INTERNATIONAL JOURNAL FOR NUMERICAL METHODS IN ENGINEERING
Int. J. Numer. Meth. Engng 2007; **71**:1297–1320
Published online 15 February 2007 in Wiley InterScience (www.interscience.wiley.com). DOI: 10.1002/nme.1986

Energetic–statistical size effect simulated by SFEM with stratified sampling and crack band model

Zdeněk P. Bažant[1,*,†,‡], Sze-Dai Pang[1,§], Miroslav Vořechovský[2,¶] and Drahomír Novák[2,‖]

[1]*Department of Civil and Environmental Engineering, Northwestern University, Evanston, IL 60208, U.S.A.*
[2]*Institute of Structural Mechanics, Brno University of Technology, Brno, Czech Republic*

SUMMARY

The paper presents a model that extends the stochastic finite element method to the modelling of transitional energetic–statistical size effect in unnotched quasibrittle structures of positive geometry (i.e. failing at the start of macro-crack growth), and to the low probability tail of structural strength distribution, important for safe design. For small structures, the model captures the energetic (deterministic) part of size effect and, for large structures, it converges to Weibull statistical size effect required by the weakest-link model of extreme value statistics. Prediction of the tail of extremely low probability such as one in a million, which needs to be known for safe design, is made feasible by the fact that the form of the cumulative distribution function (cdf) of a quasibrittle structure of any size has been established analytically in previous work. Thus, it is not necessary to turn to sophisticated methods such as importance sampling and it suffices to calibrate only the mean and variance of this cdf. Two kinds of stratified sampling of strength in a finite element code are studied. One is the Latin hypercube sampling of the strength of each element considered as an independent random variable, and the other is the Latin square design in which the strength of each element is sampled from one overall cdf of random material strength. The former is found to give a closer estimate of variance, while the latter gives a cdf with smaller scatter and a better mean for the same number of simulations. For large structures, the number of simulations required to obtain the mean size effect is greatly reduced by adopting the previously proposed method of random property blocks. Each block is assumed to have a homogeneous random material strength,

*Correspondence to: Zdeněk P. Bažant, Department of Civil and Environmental Engineering, Northwestern University, 2145 Sheridan Road, Evanston, IL 60208, U.S.A.
†E-mail: z-bazant@northwestern.edu
‡McCormick Institute Professor and W.P. Murphy Professor of Civil Engineering and Materials Science.
§Graduate Research Assistant. Currently, Assistant Professor, Engineering Science Programme and Civil Engineering Department, National University of Singapore, Singapore.
¶Assistant Professor and Research Fellow.
‖Professor.

Contract/grant sponsor: Office of Naval Research; contract/grant numbers: N00014-02-I-0622, N00014-07-1-0313
Contract/grant sponsor: U.S. National Science Foundation; contract/grant number: CMS-0556323
Contract/grant sponsor: Fulbright Foundation
Contract/grant sponsor: Grant Agency of the Czech Republic; contract/grant number: 103/06/P086
Contract/grant sponsor: Czech Academy of Science; contract/grant number: 1ET409870411

the mean and variance of which are scaled down according to the block size using the weakest-link model for a finite number of links. To check whether the theoretical cdf is followed at least up to tail beginning at the failure probability of about 0.01, a hybrid of stratified sampling and Monte Carlo simulations in the lowest probability stratum is used. With the present method, the probability distribution of strength of quasibrittle structures of positive geometry can be easily estimated for any structure size.

Received 7 September 2005; Revised 29 November 2006; Accepted 13 December 2006

KEY WORDS: stochastic simulation; size effect; extreme value statistics; quasibrittle fracture; scaling

1. INTRODUCTION

Highly developed though the stochastic finite element method (SFEM) has become [1–3], its extension to extreme value statistics remains a challenge. Engineering structures such as aircraft, bridges or ships must be designed for extremely low failure probability P_f—typically less than 1 in a million per lifetime [4–6], which is necessary to make the structural failures very rare compared to other generally accepted hazards that people face.

In the range of such extremely low probabilities, the difference between the exponentially decaying Gaussian (normal) distribution and the Weibull distribution, which has a tail decaying as a power law, is enormous; see the plots of the cumulative distribution function (cdf) in Figure 1 showing that, for the same mean and the same typical coefficient of variation (CoV), the point of $P_f = 10^{-6}$ is for Weibull distribution about twice as far from the mean than it is for the Gaussian distribution, even though the difference between the central parts of these distributions is small and hardly detectable from experimental histograms. Thus, replacing the Gaussian cdf by the Weibull cdf requires approximate doubling of the understrength part of safety factor (resistance factor), which must be applied to results of the deterministic finite element computations. Obviously, the errors of these computations are dwarfed by the uncertainty in the resistance part of safety factor.

To compute the tolerable loads of such extremely low P_f, effective SFEMs for extreme value statistics have been developed; they include the 'importance sampling' [5, 7, 8], 'subset simulation' [9, 10], 'line sampling' [11], von Neumann's 'splitting', 'Russian roulette' [11], etc. However, despite the development of these powerful methods, their practical application faces a serious obstacle: the results depend strongly on the far-out tail of the probability density function (pdf) of

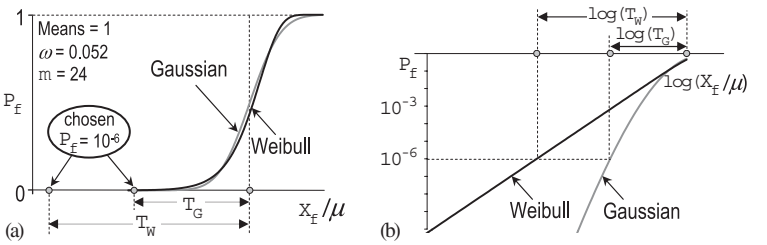

Figure 1. Large difference between points of failure probability 10^{-6} for Gaussian and Weibull distributions with mean 1 and CoV = 5.2% in (a) linear scale and (b) log scale.

DOI: 10.1002/nme

the input, but the pdf is typically verified only for the core, and when this pdf is simply extended into the tail, the extension is often incorrect [12]. If this problem is ignored, the importance sampling or any other type of SFEM for extreme value statistics is reduced to a mere mathematical exercise of no practical relevance.

This study deals with the broad class of brittle and quasibrittle structures failing at fracture initiation from a smooth surface. These are structures with neither notches nor large pre-existing cracks, having the so-called positive geometry, which is the geometry for which the maximum equilibrium load is attained right at the initiation of macroscopic fracture propagation (and is characterized by positiveness of the partial derivative of the energy release rate with respect to the crack length when the load is held constant [13–15]). The entire cdf of large structures of this kind is totally dominated by the far-left tail, $P_f < 0.0001$, of the strength of a representative volume element (RVE) of the material, while the rest of the cdf of a RVE is irrelevant. Calculation of the load of failure probability $P_f \approx 10^{-6}$ requires the cdf tail of RVE strength to be known all the way to $P_f \approx 10^{-6}$.

To determine the far-left tail directly, i.e. through a histogram, one would need to test about 10^9 identical specimens or carry out an equal number of extreme-value stochastic micromechanical simulations. Either way is practically impossible, and so a physically based theory is necessary. Such a theory, governing the type of distribution, has recently been developed on the basis of the probability of interatomic bond ruptures [16–19].

From this theory, the type of distribution is known for the case of quasibrittle structures of positive geometry failing at macro-crack initiation. So, to calibrate this distribution, one needs to obtain by computer simulations only the mean and the variance. This is a far simpler task than computer simulation of the far-left tail. By virtue of this fact, we can limit ourselves in this study to employing only simple statistical sampling techniques. Combining them with the crack band model, such techniques also suffice to capture the deterministic (or energetic) part of size effect which occurs when the RVE is not negligibly small compared to structural dimensions. This part is caused by the stress redistribution engendered by a fracture process zone (FPZ) of finite size (which is equal to the characteristic material length l).

Similar objectives are addressed in a different way in a parallel study [20], in which the SFEM with crack band model and random property blocks [21] is used to compute the mean and variance of strength of structures of any size and geometry, but the cdf extension into the tail must be specified, rather than being predicted by the theory. The computational burden is, in that study, reduced by scaling the strength of multi-element blocks.

2. EFFECTS OF RANDOM STRENGTH AND SIZE IN BRITTLE AND QUASIBRITTLE STRUCTURES

Quasibrittle structures consist of quasibrittle materials, which are materials having a brittle (non-ductile) matrix and containing inhomogeneities (grains, aggregates) that are not negligible compared to structural dimensions and cause the FPZ (and the RVE of material) to be also non-negligible. Quasibrittleness is a relative concept—all these materials become perfectly brittle on a sufficiently large scale. They include many common materials such as concretes and mortars (an archetypical paradigm, studied the earliest), rocks, masonry, sea ice, dry snow slabs, wood, paper, carton, particle board, stiff clays, silts, grouted soils, coal, refractories, as well many 'high-tech' materials such as fibre composites, fibre-reinforced concretes, rigid foams, toughened ceramics and nanocomposites,

DOI: 10.1002/nme

or bio-materials such as bone, cartilage, dentine and sea shells. All perfectly brittle materials become quasibrittle on a sufficiently small scale, e.g. the components of MEMS and NEMS, as well as thin metallic films, and nanotubes. The most important difference of quasibrittle behaviour from perfectly brittle behaviour is that the Weibull statistical size effect is accompanied, and at small sizes overshadowed, by the energetic size effect, engendered by stress redistribution due to large FPZ [13, 14].

If the geometry is positive, a brittle or quasibrittle structure fails (under load control) as soon as the first RVE fails. This means that the structure behaves as a series coupling (or chain) of a certain number of RVEs (or links), i.e. the weakest-link model applies. Rather than considering the actual number, N, of all RVEs subjected in general to different stresses, it is convenient to introduce the equivalent number N_{eq} of equally stressed RVEs for which the P_{f} of the structure is the same. A structure of positive geometry survives if and only if all the RVEs survive. So, by this classical argument, the survival probability of the structure, $1-P_{\text{f}}$, is the joint survival probability of all the RVEs, each of which has survival probability $1-P_1$; hence, $1-P_{\text{f}}=(1-P_1)^{N_{\text{eq}}}$ or [16–19]

$$P_{\text{f}}=1-[1-P_1(\sigma_N)]^{N_{\text{eq}}} \quad \underset{N_{\text{eq}}\to\infty}{\Rightarrow} \quad 1-\mathrm{e}^{-N_{\text{eq}}P_1(\sigma_N)} \tag{1}$$

Here the last expression is obtained by setting $1-P_1(\sigma_N)=1+x/N_{\text{eq}}$ where $x=-N_{\text{eq}}P_1$, and noting that $\lim_{N\to\infty}(1+x/N)^N=\mathrm{e}^x$; $P_1(\sigma_N)$ is the cdf of the strength of one RVE, which is a function of stress σ in the RVE but may more conveniently be considered as a function of the nominal strength σ_N of structure if the ratio σ/σ_N is known; σ_N is a load parameter which may be taken to represent the maximum principal stress in the most stressed RVE and, for linear elastic behaviour, may generally be written as $c_{\text{g}}F_{\max}/bD^2$, where $F_{\max}$ is the maximum load or load parameter, b the structure width, D the characteristic size, or dimension, of the structure, and c_{g} a geometry-dependent parameter. N_{eq} is a measure of structure size, proportional to the characteristic size D, and equals the actual number of RVEs only if all the RVEs are under the same stress. The relationship of N_{eq} to N depends only on structure geometry and will be discussed later.

The RVE is here defined as the smallest material volume whose failure causes the failure of the structure (of positive geometry) [16, 17]. Its diameter is typically about three inhomogeneity sizes (or maximum aggregate sizes in concrete).

Note that the classical definition of RVE used in homogenization theory is applicable only to elastic and plastic-hardening behaviours but not to strain-softening damage. In that theory, the RVE is defined as the smallest material volume for which the first few statistical moments of RVE properties remain approximately constant as the RVE is displaced through the heterogeneous material. But these moments are irrelevant to the strength of a large structure since that strength depends only on the far-left tail of the cdf of one RVE (e.g. on the RVE tail for $P_{\text{f}}<0.001$ if $N_{\text{eq}}>1000$ [16]).

The fact that, in quasibrittle materials, there exists a non-statistical size effect was experimentally demonstrated by Walsh [22, 23], though for notched specimens exhibiting a different type of size effect (type 2 [14]), which is not studied here. Theoretically, this fact was explained in [24, 25], and was numerically simulated by crack band model [24, 26], non-local models [27, 28], and gradient damage models [27, 28]. For notchless beams, considered here (type 1 size effect [14]), this fact was demonstrated by Gustafsson [29] computationally (based on simulations with the cohesive, or 'fictitious', crack model [30, 31]), by Uchida *et al.* [32] experimentally, and by Bažant and Li [33] theoretically, based on using non-linear fracture mechanics to derive, by asymptotic matching, a simple (type 1) deterministic size effect law [34]. Statistical generalization of the size effect law

DOI: 10.1002/nme

was developed in [13–15, 25, 34–36]. The combined statistical and deterministic size effects were also numerically simulated in the works of Breysse [37], Carmeliet [38], Carmeliet and Hens [39], Frantziskonis [40], Gutiérrez [41] and Vořechovský [42]. The energetic and statistical size effects were amalgamated in [20, 21, 43, 44].

Up to the 1980s, if any size effect was observed, it was automatically attributed to material strength randomness as described by the Weibull-type weakest-link statistical theory [45–54]. The basic hypothesis of this theory is that the structure fails as soon as the material strength is exhausted at one point in the structure. For quasibrittle materials, however, the classical Weibull theory is applicable only

(a) if the size of the structure is much larger than the RVE size l_0, which is a constant and roughly matches the width of the FPZ;
(b) if there is neither notch nor pre-existing macro-crack; and
(c) if the structure geometry is positive [15].

If (a) is not true, then the chain in the weakest-link model must be considered as finite (Equation (1)), which can make a huge difference for the tail (Figure 1). If (b) or (c) is not true, then the location of the FPZ of an initiating crack is almost fixed, determined mainly by structure geometry and mechanics rather than strength statistics, and thus points of different random strength, which are accessible to the crack tip and constitute the source of statistical size effect, occupy only a very small region, which makes the statistical size effect negligible. Positive geometry, which is the only case considered in this study, guarantees that the maximum load (representing failure if the load is controlled) is reached at the onset of macro-crack propagation, as soon as the FPZ is fully formed. The effective RVE size l_0 is roughly twice or thrice the size of the largest material inhomogeneities, such as the major aggregate pieces in concrete, and is roughly $10\times$ smaller than the length of the FPZ of a propagating crack, which is roughly equal to Irwin's material characteristic length $\ell = EG_F/f_t'^2$ (E the Young's modulus, G_F the fracture energy, and f_t' the tensile strength of material). The effective width of the FPZ is roughly equal to the inhomogeneity size.

Unreinforced beams subject to bending or tension, or both, generally have positive geometry. For an initiating mid-span crack in a three- or four-point-bend beam, this can easily be checked using the formula for stress intensity factor (e.g. [55]). For a transverse crack initiating under longitudinal tension at any other point, the positiveness of geometry of beams in flexure may be checked from the relation $\mathrm{d}\mathscr{G}/\mathrm{d}a = F^2 C_{,aa}/2b$ where $\mathscr{G}$ is the energy release rate, F the load, b the beam width, and $C = C(a)$ the load point compliance as a function of crack length a, the second derivative of which may be approximately calculated by finite elements as $C_{,aa}(a) \approx [C(a+\Delta a) - 2C(a) + C(a-\Delta a)]/\Delta a^2$ where Δa is a chosen small increment of a.

As discovered in the early 1980s, the cause of size effect may also be deterministic—the energy release accompanying the stress redistribution, which may be caused by

(i) either a large FPZ, which gives the type 1 size effect, which is of exclusive interest here [33, 36, 56];
(ii) or a large pre-existing crack or notch, which gives the type 2 size effect [13–15, 25, 56–58].

For type 2, material randomness affects only the scatter of nominal strength σ_N but has no effect on the mean of σ_N, i.e. causes no statistical size effect. There also exists a type 3 size effect [13, 56], which occurs in negative geometry structures with notches or large cracks but is barely distinguishable experimentally from type 2, and not of interest for the present study.

DOI: 10.1002/nme

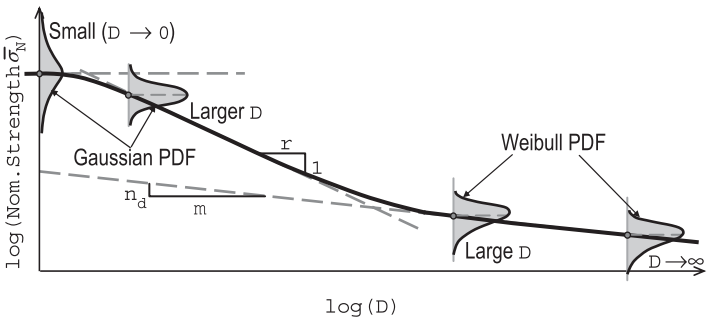

Figure 2. The curve of mean size effect for structures failing at macroscopic fracture initiation, and its probability distributions for various sizes.

The energetic (or deterministic) size effect is automatically exhibited by the cohesive crack model or the crack band model, as well as the non-local models, and also by the lattice or random particle models of concrete microstructure [13, 15]. For stochastic finite element simulations, we choose the crack band model, which is the simplest and by far the most widely used in practice. Its basic idea [26] is that softening damage localizes into a band of single-element width, and that the post-peak softening stress–strain relation must be appropriately scaled down when a different crack band width is imposed by choosing element size different from the RVE size l_0, in order to ensure that the energy dissipated per unit length (in two dimensions), or per unit area (in three dimensions), of the crack band would remain constant and equal to the fracture energy G_F of the material. If the band width h (or element size) equals the RVE size (or the FPZ width), and if the average strain across the band is multiplied by h, the resulting softening curve coincides with the softening stress-separation curve of the cohesive crack model.

The energetic size effect of type 1 dominates only for small enough cross-section sizes (not much larger than the RVE size l_0), while the statistical size effect dominates for very large sizes, far larger than the FPZ. So the type 1 size effect represents a gradual transition from energetic to statistical size effect as the structure size D increases (Figure 2). One way to capture this transition is the non-local generalization of Weibull theory [20, 21, 43, 44, 58]. That theory makes possible stochastic finite element simulations of the mean as well as the variance of the deterministic-statistical size effect in structures of arbitrary geometry.

3. REVIEW OF WEIBULL THEORY AND ITS LIMITATIONS

The Weibull-type weakest-link model is applicable if the FPZ is so small that, compared to structure size D, it can be treated as a point. For geometrically similar structures of various sizes, the dimensionless stress distribution $S(\boldsymbol{\xi})$ just before failure is then a function of only the relative coordinate vector $\boldsymbol{\xi} = \mathbf{x}/D$ of material points, and is independent of D ($\mathbf{x}$ is the actual coordinate vector). The structure may be considered as an assembly of small material elements of volume V_0 and size $l_0 = V_0^{1/n}$, where n is the number of dimensions in which fracture is scaled. In the classical Weibull theory, V_0 is arbitrary (i.e. independent of RVE size) but small enough so that the stress be approximately uniform within V_0. Volume V_0 is conveniently taken equal to the volumes

DOI: 10.1002/nme

of small specimens on which the strength distribution has been tested. The size of these small material elements to which the material statistical properties are referred is thus a matter of choice in the classical Weibull theory.

Denote P_k as the failure probability of the kth elementary volume ($k = 1, 2, \ldots, N$) and P_f the failure probability of the structure. If the failure of one elementary volume causes the whole structure to fail, then the probability of survival of the structure is the joint probability of survival of all of these elementary volumes, i.e.

$$1 - P_f = (1 - P_1)(1 - P_2) \ldots (1 - P_N) \tag{2}$$

which is a generalization of Equation (1) to unequally stressed RVEs. To obtain a continuum approximation, it is convenient to take logarithms and note that, for large N, all P_k must be small, i.e. $\ln(1 - P_k) \approx -P_k$. So,

$$\ln(1 - P_f) = \sum_{k=1}^{N} \ln(1 - P_k) \approx -\sum_{k=1}^{N} P_k \tag{3}$$

The crucial idea of Weibull [45, 46], deduced from tens of thousands of experiments, was that the left tail of the cdf of strength of any material element, denoted as $\varphi(\sigma)$, should be a power law, i.e.

$$P_k = \varphi(\sigma_k) = \langle \sigma(\mathbf{x}_k)/\sigma_0 \rangle^m \tag{4}$$

where $\langle x \rangle$ is the Macauley bracket, defined as $\max(x, 0)$, σ_0 and m are material constants called the scale parameter and Weibull modulus (or shape parameter), and $\sigma(\mathbf{x}_k)$ is the maximum principal stress at point $\mathbf{x}_k$. Recently it has been shown [16–19] that the power-law left tail, having a zero threshold, is not merely a hypothesis but a necessary consequence of Maxwell–Boltzmann distribution of atomic energies and stress dependence of activation energy barriers, and that Weibull modulus m represents the typical number of dominant cracks in the RVE required to make it fail (about 10–50).

Let Equation (4) be substituted into (3). Then, assuming N to be very large, one can replace the discrete sum by an integral over structure volume V. This leads to the following well-known Weibull probability integral:

$$-\ln(1 - P_f) = \sum_k \left\langle \frac{\sigma(\mathbf{x}_k)}{\sigma_0} \right\rangle^m \approx \int_V c[\sigma(\mathbf{x})]\, \mathrm{d}V(\mathbf{x}) \tag{5}$$

where $c[\sigma(\mathbf{x})] = \langle \sigma(\mathbf{x})/\sigma_0 \rangle^m / l_0^n$ is the spatial concentration of failure probability. Because, in geometrically similar structures of different sizes D, the elastic stress fields as functions of dimensionless coordinates $\boldsymbol{\xi} = \mathbf{x}/D$ are identical, one may set $\sigma(\mathbf{x}) = \sigma_N S(\boldsymbol{\xi})$ where σ_N is the nominal stress and $S(\boldsymbol{\xi})$ the dimensionless stress distribution, which is independent of D. Substituting this and $\mathrm{d}V(\mathbf{x}) = D^n \mathrm{d}V(\boldsymbol{\xi})$ into (5) (where n is the number of spatial dimensions in which the structure is scaled, $n = 1$, 2 or 3), we get, after rearrangements, $-\ln(1 - P_f) = (\sigma_N/\sigma_0)^m N_{eq}$ or

$$P_f(\sigma_N) = 1 - \mathrm{e}^{-N_{eq}(\sigma_N/\sigma_0)^m} \tag{6}$$

DOI: 10.1002/nme

$$N_{\mathrm{eq}} = \left(\frac{D}{l_0}\right)^n \Psi \tag{7}$$

$$\Psi = \int_V S^m(\boldsymbol{\xi})\, \mathrm{d}V(\boldsymbol{\xi}) \tag{8}$$

where Ψ is a geometry parameter characterizing the effect of the dimensionless stress field depending on structure geometry. As already stated, N_{eq} can be interpreted as the equivalent number of equally stressed material elements of a size for which the material statistical properties have been measured, i.e. equivalent to the number of identical links in a chain (since Weibull's theory is a continuum theory valid for large N_{eq}, or $D \gg l_0$, these material elements do not have to coincide with the RVE and can have any convenient size). Note that Equation (7) is exact only asymptotically, provided that N_{eq} is large enough (for small N_{eq}, or small D/l_0, one must expect deviations from Equation (7), and that is where finite element analysis will be needed). Equation (6) is the Weibull cdf, from which

$$\sigma_N = S_0\ N_{\mathrm{eq}}^{-1/m} = C_0\ (l_0/D)^{n/m} \tag{9}$$

where $S_0 = \sigma_0[-\ln(1-P_{\mathrm{f}})]^{1/m}$ and $C_0 = C_1 \Psi^{-1/m}$. This equation, in which C_0 and S_0 are independent of D, gives the scaling of nominal strength for a fixed failure probability P_{f} (e.g. for $P_{\mathrm{f}} = 0.5$, the median σ_N). The mean nominal strength is calculated as $\mu_N = \int_0^1 \sigma_N\, \mathrm{d}P_{\mathrm{f}} = \int_0^\infty \sigma_N\, p_{\mathrm{f}}(\sigma_N)\, \mathrm{d}\sigma_N$ where $p_{\mathrm{f}}(\sigma_N) = \mathrm{d}P_{\mathrm{f}}/\mathrm{d}\sigma_N$ (pdf of strength), and substitution of Equation (6) leads to the well-known Weibull scaling law

$$\mu_N = \sigma_0 \Gamma(1 + 1/m) N_{\mathrm{eq}}^{-1/m} = \sigma_0 \Psi^{-1/m} \Gamma(1 + 1/m)(l_0/D)^{n/m} \tag{10}$$

The standard deviation δ_N is calculated as ${\delta_N}^2 = \int_0^\infty {\sigma_N}^2 p_{\mathrm{f}}(\sigma_N)\, \mathrm{d}\sigma_N - \bar{\sigma}_N^2$, and substitution of Equation (6) yields for the CoV of σ_N the well-known expression

$$\omega_N = \sqrt{\frac{\Gamma(1 + 2/m)}{\Gamma^2(1 + 1/m)} - 1} \tag{11}$$

A point to note is that Weibull modulus m is, in Weibull theory, strictly a material property, and thus cannot depend on the size and shape of structure. Because Equation (11) is independent of structure size and shape, ω_N is a material property, too. To check whether the Weibull theory is applicable, the value of m obtained by fitting Equation (9) to size effect experiments must be the same for very different structure sizes and shapes, and must, for each size and shape, satisfy Equation (11) where ω_N characterizes the scatter of strength tests. In most previous studies, though, due to limited scope of experiments, the aforementioned checks were not made and m was determined from only one type of test.

In some engineering studies, in which more extensive tests were carried out, it was proposed that m and ω_N were variable, depending on structure size and shape. However, from the preceding derivation it is clear that m and ω must be constant. If they appear to be variable, the only correct conclusion is that the Weibull theory is inadequate and that some other phenomenon must have intervened in the tests—for instance a composite Weibull–Gaussian cdf or the energetic (deterministic) size effect (although the non-energetic deterministic 'wall effect' is also a possibility for very small sizes).

DOI: 10.1002/nme

In failure of quasibrittle structures (except the brittle limit of very large ones), the Weibull size effect is typically combined with the energetic size effect, arising from the finite size of FPZ (or of RVE).

The Weibull distribution, with its relationship to power-law tail and power-law scaling, was mathematically derived (11 years before Weibull) by Fisher and Tippett [59], based on a self-similarity hypothesis of extreme value distributions, known as the stability postulate of extreme value statistics, formulated by Fréchet in 1927; see [17, 59]. This original derivation, however, is not totally unrelated to Bažant and Pang's derivation from Maxwell–Boltzmann distribution because the derivation of that distribution, too, implies a certain self-similarity of the distribution of atomic energies.

In the stability postulate, one considers a chain of v identical links with random uncorrelated strength to be subdivided into N sub-chains of n links whose cdf is $\Phi(\sigma)$ ($v = Nn$). It is postulated that the cdf of the chain, which is (according to the joint probability theorem) $\Phi^N(\sigma)$, must have a stable form, which means it must be self-similar, having similar $\Phi(\sigma)$ as the sub-chains. The self-similarity of $\Phi(\sigma)$ is expressed as a form invariance for linear transformations, and so the stability postulate takes the form of the following functional equation:

$$\Phi^N(\sigma) = \Phi(a_N \sigma + b_N) \tag{12}$$

for function $\Phi(\sigma)$, where a_N, b_N are the coefficients of linear transformation depending on N.

It is easy to check that Equation (6) satisfies this equation, but much harder to prove that the Weibull distribution is one of only three distributions that satisfy this equation, the others being the Gumbel and Fréchet distributions (this proof is due to Fisher and Tippett [59]). The last two distributions extend to $-\infty$ and thus are inapplicable to tensile strength (note that the possibility of negative strength values is not an argument against a Gaussian distribution of strength, since, by virtue of the central limit theorem, the Gaussian can apply only to the central, positive, range of strength, but is not justified for a distribution intended for describing the tail). This leaves the Weibull distribution as the only possible asymptotic distribution of the strength of a chain with $N \to \infty$.

Since it was shown that the cdf of strength of a RVE of a quasibrittle material must have a power-law tail σ^m extending only up to $P_f \approx 0.001$, with the rest of cdf being necessarily Gaussian, the entire cdf becomes Weibull only for structures so large that N_{eq} is greater than approximately 5000 [16, 17], which is in the brittle limit (among concrete and fibre composite structures, only very large ones, such as a dam, large ship hull or rudder of large aircraft, are large enough).

For smaller sizes, the cdf of structure strength may be approximated as Gaussian with a Weibull tail grafted at a point. As the structure size increases from one RVE to thousands, the grafting point of the composite cdf moves gradually to higher P_f, reaching eventually $P_f > 99\%$ (see the Appendix). During this transition, Equations (10) and (11) do not apply and more complex expressions are needed for the mean and the CoV (which is no longer constant but depends on N_{eq}) [17].

To fit the experimental strength histograms of quasibrittle materials such as concrete, coarse-grained ceramics and fiber composites, it has been considered necessary to introduce into Weibull distribution a finite threshold σ_u, generalizing Equation (4) as $P_1 = \langle [\sigma(x_k) - \sigma_u]/\sigma_0 \rangle^m$ where σ_u is the strength threshold. However, it has been shown [16, 17] that a non-zero threshold is an incorrect way to improve the fit of these histograms, and is in fact inadmissible in principle, since it would conflict with Maxwell–Boltzmann distribution, a generally accepted pillar of statistical thermodynamics. It would contradict the fact that the frequency of interatomic bond ruptures is non-zero at any stress (as known from the transition state theory of chemical reactions). A much better fit of these histograms can be obtained on the basis of Equation (1) for finite N_{eq} [16, 17].

DOI: 10.1002/nme

4. CRACK BAND MODEL WITH LATIN HYPERCUBE SAMPLING (LHS) OF ELEMENT STRENGTHS AS RANDOM VARIABLES

Now it should be noted that the quasibrittleness and the positiveness of structure geometry offer an enormous advantage for applying SFEM. If the cdf of strength of one RVE is known, then the type of cdf for any structure size and any geometry is also known, as given by Equation (1). So it suffices to determine by SFEM only the mean and the CoV, from which the entire distribution follows according to Equation (1). This is a far easier task than pursuing the importance sampling or other numerical methods for extreme value statistics. We can therefore employ simple sampling techniques which are adequate only for predicting the mean and variance. What is further important is that these simple techniques can be easily combined with the crack band model for capturing the deterministic part of size effect.

The simplest and in practice by far the most widely used model for simulating fracture nonlocality and energetic size effect is the crack band model [26, 60]. Statistical sampling can be easily combined with this model. To determine the statistical properties of structural response, it suffices to generate a number of sets of random samples of input parameters, associate the generated realizations with model parameters (such as local strength), compute by finite elements the structural strength for each set, and finally evaluate the statistics of response from the simulation results.

In Monte Carlo simulation, the input parameters are generated randomly from the given probability distribution of each parameter. While the Monte Carlo approach is fundamental, it is not the most efficient. This study uses the stratified sampling—a well-known efficient technique in which the range (0, 1) of the cdf of each input parameter is subdivided into a number of strata of equal probability (i.e. equal width), from which the random samples are then drawn.

A particularly efficient and simple approach to stratified sampling, with superior stability and accuracy of estimation of the mean and variance, is the Latin hypercube sampling (LHS) which minimizes the number of simulations needed to achieve acceptable accuracy as compared to crude Monte Carlo sampling [61, 62]. LHS has been effectively applied to concrete creep [63–66], and recently also to the statistical size effect in concrete [67, 68]. The salient features of LHS are that

(i) the number of simulations, N_{sim}, equals the number of strata for each input variable X;
(ii) each stratum has the same probability, equal to $1/N_{\text{sim}}$ (Figure 3); and
(iii) each stratum, numbered k, of each random variable X is sampled by one and only one value of x_k $(k = 1, \ldots, N_{\text{sim}})$. The x_k-value may be defined so that the corresponding cdf $F(x_k)$ would lie in the middle of each stratum (Figure 3), i.e. $x_k = F^{-1}(\bar{P}_k)$ where $\bar{P}_k = (k - 0.5)/N_{\text{sim}}$.

A slightly better convergence of LHS is obtained by sampling each stratum at the centroid of the corresponding segment of the pdf [69, e.g.] $f(x) = \mathrm{d}F(x)/\mathrm{d}x$, which is given by $x_k = N_{\text{sim}} \int_{x_{\text{L}k}}^{x_{\text{R}k}} x f(x)\,\mathrm{d}x$, where $x_{\text{L}k} = F^{-1}[(k-1)/N_{\text{sim}}]$ and $x_{\text{R}k} = F^{-1}(k/N_{\text{sim}})$ are the left and right boundaries of the kth segment. As proven by McKay *et al.* [61], the convergence of the first and second statistical moments of response is then optimal, under certain typical conditions.

Each material property in each finite element represents one random variable (note that associating a random variable with each integration point of each finite element would be incorrect, and would prevent using the random block method introduced later, because the integration points within one element cannot be made to behave according to the weakest-link model; the structure geometry can be positive with respect to damage extension into the adjacent element, but not

DOI: 10.1002/nme

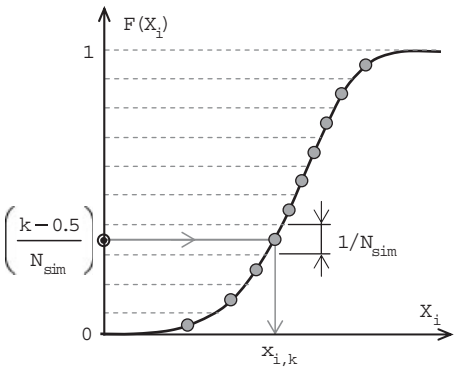

Figure 3. Stratified sampling by mid-values of the strata.

into the adjacent integration point). For n_{el} finite elements, each with n_{mp} material properties (such as the material strength and fracture energy), there will thus be $v = n_{\mathrm{el}} n_{\mathrm{mp}}$ input variables. Accordingly, the cdf of each of these variables must be divided into N_{sim} equal strata. Correlations among the random variable can be captured by modifications of the standard LHS. The algorithm to do that was shown by McKay *et al.* [61], Iman and Conover [70]. An efficient and robust algorithm using the stochastic optimization method, called simulated annealing, was proposed by Vořechovský and Novák [68].

Efficient LHS technique strategies were implemented in statistical, sensitivity and reliability software FREET [71]. This software has been combined with the commercial finite element code ATENA, which was developed at Červenka consulting in Prague, based on the crack band model, and has been used for various stochastic finite element simulations (see [71], for a summary of possible applications). This commercial software has been used here with pure Weibull statistics, Equation (6), although the finite chain statistics, Equation (1), can be implemented in it easily. Although, in view of the latest results [17], the pure Weibull scaling used in the FREET software at the time of these computations (instead of Equation (1) with a grafted Gaussian–Weibull cdf) is not very realistic for normal concrete specimens and structures, it is simple and serves the aim of this study, which is merely to demonstrate the computational method rather than predict failure of real structures. The autocorrelation of the random strength field must be captured. The simplest, though crude, approach to do that is to subdivide the structure into elements having the size of autocorrelation length, l_{a}. Clear though the necessity of a finite autocorrelation is, to ensure objective simulation, the proper value of l_{a} remains clouded and quantitative information on autocorrelations other than those imposed by the RVE size is unavailable. Therefore, it will be assumed in this study that $l_{\mathrm{a}} = l_0 =$ RVE size. This assumption is not implausible since non-local spatial averaging over a material element of RVE size, on which the statistical material properties are defined, automatically introduces autocorrelation with autocorrelation length l_0.

Finite elements of the RVE size are manageable for small structures, but lead to excessive number of elements for very large structures. As proposed by Bažant and Novák [72], this difficulty can be overcome (for structures of positive geometry) by introducing *random property blocks* (RPB) whose mean strength and CoV are scaled according to the RPB size using the appropriate scaling law. The advantage is that the number of RPBs (initially, in 2003, called 'random macro-elements')

DOI: 10.1002/nme

can be kept fixed as the structure size is increased in proportion to D. This greatly enhances the efficiency of stochastic computations for very large structures [20, 67, 72, 73]. The mean strength scaling according to Equation (10) would, of course, be realistic only for the large-size brittle limits of quasibrittle structures, which would occur if the equivalent number N_{eq} of RVEs at the lower limit of the scaling range would exceed about 1000 [16, 17] (and, of course, assuming the weakest-link model to apply, which is the case if the structural geometry is positive).

The RVE for damage and failure analysis is typically about two to three material inhomogeneities in size. It is understood as the smallest material volume whose failure causes the whole structure to fail (which is not in the classical sense of statistical homogenization). As already pointed out, the cdf of strength of one RVE must be Gaussian (or normal) except for a far-left tail grafted at $P_f = 0.0001$–0.01, which must be of Weibull (or power law) type with a zero threshold (for the precise experimental determination of RVE size and grafting probability P_f, see [16, 17]). Consequently, Equation (10) based on Weibull scaling (or the stability postulate of extreme value statistics) is realistic only within the range of random blocks larger than about 1000–10 000 RVEs. Thus, in reality, it practically never applies to concrete structures (unlike ceramics with fine enough grains). So the realistic scaling of mean strength and CoV must be obtained from the integrals for the mean and variance based on the cdf given by Equation (1), where $P_1(\sigma)$ is a Gaussian distribution onto which a power-law tail is grafted at $P_f \approx 0.0001$–0.01.

To demonstrate the RPB approach, the experiments carried out by Koide *et al.* [74, 75] have been simulated. Unfortunately, these simulations have been carried out before a parallel study [16, 17] showed that the specimens tested were not large enough to allow application of the classical Weibull theory, in which the RVE size is assumed to be negligible. Thus, the simulations that are now going to be presented are mere numerical demonstrations of the proposed numerical approach, and cannot actually provide optimal representation of Koide's data. But because the size range of Koide's data (as well as any other data in the literature) was far too small and the histograms limited, relatively little will be lost from the practical viewpoint, and the fitting exercise is not going to be altogether hypothetical. These data, unfortunately, are insufficient to clearly distinguish between optimum fits by the Weibull theory and by the weakest-link model based on Equation (1) for a finite chain of RVEs having a mostly Gaussian cdf with remote power-law tail.

Because the beam span in Koide's tests was scaled while keeping the beam depth constant, the microcracking boundary layer at maximum load, which causes stress redistribution and thus the energetic part of size effect, should have had a nearly constant thickness. Consequently, according to the beam theory with plane cross-sections, the stress redistribution in the uniform moment segment of the beam would expected to be the same, and thus the size effect to be purely statistical in nature. A previous study [43, 67], though, revealed discrepancies with the Weibull statistical theory, which means the energetic size effect due to two-dimensional stress redistributions, as well as finite chain statistics of Equation (1), must have played some role in these tests. Although the energetic size effect is not the focus of this paper, it is automatically captured by using the crack band model.

Koide *et al.*'s tests [74, 75] were performed on four-point bend beams which all had the same shear span $L_s = 20$ cm (distance from load to support) and the same square cross-sections, with sides $b = d = 10$ cm. The maximum aggregate size was $d_a = 20$ mm and the mean standard cylindrical compression strength concrete was $f'_c = 30.0$ MPa. The bending span D (i.e. the distance between the loads) varied and was 20, 40 and 60 cm in three test series. The nominal strength of these beams was defined as the largest elastically calculated stress in the beam, $\sigma_N = 6M/bd^2$, where M is the maximum bending moment.

DOI: 10.1002/nme

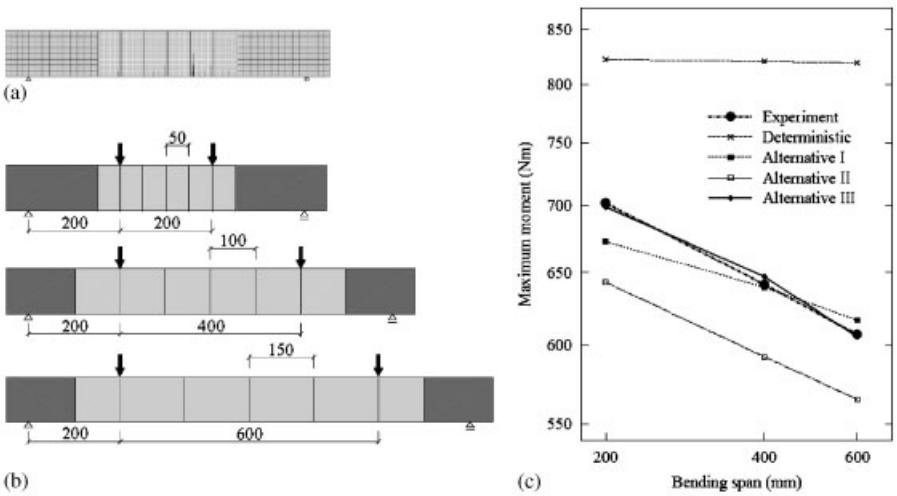

Figure 4. Simulation of Koide's test series C: (a) mesh size for the smallest beam; (b) scaling of RPBs for all sizes; and (c) comparison of mean size effect curve (MSEC).

Figure 4 shows the simulations of size effect in test series C with the RPB approach using the FREET and ATENA softwares. For each of the three lengths, Koide's beam is subdivided into six RPBs (Figure 4(a)); 16 LHS simulations are performed for six random tensile strengths f_t' and random fracture energies G_F of the material. These random variables, statistically independent (or correlated), are sampled according to the optimized techniques of [68].

Three alternatives [67] are simulated (Figure 4(b)). They are

- Alt. I with tensile strength as the only random property.
- Alt. II with high statistical correlation between tensile strength and fracture energy.
- Alt. III with the material parameters from Alt. II adjusted to shift the size effect curve upward in order to match the experimental data.

Comparison of these three alternatives reveals how the material parameters influence the results. A decrease of Weibull modulus m, as well as a decrease of the autocorrelation length l_a, makes the size effect stronger (note in Figure 1(f) the steeper slope of the straight line of mean size effect). An increase of tensile strength f_t', plausibly, shifts this line upwards. A decrease of G_F, as well as correlation between f_t' and G_F (introduced by correlation factor $r = 0.99$), makes the size effect stronger (i.e. the straight line steeper) and, at the same time, shifts this line downwards (and thus closer to chain-like statistical behaviour). Alt. III is able to fit the size effect curve very well, by virtue of changing the mean f_t' and G_F of the finite elements (which is admissible for Koide's tests because f_t' and G_F were not measured).

Although the finite element stress analysis of Koide's beams was two-dimensional, the parameters of the RPBs were scaled only in one dimension. The reason is that, because of constant beam depth, the weakest-link model is represented essentially by the one-dimensional statistical variation of strength along the bottom layer of highly stressed finite elements.

Despite a good match with such one-dimensional statistical modelling, the material parameters used in these computations could not be reproduced on a different set of experiments. The likely

DOI: 10.1002/nme

cause of this problem is that the more complex RPB scaling based on Equation (1) should have been used instead of the simple Weibull scaling, as already discussed. Nevertheless, despite this problem with comprehensive fitting of Koide *et al.*'s tests (which are planned to be addressed in a follow-up study), the foregoing analysis serves as a demonstration of the RPB approach.

5. MODIFIED LHS WITH EACH ELEMENT STRENGTH TAKEN AS A RANDOM SAMPLE FROM ONE MATERIAL STRENGTH DISTRIBUTION

Another stratified sampling approach for SFEM will now be proposed. It reduces the amount of computational work by combining stratified sampling with a modified Latin square design (LSD) [76–79]. In the LSD, the number of simulations is again the same as the number of strata. The difference from LHS lies in an additional constraint on the number n_{RV} of RVEs: If $N_{\mathrm{sim}} \geqslant n_{\mathrm{RV}}$, the stratum value is not sampled more than once in each row and column of the input table of random variables (Table I), and if $N_{\mathrm{sim}} < n_{\mathrm{RV}}$, each stratum value must appear for equal number of times in each row and each column (Table II). The LSD can simply be viewed as a subset of all possible random configurations in standard LHS, with the advantage of reducing the variance in the moment estimates before correlations are introduced, by virtue of optimum adjustment of initial permutations.

Multiple random variables can be obtained by adopting an approach similar to the Graeco-LSD [78], which is a generalization of LSD to multiple variables using multiple Latin square tables. Desired correlations between multiple variables may be achieved by certain optimum adjustments of initially random permutations. But correlations will not be studied here, and f_{t}' and G_{F} will be assumed to be fully correlated by a functional relationship.

In LSD, the variability in the estimates of statistical moments, for a small limited number of realizations, is effectively suppressed by enforcing equal likelihood of sampling from a strata in an exhaustive list of all possible combinations (Table I). The variability of the first and second

Table I. Input table for LSD with four RVEs and six simulations.

Simulation	Var X_1	Var X_2	Var X_3	Var X_4
1	$x_{1,1}$	$x_{2,2}$	$x_{3,5}$	$x_{4,3}$
2	$x_{1,2}$	$x_{2,5}$	$x_{3,4}$	$x_{4,6}$
3	$x_{1,6}$	$x_{2,3}$	$x_{3,2}$	$x_{4,1}$
4	$x_{1,5}$	$x_{2,6}$	$x_{3,1}$	$x_{4,4}$
5	$x_{1,3}$	$x_{2,4}$	$x_{3,6}$	$x_{4,5}$
6	$x_{1,4}$	$x_{2,1}$	$x_{3,3}$	$x_{4,2}$

Table II. Input table for LSD with four RVEs and two simulations.

Simulation	Var X_1	Var X_2	Var X_3	Var X_4
1	$x_{1,1}$	$x_{2,2}$	$x_{3,2}$	$x_{4,1}$
2	$x_{1,2}$	$x_{2,1}$	$x_{3,1}$	$x_{4,2}$

DOI: 10.1002/nme

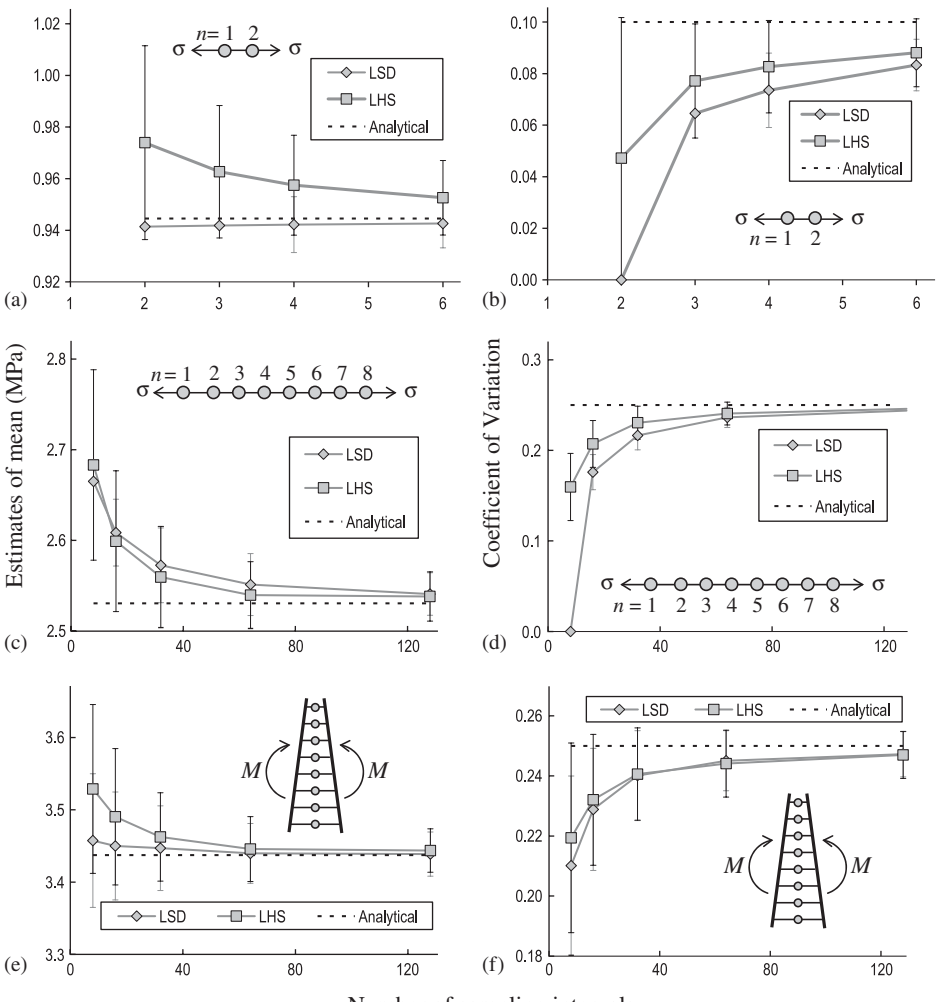

Figure 5. Comparison of: (a) first statistical moment and (b) CoV for two-element rod structure; (c) first statistical moment* and (d) CoV for eight-element rod structure; and (e) first statistical moment* and (f) CoV for eight-element structure under linearly distributed stresses. (*LSD curve mirrored about analytical curve for more obvious comparison.)

moment estimates and their rapid convergence may be demonstrated on a uniformly tensioned one-dimensional rod which fails at its weakest point. First, a rod subdivided into only two elements is analysed. All the possible combinations of the sampling intervals are considered to determine the theoretical means and bounds for the first two statistical moments of the sampling methods. Again, as a hypothesis of this exercise (though not in agreement with [17]), the element strength values are sampled from the Weibull distribution. A comparison of the moment estimates for the two sampling methods for the weakest-link model is shown in Figure 5(a).

DOI: 10.1002/nme

The objective of a good sampling method is to generate close estimates of the statistical behaviour with low variability while using the least possible number of simulations. The advantage of the LSD is clear from Figure 5(a) where a close approximation of the first statistical moment (i.e. the mean) is obtained even when the number of simulations is close to the number of elements. Figure 5(a) shows a significantly better mean (or first moment) estimate with low variability for the LSD method, but the estimate of CoV (or second moment) is not better in comparison to the LHS method.

When the number of elements is large, the means and error bars (showing the mean $\pm$ standard deviation) can be based on a large number of computer runs (300 in Figures 5(b) and (c)). The two sampling methods are compared with this approach using the example of a rod with eight elements under uniform uniaxial tensile stress (Figure 5(b)) and a stepwise linearly decreasing uniaxial stress (Figure 5(c)). The rod is assumed to fail when the strength limit is exhausted in any one of the elements. In Figure 5(b), both sampling methods are compared for the first statistical moment and it is seen that the LSD method achieves a smaller variability of the moment estimates. The benefit of the LSD becomes more evident for the simple structures in Figure 5(c) where the elements are subjected to different stresses. The LSD method yields an estimate of the first statistical moment that is significantly closer to the analytical mean.

Improvement in LHS by enforcing desired correlation or by removing undesirable correlation has been widely discussed [68–70]. The goal is to create correlation between material parameters reflecting empirical correlations apparent from experiments. A correlation matrix that samples the material properties can be established for each element. The approach can be extended to a larger correlation matrix that simultaneously enforces desired correlation between different material parameters for the same element, and removes undesirable correlation between different elements having the same material parameters. The procedure to remove undesirable correlation can lead to improvements in the statistical response for a small structure with few RVEs. But the gain can be minor for a large structure in comparison with the errors incurred due to discretization of the random field and the solid continuum. In large structures, the total number of RVEs can be enormous and sampling cumbersome. For example, a structure with $n_{\mathrm{el}} = 1000$ finite elements and with $n_{\mathrm{mp}} = 3$ material parameters will require sampling of the material properties that satisfies a 3000×3000 matrix of desired correlation.

It has been demonstrated that, compared to the LHS, the LSD gives a better estimate of the first statistical moment, with smaller variability. Currently, the premise of LSD is zero correlation between the elements of the same material property, but this premise is valid only as long as the size of the element is approximately the same as the autocorrelation length l_{a}.

Finite element modelling with crack band model is used to compare the LSD method with Koide *et al.*'s [74, 75] beams of Series B and C (with the same reservations to practical relevance to these data as note before). The crack band model with $l_0 \approx 2.5 d_{\mathrm{a}}$ is used to describe tensile fracturing of concrete and $l_{\mathrm{a}} \approx l_0$ is assumed. The tensile strength f_{t}' of each element is sampled from the Weibull pdf with the Weibull modulus of $m = 7.6$. The post-peak softening stress–strain curve in the crack band model is assumed to be linear. The fracture energy G_{F} and tensile strength f_{t}' are assumed to be perfectly correlated and G_{F} is estimated from a formula recommended by CEB90 CEB [80]: $G_{\mathrm{F}} = G_{\mathrm{F0}}(f_{\mathrm{cm}}/f_{\mathrm{cm0}})^{0.7}$ where $f_{\mathrm{cm0}} = 10$ MPa, f_{cm} is the mean compressive strength, G_{F0} the base value of fracture energy which depends on D_{agg}, and can be read from Table 2.1.3 of [80].

A subdivision of cdf range (0, 1) into 16 equal sampling intervals is used for direct comparison of the two sampling methods. The peak load is computed with the help of arc-length control using a commercial finite element software, FEAP, and a user-supplied subroutine for the material

DOI: 10.1002/nme

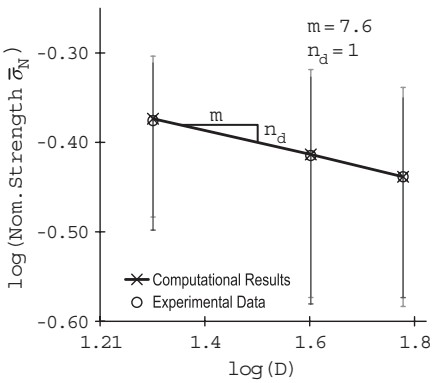

Figure 6. Comparison of experimental and computational mean size effect curves for Koide *et al.*'s beam (experimental and computational scatter bands in black and grey, respectively).

softening model. The mean values and the scatter bands from the simulations of specimens of each size are plotted on the mean size effect curve (MSEC) in Figure 6. In spite of using an unrealistic cdf of the material strength, the size effect curve is in satisfactory agreement with the experimental data and the scatter bands are matched quite well (doubtless because of the limited scope of Koide's data, which could be fitted by different theories almost equally well).

6. HYBRID LHS AND MONTE CARLO SAMPLING WITH EXTRA SIMULATIONS IN LOWEST STRATUM

Figure 7 shows three plots of data points representing the cdf of strength of four-point bend beam obtained by computer finite element simulations based on LHS, in which the number of strata of material strength distribution (and thus the number of simulations) was 25, 40 and 100. The simulated cdf plots are seen to be clustered and stair-like, and this undesirable feature does not disappear even when 100 strata (and simulations) are used. The simulated left tail seems totally useless for estimating the low probability tail of cdf, which is the main information needed for judging structural safety. Each of these three simulated cdf's is optimally fitted by Weibull cdf (solid curves) in Figure 7.

Are these smooth fits of the rugged simulated histograms of any use? Do they give realistic approximate information on the left cdf tail? Despite the clustering and ruggedness of the simulated cdf plots, the answer is in the affirmative. The simple sampling technique used is perfectly sufficient for determining the mean and the CoV, and this is enough because the type of distribution is known. It is given in Equation (1), and in the special case of the present example it is simply the Weibull distribution.

The distribution according to Equation (1) can, of course, be determined only if N_{eq} is known. Equations (7) and (8) suffice for determining N_{eq} only if the state just prior to the maximum load is elastic, which can be true only for perfectly brittle behaviour (the large-size limit of quasibrittle structure). Otherwise, the stress field influenced by cracking damage before the maximum load should be used. So N_{eq} must generally be obtained by numerical integration of Equation (8), in which one must substitute the stress field obtained by non-linear finite element analysis, e.g. with the crack band model. This is where numerical simulation is irreplaceable.

DOI: 10.1002/nme

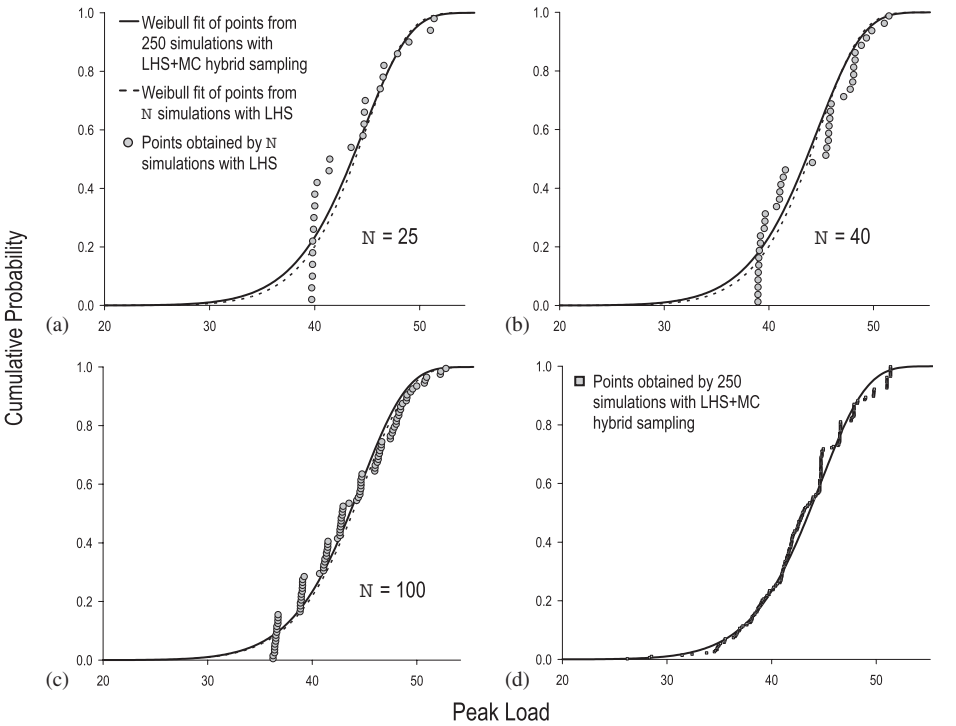

Figure 7. Response of four-point bend specimens with: (a) 25 LHS; (b) 40 LHS; (c) 100 LHS; and (d) 250 LHS+MC hybrid sampling.

To determine the stress field for calculating N_{eq}, the non-linear finite element analysis can be deterministic. So, is not the stochastic simulation merely an alternative? Not really. Once formulated and demonstrated, the present simulation approach can be extended to structures that behave like a generalization of the weakest-link model, for instance, those where correlations among adjacent RVEs exist, causing deviations from positive structural geometry (correlations among adjacent RVEs have essentially the effect of creating a large super-RVE, but if all the RVEs in the structure are correlated then the present approach does not apply and importance sampling or the like becomes probably necessary).

Although complete numerical verification of the far-left (say, up to $P_{\text{f}} = 10^{-6}$) would necessitate techniques such as importance sampling [7], knowledge of the full distribution permits us to verify only the near tail, up to only about $P_{\text{f}} = 1\%$. This can be done by a simple refinement of the LHS. It would be ineffective and wasteful to increase the number of strata. Rather, what is necessary is to conduct many more simulations within the lowest stratum (which is in the spirit of importance sampling).

One effective way to do that is a hybrid sampling, in which the LHS samples are taken from all the strata except the lowest stratum and n_{t} Monte Carlo simulations are conducted with the finite element program within the lowest stratum. The Monte Carlo simulations sample the probability interval across the lowest stratum according to the cdf of RVE strength of the material (which was taken, in the present example, as the Weibull cdf for the lowest stratum). The weight ascribed to

DOI: 10.1002/nme

the structural strength obtained from each of these Monte Carlo simulations is $(1/n_t) \times$ the weight of each of the upper strata used for LHS. For 25 strata overall, and for $n_t = 10$ random simulations within the lower stratum, the cdf plot obtained from such combined hybrid LHS—Monte Carlo sampling method is shown in Figure 7(a).

As can be seen, the clustering and ruggedness of the plot are eliminated and the plot closely matches the Weibull cdf optimally fitted to the simulated plot, as far as the eye can discern. Figures 7(b) and (c) also show by the dashed curve the Weibull cdf from Figure 7(a), which was obtained by least-squares fitting of the cdf plot simulated by pure LHS sampling. Remarkably, the dashed and solid curves of Weibull cdf are very close. Of course, this is not a verification of the far-left tail up to $P_f = 10^{-6}$, but it is an indication that numerical refinement tends to agree with what is expected theoretically.

7. CONCLUSIONS

1. The proposed formulation of stochastic finite element method for predicting load capacity of concrete and other quasibrittle structures failing at macro-crack initiation meets the following three requirements:
   (a) For small structure size, it simulates in an objective manner damage localization with the energetic (deterministic) part of size effect.
   (b) For the large-size limit, it converges to Weibull statistical theory, as required by extreme value statistics.
   (c) In the intermediate size range, it simulates the combined statistical–energetic size effect.
2. The mean and variance of structural response can be effectively computed by stratified sampling in the form of either Latin hypercube sampling or Latin square design. The former, in which the strength of each finite element is considered as one random variable to be sampled, gives closer estimates for the variance of response when the number of simulations is small. The latter, in which each element strength is sampled from one overall material strength distribution, gives a smaller scatter in estimating the variance and a closer estimate for the mean response.
3. For quasibrittle structures of positive geometry (i.e. those failing at macro-crack initiation), the statistical model is a chain with a finite number of links, each of which corresponds to one representative volume element (RVE) of material, which has previously be shown to have a strength with Gaussian (normal) distribution onto which a Weibull tail is grafted at the cumulative probability of roughly 0.001. Thanks to this fact, the type of the entire distribution of structural strength can be obtained analytically for any structure size, depending on the effective (geometry adjusted) number of RVEs in the structure. This means that a failure probability such as $P_f = 10^{-6}$, which must not be exceeded in design, can be determined by computing merely the mean and variance of this distribution (and thus the need for more complex methods such as importance sampling is obviated). The mean and variance can be easily obtained either analytically, from the weakest-link model for a finite chain, or by simple stochastic simulation techniques such as stratified sampling. The latter is shown to give good results, which makes it a candidate for future extensions to the case of correlations among adjacent RVEs.

DOI: 10.1002/nme

4. For large structures failing at macro-crack initiation from one RVE, the simulations confirm the previous finding that the number of structural simulations required to obtain the mean size effect can be greatly reduced by introducing 'random property blocks' (RPB)—blocks that are scaled with the structure size, to keep their number constant, and are small enough for having in each of them a nearly homogeneous field of random material strength. The mean and variance of each random block strength are scaled according to the block size using the weakest-link model with a finite number of links.
5. Numerical examples of simple sampling approaches and the use of random block method are computed, for the sake of simplicity, under the hypothesis that the strength of each RVE is Weibullian. Although not realistic, this hypothesis suffices for numerical demonstration. It has advantage that the structural strength is Weibullian for every size, and that the random block strength can be scaled simply with a constant coefficient of variation and a mean reduced according to the Weibull power-law size effect. The scaling must generally be based on the equivalent number of RVEs (corresponding to the number of links in the weakest-link model) which must be corrected according to the stress field just before collapse, depending on structure geometry. The RVE cannot be here defined by homogenization theory but must be understood as the smallest material element whose failure causes the whole structure to fail (thus typically the RVE size $\approx$ two to three grains or inhomogeneity sizes $\approx$ effective width of the fracture process zone).
6. The cumulative histogram of structural strength values obtained with the stratified sampling methods is rugged, clustered and stair-like at the left margin. In spite of that, least-squares fitting of the computed histogram suffices to give the correct mean and variance. Refinement in which the stratified sampling is combined with extensive Monte Carlo sampling of material strength in the lowest stratum is shown to make the histogram visually smooth, agreeing with the theoretically expected cumulative distribution in the central part up to the tail starting at the probability of the order of 1%.

APPENDIX: DISTRIBUTION OF RVE STRENGTH

In a study subsequent to the present one, reported briefly in [16] and in full detail in [17] (and summarized in conference papers [18, 19]), the Maxwell–Boltzmann distribution of atomic energies and the stress dependence of the activation energy barriers was used to show that the core of distribution $P_1(\sigma_N)$ must be Gaussian, while the far-left tail must be Weibullian. This means that the tail must have the form of a power law of stress with a zero threshold, i.e. $P_1 = {\sigma_N}^m$ where m represents the Weibull modulus (see also [18, 19] for a summary). The transition between the core and tail is relatively abrupt; in other words, one may consider a Weibullian tail to be grafted onto the Gaussian core at a certain transitional probability $P_f = P_{gr}$.

It has further been shown that, for the cdf of one RVE of a quasibrittle structure, $P_{gr} \approx 0.001$. Such a tail is normally undetectable from experimental histograms (about 10^5 tests of identical specimens would be needed to see it), yet it totally controls the failure probability of a large structure. If P_{gr} were much larger, it would imply a RVE to behave as a chain of sub-RVEs, but then the fracturing would have to localize into just one sub-RVE, and so the assumed RVE would not be a RVE. Equation (1) indicates [18, 19] that, as N_{eq} or structure size D is increased, the transitional probability P_{gr} increases and for $N_{eq} \approx 3000$ one has $P_{gr} \approx 0.95$, i.e. the entire cdf is Weibullian for all practical purposes. But if P_{gr} were much smaller than 0.001, it

DOI: 10.1002/nme

would incorrectly imply that extremely large quasibrittle structures (with $N_{eq} \gg 3000$) have a Gambel cdf and fail in a ductile manner (note that, in particular, the lognormal distribution cannot apply to strength, for if it did then the failure load probability would have to be a product, rather than a sum, of the failure probabilities of all the RVEs along the failure surface, which is inconceivable).

According to the central limit theorem of probability, a Gaussian cdf is characteristic of ductile failures. Indeed, for such failures, the strength of all the RVEs along the failure surface is at maximum load fully mobilized. This means that the load is a (weighted) sum of many independent random variables, which is known to converge to the Gaussian distribution.

In general, no strength distribution can be other than Gaussian, Weibullian, or a transition between these two. The threshold of the Weibullian part must be zero, i.e. the tail $P_1 = (\sigma_N - \sigma_u)^m$ with non-zero σ_u is physically impossible, or else the Maxwell–Boltzmann distribution of atomic energies would be contradicted [16, 17].

ACKNOWLEDGEMENTS

Financial support for the work of all authors on the basic theory applicable to composites was obtained under grants CMS-0556323 from the U.S. National Science Foundation (NSF), and grants N00014-02-I-0622 and N00014-07-01-0313 from the Office of Naval Research, all to Northwestern University. M. Vořechovský worked at Northwestern University as a Post-Doctoral Associate, supported mainly by Fulbright Foundation and in the final stage by the aforementioned grants. D. Novák and M. Vořechovský also received support under project No. 103/06/P086 from the Grant Agency of the Czech Republic and, together with D. Novák, under project VITESPO, no. 1ET409870411, from the Czech Academy of Science.

REFERENCES

1. Ghanem RG, Spanos PD. *Stochastic Finite Elements*: *A Spectral Approach*. Springer: New York, 1991.
2. Kleiber M, Hien TD. *The Stochastic Finite Element Method*, *Basic Perturbation Technique and Computer Implementation*. Wiley: Chichester, 1992.
3. Haldar A, Mahadevan S. *Reliability Assessment Using Stochastic Finite Element Analysis*. Wiley: New York, 2000.
4. Nordic Committee for Building Structures (NKB). Recommendation for loading and safety regulations for structural design. *NKB Report*, *No. 36*, 1978.
5. Melchers RE. 1987 *Structural Reliability*, *Analysis and Prediction* (2nd edn). Wiley: New York, 1999.
6. Duckett W. Risk analysis and the acceptable probability of failure. *Structural Engineer* 2005; **83**(15):25–26.
7. Hammersley JM, Handscomb DC. *Monte Carlo Methods*. Methuen and Co. Ltd.: London, 1964.
8. Bucher CG. Adaptive sampling—an iterative fast Monte-Carlo procedure. *Structural Safety* 1988; **5**(2):119–126.
9. Liu JS. *Monte Carlo Strategies in Scientific Computing*. Springer Series in Statistics. Springer: New York, 2001.
10. Au S, Beck J. Estimation of small failure probabilities in high dimensions by subset simulation. *Probabilistic Engineering Mechanics* 2001; **16**:263–277.
11. Koutsourelakis PS, Pradlwarter HJ, Schuëller GI. Reliability of structures in high dimensions, part I: algorithms and applications. *Probabilistic Engineering Mechanics* 2004; **19**(4):409–417.
12. Bouchaud J-P, Potters M. *Theory of Financial Risks*: *From Statistical Physics to Risk Management*. Cambridge University Press: Cambridge, U.K., 2000.
13. Bažant ZP. *Scaling of Structural Strength*. Hermes Penton Science, London, 2002; Elsevier, 2005 (2nd updated edn.) (French translation, Hermès Science, Paris, 2004).
14. Bažant ZP. Probability distribution of energetic–statistical size effect in quasibrittle fracture. *Probabilistic Engineering Mechanics* 2004; **19**(4):307–319.
15. Bažant ZP, Planas J. *Fracture and Size Effect in Concrete and Other Quasibrittle Materials*. CRC Press: Boca Raton, FL, 1998.

DOI: 10.1002/nme

16. Bažant ZP, Pang S-D. Mechanics based statistics of failure risk of quasibrittle structures and size effect on safety factors. *Proceedings of the National Academy of Sciences* 2006; **103**(25):9434–9439.
17. Bažant ZP, Pang S-D. Activation energy based extreme value statistics and size effect in brittle and quasibrittle fracture. *Journal of the Mechanics and Physics of Solids* 2007; **55**:91–131.
18. Bažant ZP, Pang SD. Revision of reliability concepts for quasibrittle structures and size effect on probability distribution of structural strength. In *Proceedings of the Ninth International Conference on Structural Safety and Reliability* (*ICOSSAR*), Rome, Augusti G, Schuëller GI, Ciampoli M (eds). Milpress: Rotterdam, 2005; 377–386.
19. Bažant ZP, Pang S-D. Effect of size on safety factors and strength of quasibrittle structures: beckoning reform of reliability concepts. In *Proceedings of the Structural Engineering Convention* (*SEC 2005*), Chandra Kishen JM, Roy D (eds). Indian Institute of Science: Bangalore, India, 2005; 2–20.
20. Bažant ZP, Vořechovský M, Novák D. Asymptotic prediction of energetic–statistical size effect from deterministic finite element solutions. *Journal of Engineering Mechanics* 2007; **133**(2):153–162.
21. Bažant ZP, Xi Y. Statistical size effect in quasi-brittle structures: II. Nonlocal theory. *Journal of Engineering Mechanics* (ASCE) 1991; **117**(17):2623–2640.
22. Walsh PF. Fracture of plain concrete. *Indian Concrete Journal* 1972; **46**(11).
23. Walsh PF. Crack initiation in plain concrete. *Magazine of Concrete Research* 1976; **28**:37C41.
24. Bažant ZP. Instability, ductility, and size effect in strain-softening concrete. *Journal of Engineering Mechanics Division*, *American Society of Civil Engineers* 1976; **102**(EM2):331–344; disc. **103**:357–358, 775–777, **104**: 501–502.
25. Bažant ZP. Size effect in blunt fracture: concrete, rock, metal. *Journal of Engineering Mechanics* (ASCE) 1984; **110**(4):518–535.
26. Bažant ZP, Oh B-H. Crack band theory for fracture of concrete. *Materials and Structures* (RILEM, Paris) 1983; **16**:155–177.
27. Bažant ZP, Belytschko TB, Chang T-P. Continuum model for strain softening. *Journal of Engineering Mechanics* (ASCE) 1984; **110**(12):1666–1692.
28. Bažant ZP, Jirásek M. Nonlocal integral formulations of plasticity and damage: survey of progress. *Journal of Engineering Mechanics* (ASCE) 2002; **128**(11):1119–1149.
29. Gustafsson PJ. Fracture mechanics studies of non-yielding materials like concrete: modeling of tensile fracture and applied strength analyses. *Report No. TVBM-1007*, Division of Building Materials, Lund Institute of Technology, Lund, Sweden, 1985.
30. Hillerborg A, Modéer M, Petersson PE. Analysis of crack formation and crack growth in concrete by means of fracture mechanics and finite elements. *Cement and Concrete Research* 1976; **6**:773–782.
31. Petersson PE. Crack growth and development of fracture zone in plain concrete and similar materials. *Report No. TVBM-1006*, Division of Building Materials, Lund Institute of Technology, Lund, Sweden, 1981.
32. Uchida Y, Rokugo K, Koyanagi W. Application of fracture mechanics to size effect on flexural strength of concrete. *Proceedings of JSCE*, *Concrete Engineering and Pavements* 1992; **442**:101–107.
33. Bažant ZP, Li Z. Zero-brittleness size-effect method for one-size fracture test of concrete. *Journal of Engineering Mechanics* (ASCE) 1996; **122**(5):458–468.
34. RILEM Techn. Com. QFS. Quasibrittle fracture scaling and size effect—final report. *Materials and Structures* (RILEM, Paris) 2004; **37**(272):547–586.
35. Bažant ZP, Kazemi MT. Size effect in fracture of ceramics and its use to determine fracture energy and effective process zone length. *Journal of American Ceramic Society* 1990; **73**(7):1841–1853.
36. Bažant ZP, Li Z. Modulus of rupture: size effect due to fracture initiation in boundary layer. *Journal of Structural Engineering* (ASCE) 1995; **121**(4):739–746.
37. Breysse D. A probabilistic formulation of the damage evaluation law. *Structural Safety* 1990; **8**:311–325.
38. Carmeliet J. On stochastic descriptions for damage evolution in quasibrittle materials. In *DIANA Comp. Mech.*, Kusters GMA, Hendriks MAN (eds). DIANA: Delft, Netherlands, 1994.
39. Carmeliet J, Hens H. Probabilistic nonlocal damage model for continua with random field properties. *Journal of Engineering Mechanics* (ASCE) 1994; **120**:2013–2027.
40. Frantziskonis GN. Stochastic modeling of heterogeneous materials—a process for the analysis and evaluation of alternative formulations. *Mechanics of Materials* 1998; **27**:165–175.
41. Gutiérrez MA. Objective simulation of failure in heterogeneous softening solids. *Dissertation*, Delft University of Technology, Delft, Netherlands, 1999.
42. Vořechovský M. Interplay of size effects in concrete specimens under tension studied via computational stochastic fracture mechanics. *International Journal of Solids and Structures* 2006, in press. http://dx.doi.org/10.1016/j.ijsolstr.2006.08.019

DOI: 10.1002/nme

43. Bažant ZP, Novák D. Probabilistic nonlocal theory for quasibrittle fracture initiation and size effect. I. Theory. *Journal of Engineering Mechanics* (ASCE) 2000; **126**(2):166–174.
44. Bažant ZP, Novák D. Probabilistic nonlocal theory for quasibrittle fracture initiation and size effect. II. Application. *Journal of Engineering Mechanics* (ASCE) 2000; **126**(2):175–185.
45. Weibull W. The phenomenon of rupture in solids. *Proceedings of the Royal Swedish Institute of Engineering Research* (Ingenioersvetenskaps Akad. Handl.), vol. 153, Stockholm, 1939; 1–55.
46. Weibull W. A statistical representation of fatigue failures in solids. *Proceedings of the Royal Institute of Technology* 1949; **27**.
47. Weibull W. A statistical distribution function of wide applicability. *Journal of Applied Mechanics* (ASME) 1951; **18**:292–297.
48. Weibull W. Basic aspects of fatigue. *Proceedings of the Colloquium on Fatigue*, Stockholm. Springer: Berlin, 1956; 289–298.
49. Epstein B. Statistical aspects of fracture problems. *Journal of Applied Physics* 1948; **19**:140–147.
50. Freudenthal AM. Physical and statistical aspects of fatigue. *Advance in Applied Mechanics*, vol. 4. Academic Press: New York, 1956; 117–157.
51. Freudenthal AM. Statistical approach to brittle fracture. In *Fracture*, vol. 2, Liebowitz H (ed.). Academic Press: New York, 1968; 591–619.
52. Freudenthal AM, Gumbel EJ. On the statistical interpretation of fatigue tests. *Proceedings of the Royal Society* (London) 1953; **A21**:309–332.
53. Gumbel EJ. *Statistics of Extremes*. Columbia University Press: New York, 1958.
54. Saibel E. Size effect in structural safety. *Proceedings of the International Conference on Structures, Solid Mechanics and Material Engineering*, Southampton, Part I, 1969; 125–130.
55. Tada H, Paris PC, Irwin GR. *The Stress Analysis of Cracks Handbook*. ASME Press: New York, 2000.
56. Bažant ZP. Size effects in quasibrittle fracture: Apercu of recent results. In *Fracture Mechanics of Concrete Structures* (*Proceedings*, *FraMCoS-4 International Conference*, Paris), de Borst R *et al.* (eds). A.A. Balkema Publishers: Lisse, Netherlands, 2001; 651–658.
57. Bažant ZP. Scaling of quasibrittle fracture: asymptotic analysis. *International Journal of Fracture* 1997; **83**(1): 19–40.
58. Bažant ZP. Probabilistic modeling of quasibrittle fracture and size effect. *Proceedings of Eighth International Conference on Structural Safety and Reliability* (*ICOSSAR*), Newport Beach, CA, 2001, Corotis RB (ed.). Swets & Zeitinger, Balkema: 2001; 1–23.
59. Fisher RA, Tippett LHC. Limiting forms of the frequency distribution of the largest and smallest member of a sample. *Proceedings of Cambridge Philosophical Society* 1928; **24**:180–190.
60. Pijaudier-Cabot G, Bažant ZP. Nonlocal damage theory. *Journal of Engineering Mechanics* (ASCE) 1987; **113**(10):1512–1533.
61. McKay MD, Conover WJ, Beckman RJ. A comparison of three methods for selecting values of input variables in the analysis of output from a computer code. *Technometrics* 1979; **21**:239–245.
62. Iman RL, Conover WJ. Small sample sensitivity analysis techniques for computer models, with an application to risk assessment. *Communications in Statistics—A* 1980; **9**:1749–1842.
63. Bažant ZP, Liu K-L. Random creep and shrinkage in structures: sampling. *Journal of Structural Engineering* (ASCE) 1985; **111**:1113–1134.
64. Bažant ZP, Kim J-K, Wittmann FH, Alou F. Statistical extrapolation of shrinkage data—Part II: Bayesian updating. *ACI Materials Journal* 1987; **84**:83–91.
65. Xi Y, Bažant ZP. Sampling analysis of concrete structures for creep and shrinkage with correlated random material parameters. *Probabilistic Engineering Mechanics* 1989; **4**(4):174–186.
66. Novák D, Teplý B, Keršner Z. The role of latin hypercube sampling method in reliability engineering. In *Proceedings of the Seventh International Conference on Structural Safety and Reliability* (*ICOSSAR*), Kyoto, Japan, Shiraishi, Shinozuka, Wen (eds). Balkema: Rotterdam, 1998; 403–409.
67. Novák D, Bažant ZP, Vořechovský M. Computational modeling of statistical size effect in quasibrittle structures. In *Applications of Statistics and Probability* (*Proceedings of Ninth International Conference*, *ICASP-9*, San Francisco), Der Kiureghian A, Madanat S, Pestana JM (eds). Millpress: Rotterdam, 2003; 621–628.
68. Vořechovský M, Novák D. Statistical correlation in stratified sampling. In *Applications of Statistics and Probability* (*Proceedings of Ninth International Conference*, *ICASP-9*, Berkeley, CA, 2003), Der Kiureghian A, Madanat S, Pestana JM (eds), Rotterdam, 2003; 119–124.
69. Huntington DE, Lyrintzis CS. Improvements to and limitations of latin hypercube sampling. *Probabilistic Engineering Mechanics* 1998; **13**(4):245–253.

DOI: 10.1002/nme

70. Iman RL, Conover WJ. A distribution free approach to inducing rank correlation among input variables. *Communications in Statistics* 1982; **B**(11):311–334.
71. Novák D, Vořechovský M, Lehký D, Rusina R, Pukl R, Červenka V. Stochastic nonlinear fracture mechanics finite element analysis of concrete structures. In *Proceedings of Ninth International Conference on Structural Safety and Reliability* (*ICOSSAR*), Rome, Augusti G, Schuëller GI, Ciampoli M (eds). Milpress: Rotterdam, 2005.
72. Bažant ZP, Novák D. Stochastic models for deformation and failure of quasibrittle structures: recent advances and new directions. In *Computational Modelling of Concrete Structures* (*Proceedings of EURO-C Conference*, St. Johann im Pongau, Austria), Bićanić N, de Borst R, Mang H, Meschke G (eds). A.A. Balkema Publishers: Lisse, Netherlands, 2003; 583–598.
73. Bažant ZP, Pang SD, Vořechovský M, Novák D, Pukl R. Statistical size effect in quasibrittle materials: computation and extreme value theory. In *Fracture Mechanics of Concrete Structures* (*Proceedings*, *FraMCoS-5, Fifth International Conference on Fracture Mechanics of Concrete and Concr. Structures*, vol. 1. Vail, Colo.), Li VC, Leung KY, Willam KJ, Billington SL (eds). IA-FraMCoS, 2004; 189–196.
74. Koide H, Akita H, Tomon M. Size effect on flexural resistance due to bending span of concrete beams. In *Fracture Mechanics of Concrete Structures* (*Proceedings of Third International Conference*, *FraMCoS-3* Gifu, Japan), Mihashi H, Rokugp K (eds). Aedificatio Publishers: Freiburg, Germany, 1998; 2121–2130.
75. Koide H, Akita H, Tomon M. Probability model of flexural resistance on different lengths of concrete beams. In *Applications of Statistics and Probability* (*Proceedings of Eighth International Conference*, *ICASP-8*, vol. 2, Sydney, Australia, 1999), Melchers RE, Stewart MG (eds). Balkema: Rotterdam, 2003; 1053–1057.
76. Hald A. *Statistical Theory with Engineering Applications*. Wiley: New York, 1952.
77. Box GEP, Hunter WG, Hunter SJ. *Statistics for Experimenters*. Wiley: New York, 1978.
78. Fisher RA, Yates F. The 6×6 latin squares. *Proceedings of the Cambridge Philosophical Society* 1934; **30**: 492–507.
79. Winer BJ. *Statistical Principles in Experimental Design*. McGraw-Hill: New York, 1962.
80. CEB90 CEB. *CEB-FIP Model Code 1990.* Thomas Telford: Heron Quay, London, 1998.

DOI: 10.1002/nme