Energy-based Lyapunov functions for forced Hamiltonian systems with dissipation

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Abstract

It is well known that the total energy is a suitable Lyapunov function to study the stability of the trivial equilibrium of an isolated standard Hamiltonian system. In many practical instances, however, the system is in interaction with its environment through some constant forcing terms. This gives rise to what we call forced Hamiltonian systems, for which the equilibria of interest are now different from zero. When the system is linear a Lyapunov function can be immediately obtained by simply shifting the coordinates in the total energy. However, for nonlinear systems there is no guarantee that this incremental energy is, not even locally, a Lyapunov function. In this paper we propose a constructive procedure to modify the total energy function of forced Hamiltonian systems with dissipation in order to generate Lyapunov functions for non–zero equilibria. A key step in the procedure, which is motivated from energy–balance considerations standard in network modeling of physical systems, is to embed the system into a larger Hamiltonian system for which a series of Casimir functions (i.e., first integrals) can be easily constructed. Interestingly enough, for linear systems the resulting Lyapunov function is the incremental energy, thus our derivations provide a physical explanation to it. An easily verifiable necessary and sufficient condition for the applicability of the technique in the general nonlinear case is given. Some examples that illustrate the method are given.

1 Problem formulation

Network modelling of energy-conserving lumped-parameter physical systems [1] [10] with independent storage elements leads to models of the form (called port controlled Hamiltonian systems [3] [16])

$$\Sigma \; : \; \begin{cases} \dot{x} & = \; J(x)\frac{\partial H}{\partial x}(x) + g(x)u \\ y & = \; g^{\mathsf{T}}(x)\frac{\partial H}{\partial x}(x) \end{cases} \qquad (1)$$

where $x \in \Xi$, an n-dimensional manifold, $u, y \in \mathcal{R}^m$, $x = [x_1, \cdots, x_n]^{\mathsf{T}}$ are the energy variables, the smooth function $H(x_1, \cdots, x_n) : \mathcal{R}^n \to \mathcal{R}$ represents the total stored energy,

and u, y are the port power variables. The interconnection structure is captured in the $n \times n$ matrix $J(x)$ and the $n \times m$ matrix $g(x)$, both depending smoothly on the state x. Because of the assumption of energy-conservation, the matrix $J(x)$ is skew-symmetric, that is,

$$J(x) = -J^{\mathsf{T}}(x), \quad \forall \, x \in \Xi \qquad (2)$$

and defines a generalized Poisson bracket on Ξ (generalized because it need not satisfy the Jacobi-identity [15]).

From (1) and (2), we immediately obtain the power-balance

$$\frac{d}{dt}H = u^{\mathsf{T}}y \qquad (3)$$

with $u^{\mathsf{T}}y$ the power externally supplied to the system. *Energy-dissipation* is included by terminating some of the ports by resistive elements, see e.g. [16], [5], [4]. Indeed, consider instead of $g(x)u$ in (1) a term

$$\begin{bmatrix} g(x) & g_R(x) \end{bmatrix} \begin{bmatrix} u \\ u_R \end{bmatrix} = g(x)u + g_R(x)u_R$$

and extend correspondingly $y = g^{\mathsf{T}}(x)\frac{\partial H}{\partial x}(x)$ to

$$\begin{bmatrix} y \\ y_R \end{bmatrix} = \begin{bmatrix} g^{\mathsf{T}}(x)\frac{\partial H}{\partial x}(x) \\ g_R^{\mathsf{T}}(x)\frac{\partial H}{\partial x}(x) \end{bmatrix}$$

Here u_R, y_R denote the power variables at the ports which are terminated by (linear) resistive elements

$$u_R = -Sy_R$$

for some non–negative definite symmetric matrix S. Substitution in (1) leads to models of the form

$$\Sigma \; : \; \begin{cases} \dot{x} & = \; [J(x) - R(x)]\frac{\partial H}{\partial x}(x) + g(x)u \\ y & = \; g^{\mathsf{T}}(x)\frac{\partial H}{\partial x}(x) \end{cases} \qquad (4)$$

where

$$R(x) \triangleq g_R(x)Sg_R^{\mathsf{T}}(x)$$

which is a non-negative *symmetric* matrix depending smoothly on x, *i.e.*

$$R(x) = R^{\mathsf{T}}(x) \geq 0, \quad \forall \, x \in \Xi \qquad (5)$$

The matrix $R(x)$ actually defines a symmetric bracket on the state manifold in the same way as the skew-symmetric matrix $J(x)$ defines a generalized Poisson bracket.

For systems Σ given by (4) the power-balance (3) extends to

$$\frac{d}{dt}H = -\frac{\partial^\top H}{\partial x}(x)R(x)\frac{\partial H}{\partial x}(x) + u^\top y \qquad (6)$$

where the first term on the right-hand side (which is non-positive by (5)) represents the *energy-dissipation* due to the resitive elements in the system.

Now let us approach the *stability analysis* of the system Σ. Towards this end, we recall here the following [7]

Definition 1.1 Consider the system $\dot{x} = f(x)$, with an equilibrium point $\bar{x}$, i.e., $f(\bar{x}) = 0$. We say that a function $V(x) : \mathcal{R}^n \to \mathcal{R}$ is a Lyapunov function for the equilibrum $\bar{x}$ if:

1. it has a local minimum at $\bar{x}$ and
2. satisfies $\frac{d}{dt}V \le 0$.

□□□

Let us first consider the case $u = 0$ (the *uncontrolled* or *unforced* case). If x^* is a *minimum* of the energy $H(x)$, then necessarily $\frac{\partial H}{\partial x}(x^*) = 0$, and thus x^* is an equilibrium of the unforced dynamics

$$\dot{x} = [J(x) - R(x)]\frac{\partial H}{\partial x}(x) \qquad (7)$$

Furthermore by (5), (6) for $u = 0$

$$\frac{d}{dt}H = -\frac{\partial^\top H}{\partial x}(x)R(x)\frac{\partial H}{\partial x}(x) \le 0 \qquad (8)$$

and thus the total stored energy H is a Lyapunov function for investigating the stability of the equilibrium x^*. Of course, this is common practice in the stability analysis of physical systems, and in some sense is the *origin* of Lyapunov stability theory.

On the other hand, in many application areas one wants to investigate the stability of Σ for a *constant*, but *non-zero*, input $\bar{u} \in \mathcal{R}^m$. (An example where this scenario arises, which actually motivated our research, is in studies of transient stability of synchronous generators in power systems [8], see also [14], [13].) Corresponding to $u = \bar{u}$ one considers *forced* equilibria $\bar{x}$, which are solutions of

$$[J(\bar{x}) - R(\bar{x})]\frac{\partial H}{\partial x}(\bar{x}) + g(\bar{x})\bar{u} = 0 \qquad (9)$$

Now, let $\bar{x}$ be a forced equilibrium. Then in general, $\bar{x}$ will need not be a minimum (or an extremum) of H. Furthermore, inserting $u = \bar{u}$ in (6) yields

$$\frac{d}{dt}H = -\frac{\partial^\top H}{\partial x}(x)R(x)\frac{\partial H}{\partial x}(x) + \bar{u}^\top g^\top(x)\frac{\partial H}{\partial x}(x) \qquad (10)$$

and in general the right-hand side of (10) will not be non-positive. Thus, in general, the total stored energy can *not* be used as a Lyapunov function for investigating the stability of $\bar{x}$. Hence the question comes up if, and how, we can construct *physically-based* Lyapunov functions for *equilibria of forced physical systems* Σ. Providing some (partial) answers to this question are the main contributions of our work.

Remark 1.1
Note that we may take $J(x)$ and $R(x)$ together in a single matrix $I(x) \triangleq J(x) - R(x)$, and that conversely *every* $n \times n$ matrix $I(x)$ can be decomposed as the difference of a skew-symmetric matrix $J(x)$ and a symmetric matrix $R(x)$ (not necessarily non-negative).

Remark 1.2
The question above is of interest only for nonlinear systems. Since, as it is well-known, for linear systems (with J, R and g constant) with quadratic energy function $H(x) = \frac{1}{2}x^\top Qx$, we can simply shift the coordinates to obtain the so-called incremental Lyapunov function $H(x) = \frac{1}{2}(x - \bar{x})^\top Q(x - \bar{x})$. This procedure, of course, will not generate a Lyapunov function for nonlinear systems in general.

2 A Lyapunov function based on energy–balance

One way of approaching this question is to start from the power balance of the forced system (10), and to bring the second term on the right-hand side to the left-hand side as

$$\frac{d}{dt}\left(H(x(t)) - \bar{u}^\top\int_0^t g^\top(x(\tau))\frac{\partial H}{\partial x}(x(\tau))d\tau\right) =$$
$$-\frac{\partial^\top H}{\partial x}(x(t))R(x(t))\frac{\partial H}{\partial x}(x(t))$$

suggesting as candidate Lyapunov "function"

$$H(x(t)) - \bar{u}^\top\int_0^t g^\top(x(\tau))\frac{\partial H}{\partial x}(x(\tau))d\tau =$$
$$H(x(t)) - \bar{u}^\top\int_0^t y(\tau)d\tau \qquad (11)$$

where we have replaced y from (1). Notice that the term $\bar{u}^\top\int_0^t y(\tau)d\tau$ is precisely the energy externally supplied to the system Σ. Hence the new function (11) that we propose is exactly the *difference between the energy of the system and the supplied energy*.

To check whether (11) can be used as a Lyapunov function, the first basic question is, of course, if we can write $\bar{u}^\top\int_0^t y(\tau)d\tau$ as a function of the state $x(t)$. The main emphasis in this paper will be in trying to answer this question. Condition 2 of the Definition 1.1 is clearly satisfied by construction. And there will only remain condition 1 to be verified.

From a *control theoretic point of view* the question posed above suggests to consider a cascade of Σ with input $\bar{u}$, followed by the integration of y, as depicted in Fig. 1, and to look for Lyapunov functions of the composed system

$$\dot{x} = [J(x) - R(x)]\frac{\partial H}{\partial x}(x) + g(x)\bar{u}$$
$$\zeta = g^\top(x)\frac{\partial H}{\partial x}(x), \quad \zeta \in \mathcal{R}^m \qquad (12)$$
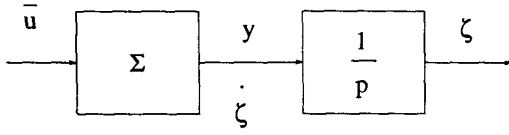
Figure 1: Cascaded system.

We note that (12) can be rewritten into the form (7) as

$$
\begin{bmatrix} \dot{x} \\ \dot{\zeta} \end{bmatrix} = \left(\begin{bmatrix} J(x) & -g(x) \\ g^{\mathsf{T}}(x) & 0 \end{bmatrix} - \begin{bmatrix} R(x) & 0 \\ 0 & 0 \end{bmatrix} \right) \begin{bmatrix} \frac{\partial H_a}{\partial x} \\ \frac{\partial H_a}{\partial \zeta} \end{bmatrix}
\tag{13}
$$

with $H_a(x, \zeta)$ the *augmented* energy function

$$
H_a(x, \zeta) \triangleq H(x) + H_s(\zeta), \quad H_s(\zeta) \triangleq -\bar{u}^{\mathsf{T}}\zeta
\tag{14}
$$

Writing $\bar{u}^{\mathsf{T}} \int_0^t y(\tau)d\tau$ as a function of $x(t)$, then corresponds to expressing $\zeta(t)$ as a function of $x(t)$ along the dynamics (13). This is the starting point of our approach.

In the full version of this paper it is shown that even for simple linear systems, e.g. a parallel RLC circuit, (11) does not qualify as a Lyapunov function candidate because the energy extracted from the source is infinite, thus (11) is not bounded. To overcome this problem, preserving the energy-balance Lyapunov function construction, we have to extend the dynamical system (13) to a more general form.

Remark 2.1
Note firstly that the system (13) is an $n + m$-dimensional generalized Hamiltonian system with dissipation, embedding the forced system (4). From a modeling perspective, (13) corresponds to viewing Σ for constant $u = \bar{u}$ as the *interconnection* of Σ with a *source system*

$$
\begin{aligned}
\dot{\zeta} &= u_s \\
y_s &= \frac{\partial H_s}{\partial \zeta}
\end{aligned}
\tag{15}
$$

with $H_s(\zeta) = -\bar{u}^{\mathsf{T}}\zeta$ the (unbounded) energy of the source system, via the interconection constraints

$$
\begin{aligned}
u_s &= y \\
u &= -y_s
\end{aligned}
\tag{16}
$$

3 System embedding

Key to our developments is the static relation (9) describing the forced equilibria. Since we want to consider forced equilibria for every $\bar{u}$ it is logical to assume that

$$
Im\{g(x)\} \subset Im\{J(x) - R(x)\}
$$

For simplicity we will make throughout the following stronger assumption

Assumption A $[J(x) - R(x)]$ is invertible for every $x \in \Xi$.

□□□

We now consider the equation (9) in $v = \frac{\partial H}{\partial x}(x)$, that is

$$
[J(x) - R(x)]v + g(x)\bar{u} = 0
\tag{17}
$$

By Assumption A, (17) has a unique solution $v = K(x)\bar{u}$, with

$$
K(x) = -[J(x) - R(x)]^{-1} g(x)
\tag{18}
$$

Let us now define, as an extension to (13), the system

$$
\begin{bmatrix} \dot{x} \\ \dot{\zeta} \end{bmatrix} = \left(\begin{bmatrix} J(x) & R(x)K(x) - g(x) \\ -(R(x)K(x) - g(x))^{\mathsf{T}} & J_s(x) \end{bmatrix} - \begin{bmatrix} R(x) & R(x)K(x) \\ (R(x)K(x))^{\mathsf{T}} & R_s(x) \end{bmatrix} \right) \begin{bmatrix} \frac{\partial H_a}{\partial x} \\ \frac{\partial H_a}{\partial \zeta} \end{bmatrix}
\tag{19}
$$

with $H_a(x, \zeta)$ defined by (14), and with $J_s(x) = -J_s^{\mathsf{T}}(x)$, and $R_s(x) = R_s^{\mathsf{T}}(x)$ yet to determined. Note that the x-dynamics are unaffected, thus the x-dynamics of Σ for $u = \bar{u}$ have been *embedded* in the dynamics (19).

By (18) we have

$$
R(x)K(x) = J(x)K(x) + g(x)
$$

and hence we can rewrite (19) into the simpler form

$$
\begin{bmatrix} \dot{x} \\ \dot{\zeta} \end{bmatrix} = [J_a(x) - R_a(x)] \begin{bmatrix} \frac{\partial H_a}{\partial x} \\ \frac{\partial H_a}{\partial \zeta} \end{bmatrix}
\tag{20}
$$

where $(x, \zeta) \in \Xi \times \mathcal{R}^m$, and we have defined

$$
J_a(x) \triangleq \begin{bmatrix} J(x) & J(x)K(x) \\ -(J(x)K(x))^{\mathsf{T}} & J_s(x) \end{bmatrix}
$$

$$
R_a(x) \triangleq \begin{bmatrix} R(x) & R(x)K(x) \\ (R(x)K(x))^{\mathsf{T}} & R_s(x) \end{bmatrix}
$$

Remark 3.1
Note that (20) is of the same form as (4), with J_a defining a generalized Poisson bracket and R_a a symmetric bracket on the augmented state space $\Xi \times \mathcal{R}^m$.

Remark 3.2
The precise form of the proposed embedding (19) was motivated by a network analysis of (1) interconnected with the "source system" (15), together with the equilibrium equation (17). See Remark 2.1. It is based on modifying, on the one hand, the dynamics of the source system using a non zero bracket; and, on the other hand, by choosing a more general interconnection than (16). Further details on this perspective of the problem, as well as its application to controller synthesis, may be found in [11].

Remark 3.3
In some cases it is convenient to allow for feedthrough terms

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in the description of physical systems and to consider instead of (4) models of the form

$$\Sigma_D \; : \; \begin{cases} \dot{x} & = & [J(x) - R(x)]\frac{\partial H}{\partial x}(x) + g(x)u \\[2mm] y & = & g^\top(x)\frac{\partial H}{\partial x}(x) + D(x)u \end{cases}$$

with $D(x) = -D^\top(x)$. Because of the latter these models satisfy the power balance (6). In this case the composed system (12) is replaced by

$$\dot{x} = [J(x) - R(x)]\frac{\partial H}{\partial x}(x) + g(x)\bar{u}$$

$$\dot{\zeta} = g^\top(x)\frac{\partial H}{\partial x}(x) + D(x)u$$

and therefore (13) is replaced by

$$\begin{bmatrix} \dot{x} \\ \dot{\zeta} \end{bmatrix} = \left(\begin{bmatrix} J(x) & -g(x) \\ g^\top(x) & D(x) \end{bmatrix} - \begin{bmatrix} R(x) & 0 \\ 0 & 0 \end{bmatrix} \right) \begin{bmatrix} \frac{\partial H_a}{\partial x} \\ \frac{\partial H_a}{\partial \zeta} \end{bmatrix}$$

This suggests a way to interpret the skew–symmetric matrix $J_s(x)$ in the general construction (20). Indeed, we may regard

$$J_s(x)\frac{\partial H_a}{\partial \zeta} = -J_s(x)\bar{u}$$

as an additional feedthrough term, which does not disturb the power balance of the system.

4 Construction of the Lyapunov function

Next question is how we determine $J_s(x) = -J_s^\top(x)$, and $R_s(x) = R_s^\top(x)$. This is guided by (17), (18). Indeed, the m–dimensional linear spaces

$$P(x) = \left\{ \begin{bmatrix} -K(x)v \\ v \end{bmatrix} \;\middle|\; v \in \mathcal{R}^m \right\} \tag{21}$$

are, by construction, in the kernel of the matrix $[J(x), J(x)K(x)]$ defined by the first n rows of J_a in (20). We now define $J_s(x)$ in such manner that $P(x)$ is in the kernel of the *whole* matrix J_a, by setting

$$J_s(x) \overset{\Delta}{=} K^\top(x)J(x)K(x) \tag{22}$$

Clearly J_s satisfies $J_s(x) = -J_s^\top(x)$. In the same way, we note that $P(x)$ is in the kernel of the first n rows of R_a in (20), while it is in the kernel of the whole matrix R_a if we choose

$$R_s(x) \overset{\Delta}{=} K^\top(x)R(x)K(x) \tag{23}$$

Now, $R_s(x) = R_s^\top(x)$, and thus, since by assumption $R(x) \geq 0$, also $R_a(x) \geq 0$

We are ready to deliver the *coup de grâce*.

Assume that there exist smooth functions $C_j \; : \; \Xi \to \mathcal{R}$, $j \in \bar{m} \overset{\Delta}{=} \{1, \cdots, m\}$, such that

$$K_{ij}(x) = \frac{\partial C_j}{\partial x_i}(x), \; i \in \bar{n} \overset{\Delta}{=} \{1, \cdots, n\}, \; j \in \bar{m} \tag{24}$$

Then it immediately follows that the functions

$$\zeta_j - C_j(x), \; i \in \bar{m} \tag{25}$$

are *constant* along the trajectories of (20), with J_s and R_s as defined in (22), respectively (23). Indeed, we can write

$$\frac{d}{dt}[\zeta_j - C_j(x)] = \begin{bmatrix} -\frac{\partial^\top C_i}{\partial x}(x), & e_j^\top \end{bmatrix} (J_a(x) - R_a(x)) \begin{bmatrix} \frac{\partial H_a}{\partial x} \\ \frac{\partial H_a}{\partial \zeta} \end{bmatrix} \tag{26}$$

with e_j the j-th basis vector in $\mathcal{R}^m$. Since the $(n + m)$-dimensional column vector $[\frac{\partial C_j^\top}{\partial x}(x), -e_j^\top]^\top$ is by (24) contained in $P(x)$, it is by construction and definition of J_s and R_s contained in the kernels of J_a and R_a. Thus the expression in (26) is zero. Hence, along trajectories of (20) we can express

$$\zeta_j = C_j(x) + c_j, \; j \in \bar{m} \tag{27}$$

where the constants $c_1, \cdots, c_m$ depend on the initial conditions of ζ (and can be set to zero). Thus the dynamics of

$$\dot{x} = [J(x) - R(x)]\frac{\partial H}{\partial x}(x) + g(x)\bar{u}$$

is copied on every submanifold of $\Xi \times \mathcal{R}^m$ defined by (27). The total energy of the augmented system

$$H_a(x, \zeta) = H(x) - \bar{u}^\top \zeta$$

restricted to such a submanifold is given as

$$H_r(x) \overset{\Delta}{=} H_a(x, C(x)+c) = H(x) - \Sigma_{j=1}^m \bar{u}_j(C_j(x)+c_j) \tag{28}$$

while the dynamics restricted to such a submanifold is given by

$$\dot{x} = [J(x) - R(x)]\frac{\partial H_r}{\partial x}(x) \tag{29}$$

Note that by (24)

$$\frac{\partial H_r}{\partial x}(x) = \frac{\partial H}{\partial x}(x) - \Sigma_{j=1}^m \bar{u}_j \frac{\partial C_j}{\partial x}(x) = \frac{\partial H}{\partial x}(x) - K(x)\bar{u} \tag{30}$$

Hence, premultiplying by $[J(x) - R(x)]$, and using (18)

$$[J(x) - R(x)]\frac{\partial H_r}{\partial x}(x) = [J(x) - R(x)]\frac{\partial H}{\partial x}(x) + g(x)\bar{u} \tag{31}$$

Consequently, by (9) and assumption A, the unique forced equilibrium $\bar{x}$ corresponding to $\bar{u}$ is an *extremum* of H_r.

Remark 4.1
From the derivations above it follows that the functions $\zeta_j - C_j(x)$ defined on the augmented state space $\Xi \times \mathcal{R}^m$ are *Casimirs* of the generalized Poisson bracket defined by J_a (see for this notion [9], and the references quoted in there). Furthermore, the functions $\zeta_j - C_j(x)$ are also "Casimirs" with respect to the symmetric bracket corresponding to R_a, see [11].

5 Main result

Let us summarize the developments above in the following theorem.

Theorem 5.1 Consider Σ for constant $u = \bar{u}$ that is,

$$\Sigma \; : \; \dot{x} = [J(x) - R(x)]\frac{\partial H}{\partial x}(x) + g(x)\bar{u} \tag{32}$$

with **Assumption A**. Define $K(x)$ by (18) and assume its column vectors are closed 1-forms, that is, the functions K_{ij} satisfy

$$\frac{\partial K_{ij}}{\partial x_k} = \frac{\partial K_{kj}}{\partial x_i}, \quad i,k \in \bar{n} \qquad (33)$$

for $j \in \bar{m}$. Then, there exist *locally* smooth functions $C_1, \cdots, C_m$ satisfying (24), and the dynamics (32) can be alternatively represented by

$$\dot{x} = [J(x) - R(x)]\frac{\partial H_r}{\partial x}(x) \qquad (34)$$

where

$$H_r(x) \stackrel{\triangle}{=} H(x) - \Sigma_{j=1}^m \bar{u}_j(C_j(x) + c_j)$$

The function $H_r(x)$ has an extremum at $\bar{x}$, which is an equilibrium of (32). Further, we have

$$\frac{d}{dt}H_r = -\frac{\partial^\top H_r}{\partial x}(x)R(x)\frac{\partial H_r}{\partial x}(x) \le 0 \qquad (35)$$

and thus H_r qualifies as *Lyapunov function* for the forced dynamics (32) provided we can show that H_r not only has an extremum at $\bar{x}$ but even a *minimum*.

Proof
In view of the developments of the previous section, to complete the proof it only remains to show that, under the given conditions, there exist smooth functions $C_1, \cdots, C_m$, satisfying (24). This follows immediately from (33) and Poincare's lemma.

□□□

The corollary below follows immediately from Theorem 5.1 and standard Lyapunov stability theory, e.g. [7].

Corollary 5.1 Assume that H_r has a strict local minimum at $\bar{x}$, that is, there exists an open neighborhood $\mathcal{B}$ of $\bar{x}$ such that $H_r(x) > H_r(\bar{x})$ for all $x \in \mathcal{B}$. Furthermore, assume that the largest invariant set under the dynamics (34) contained in

$$\left\{ x \in \Xi \cap \mathcal{B} \mid \frac{\partial^\top H_r}{\partial x}(x)R(x)\frac{\partial H_r}{\partial x}(x) = 0 \right\}$$

equals $\{\bar{x}\}$. Then, $\bar{x}$ is a locally asymptotically stable equilibrium of the forced system (32).

Remark 5.1
Note that if Ξ is simply connected then $C_1, \cdots, C_m$ exist *globally*.

Remark 5.2
An equivalent way to analyse the stability of the equilibrium $\bar{x}$ of the forced system (32) by means of the Lyapunov function H_r is to look at the stability of the equilibrium $(\bar{x}, \bar{\zeta})$ with $\bar{\zeta}_j = C_j(\bar{x})$, $j \in \bar{m}$ of the embedding system (20) by means of a candidate Lyapunov function of the form

$$\tilde{H}(x,\zeta) \stackrel{\triangle}{=} H(x) - \bar{u}^\top\zeta + \Phi(\zeta_1 - C_1(x), \cdots, \zeta_m - C_m(x))$$

where the function Φ, depending on the Casimirs $\zeta_j - C_j(x)$, $j \in \bar{m}$, is still to be determined. This approach is similar to what is called the Energy–Casimir method in mechanics (see e.g. [9]). Note that restricted to any submanifold given by (27) the function $\tilde{H}(x,\zeta)$ reduces to the function $H_r(x)$.

6 Examples

6.1 Linear systems
If J, R and g are *constant* matrices, then also K is a constant matrix, and the existence of functions $C_1, \cdots, C_m$ satisfying (24) is automatic. (In fact $C_j(x)$ is given as the *linear* function $K_{1j}x_1 + \cdots + K_{nj}x_n$.) In particular, for linear systems Σ with

$$H(x) = \frac{1}{2}x^\top Qx, \quad Q = Q^\top$$

Theorem 5.1 results in a linear forced dynamics.

$$\dot{x} = (J - R)\frac{\partial H_r}{\partial x}(x)$$

with (since $K\bar{u} = Q\bar{x}$)

$$H_r(x) = \frac{1}{2}x^\top Qx - x^\top K\bar{u} + c = \frac{1}{2}(x - \bar{x}^\top)Q(x - \bar{x}) + c \quad (36)$$

Hence we have recovered in this special case the incremental Lyapunov function which is normally used. More importantly, we have given an *energy interpretation* for it!

6.2 Mechanical systems
Consider a mechanical system with damping $D(q) = D^\top(q) \ge 0$ and actuated by external forces u

$$\begin{bmatrix} \dot{q} \\ \dot{p} \end{bmatrix} = \begin{bmatrix} 0 & I_k \\ -I_k & D(q) \end{bmatrix} \begin{bmatrix} \frac{\partial H}{\partial q} \\ \frac{\partial H}{\partial p} \end{bmatrix} + \begin{bmatrix} 0 \\ B(q) \end{bmatrix} u \quad (37)$$

$$y = B^\top(q)\frac{\partial H}{\partial p}$$

with generalized configuration coordinates $q = [q_1, \cdots, q_k]^\top$, and generalized momenta $p = [p_1, \cdots, p_k]^\top$. The outputs $y \in \mathcal{R}^m$ are the generalized velocities corresponding to the generalized external forces $u \in \mathcal{R}^m$. Let $\bar{u}$ be a constant actuating force. It follows that

$$K(q,p) = -\begin{bmatrix} 0 & I_k \\ -I_k & -D(q) \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ B(q) \end{bmatrix} = \begin{bmatrix} B(q) \\ 0 \end{bmatrix} \quad (38)$$

and hence

$$J_s(q,p) = [B^\top(q), \ 0]\begin{bmatrix} 0 & I_k \\ -I_k & 0 \end{bmatrix}\begin{bmatrix} B(q) \\ 0 \end{bmatrix} = 0 \quad (39)$$

$$R_s(q,p) = [B^\top(q), \ 0]\begin{bmatrix} 0 & 0 \\ 0 & D(q) \end{bmatrix}\begin{bmatrix} B(q) \\ 0 \end{bmatrix} = 0 \quad (40)$$

Hence the embedding system (19) or (20) is given as in (13)
:

$$\begin{bmatrix} \dot{q} \\ \dot{p} \\ \dot{\zeta} \end{bmatrix} = \begin{bmatrix} 0 & I_k & 0 \\ -I_k & D(q) & -B(q) \\ 0 & B^\top(q) & 0 \end{bmatrix} \begin{bmatrix} \frac{\partial H_a}{\partial q} \\ \frac{\partial H_a}{\partial p} \\ \frac{\partial H_a}{\partial \zeta} \end{bmatrix} \quad (41)$$

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with $H_a(q,p,\zeta) = H(q,p) - \bar{u}^T\zeta$. Furthermore, the integrability conditions (24) boil down to the existence of functions $C_1, \cdots, C_m$ such that

$$B_{ij}(q) = \frac{\partial C_j}{\partial q_i}(q), \quad i,j \in \bar{m} \tag{42}$$

Condition (42) means that the input vector fields formed by the colums of $\begin{bmatrix} 0 \\ B(q) \end{bmatrix}$ are actually *Hamiltonian* vector fields with Hamiltonians $C_1, \cdots, C_m$.

Considering the output equation of the model and the dynamics, this simply means that the outputs are geometric functions of the generalized coordinates:

$$y_j = \frac{\partial C_j}{\partial q_i}(q)\frac{\partial H}{\partial p} = \frac{\partial C_j}{\partial q_i}\dot{q} = \frac{d}{dt}C_j$$

Assume moreover that the number of generalized forces is exactly the number of generalized coordinates: $m = k$. Then one may choose y as generalized coordinates and their conjugated momenta as generalized momenta and the dynamics has the form (37) with $B(q)$ the identity matrix. Assume moreover that the mechanical system is a simple mechanical system with energy:

$$H(q,p) = \frac{1}{2}p^T Y(q)p + V(q)$$

where $Y(q)$ is the mobility tensor of the system (which is positive definite) and $V(q)$ is the potential energy of the system. The stability of the equilibria ($\bar{p} = 0$ and $\frac{\partial V}{\partial q}(\bar{q}) = \bar{u}$) of the forced system comes then down to analyze the positive definitness of the function $V(q) - \bar{u}^T q$.

7 Conclusions

In this paper we have shown that, under certain integrability conditions on the input vector fields, we can construct a Lyapunov function candidate to study the stability of forced Hamiltonian systems with dissipation. The main feature of this construction is that the Lyapunov function is directly derived from the Hamiltonian function of the system and the energy associated with the source. An interpretation which is possible provided the system is embedded in a higher dimensional system where the infinite energy reservoir is connected in a particular way to the forced system.

In a subsequent paper [11] we show how this construction can be used to derive stabilizing controllers for a large class of physical systems including mechanical, electrical and electromechanical dynamics. In some cases, the resulting controllers are new, while in others we find again some well-known schemes that have been derived using a Lagrangian formalism. In all cases, however, the proposed formulation is quite systematic and provides deep physical insight into the obstacles for stabilization with energy-shaping methods, features whose importance can hardly be overestimated and which are conspicuous by their absence in Lagrangian derivations.

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