Energy and velocity of a forming vortex ring

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It is known that vortex rings formed by large stroke ratios (in a piston/cylinder arrangement) pinch off from their generating jets at a fairly constant universal time scale. In this paper we show that the hypothesis that at the pinch off the translational velocity of the ring equals the jet flow velocity near the ring is equivalent to the recently proposed idea based on a variational principle by Kelvin and Benjamin that the pinch off occurs when the apparatus is no longer able to deliver energy at a rate required for steady vortex ring existence. A formula for the propagation velocity of a thick vortex ring is also proposed and compared with available experimental data and empirical relations. © *2000 American Institute of Physics.* [S1070-6631(00)00703-0]

I. INTRODUCTION

Recently Gharib *et al.*,[1] investigating the phenomenon of vortex ring formation in a piston/cylinder arrangement, demonstrated the existence of a universal time scale for vortex ring formation. This time scale (the ''formation number'') is the time when the vortex ring reaches its maximum circulation and energy, which physically coincide with the pinch off of the vortex ring in terms of its entraiment, velocity, and vorticity fields from its generating axisymmetric jet.

Gharib *et al.*[1] showed that the existence of the formation number was based on the Kelvin–Benjamin variational principle for steady axis-touching vortex rings.[2] It follows from this principle that the pinch off occurs when the apparatus is no longer able to deliver energy at a rate compatible with the requirement that a steady translating vortex ring have maximum energy with respect to impulse-preserving iso-vortical perturbations.

Mohseni and Gharib[4] used this idea to predict the formation number analytically by considering the dimensionless energy of a vortex ring. It follows from their analysis that the translational velocity of the ring $W=0.5U_P$, where U_P is the piston velocity. The predictions of the model were in reasonable agreement with experiment, though the authors mention that in practice the translational velocity of a vortex ring is higher.

An alternative approach was used by Shusser and Gharib,[3] who modeled inviscid vortex ring formation in a nozzle flow generator by using a hypothesis that a vortex ring completes its formation and pinches off from its generating axisymmetric jet when the translational velocity of the ring becomes equal to the jet flow velocity near the ring. The flow in the jet was approximated as a one-dimensional round jet of a variable cross section, its diameter being equal to the nozzle diameter near the nozzle and to the diameter of the vortex ring in the immediate vicinity of the ring (see Fig. 1). Relating the jet flow velocity to the piston velocity by conservation of mass and choosing the thin vortex ring approximation for the radius and the velocity of the vortex ring the authors were able to reasonably estimate the formation number. This theory gave good predictions for a broad range of flow conditions.

While the Kelvin–Benjamin principle offers an elegant theory for the mere existence of a limiting process, it fails to suggest a process for the dynamics of pinch off. In this paper we will try to show that the kinematic model suggested by Shusser and Gharib[3] is the physical manifestation of the Kelvin–Benjamin principle.

II. VORTEX RING FORMATION

We start by considering the formation of a vortex ring in a piston/cylinder arrangement. In order to concentrate on the main physical factors, we first study the basic case of a constant piston velocity (an impulse velocity program) and assume a uniform velocity profile within the cylinder and across the ring generating axisymmetric jet. Time-dependent piston velocities are considered by Shusser and Gharib.[3] We also adopt the slug flow approximation[1,4]

$$\Gamma=\tfrac{1}{2}LU_P, \tag{1}$$

$$I=\tfrac{1}{4}\pi D^2\rho LU_P, \tag{2}$$

$$E=\tfrac{1}{8}\pi D^2\rho LU_P^2. \tag{3}$$

Here E is the energy, I is the impulse, Γ is the circulation, ρ is the density of the fluid, D is the diameter of the piston, L is the piston stroke and U_P is the piston velocity. The advantages and limitations of this approach are discussed elsewhere.[5,6]

To apply the Kelvin–Benjamin variational principle, one needs to compare the energy provided by the piston E, as given by (3), with the energy required for steady vortex ring E_R. The formation process will not be complete as long as $E>E_R$. We now show that this will result in a limitation on the value of a dimensionless coefficient α defined by Gharib *et al.*[1] as

$$\alpha=\frac{E}{\sqrt{\rho I\Gamma^3}}. \tag{4}$$

Generally, a vortex ring can be described by its radius R, radius of the core a, circulation Γ, and the distribution of the circulation within the core.[5,6] Mohseni and Gharib (Ref. 4, p.

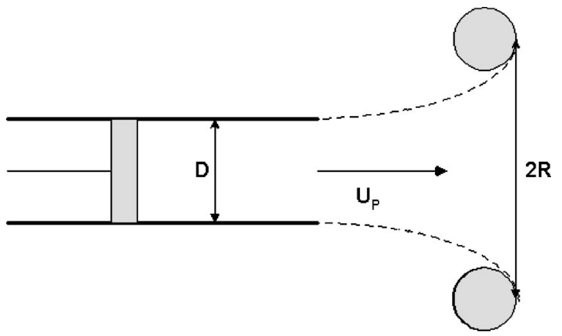

FIG. 1. Vortex ring formation in a piston/cylinder arrangement as modeled by Shusser and Gharib (Ref. 3).

2436) pointed out that the experimental results provided by Gharib *et al.*[1] for different piston velocity programs and exit conditions suggest that the limiting formation number is not very sensitive to the vorticity distribution. We will therefore assume that the functional form of the circulation distribution (but not the absolute values) does not change during the formation process. Then the properties of the ring are determined by choosing the values of three independent parameters. These can be taken kinematically as R, a, and the translation velocity W or dynamically as E/ρ, I/ρ, Γ.

On the other hand, these three quantities are not entirely independent during the formation. Experimental evidence (see Figs. 4.4.1, 4.4.3, 4.4.4 of Ref. 6) has clearly demonstrated that after the initial period of growth the vortex ring radius R remains constant during the formation. This is caused probably by the inability of the cylindrical vortex sheet having a radius of $D/2$ near the nozzle exit to increase this radius substantially during the roll up.

The fact that the ring radius remains constant is a constraint that limits the number of independent parameters to two, which can be taken as I/ρ and Γ. Therefore the energy of a steady vortex ring is a function of these two parameters,

$$\frac{E_R}{\rho}=f\left(\frac{I}{\rho},\Gamma\right). \tag{5}$$

Only one dimensionless combination can be obtained from these quantities and hence according to the *Pi* theorem[7]

$$\frac{E_R}{\sqrt{\rho I \Gamma^3}}=\text{const.} \tag{6}$$

Therefore the energy of a steady vortex ring

$$E_R=A\sqrt{\rho I\Gamma^3}, \tag{7}$$

where A is an unknown constant.

Substituting (7) into the requirement $E \geqslant E_R$ and using (4), we obtain

$$\alpha \geqslant A. \tag{8}$$

A is the limit value of α that was found experimentally by Gharib *et al.*[1] as 0.33. One sees that the formation process continues as long as the value of α exceeds A. To illustrate the evolution of the jet and ring energies during vortex ring formation we define dimensionless energy

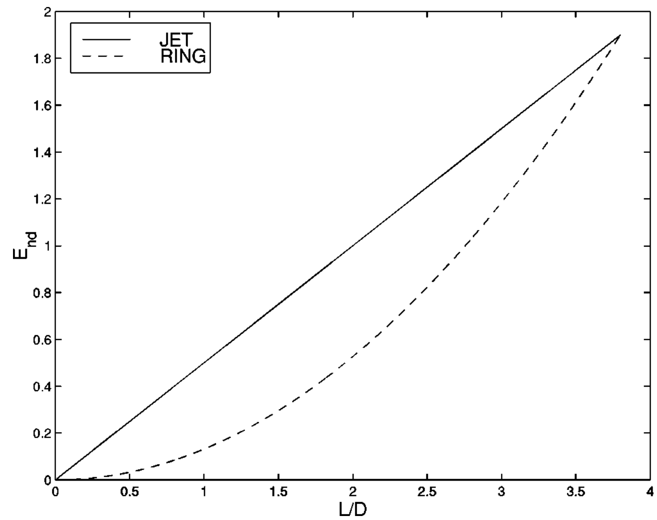

FIG. 2. Vortex ring and jet energy.

$$E_{\text{nd}}=\frac{4E}{\rho\pi D^3 U_P^2}. \tag{9}$$

In Fig. 2 we plotted E_{nd} for the jet and the ring as a function of the formation number L/D. The energy provided by the piston was calculated using the slug model (3), while for the ring energy we used (7) with $A=0.33$. One sees that initially the jet energy is larger but the ring energy grows faster. The pinch off is predicted at $L/D\approx 3.8$, in agreement with the results of Gharib *et al.*[1]

Next, we estimate the thickness of the formed vortex ring using the experimental value of α. Following Mohseni and Gharib[4] we approximate the ring as a member of Norbury's family of vortex rings[8] and utilize Fraenkel's[9] second-order formulas for these vortices:

$$E=\frac{1}{2}\rho R\Gamma^2\left[\ln\frac{8}{\epsilon}-\frac{7}{4}+\frac{3}{8}\epsilon^2\ln\frac{8}{\epsilon}\right], \tag{10}$$

$$I=\rho\pi\Gamma R^2(1+\tfrac{3}{4}\epsilon^2). \tag{11}$$

Here $\epsilon=a/R$, i.e., the dimensionless core radius. Substituting (10) and (11) into (4) we obtain

$$\alpha=\frac{\ln\frac{8}{\epsilon}-\frac{7}{4}+\frac{3}{8}\epsilon^2\ln\frac{8}{\epsilon}}{2\sqrt{\pi}\sqrt{1+\frac{3}{4}\epsilon^2}}. \tag{12}$$

Then from $\alpha=0.33$ follows $\epsilon\approx 0.50$. It should be noted that Mohseni and Gharib,[4] considering the propagation velocity of a vortex ring, arrived at a lower value of $\epsilon\approx 0.3$. On the other hand, the dependence of propagation velocity on ϵ is relatively weak. Indeed, the propagation velocity W can be calculated as[9]

$$W=\frac{\Gamma}{4\pi R}\left[\ln\frac{8}{\epsilon}-\frac{1}{4}+\frac{3\epsilon^2}{8}\left(\frac{5}{4}-\ln\frac{8}{\epsilon}\right)\right]. \tag{13}$$

Eliminating the vortex ring radius R from (11) and (13) one obtains

$$W = B\sqrt{\frac{\rho\Gamma^3}{\pi I}}. \tag{14}$$

Here

$$B = \frac{1}{4}\sqrt{1+\frac{3}{4}\epsilon^2}\left[\ln\frac{8}{\epsilon} - \frac{1}{4} + \frac{3\epsilon^2}{8}\left(\frac{5}{4} - \ln\frac{8}{\epsilon}\right)\right]. \tag{15}$$

One sees from (15) that when $\epsilon = 0.3$, $B \approx 0.7658$ and when $\epsilon = 0.5$, $B \approx 0.6483$. Thus the dependence is indeed weak.

We now use (14) to compare the vortex ring velocity with the jet flow velocity near the ring, as done by Shusser and Gharib.[3]

Shusser and Gharib[3] pointed out that the jet velocity differs from the piston velocity due to the inapplicability of the slug model in the immediate vicinity of the ring, as shown in Fig. 1. We denote the ratio of jet to piston velocities N without specifying its value at this stage. Then according to Shusser and Gharib[3] the formation process continues as long as

$$W \leqslant N U_P. \tag{16}$$

From (2) and (3) it follows that

$$U_P = \frac{2E}{I}. \tag{17}$$

Substituting (14) and (17) into (16) one obtains

$$E \geqslant \frac{B}{2N}\sqrt{\frac{\rho\Gamma^3 I}{\pi}}. \tag{18}$$

That is,

$$\alpha \geqslant \frac{B}{2N\sqrt{\pi}}. \tag{19}$$

To compare the approaches quantitatively, one must estimate the value of N. Following Shusser and Gharib[3] we assume that in the vicinity of the ring the radius of the generating jet increases to that of the ring R. Then in the one-dimensional approximation (see Fig. 1) it follows immediately from conservation of mass that

$$N = \frac{D^2}{4R^2}. \tag{20}$$

Substituting the slug-model formulas (1) and (2) into (11) one can calculate the ring radius as

$$R = \frac{D}{\sqrt{2(1+\frac{3}{4}\epsilon^2)}}. \tag{21}$$

Therefore from (20),

$$N = \tfrac{1}{2}(1+\tfrac{3}{4}\epsilon^2). \tag{22}$$

It follows from (15) and (22) that for $\epsilon = 0.50$, $N \approx 0.59$ and $B = 0.6483$. For comparison, Mohseni and Gharib's[4] analysis predicts $N = 0.5$. Substituting the obtained values of N and B into the right-hand side of (19) we obtain the value of 0.31, which is very close to the experimental value of 0.33 previously obtained by Gharib *et al.*[1]

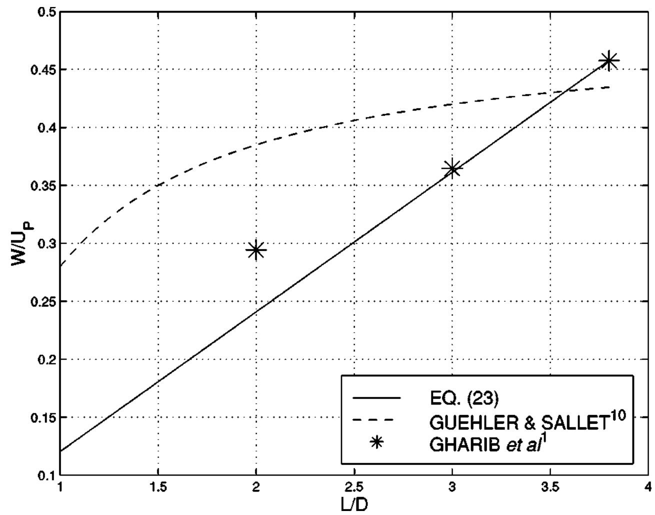

FIG. 3. Vortex ring velocity.

We see that both dynamical (based on the Kelvin–Benjamin variational principle) and kinematic (based on comparison of the velocities) approaches results in the same limitation on the value of α (8) and (19), respectively.

III. VORTEX RING VELOCITY

The fact that the piston velocity is larger than the flow velocity in the ring generating jet ($N<1$) can be verified by using numerical calculations of Rosenfeld *et al.*[10] Though in a real flow the velocity is not uniform in the cross section, the physically important part is the velocity in the boundary layer because it is where the vortex ring vorticity is created. One can see from Fig. 13 of Rosenfeld *et al.*[10] that this velocity, which is initially the highest in the cross section and slightly higher than the piston velocity, decreases with time and can become substantially smaller than the piston velocity. For example, we can retrieve from Fig. 13(b) of Ref. 10 that the initial peak velocity decreases to approximately 17.2 cm/s at later times. For the piston velocity of 30 cm/s, as reported by Rosenfeld *et al.* (Ref. 10, p. 301), this corresponds to $N = 0.57$, which is in good agreement with the value of 0.59 obtained in the present paper.

It is interesting to note that one can use (19) to utilize the experimental value of N to calculate B, in order to obtain a formula for the propagation velocity of thick vortex rings. For example, if we use the vortex ring propagation velocity at the pinch off that can be retrieved from Fig. 11 of Gharib *et al.*,[1] we will obtain that $N \approx 0.4575$, which by substituting (19) will correspond to $B = 0.5352$. Therefore the following relation for the propagation velocity of a vortex ring can be recommended:

$$W = 0.5352\sqrt{\frac{\rho\Gamma^3}{\pi I}}. \tag{23}$$

This formula is supposed to be good close to the pinch off, i.e., for thick vortex rings. On the other hand, Gühler and Sallet[11] proposed the following empiric relationship for a vortex ring propagation velocity:

$$\frac{W}{U_P} = \frac{3+\pi}{4\pi} - \frac{3+\pi}{3\pi^2}\frac{D}{L}. \tag{24}$$

In Fig. 3 we compared relations (23) and (24) with three experimental values calculated from Fig. 11 of Gharib *et al.*[1] These values correspond to the range of $2 \leqslant L/D \leqslant 3.8$ covering the second part of the formation process.

It is seen from Fig. 3 that though (23) was obtained using the pinch off velocity its predictions are very good not only for $L/D = 3.8$, but also for $L/D = 3$. Even for as low values as $L/D = 2$ the accuracy of the prediction remains fair. On the other hand, Gühler and Sallet's[11] formula (24) is good only close to the pinch off because it overpredicts the velocity for shorter piston strokes.

One can conclude that Eq. (23) is a good approximation for the propagation velocity of a thick vortex ring.

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