nu-1})$  (rather than the expected rate  $\mathcal{O}(r^{\nu-2})$ ). We let  $\Gamma(\Delta_A)$  denote the set of indicial roots of  $\Delta_A$ .

Thus  $\nu$  is an indicial root provided there is some leading order cancellation. It is not hard to see that  $\nu \in \Gamma(\Delta_A)$  if and only if  $-\nu^2$  is an eigenvalue for the tangential operator of  $\Delta_A$  and  $\zeta$  is the corresponding eigenfunction, i.e.,  $(\nabla_{\partial_\theta}^2 + \nu^2)\zeta(\theta) = 0$ . Proposition 4.6 below indicates the importance of this notion. Before turning to this, however, we compute the indicial roots for the connection Laplacian.

**Lemma 4.5.** *The set of indicial roots of  $\Delta_A$  on sections of  $i\mathfrak{su}(E)$  is  $\Gamma(\Delta_A) = \frac{1}{2}\mathbb{Z}$ . On the other hand,  $\Gamma(\Delta_A|_{iL_\Phi}) = \frac{1}{2} + \mathbb{Z}$ , and in this latter case, all indicial roots are simple.*

*Proof.* This is a local computation near each  $p_i$ , so we work in the fixed fiducial frame near any such point. Let  $\{\tau_1, \tau_2, \tau_3\}$  be the standard basis of  $\mathfrak{su}(2)$ , i.e.

$$(34) \quad \tau_1 = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \quad \tau_2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \tau_3 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}.$$

Then  $[\tau_1, \tau_2] = 2\tau_3$ ,  $[\tau_2, \tau_3] = 2\tau_1$ ,  $[\tau_3, \tau_1] = 2\tau_2$  and the connection matrix  $\alpha$  in (32) equals  $\tau_1/4$ . Thus writing

$$\zeta = i\zeta^1\tau_1 + i\zeta^2\tau_2 + i\zeta^3\tau_3,$$

then

$$[\alpha, \partial_\theta \zeta] = \frac{1}{2}(-\partial_\theta \zeta^3 i\tau_2 + \partial_\theta \zeta^2 i\tau_3), \quad [\alpha, [\alpha, \zeta]] = -\frac{1}{4}(\zeta^2 i\tau_2 + \zeta^3 i\tau_3),$$

and hence

$$\nabla_{\partial_\theta}^2 \begin{pmatrix} \zeta^1 \\ \zeta^2 \\ \zeta^3 \end{pmatrix} = \begin{pmatrix} \partial_\theta^2 \zeta^1 \\ \partial_\theta^2 \zeta^2 - \partial_\theta \zeta^3 - \frac{1}{4} \zeta^2 \\ \partial_\theta^2 \zeta^3 + \partial_\theta \zeta^2 - \frac{1}{4} \zeta^3 \end{pmatrix}.$$

Thus  $\nabla_{\partial_\theta}^2 \zeta + \nu^2 \zeta = 0$  if and only if

$$(35) \quad (\partial_\theta^2 + \nu^2)\zeta^1 = 0, \quad \text{and} \quad \begin{aligned} (\partial_\theta^2 - \frac{1}{4} + \nu^2)\zeta^2 - \partial_\theta \zeta^3 &= 0, \\ (\partial_\theta^2 - \frac{1}{4} + \nu^2)\zeta^3 + \partial_\theta \zeta^2 &= 0. \end{aligned}$$

The first, uncoupled, equation has  $\mathbb{Z}$  as its set of indicial roots. On the other hand, restricting the coupled system to the span of  $\zeta_\ell(\theta) = e^{i\ell\theta}/\sqrt{2\pi}$ ,  $\ell \in \mathbb{Z}$ , then there is a homogeneous solution if and only if

$$\det \begin{pmatrix} -\ell^2 - \frac{1}{4} + \nu^2 & -i\ell \\ i\ell & -\ell^2 - \frac{1}{4} + \nu^2 \end{pmatrix} = 0,$$

which occurs precisely when  $\nu = \pm|\ell \pm 1/2|$ . Putting these two cases together shows that every  $\ell/2$ ,  $\ell \in \mathbb{Z}$ , is an indicial root.

Let us now compute the indicial roots for the restriction of  $\Delta_A$  to sections of  $iL_\Phi$ . On  $U$ , where  $\Phi$  is in fiducial form,  $iL_\Phi$  is spanned by  $\sigma(\theta) = \sin(\theta/2)i\tau_2 + \cos(\theta/2)i\tau_3$  (which equals  $-e^{-i\theta/2}\gamma_1$  in the notation of Section 3.1). Write  $\zeta(\theta) = f(\theta)\sigma(\theta)$  with  $f(2\pi) = -f(0)$ . Then  $\nabla_{\partial_\theta}^2 \zeta + \nu^2 \zeta = 0$  if and only if  $\partial_\theta^2 f + \nu^2 f = 0$ . The space  $\{f \in H^2(\mathbb{R}) \mid f(\theta + 2\pi) = -f(\theta)\}$  is spanned by the functions  $\{\zeta_{\ell+1/2}\}_{\ell \in \mathbb{Z}}$ , so this equation has a nontrivial solution if and only if  $\nu \in \mathbb{Z} + 1/2$ .  $\square$

We finally turn to the solvability of (30). To state the main result, let us first introduce appropriate function spaces. Let  $\mathcal{V}_b$  denote the span over  $\mathcal{C}^\infty$  of the vector fields  $r\partial_r$  and  $\partial_\theta$ . The corresponding  $L^2$ -based weighted  $b$ -Sobolev spaces are defined as follows. First, for  $\ell \in \mathbb{N}$ , set

$$H_b^\ell(\mathfrak{su}(E)) = \{u \in L^2(X) \mid V_1 \dots V_j u \in L^2(\mathfrak{su}(E)) \text{ for all } j \leq \ell, V_i \in \mathcal{V}_b\},$$

and then define, for  $\delta \in \mathbb{R}$ ,

$$r^\delta H_b^\ell(\mathfrak{su}(E)) = \{r^\delta u \mid u \in H_b^\ell(\mathfrak{su}(E))\}.$$

Since the area form is  $rdrd\theta$ , then locally near  $r = 0$ ,

$$r^\nu \in r^\delta H_b^\ell \Leftrightarrow \nu > \delta - 1.$$

This explains various index shifts below. We note, in particular, that

$$-1/2 < \nu < 1/2 \Leftrightarrow 1/2 < \delta < 3/2.$$

From the basic definitions,

$$\Delta_A: r^\delta H_b^{\ell+2}(iL_\Phi) \rightarrow r^{\delta-2} H_b^\ell(iL_\Phi)$$

is bounded for every  $\delta$  and  $\ell$ . The main result shows when this map is Fredholm.

**Proposition 4.6.** *Fix a real number  $\nu \notin \Gamma(\Delta_A|_{iL_\Phi})$  and define  $\delta = \nu + 1$ .*

*i) The operator*

$$\Delta_A: r^\delta H_b^{\ell+2}(\mathfrak{su}(E)) \rightarrow r^{\delta-2} H_b^\ell(\mathfrak{su}(E))$$

*is Fredholm, with index and nullspace remaining constant as  $\delta$  varies over each connected component of  $1 + (\mathbb{R} \setminus \Gamma(\Delta_A))$ .*

*ii) Suppose that  $\Delta_A \zeta = \eta \in r^{\delta-2} H_b^\ell(\mathfrak{su}(E))$ , where  $\zeta \in r^\delta L^2(\mathfrak{su}(E))$ . Then  $\zeta \in r^\delta H_b^{\ell+2}(\mathfrak{su}(E))$ . If  $\eta$  is polyhomogeneous, then so is  $\zeta$ , and the exponents in the expansion of  $\zeta$  are determined by the exponents in the expansion for  $\eta$  and the indicial roots  $\nu_i \in \Gamma(\Delta_A)$  with  $\nu_i > \delta - 1$ . In particular, any element of the nullspace of  $\Delta_A$  is polyhomogeneous, with terms in its expansion determined entirely by the indicial roots in this range.*

This is a straightforward adaptation of [MaMo11, Proposition 5 and 6], see [Ma91] for the proof. The particular case needed here is the

**Proposition 4.7.** *The mapping*

$$(36) \quad \Delta_A: r^\delta H_b^{\ell+2}(iL_\Phi) \rightarrow r^{\delta-2} H_b^\ell(iL_\Phi)$$

*is an isomorphism when  $1/2 < \delta < 3/2$ .*

*Proof.* By Proposition 4.6, this map is Fredholm since  $(-1/2, 1/2)$  contains no indicial roots. Next, as a general remark, the adjoint of (36) with weight  $\delta$  can be identified with the corresponding map with weight  $2 - \delta$ . Noting that the interval  $(1/2, 3/2)$  is invariant under this reflection, we see that it suffices to show that the nullspace of (36) is trivial for every  $\delta \in (1/2, 3/2)$ , for then the cokernel is also trivial and this map is an isomorphism.

By the final statement of Proposition 4.6, if  $\Delta_A \gamma = 0$  with  $\gamma \in r^\delta L_b^2$ ,  $1/2 < \delta < 3/2$ , then  $\gamma$  is polyhomogeneous with leading term  $r^{1/2}$ . Then

$$0 = \int_{X_\varepsilon^\times} \langle \Delta_A \gamma, \gamma \rangle = \int_{X_\varepsilon^\times} |d_A \gamma|^2 + \int_{\partial X_\varepsilon^\times} \langle \partial_\nu \gamma, \gamma \rangle,$$

where  $X_\varepsilon^\times = X^\times \setminus \bigcup B_\varepsilon(p_i)$ . Since  $\gamma \sim r^{1/2}$  and  $\partial_\nu \gamma \sim r^{-1/2}$ , and the length of  $\partial X_\varepsilon^\times$  is of order  $\varepsilon$ , the boundary term tends to zero. This proves that  $\gamma$  is parallel with respect to  $A$ ; however, since it vanishes as  $r \rightarrow 0$ , we must have  $\gamma \equiv 0$ . This proves the result.  $\square$

We apply this as follows. Our connection  $A$  is flat near each  $p_i$ , so the right hand side of (30) vanishes in each  $U_i^\times$ . Hence by the preceding proposition, there is a solution  $\gamma$  to this equation and it is polyhomogeneous and vanishes like  $r^{1/2}$  at these points. We obtain, therefore, a complex gauge

transformation  $g_1 = \exp \gamma$  such that  $\Phi^{g_1} = \Phi$  and the trace-free part of the curvature of  $A^{g_1}$  vanishes.

Resetting notation back to the initial Higgs pair  $(A, \Phi)$ , we have now produced a gauge-equivalent Higgs pair  $(A, \Phi)^{g_0 g_1}$  where the connection is projectively flat and the Higgs field is normal on all of  $X^\times$  and fiducial near each  $p_i$ . Note that  $A^{g_0 g_1}$  may not be in fiducial form in these neighbourhoods yet, but by Proposition 3.2 there is a further unitary gauge transformation  $g_2 \in \Gamma(\cup U_i^\times, \mathbf{U}(E))$  which stabilizes  $\Phi^{g_0}$  and puts the connection into fiducial form here as well. Extending  $g_2$  to a global unitary gauge transformation over  $X^\times$ , then  $g_\infty = g_2 g_1 g_0 \in \Gamma(X^\times, \mathbf{SL}(E))$  is the complex gauge transformation for which we have been searching. This finishes the proof of Theorem 4.1.

**4.4. Deformation theory of limiting configurations.** In this final section we examine the moduli space of limiting configurations up to unitary gauge transformations.

First, by Lemma 3.1, the second component  $\Phi_\infty$  of a limiting configuration is completely determined modulo unitary gauge by the holomorphic quadratic differential  $q$ .

Now consider the space of unitary connections solving

$$(37) \quad \bar{\partial}_A \Phi_\infty = 0, \quad F_A^1 = 0;$$

the gauge freedom is the stabilizer of  $\Phi_\infty$  in  $\Gamma(X^\times, \mathbf{SU}(E))$ . Fix a base solution  $A_\infty$  and write  $A = A_\infty + \alpha$ , where  $\alpha \in \Omega^1(X^\times, \mathfrak{su}(E))$ . The first equation of (37) gives that  $[\alpha^{0,1} \wedge \Phi_\infty] = [\alpha \wedge \Phi_\infty] = 0$ , so  $\alpha$  takes values in the real line bundle  $L_{\Phi_\infty} = L_{\Phi_\infty}^{\mathbb{C}} \cap \mathfrak{su}(E)$ . This implies in particular that  $[\alpha \wedge \alpha] = 0$ . From the second equation of (37) we obtain  $d_{A_\infty} \alpha = 0$ , hence the ungauged deformation space at  $(A_\infty, \Phi_\infty)$  is identified with

$$Z^1(X^\times; L_{\Phi_\infty}) := \{\alpha \in \Omega^1(X^\times, L_{\Phi_\infty}) \mid d_{A_\infty} \alpha = 0\}.$$

Next consider the subgroup  $\text{Stab}_{\Phi_\infty}$  of unitary gauge transformations which fix  $\Phi_\infty$ . If  $g \in \text{Stab}_{\Phi_\infty}$  lifts to a section of  $L_\Phi$ , i.e.,  $g = \exp(\gamma)$ ,  $\gamma \in \Omega^0(X^\times, L_{\Phi_\infty})$ , then  $g$  acts on  $\alpha \in \Omega^1(X^\times, L_{\Phi_\infty})$  by

$$\alpha^g = g^{-1} \circ \alpha \circ g + g^{-1}(d_{A_\infty} \gamma) = \alpha + d_{A_\infty} \gamma$$

(recall that  $L_{\Phi_\infty}$  is an  $A_\infty$ -parallel line subbundle of  $\mathfrak{su}(E)$ , so  $g^{-1} \circ \alpha \circ g = \alpha$  and  $d_{A_\infty} \exp(\gamma) = \exp(\gamma) d_{A_\infty} \gamma$ ). Hence the infinitesimal deformation space is

$$H^1(X^\times; L_{\Phi_\infty}) = Z^1(X^\times; L_{\Phi_\infty}) / B^1(X^\times; L_{\Phi_\infty}),$$

where

$$B^1(X^\times; L_{\Phi_\infty}) := \{d_{A_\infty} \gamma \mid \gamma \in \Omega^0(X^\times, L_{\Phi_\infty})\}.$$

**Lemma 4.8.** *If all zeroes of  $q$  are simple, then*

$$\dim_{\mathbb{R}} H^1(X^\times; L_{\Phi_\infty}) = 6\gamma - 6$$

where  $\gamma$  is the genus of  $X$ .

*Proof.* Working either with  $X^\times$  or the homotopy equivalent  $M = X \setminus B_\varepsilon(\mathfrak{p})$  (so  $\partial M$  is a union of  $k$  circles,  $k = |\mathfrak{p}|$ ), we note the following. First, there are no nontrivial parallel sections since  $L_{\Phi_\infty}$  is twisted near each  $p_i$ , so  $H^0(X^\times; L_{\Phi_\infty}) = 0$ ; by Poincaré duality,  $H^2(X^\times; L_{\Phi_\infty}) = H^0(M, \partial M; L_{\Phi_\infty}) = 0$  as well. Recall also that since  $L_\infty$  is a flat real line bundle, i.e., a local system of rank 1, it has Euler-Poincaré characteristic

$$\chi(X^\times; L_{\Phi_\infty}) = \chi(X^\times) = 2 - 2\gamma - k.$$

These facts together give that

$$\dim_{\mathbb{R}} H^1(X^\times; L_{\Phi_\infty}) = k + 2\gamma - 2 = 4\gamma - 4 + 2\gamma - 2 = 6\gamma - 6.$$

□

To obtain the moduli space, we must also divide the infinitesimal deformation space by the residual action of the component group  $\pi_0(\text{Stab}_{\Phi_\infty})$ . This is very similar to the determination of the moduli space of flat  $U(1)$ -connections on a principal  $U(1)$ -bundle, cf. [GoXi08, Section 2.2].

**Corollary 4.9.** *The moduli space of limiting configurations with determinant equal to a fixed holomorphic quadratic differential  $q$  with simple zeroes is a torus of dimension  $6\gamma - 6$ .*

*Proof.* An element  $g \in \text{Stab}_{\Phi_\infty}$  lifts to a multivalued section  $\log g: X^\times \rightarrow L_{\Phi_\infty}$ ; this has a well defined differential, and

$$\alpha^g = \alpha + d_{A_\infty} \log g.$$

If we now define

$$Z_{\mathbb{Z}}^1(X^\times; L_{\Phi_\infty}) = \left\{ \alpha \in Z^1(X^\times; L_{\Phi_\infty}) \mid \frac{1}{2\pi} \int_C \alpha \in \mathbb{Z} \text{ for any cycle } C \subset X^\times \right\},$$

then the action of  $\pi_0(\text{Stab}_{\Phi_\infty})$  corresponds to the action by translation of

$$H_{\mathbb{Z}}^1(X^\times; L_{\Phi_\infty}) = Z_{\mathbb{Z}}^1(X^\times; L_{\Phi_\infty}) / B^1(X^\times; L_{\Phi_\infty}).$$

In other words, the space of solutions  $\alpha$  modulo unitary gauge transformations in  $\text{Stab}_{\Phi_\infty}$  is the quotient of  $H^1(X^\times; L_{\Phi_\infty})$  by the lattice of classes with integer periods. □

**Remark.** This is consistent with [Hi87, Theorem 8.1], where it is shown that the space of Higgs bundles  $(E, \bar{\partial}, \Phi)$  with fixed determinant and where  $\Phi$  is simple is a  $(3\gamma - 3)$ -dimensional Prym variety (and thus a  $(6\gamma - 6)$ -dimensional real torus).

## 5. THE LINEARIZED PROBLEM

**5.1. Linearization of the Hitchin operator.** For any Hermitian vector bundle  $V \rightarrow X$  with connection  $\nabla$ , denote by  $H^k = H^k(V)$  the usual Sobolev space of sections  $s$  with  $\nabla^j s \in L^2$ ,  $j \leq k$ . More generally, we also consider

$H^k$  sections of fibre bundles. Since we are in the fixed determinant case, we fix a background connection  $A_0$  now and consider the Hitchin operator

$$\mathcal{H}_t(A, \Phi) = (F_A^\perp + t^2[\Phi \wedge \Phi^*], \bar{\partial}_A \Phi)$$

for connections  $A$  which are trace-free relative to  $A_0$  and trace-free Higgs fields  $\Phi$ . We further consider the orbit map

$$(38) \quad \mathcal{O}_{(A, \Phi)}(\gamma) = (A, \Phi)^g = (A^g, \Phi^g), \quad g = \exp(\gamma).$$

Our ultimate goal is to find a point in the complex gauge orbit of a given Higgs pair  $(A, \Phi)$  which is a solution of  $\mathcal{H}_t(A, \Phi) = 0$ . Since the condition that  $\bar{\partial}_A \Phi = 0$  is preserved under the complex gauge group, we in fact only need to find a solution of

$$(39) \quad \mathcal{F}_t(\gamma) := \text{pr}_1 \circ \mathcal{H}_t \circ \mathcal{O}_{(A, \Phi)}(\exp(\gamma)) = 0.$$

More explicitly, we wish to solve

$$F_{A^g}^\perp + t^2[\Phi^g \wedge (\Phi^g)^*] = 0, \quad g = \exp(\gamma).$$

Using the continuity of the multiplication maps  $H^1 \cdot H^1 \rightarrow L^2$  and  $H^2 \cdot H^1 \rightarrow H^1$ , it is straightforward that the three maps

$$(40) \quad \begin{aligned} \mathcal{H}_t: H^1(\Lambda^1 \otimes \mathfrak{su}(E) \oplus \Lambda^{1,0} \otimes \mathfrak{sl}(E)) &\rightarrow L^2(\Lambda^2 \otimes \mathfrak{su}(E) \oplus \Lambda^{1,1} \otimes \mathfrak{sl}(E)), \\ \mathcal{O}_{(A, \Phi)}: H^2(i\mathfrak{su}(E)) &\rightarrow H^1(\Lambda^1 \otimes \mathfrak{su}(E) \oplus \Lambda^{1,0} \otimes \mathfrak{sl}(E)), \\ \mathcal{F}_t: H^2(i\mathfrak{su}(E)) &\rightarrow L^2(\Lambda^2 \otimes \mathfrak{su}(E)) \end{aligned}$$

are well-defined and smooth. Here we identified the space of unitary connections on  $E$  with the linear space  $\Gamma(\Lambda^1 \otimes \mathfrak{su}(E))$  using the background connection  $A_0$ .

We now compute the linearizations of these mappings. First, the differential at  $g = \text{Id}$  of (38) is

$$\Lambda_{(A, \Phi)}\gamma = (\Lambda_A(\gamma), \Lambda_\Phi(\gamma)) = (\bar{\partial}_A \gamma - \partial_A \gamma^*, [\Phi, \gamma]),$$

so when  $\gamma \in \Omega^0(i\mathfrak{su}(E))$ ,

$$\Lambda_{(A, \Phi)}\gamma = (\bar{\partial}_A \gamma - \partial_A \gamma, [\Phi, \gamma]).$$

Next,

$$D\mathcal{H}_t \begin{pmatrix} \dot{A} \\ \dot{\Phi} \end{pmatrix} = \begin{pmatrix} d_A & t^2([\Phi \wedge \cdot]^* + [\Phi^* \wedge \cdot]) \\ [\Phi \wedge \cdot] & \bar{\partial}_A \end{pmatrix} \begin{pmatrix} \dot{A} \\ \dot{\Phi} \end{pmatrix},$$

whence

$$(D\mathcal{H}_t \circ \Lambda_{(A, \Phi)})(\gamma) = \begin{pmatrix} (\partial_A \bar{\partial}_A - \bar{\partial}_A \partial_A)\gamma + t^2([\Phi \wedge [\Phi, \gamma]^*] + [\Phi^* \wedge [\Phi, \gamma]]) \\ [\Phi \wedge (\bar{\partial}_A \gamma - \partial_A \gamma)] + \bar{\partial}_A [\Phi, \gamma] \end{pmatrix}.$$

The first component is precisely  $D\mathcal{F}_t(\gamma)$ . Using that  $\bar{\partial}_A \Phi = 0$ , as well as the fact that  $[\Phi \wedge \partial_A \gamma] = 0$  for dimensional reasons, the entire second component vanishes. Now recall from [Ni00, Prop. 1.4.21 and 1.4.22] the identities

$$2\bar{\partial}_A \partial_A = F_A - i * \Delta_A, \quad 2\partial_A \bar{\partial}_A = F_A + i * \Delta_A,$$



as well as

$$[\Phi \wedge [\Phi, \gamma]^*] = -[\Phi \wedge [\Phi^*, \gamma]],$$

to rewrite

$$(41) \quad D\mathcal{F}_t(\gamma) = i * \Delta_A \gamma + t^2 M_\Phi \gamma,$$

where

$$M_\Phi \gamma := [\Phi^* \wedge [\Phi, \gamma]] - [\Phi \wedge [\Phi^*, \gamma]].$$

Applying  $-i * : \Omega^2(\mathfrak{su}(E)) \rightarrow \Omega^0(i\mathfrak{su}(E))$  finally yields the operator

$$L_t \gamma := \Delta_A \gamma - i * t^2 M_\Phi \gamma.$$

Observe that

$$\begin{aligned} \Lambda_{(A, \Phi)} : \Omega^0(i\mathfrak{su}(E)) &\rightarrow \Omega^1(\mathfrak{su}(E)) \oplus \Omega^{1,0}(\mathfrak{sl}(E)), \\ D\mathcal{H}_t \circ \Lambda_{(A, \Phi)} : \Omega^0(i\mathfrak{su}(E)) &\rightarrow \Omega^2(\mathfrak{su}(E)) \oplus \Omega^{1,1}(\mathfrak{sl}(E)), \\ \text{and } L_t : \Omega^0(i\mathfrak{su}(E)) &\rightarrow \Omega^0(i\mathfrak{su}(E)) \end{aligned}$$

are all bounded from  $H^1$  to  $L^2$ , or  $H^2$  to  $L^2$  respectively.

Now define, for  $t > 0$ , the operators

$$D'_t = \partial_A + t\Phi^* \quad \text{and} \quad D''_t = \bar{\partial}_A + t\Phi,$$

and also set  $D_t = D'_t + D''_t$ . These appear in Simpson's work [Si88, Si92] (with  $t = 1$ ), and it is shown there that these satisfy analogues of the Kähler identities. Those calculations show that

$$L_t = D_t^* D_t = 2(D'_t)^* D'_t = 2(D''_t)^* D''_t,$$

which immediately implies the following proposition.

**Proposition 5.1.** *If  $\gamma \in \Omega^0(i\mathfrak{su}(E))$ , then*

$$\langle L_t \gamma, \gamma \rangle_{L^2} = \|d_A \gamma\|_{L^2}^2 + 2t^2 \|[\Phi, \gamma]\|_{L^2}^2 \geq 0.$$

*In particular,  $L_t \gamma = 0$  if and only if  $d_A \gamma = [\Phi, \gamma] = 0$ .*

**5.2. Local analysis of the linearization at a fiducial solution.** In this section we analyze the linear operator  $L_t$  on the unit disk  $D$ , computed relative to a fiducial solution  $(A_t^{\text{fid}}, \Phi_t^{\text{fid}})$ , with the goal of determining sharp bounds for the norm of its inverse  $G_t$ . In what follows, we often omit the bundles from the function spaces. We also replace the  $H^2$  norm with the equivalent graph norm for the standard Laplacian  $\Delta = -((r\partial_r)^2 + \partial_\theta^2)/r^2$ , i.e.

$$\|u\|_\Delta^2 = \|u\|_{L^2}^2 + \|\Delta u\|_{L^2}^2.$$

Consider

$$L_t = \Delta_{A_t^{\text{fid}}} - i * t^2 M_{\Phi_t^{\text{fid}}}$$

with either Dirichlet or Neumann boundary conditions. Because of the non-negativity of  $t^2 M_{\Phi_t^{\text{fid}}}$  and the positivity of the leading part on the Dirichlet domain, it is clear that the Dirichlet extension  $L_t : H^2(D) \cap H_0^1(D) \rightarrow L^2(D)$  is injective, and since it is also self-adjoint, it is an isomorphism. We show

below that the same is true for the Neumann extension  $L_t: H^2(D) \rightarrow L^2(D)$ , but note that this requires the positivity of the potential term on the nullspace of  $\Delta_{A_t^{\text{fid}}}$ . We explain this below. In both cases, therefore, we can consider the inverse

$$G_t := L_t^{-1}: L^2(D) \rightarrow H^2(D),$$

and our goal is to understand the norm of this operator as  $t \nearrow \infty$ .

To this end, trivialize the bundle  $i\mathfrak{su}(E)$  by the constant sections  $\{\sigma_1 = i\tau_1, \sigma_2 = i\tau_2, \sigma_3 = i\tau_3\}$ , cf. (34), so  $[\tau_1, \sigma_1] = 0$ ,  $[\tau_1, \sigma_2] = 2\sigma_3$  and  $[\tau_1, \sigma_3] = -2\sigma_2$ , and consider the splitting

$$(42) \quad i\mathfrak{su}(E) = \langle \sigma_1 \rangle \oplus \langle \sigma_2, \sigma_3 \rangle =: iV \oplus iV^\perp,$$

which is orthogonal with respect to the Killing metric  $\langle A, B \rangle = \text{Tr}(AB)$  and parallel for the connection  $A_t^{\text{fid}} = 2f_t\tau_1 d\theta$ . The restriction of  $\Delta_{A_t^{\text{fid}}}$  to  $iV$  is the scalar Laplacian, while

$$\Delta_{A_t^{\text{fid}}}\big|_{iV^\perp} = -\frac{1}{r^2} \left( (r\partial_r)^2 + \partial_\theta^2 + \begin{pmatrix} -16f_t^2 & -8f_t\partial_\theta \\ 8f_t\partial_\theta & -16f_t^2 \end{pmatrix} \right)$$

acting on pairs  $(a_2, a_3)^\top = a_2\sigma_2 + a_3\sigma_3$ . Conjugating by the unitary matrix

$$U = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ i & -i \end{pmatrix}$$

provides a decoupling:

$$(43) \quad U^{-1} \circ \Delta_{A_t^{\text{fid}}}\big|_{iV^\perp} \circ U = -\frac{1}{r^2} \left( (r\partial_r)^2 + \begin{pmatrix} (\partial_\theta - 4if_t)^2 & 0 \\ 0 & (\partial_\theta + 4if_t)^2 \end{pmatrix} \right).$$

**Proposition 5.2.** *Consider  $L_t$  on  $D$  with either Dirichlet or Neumann boundary conditions. Then there exists a constant  $C > 0$  which does not depend on  $t \geq 1$  such that*

- (i)  $\|G_t\|_{\mathcal{L}(L^2, L^2)} \leq C$ ;
- (ii)  $\|G_t\|_{\mathcal{L}(L^2, H^2)} \leq Ct^2$ .

*Proof.* We continue to work with the decomposition (42), and set  $(A_t, \Phi_t) := (A_t^{\text{fid}}, \Phi_t^{\text{fid}})$ , to simplify notation.

To prove (i), consider the restriction of  $L_t$  to  $iV$  first. The restriction of  $\Delta_{A_t}$  to  $iV$  is the scalar Laplacian, which is  $t$ -independent and clearly has a fixed positive lower bound in the Dirichlet case. With Neumann boundary conditions, we use that

$$\langle -i * M_{\Phi_t} \sigma_1, \sigma_1 \rangle = 16r \cosh 2h_t(r) \geq 16r$$

which is strictly positive on  $D^\times$ . Hence  $L_t = \Delta - i * t^2 M_{\Phi_t}$  has a positive lower bound which is uniform in  $t$  with Neumann boundary conditions as well.

Turn now to  $L_t$  on  $iV^\perp$ ; since  $M_{\Phi_t}$  is nonnegative, it is sufficient to estimate  $\Delta_{A_t}$  from below. This requires the orthogonal splitting  $L^2(D) =$

$\mathcal{H}_0 \oplus \mathcal{H}^\perp$ , where  $\mathcal{H}_0 = \{u = u(r)\}$ . By (43), the restriction of  $\Delta_{A_t}$  to  $\mathcal{H}_0$  satisfies

$$-\frac{1}{r^2}((r\partial_r)^2 + (\partial_\theta \pm 4if_t)^2) = \frac{1}{r^2}(-(r\partial_r)^2 + 16f_t^2) \geq \frac{1}{r^2}(-(r\partial_r)^2 + 16f_1^2)$$

since  $f_t \geq f_1$  when  $t \geq 1$ . The operator on the right certainly has a  $t$ -independent positive lower bound. On  $\mathcal{H}^\perp$ , both terms in

$$-\frac{1}{r^2}((r\partial_r)^2 + (\partial_\theta \pm 4if_t)^2) = -\frac{1}{r^2}(r\partial_r)^2 + \frac{1}{r^2}(-i\partial_\theta \pm 4f_t)^2$$

are nonnegative operators with either type of boundary condition. Since  $0 \leq 4f_t \leq \frac{1}{2}$  and the spectrum of  $-i\partial_\theta$  equals  $\mathbb{Z} \setminus \{0\}$ , the operators  $\frac{1}{r^2}(-i\partial_\theta \pm 4f_t)^2$  have a uniform positive lower bound. This proves (i).

As for (ii), let  $t \geq 1$ . Setting  $A_t = 2f_t\tau_1 d\theta$ , we have

$$\begin{aligned} \Delta_{A_t}\gamma &= -\frac{1}{r^2}((r\partial_r)^2\gamma + \partial_\theta^2\gamma + 2[\alpha_t, \partial_\theta\gamma] + [\alpha_t, [\alpha_t, \gamma]]) \\ &= \Delta\gamma + V_t^1\gamma + V_t^2\gamma, \end{aligned}$$

where

$$V_t^1\gamma = -\frac{2[\alpha_t, \partial_\theta\gamma]}{r^2}, \quad V_t^2\gamma = -\frac{[\alpha_t, [\alpha_t, \gamma]]}{r^2}.$$

Now

$$L_t = \Delta_{A_t} + t^2 M_{\Phi_t} =: \Delta + V_t^1 + V_t^2 + t^2 M_t,$$

so that

$$(44) \quad \frac{1}{2} \|\Delta G_t v\|_{L^2}^2 \leq \|L_t G_t v\|_{L^2}^2 + \|V_t^1 G_t v\|_{L^2}^2 + \|V_t^2 G_t v\|_{L^2}^2 + t^4 \|M_t v\|_{L^2}^2.$$

Then for any  $\varepsilon > 0$ ,

$$\begin{aligned} \|V_t^1 G_t v\|_{L^2}^2 &= \left\langle \frac{4}{r^4} [\alpha_t, \partial_\theta G_t v], [\alpha_t, \partial_\theta G_t v] \right\rangle \\ &\leq C \left( \sup_{0 < r < 1} \left| \frac{\alpha_t}{r} \right|^2 \right) \left\| \frac{1}{r} \partial_\theta G_t v \right\|_{L^2}^2 \\ &\leq C \left( \sup_{0 < r < 1} \left| \frac{\alpha_t}{r} \right|^2 \right) \left( \frac{\varepsilon^2}{2} \|\Delta G_t v\|_{L^2}^2 + \frac{1}{2\varepsilon^2} \|v\|_{L^2}^2 \right). \end{aligned}$$

By Lemma 3.4 (iii)

$$\sup_{0 < r < 1} \left| \frac{\alpha_t}{r} \right|^2 \leq Ct^{\frac{4}{3}},$$

so choosing  $\varepsilon^2 < C^{-1}t^{-\frac{4}{3}}$ , we can absorb the term involving  $\Delta G_t v$  into the left side, yielding

$$(45) \quad \|V_t^1 G_t v\|_{L^2} \leq Ct^{\frac{4}{3}} \|v\|_{L^2}.$$

Similarly,

$$\sup_{r \in (0,1)} |V_t^2(r)| \leq Ct^{\frac{4}{3}} \Rightarrow \|V_t^2 G_t v\|_{L^2} \leq Ct^{\frac{4}{3}} \|G_t v\|_{L^2} \leq Ct^{\frac{4}{3}}.$$

Finally, since  $M_t$  is quadratic in the entries of  $\Phi_t$ , Lemma 3.4 (v) gives

$$\sup_{r \in (0,1)} |M_t(r)| \leq C \Rightarrow \|t^2 M_t v\|_{L^2} \leq Ct^2 \|G_t v\|_{L^2} \leq Ct^2.$$

This proves the bound.  $\square$

**Corollary 5.3.** *There is a constant  $C > 0$  such that  $\|u\|_{H^2} \leq Ct^2 \|u\|_{L_t}$ , where  $\|u\|_{L_t}$  is the graph norm for the operator  $L_t$ , for all  $u \in H^2(D) \cap H_0^1(D)$ .*

## 6. GLUING CONSTRUCTION

We are now in a position to prove the main gluing theorem. The strategy is the standard one: we construct a family of approximate solutions to  $\mathcal{F}_t(\gamma) = 0$ , then use the invertibility of the linearized operator to perturb these approximate solutions to exact solutions.

**6.1. Approximate solutions.** Let  $H(E)$  denote the bundle of Hermitian elements in  $\mathrm{SL}(E)$ . Now consider the map

$$\begin{aligned} \mathcal{F}_t: H^2(H(E)) &\rightarrow L^2(\Lambda^2 \otimes \mathfrak{su}(E)), \\ \mathcal{F}_t(g) &= F_{A_\infty^g}^\perp + t^2[\Phi_\infty^g \wedge (\Phi_\infty^g)^*], \end{aligned}$$

computed at a limiting configuration  $(A_\infty, \Phi_\infty)$ . Write  $X^{\mathrm{int}} = \bigcup_{p \in \mathfrak{p}} D_1^\times(p)$  for the union of the punctured disks, and assume that  $(A_\infty, \Phi_\infty)$  is in fiducial form in each of these. We also set  $X^{\mathrm{ext}} = X \setminus \overline{X^{\mathrm{int}}}$ .

Define the family of complex gauge transformations

$$g_t = \exp(\gamma_t), \quad \gamma_t = \begin{pmatrix} -\frac{1}{2}h_t & 0 \\ 0 & \frac{1}{2}h_t \end{pmatrix}$$

on  $X^{\mathrm{int}}$ ; by Lemma 3.7,

$$(A_t^{\mathrm{fid}}, \Phi_t^{\mathrm{fid}}) = (A_\infty^{\mathrm{fid}}, \Phi_\infty^{\mathrm{fid}})^{g_t}$$

on  $X^{\mathrm{int}}$ . Our approximate solution is obtained by gluing  $(A_t^{\mathrm{fid}}, \Phi_t^{\mathrm{fid}})$  on  $X^{\mathrm{int}}$  to  $(A_\infty, \Phi_\infty)$  on  $X^{\mathrm{ext}}$ . Thus, choose a smooth cut-off function  $\chi: X \rightarrow [0, 1]$  with  $\mathrm{supp} \chi \subseteq X^{\mathrm{int}}$  and  $\chi(z) \equiv 1$  for  $z \in \bigcup_{p \in \mathfrak{p}} D_{1/2}(p)$ . Then

$$(46) \quad g_t^{\mathrm{app}}(z) := \exp(\chi \gamma_t)$$

is a family of smooth gauge transformations on  $X^\times$  with

$$g_t^{\mathrm{app}} = g_t \text{ on } \bigcup_{p \in \mathfrak{p}} D_{1/2}(p) \text{ and } g_t^{\mathrm{app}} = \mathrm{Id} \text{ on } X^{\mathrm{ext}}.$$

The new pair

$$(A_t^{\mathrm{app}}, \Phi_t^{\mathrm{app}}) := (A_\infty, \Phi_\infty)^{g_t^{\mathrm{app}}}$$

is smooth and coincides with the fiducial solution  $(A_t^{\mathrm{fid}}, \Phi_t^{\mathrm{fid}})$  on  $\bigcup_{p \in \mathfrak{p}} D_{1/2}(p)$ , and with  $(A_\infty, \Phi_\infty)$  on  $X^{\mathrm{ext}}$ .

We claim that if the limiting configuration  $(A_\infty, \Phi_\infty)$  is constructed from an initial Higgs pair  $(A, \Phi)$  as in Section 4, then  $(A_t^{\mathrm{app}}, \Phi_t^{\mathrm{app}})$  is complex gauge equivalent to  $(A, \Phi)$  by a smooth gauge transformation *defined over*

all of  $X$ . Indeed, recall from Section 4 that in a suitable holomorphic frame around a zero  $p \in \mathfrak{p}$  of  $\det \Phi$ , the connection matrix of  $A$  vanishes and  $\Phi$  is of the form of Lemma 3.6. To transform  $(A, \Phi)$  into  $(A_t^{\text{app}}, \Phi_t^{\text{app}})$  we apply the gauge transformation

$$G_t = g_\infty g_{\mu_p} g_{\mu_f} g_t^{\text{app}}$$

where

- $g_\infty$  is a normalizing gauge transformation which puts  $(A, \Phi)$  into fiducial form on a neighbourhood of the zeroes of  $\det \Phi$ . It is obtained by using Lemma 4.2 to extend the locally defined gauge transformation  $g_\infty$  from Lemma 3.7 to a smooth normalizing gauge transformation on  $X^\times$ .
- $g_{\mu_p} = \exp(\gamma_{\mu_p})$  is the Hermitian gauge transformation in the stabilizer of  $\Phi_\infty^{\text{fid}}$  which gauges away the trace-less part of the curvature. This is obtained by solving the Poisson equation for  $\gamma_{\mu_p}$  (cf. Proposition 4.4 and Proposition 4.7).
- $g_{\mu_f} = \exp(\gamma_{\mu_f})$  is the unitary gauge transformation which fiducializes  $A_\infty^{g_{\mu_p}}$  (cf. Proposition 3.2).
- $g_t^{\text{app}}$  is the complex gauge transformation from (46).

**Proposition 6.1.** *The complex gauge transformation  $G_t$  admits a smooth extension across any point  $p \in \mathfrak{p}$ . In particular,  $(A_t^{\text{app}}, \Phi_t^{\text{app}})$  is complex gauge equivalent to  $(A, \Phi)$  over  $X$ .*

*Proof.* First note that we only need to prove continuity of the extension. Indeed, we can bootstrap the identity

$$dG_t = G_t A_t^{\text{app}} - A G_t$$

since  $A_t^{\text{app}}$  and  $A$  are smooth connections. Since  $G_t$  is smooth on  $X^\times$ , the discussion is completely local. We proceed in three steps.

*Step 1.* The coefficient  $\mu_p$  of the solution  $\gamma_{\mu_p}$  (as in Eq. (9)) of the Poisson equation has an expansion of the form

$$\mu_p \sim (C_0 + C_1 e^{-i\theta}) r^{\frac{1}{2}} + \mathcal{O}(r^{\frac{3}{2}}).$$

This follows directly from the indicial root calculation for the Laplacian  $\Delta_A$  in Section 4.3.

*Step 2.* The coefficient  $\mu_f$  of  $\gamma_f$  has

$$\mu_f \sim (C_0 - C_1 e^{-i\theta}) r^{\frac{1}{2}} + \mathcal{O}(r^{\frac{3}{2}}).$$

In particular,  $\mu_p + \mu_f$  decays like  $r^{\frac{1}{2}}$  as  $r \rightarrow 0$ .

Indeed,  $\mu_f$  is the solution of

$$P\mu_f := (-i\partial_\theta + \frac{1}{2})\mu_f = iv,$$

where  $v$  is the upper right entry of the  $d\theta$ -component of  $A_\infty^{g_{\mu_p}}$  (see Section 3.1 for the notation and calculations). Using the transformation formula (17) for the  $(0, 1)$ -component of the connection shows that

$$iv = re^{-2i\theta} D\mu_p + re^{i\theta} \overline{D\mu_p},$$

where

$$D = \frac{1}{2} e^{2i\theta} \left( \partial_r + \frac{i}{r} \partial_\theta - \frac{1}{2r} \right).$$

Furthermore, since  $\gamma_{\mu_p}$  is Hermitian,  $\bar{\mu} = e^{i\theta} \mu$ . It follows that

$$re^{-2i\theta} D\mu + re^{i\theta} \overline{D\mu} = r\partial_r \mu,$$

so that  $\mu_f$  is the solution of the ODE

$$P\mu_f = r\partial_r \mu_p.$$

This implies that  $\mu_f$  has an expansion in powers of  $r^{1/2}$  and Step 2 follows from a comparison of coefficients.

*Step 3.* We now check continuity of the gauge transformation  $G_t$  at  $r = 0$ .

By Lemma 3.7 we know that

$$g_\infty = \begin{pmatrix} r^{\frac{1}{4}} & 0 \\ 0 & r^{-\frac{1}{4}} \end{pmatrix}.$$

Furthermore,  $g_t^{\text{app}} = g_\infty^{-1}$  up to multiplication by a smooth gauge transformation, which can be ignored here. By Step 2,  $\mu = \mu_p + \mu_f = 2C_0 r^{1/2} + \mathcal{O}(r^{3/2})$ , so that

$$g_{\mu_p} g_{\mu_f} = g_\mu = \begin{pmatrix} \cosh(e^{i\theta/2} \mu) & e^{-i\theta/2} \sinh(e^{i\theta/2} \mu) \\ e^{i\theta/2} \sinh(e^{i\theta/2} \mu) & \cosh(e^{i\theta/2} \mu) \end{pmatrix}$$

and finally

$$\begin{aligned} & \begin{pmatrix} r^{-\frac{1}{4}} & 0 \\ 0 & r^{\frac{1}{4}} \end{pmatrix} \begin{pmatrix} \cosh(e^{i\theta/2} \mu) & e^{-i\theta/2} \sinh(e^{i\theta/2} \mu) \\ e^{i\theta/2} \sinh(e^{i\theta/2} \mu) & \cosh(e^{i\theta/2} \mu) \end{pmatrix} \begin{pmatrix} r^{\frac{1}{4}} & 0 \\ 0 & r^{-\frac{1}{4}} \end{pmatrix} \\ &= \begin{pmatrix} \cosh(e^{i\theta/2} \mu) & r^{-\frac{1}{2}} e^{-i\theta/2} \sinh(e^{i\theta/2} \mu) \\ r^{\frac{1}{2}} e^{i\theta/2} \sinh(e^{i\theta/2} \mu) & \cosh(e^{i\theta/2} \mu) \end{pmatrix}. \end{aligned}$$

This is easily seen to have a limit as  $r \rightarrow 0$ .  $\square$

Starting from the initial Higgs pair  $(A, \Phi)$  with simple Higgs field, we have thus arrived at a complex gauge equivalent pair  $(A_t^{\text{app}}, \Phi_t^{\text{app}})$ . The latter can be regarded as an approximate solution in the following sense.

**Lemma 6.2.** *There exist  $C, \delta > 0$  such that for  $t > 1$ ,*

$$(47) \quad \|\mathcal{F}_t(g_t^{\text{app}})\|_{L^2} \leq C e^{-\delta t}.$$

*Proof.* By the definition of  $(A_t^{\text{app}}, \Phi_t^{\text{app}})$ , it suffices to estimate the error on  $X^{\text{int}} \setminus \bigcup_{p \in \text{ep}} D_{1/2}(p)$ . From the properties of  $h_t$  in Lemma 3.4 we see that  $g_t$  converges to the identity on  $X^{\text{int}} \setminus \bigcup_{p \in \text{ep}} D_{1/2}(p)$  like  $e^{-\delta t}$  as  $t \rightarrow \infty$ . In particular, both terms on the right in

$$\mathcal{F}_t(g_t^{\text{app}}) = F_{(A^\infty)g_t^{\text{app}}}^\perp + t^2[(g_t^{\text{app}})^{-1}\Phi_\infty g_t^{\text{app}} \wedge ((g_t^{\text{app}})^{-1}\Phi_\infty g_t^{\text{app}})^*]$$

converge exponentially in  $t$  to 0 (cf. Eq. (4) for the curvature term). This gives (47).  $\square$

**6.2. Global linear estimates.** We now consider, as in Section 5.1, the operator  $L_t = \Delta_{A_t^{\text{app}}} - i * t^2 M_{\Phi_t^{\text{app}}}$  taken with respect to  $(A_t^{\text{app}}, \Phi_t^{\text{app}})$  on all of  $X$ . Our goal is to establish estimates for  $\|G_t\|_{L^2 \rightarrow L^2}$  and  $\|G_t\|_{L^2 \rightarrow H^2}$ , where  $G_t = L_t^{-1}$ . Let  $\lambda_t(X)$ ,  $\lambda_t(X^{\text{int}})$  and  $\lambda_t(X^{\text{ext}})$  denote the lowest eigenvalues of  $L_t$  on  $L^2(X)$ ,  $L^2(X^{\text{int}})$  and  $L^2(X^{\text{ext}})$ , respectively, the latter two with respect to Neumann boundary conditions  $(d_{A_t} u)\nu = 0$ , where  $\nu$  is a unit normal to  $\partial X^{\text{int}} = \partial X^{\text{ext}}$ . The key is the *domain decomposition principle*, see for instance [Bä, Proposition 3], which gives the estimate

$$\lambda_t(X) \geq \min\{\lambda_t(X^{\text{int}}), \lambda_t(X^{\text{ext}})\}.$$

**Lemma 6.3.** *For  $t \geq 1$ , there is a uniform lower bound  $\lambda_t(X) \geq \lambda > 0$ , so  $\|G_t v\|_{L^2} \leq C\|v\|_{L^2}$ , for  $t \geq 1$ .*

*Proof.* As noted above, we need only estimate the lowest eigenvalues on the interior and exterior regions. For the former, noting that  $A_t^{\text{app}} = 2f_{\chi,t}\sigma_1 d\theta = \frac{1}{4}(1 + 2r\partial_r(\chi h_t))\sigma_1 d\theta$ , so Proposition 5.2 (i) gives  $\lambda_t^{\text{int}} \geq C > 0$  for all  $t \geq 1$ .

Note that  $L_t \geq L_1$ , so  $\lambda_t^{\text{ext}} \geq \lambda_1^{\text{ext}}$ , when  $t \geq 1$ . Now if  $\gamma$  satisfies Neumann conditions, then by Proposition 5.1,

$$\int_{X^{\text{ext}}} \langle (\Delta_{A_\infty} - i * M_{\Phi_\infty})\gamma, \gamma \rangle = \int_{X^{\text{ext}}} |d_{A_\infty} \gamma|^2 + 2 \int_{X^{\text{ext}}} |[\Phi_\infty, \gamma]|^2;$$

thus any element of the kernel of the Neumann extension is a parallel section  $\gamma$  of  $iL_{\Phi_\infty}$ . As explained in Section 4.2 and Section 3.1, this is a twisted line bundle, so  $\gamma = 0$ . We conclude that  $\lambda_t(X^{\text{ext}}) \geq C > 0$  as well.  $\square$

We now introduce the  $t$ -dependent Sobolev space  $H_t^2 := \text{dom } L_t$  with the graph norm

$$\|u\|_{L_t}^2 = \|u\|_{L^2}^2 + \|L_t u\|_{L^2}^2.$$

Note that  $H_t^2 = H^2$  for all  $t$ , but the norms are not uniformly equivalent as  $t \nearrow \infty$ . The ordinary Sobolev space  $H^2$  corresponds to  $t = 0$ . Clearly,  $\|G_t v\|_{L_t} \leq C\|v\|_{L^2}$  uniformly in  $t \geq 1$ .

**Lemma 6.4.** *If  $u \in H^2(i\text{su}(E))$ , then  $\|u\|_{H^2} \leq Ct^2\|u\|_{L_t}$  for  $t \geq 1$ .*

*Proof.* Use a cut-off function to write  $u = u^{\text{int}} + u^{\text{ext}}$  with  $\text{supp } u^{\text{int}} \subset X^{\text{int}}$  and  $\text{supp } u^{\text{ext}} \subset X \setminus \bigcup_{p \in \text{ep}} D_{1/2}(p)$ . By Corollary 5.3,

$$\|u^{\text{int}}\|_{H^2} \leq Ct^2\|u\|_{L_t}.$$

On  $X \setminus \bigcup_{p \in \text{ep}} D_{1/2}(p)$ , consider the linear operator

$$\tilde{L}_t := \Delta_{A_\infty} - it^2 * M_{\Phi_\infty}$$

with Dirichlet boundary conditions. Then  $\tilde{L}_t$  is invertible and we write  $\tilde{G}_t := \tilde{L}_t^{-1}$ . Now

$$\|\tilde{L}_t u\|_{L^2} \leq \|L_t u\|_{L^2} + \|(\tilde{L}_t - L_t)u\|_{L^2}$$

and since  $A_t$  converges to  $A_\infty$  and  $\Phi_t$  converges to  $\Phi_\infty$  exponentially in  $t$ ,

$$\|(\tilde{L}_t - L_t)u\|_{L^2} \leq C e^{-\delta t} \|u\|_{L^2}.$$

In addition,

$$\begin{aligned} \|\Delta_{A_\infty} u\|_{L^2} &= \|\Delta_{A_\infty} u - it^2 * M_{\Phi_\infty} u + it^2 * M_{\Phi_\infty} u\|_{L^2} \\ &\leq \|\tilde{L}_t u\|_{L^2} + t^2 \|M_{\Phi_\infty} u\|_{L^2} \\ &\leq \|\tilde{L}_t u\|_{L^2} + t^2 \sup |M_{\Phi_\infty}| \|u\|_{L^2}, \end{aligned}$$

which leads to the estimate

$$\|\Delta_{A_\infty} u\|_{L^2} \leq \|L_t u\|_{L^2} + C e^{-\delta t} \|u\|_{L^2} + C t^2 \|u\|_{L^2} \leq C t^2 \|u\|_{L_t}.$$

This gives the claim since the graph norm of  $\Delta_{A_\infty}$  is equivalent to  $\|\cdot\|_{H^2}$ .  $\square$

Summarizing we proved the following global linear estimate.

**Proposition 6.5.** *Let  $(A_t^{\text{app}}, \Phi_t^{\text{app}})$  be the approximate solution from Section 6.1. Then the inverse  $G_t$  to  $L_t = \Delta_{A_t^{\text{app}}} - it^2 * M_{\Phi_t^{\text{app}}}$  satisfies*

$$\|G_t v\|_{H^2} \leq C t^2 \|v\|_{L^2}$$

for all  $v \in L^2$ .

**6.3. Deforming the approximate solutions.** We are now finally prepared to give the argument which shows how to perturb the approximate solution  $(A_t^{\text{app}}, \Phi_t^{\text{app}})$  to an exact solution of Hitchin's equations when  $t \gg 1$ .

**Theorem 6.6.** *Let  $B_\rho$  be the closed ball of radius  $\rho$  around  $0 \in H^2(\text{isu}(E))$ . Then there is a value  $m > 0$  and a unique  $\gamma_t \in B_{t^{-m}}$  such that, when  $t$  is sufficiently large,  $(A_t, \Phi_t) := (A_t^{\text{app}}, \Phi_t^{\text{app}})^{\exp(\gamma_t)}$  solves the rescaled Hitchin equations.*

**Remark.** Theorem 6.6 gives a solution to Hitchin's equation, when the parameter  $t$  is large, which is complex gauge equivalent to the initial Higgs pair  $(A, t\Phi)$  as shown by Proposition 6.1. In this way, Theorem 6.6 provides a constructive proof of Hitchin's existence theorem for large Higgs fields. We can regard Theorem 6.6 as a desingularization theorem for limiting configurations. This shows in particular that any limiting configuration arises from a Higgs pair.

The solution  $\gamma_t$  is obtained using a standard contraction mapping argument. To carry this out, we need to study the linearization  $L_t$ , computed at  $(A_t^{\text{app}}, \Phi_t^{\text{app}})$ . The argument relies on controlling the following quantities:



- the norm of the inverse  $G_t = L_t^{-1}$ , and
- the Lipschitz constants of the linear and higher order terms in the Taylor expansion of  $\mathcal{F}_t$ .

The first of these was handled by Proposition 6.5, but we must now study the nonlinear terms in  $\mathcal{F}_t$  in greater detail.

For  $g = \exp(\gamma)$ ,  $\gamma \in \Omega^0(\text{isu}(E))$ , we have

$$\mathcal{O}_{(A,\Phi)}(g) = (A, \Phi)^g = (A + g^{-1}(\bar{\partial}_A g) - (\partial_A g)g^{-1}, g^{-1}\Phi g),$$

and consequently,

$$\begin{aligned} A^{\exp \gamma} &= A + (\bar{\partial}_A - \partial_A)\gamma + R_A(\gamma), \\ \Phi^{\exp \gamma} &= \Phi + [\Phi, \gamma] + R_\Phi(\gamma). \end{aligned}$$

The explicit expressions of these remainder terms are

$$(48) \quad R_A(\gamma) = \exp(-\gamma)(\bar{\partial}_A(\exp \gamma)) - (\partial_A(\exp \gamma))\exp(-\gamma) - (\bar{\partial}_A - \partial_A)\gamma,$$

$$(49) \quad R_\Phi(\gamma) = \exp(-\gamma)\Phi \exp \gamma - [\Phi, \gamma] - \Phi.$$

We then calculate that

$$(50) \quad \begin{aligned} \mathcal{F}_t(\exp \gamma) &= F_{(A_t^{\text{app}})^{\exp(\gamma)}}^\perp + t^2[(\Phi_t^{\text{app}})^{\exp(\gamma)} \wedge ((\Phi_t^{\text{app}})^{\exp(\gamma)})^*] \\ &= \text{pr}_1 \mathcal{H}_t(A_t^{\text{app}}, \Phi_t^{\text{app}}) + L_t \gamma + Q_t(\gamma), \end{aligned}$$

where, in full detail,

$$\begin{aligned} Q_t(\gamma) &= d_{A_t^{\text{app}}}(R_{A_t^{\text{app}}}(\gamma)) + t^2[R_{\Phi_t^{\text{app}}}(\gamma) \wedge (\Phi_t^{\text{app}})^*] + t^2[\Phi_t^{\text{app}} \wedge R_{\Phi_t^{\text{app}}}(\gamma)^*] \\ &\quad + \frac{1}{2}[((\bar{\partial}_{A_t^{\text{app}}} - \partial_{A_t^{\text{app}}})\gamma + R_{A_t^{\text{app}}}(\gamma)) \wedge ((\bar{\partial}_{A_t^{\text{app}}} - \partial_{A_t^{\text{app}}})\gamma + R_{A_t^{\text{app}}}(\gamma))] \\ &\quad + t^2[[\Phi_t^{\text{app}}, \gamma] + R_{\Phi_t^{\text{app}}}(\gamma)] \wedge [(\Phi_t^{\text{app}}, \gamma) + R_{\Phi_t^{\text{app}}}(\gamma)]^*. \end{aligned}$$

**Lemma 6.7.** *The approximate solution satisfies*

$$\|A_t^{\text{app}}\|_{C^1} \leq Ct$$

on the unit disk, so that for every  $H^{k+1}$  section  $\gamma$ ,  $k = 0, 1$ ,

$$\|d_{A_t^{\text{app}}}\gamma\|_{H^k} \leq Ct\|\gamma\|_{H^{k+1}},$$

and moreover,

$$\|L_t \gamma\|_{L^2} \leq Ct^2\|\gamma\|_{H^2}.$$

*Proof.* We have

$$A_t^{\text{app}} = 2f_{\chi,t} \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} d\theta,$$

where  $f_{\chi,t}(r) = \frac{1}{8} + \frac{1}{4}r\partial_r(\chi h_t)(r)$ , see Section 6.1. Clearly  $f_{\chi,t}$  has the same asymptotics as  $f_t$ ; thus  $f_{\chi,t}$  is uniformly bounded in  $t$ .

Now recall from Section 3.2 that  $f_t(r) = \eta(\rho)$ ,  $\rho = \frac{8t}{3}r^{3/2}$ , where  $\eta(\rho) = \frac{1}{8} + \frac{3}{8}\rho\psi'(\rho)$ . Then

$$\partial_r f_t(r) = 4tr^{1/2}\eta'(\rho),$$

and we already know that  $\eta'(\rho) = \frac{3}{16}\rho \sinh(2\psi(\rho))$ . Since  $\psi(\rho) \sim -\frac{1}{3}\log \rho$  as  $\rho \rightarrow 0$  and  $\psi(\rho) \sim e^{-\rho}$  as  $\rho \rightarrow \infty$ , we see that  $\lim_{\rho \rightarrow 0} \eta'(\rho) = \lim_{\rho \rightarrow \infty} \eta'(\rho) = 0$ . This gives that  $0 \leq \eta'(\rho) \leq C_0$  for some constant  $C_0 > 0$ . Altogether,

$$|\partial_r f_t| = |4tr^{1/2}\eta'(\rho)| \leq C_1 t$$

for some constant  $C_1 > 0$ , which also yields the desired estimate for  $|\partial_r f_{\chi,t}|$ .  $\square$

**Lemma 6.8.** *There exists a constant  $C > 0$  such that*

$$(51) \quad \|Q_t(\gamma_1) - Q_t(\gamma_0)\|_{L^2} \leq C\rho t^2 \|\gamma_1 - \gamma_0\|_{H^2}$$

for all  $0 < \rho \leq 1$  and  $\gamma_0, \gamma_1 \in B_\rho$ .

*Proof.* The proof has two steps. To simplify notation, we write  $(A, \Phi)$  for  $(A_t^{\text{app}}, \Phi_t^{\text{app}})$ .

*Step 1.* We first check that if  $\rho \in (0, 1]$  and  $\gamma_0, \gamma_1 \in B_\rho$ , then

$$\begin{aligned} \|R_A(\gamma_1) - R_A(\gamma_0)\|_{H^1} &\leq Ct\rho \|\gamma_1 - \gamma_0\|_{H^2} \\ \|R_\Phi(\gamma_1) - R_\Phi(\gamma_0)\|_{H^1} &\leq Ct\rho \|\gamma_1 - \gamma_0\|_{H^2}. \end{aligned}$$

We begin by estimating the difference of the first two terms on the right in (48):

$$\begin{aligned} &\|\exp(-\gamma_1)(\bar{\partial}_A(\exp \gamma_1)) - \exp(-\gamma_0)(\bar{\partial}_A(\exp \gamma_0)) - \bar{\partial}_A(\gamma_1 - \gamma_0)\|_{H^1} \\ &\leq \|(\exp(-\gamma_1) - \exp(-\gamma_0))\bar{\partial}_A(\exp(\gamma_1))\|_{H^1} \\ &\quad + \|\exp(-\gamma_0)(\bar{\partial}_A(\exp(\gamma_1) - \exp(\gamma_0))) - \bar{\partial}_A(\gamma_1 - \gamma_0)\|_{H^1} =: \text{I} + \text{II}. \end{aligned}$$

Writing  $\exp(\gamma) = 1 + \gamma + S(\gamma)$ , then we have

$$\begin{aligned} \|\text{I}\|_{H^1} &\leq C_0 \|\exp(-\gamma_1) - \exp(-\gamma_0)\|_{H^2} \|\bar{\partial}_A(\exp(\gamma_1))\|_{H^1} \\ &\leq C_1 t \|\gamma_1 - \gamma_0\|_{H^2} \|\gamma_1 + S(\gamma_1)\|_{H^2} \\ &\leq C_2 t \rho \|\gamma_1 - \gamma_0\|_{H^2}, \end{aligned}$$

and similarly,

$$\begin{aligned} \|\text{II}\|_{H^1} &= \|(1 - \gamma_0 + S(-\gamma_0))(\bar{\partial}_A(\gamma_1 - \gamma_0 + S(\gamma_1) - S(\gamma_0))) - \bar{\partial}_A(\gamma_1 - \gamma_0)\|_{H^1} \\ &\leq \|\bar{\partial}_A(S(\gamma_0) - S(\gamma_1))\|_{H^1} \\ &\quad + \|(-\gamma_0 + S(-\gamma_0))\bar{\partial}_A(\gamma_0 - \gamma_1 + S(\gamma_0) - S(\gamma_1))\|_{H^1} \\ &\leq C_0 t \|S(\gamma_0) - S(\gamma_1)\|_{H^2} \\ &\quad + C_0 t \|-\gamma_0 + S(-\gamma_0)\|_{H^2} \|\gamma_0 - \gamma_1 + S(\gamma_0) - S(\gamma_1)\|_{H^2} \\ &\leq C_1 t \rho \|\gamma_1 - \gamma_0\|_{H^2}, \end{aligned}$$

where we have estimated  $\|S(\gamma_0) - S(\gamma_1)\|_{H^2} \leq \|\gamma_0 - \gamma_1\|_{H^2} \sum_{k \geq 1} \rho^k / k! \leq C\rho \|\gamma_0 - \gamma_1\|_{H^2}$ . These estimates together with analogous ones for the terms

involving  $\partial_A$  give the stated Lipschitz estimate for  $R_A$ . The corresponding estimate for

$$R_\Phi = \exp(-\gamma)\Phi \exp \gamma - [\Phi, \gamma] - \Phi$$

and the estimates

$$\|R_A(\gamma)\|_{H^1} \leq Ct\rho, \quad \|R_\Phi(\gamma)\|_{H^1} \leq C\rho$$

for  $\gamma \in B_\rho$  follow in the same way.

*Step 2.* We can now prove the claim. First,

$$(52) \quad \begin{aligned} Q_t(\gamma_1) - Q_t(\gamma_0) &= d_A(R_A(\gamma_1) - R_A(\gamma_0)) \\ &\quad + t^2[(R_\Phi(\gamma_1) - R_\Phi(\gamma_0)) \wedge \Phi^*] + t^2[\Phi \wedge (R_\Phi(\gamma_1) - R_\Phi(\gamma_0))^*] \\ &\quad + \frac{1}{2}[(\bar{\partial}_A - \partial_A)\gamma_1 + R_A(\gamma_1)] \wedge ((\bar{\partial}_A - \partial_A)\gamma_1 + R_A(\gamma_1)) \\ &\quad - \frac{1}{2}[(\bar{\partial}_A - \partial_A)\gamma_0 + R_A(\gamma_0)] \wedge ((\bar{\partial}_A - \partial_A)\gamma_0 + R_A(\gamma_0)) \\ &\quad + t^2[(\Phi, \gamma_1) + R_\Phi(\gamma_1)] \wedge ((\Phi, \gamma_1) + R_\Phi(\gamma_1))^* \\ &\quad - t^2[(\Phi, \gamma_0) + R_\Phi(\gamma_0)] \wedge ((\Phi, \gamma_0) + R_\Phi(\gamma_0))^*. \end{aligned}$$

By Lemma 6.7,

$$\|d_A(R_A(\gamma_1) - R_A(\gamma_0))\|_{L^2} \leq C(t+1)\|R_A(\gamma_1) - R_A(\gamma_0)\|_{H^1},$$

and we then apply Step 1. The remaining terms are bilinear combinations  $B(\psi, \tau)$  of functions  $\psi$  and  $\tau$  with fixed coefficients, which can be estimated as

$$\begin{aligned} \|B(\psi_1, \tau_1) - B(\psi_0, \tau_0)\|_{L^2} &\leq \|B(\psi_1 - \psi_0, \tau_1)\|_{L^2} + \|B(\psi_0, \tau_1 - \tau_0)\|_{L^2} \\ &\leq C\|\psi_1 - \psi_0\|_{H^1}\|\tau_1\|_{H^1} + C\|\psi_0\|_{H^1}\|\tau_1 - \tau_0\|_{H^1}. \end{aligned}$$

The desired estimate follows from Step 1 again.  $\square$

*Proof of Theorem 6.6.* From (50),

$$\mathcal{F}_t(\exp(\gamma)) = \text{pr}_1 \mathcal{H}_t(A_t^{\text{app}}, \Phi_t^{\text{app}}) + L_t \gamma + Q_t(\gamma),$$

and since  $L_t$  is invertible, the solutions of this equation are the same as the solutions of

$$\gamma = -L_t^{-1}(\text{pr}_1 \mathcal{H}_t(A_t^{\text{app}}, \Phi_t^{\text{app}}) + Q_t(\gamma)).$$

Thus consider the map

$$T: B_\rho \rightarrow H^2(\text{isu}(E)), \quad \gamma \mapsto -L_t^{-1}(\text{pr}_1 \mathcal{H}_t(A_t^{\text{app}}, \Phi_t^{\text{app}}) + Q_t(\gamma)).$$

We claim that for  $\rho$  sufficiently small,  $T$  is a contraction of  $B_\rho$ , from which we immediately obtain a unique fixed point  $\gamma \in B_\rho$ . To prove this, use Proposition 6.5 and (51) to get

$$\begin{aligned} \|T(\gamma_1 - \gamma_0)\|_{H^2} &= \|G_t(Q_t(\gamma_1) - Q_t(\gamma_0))\|_{H^2} \\ &\leq Ct^2\|Q_t(\gamma_1) - Q_t(\gamma_0)\|_{L^2} \leq C\rho t^4\|\gamma_1 - \gamma_0\|_{H^2}. \end{aligned}$$

Thus  $T$  is a contraction on the ball of radius  $\rho_t = t^{-4-\epsilon}$  for any  $\epsilon > 0$ . Furthermore, since  $Q_t(0) = 0$ , then by Proposition 6.5 and (47),

$$\|T(0)\|_{H^2} = \|G_t(\text{pr}_1 \mathcal{H}_t(A_t^{\text{app}}, \Phi_t^{\text{app}}))\|_{H^2} \leq Ct^2 e^{-\delta t}.$$

Thus when  $t \gg 0$ ,  $\|T(0)\|_{H^2} < \frac{1}{10}\rho_t$ , so the ball  $B_{\rho_t}$  is mapped to itself by  $T$ .  $\square$

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DEPARTMENT OF MATHEMATICS, STANFORD UNIVERSITY, STANFORD, CA 94305 USA  
*E-mail address:* mazzeeo@math.stanford.edu

MATHEMATISCHES INSTITUT DER LMU MÜNCHEN, THERESIENSTRASSE 39, D-80333 MÜNCHEN, GERMANY  
*E-mail address:* swoboda@math.lmu.de

MATHEMATISCHES SEMINAR DER UNIVERSITÄT KIEL, LUDEWIG-MEYN-STRASSE 4, D-24098 KIEL, GERMANY  
*E-mail address:* weiss@math.uni-kiel.de

INSTITUT FÜR GEOMETRIE UND TOPOLOGIE DER UNIVERSITÄT STUTTGART, PFAFFENWALDRING 57, D-70569 STUTTGART, GERMANY  
*E-mail address:* frederik.witt@mathematik.uni-stuttgart.de